Variational symmetries and pluri-Lagrangian structures for integrable hierarchies of PDEs

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Abstract

We investigate the relation between pluri-Lagrangian hierarchies of 2-dimensional partial differential equations and their variational symmetries. The aim is to generalize to the case of partial differential equations the recent findings in Petrera and Suris (Nonlinear Math. Phys. 24(suppl. 1):121–145, 2017) for ordinary differential equations. We consider hierarchies of 2-dimensional Lagrangian PDEs (many of which have a natural $(1 + 1)$-dimensional space-time interpretation) and show that if the flow of each PDE is a variational symmetry of all others, then there exists a pluri-Lagrangian 2-form for the hierarchy. The corresponding multi-time Euler–Lagrange equations coincide with the original system supplied with commuting evolutionary flows induced by the variational symmetries.

Keywords Integrable PDEs · Variational principles · Variational symmetries

Mathematics Subject Classification 37K05 · 37K10

1 Introduction

In the last decade a variational perspective on integrable systems has emerged under the name of *pluri-Lagrangian* systems (or *Lagrangian multiform* systems). The the-

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ory was initiated in [13] in the discrete setting, more specifically in the context of multidimensionally consistent lattice equations on a quadrilateral stencil, called *quad equations*. Multidimensional consistency means that the equation can be imposed on all elementary squares in a higher-dimensional lattice without leading to contradictions. Analogously to commutativity of differential equations, multidimensional consistency is a key feature of integrability for difference equations.

In [13] it was shown that the property of multidimensional consistency can be combined with a variational formulation for quad equations. Solutions of integrable quad equations are critical points of an action functional obtained by integrating a suitable discrete Lagrangian 2-form over an arbitrary 2-dimensional surface in a higher-dimensional lattice. If the 2-dimensional surface is a plane, we recover a traditional variational principle for a 2-dimensional difference equation where the action is the sum over a plane of evaluations of the Lagrange function. The pluri-Lagrangian property requires the action to be critical also when this plane is replaced by any other 2-dimensional discrete surface in a higher-dimensional lattice. This remarkable property has been considered as a defining feature of integrability of 2-dimensional discrete equations [2,4,6,9,13–15,32] as well as in the 1-dimensional [5,7,33] and 3-dimensional [8,16] cases.

The pluri-Lagrangian property can also be formulated in the continuous case, where solutions of (hierarchies of) integrable 2-dimensional partial differential equations (PDEs) are critical points of an action functional obtained by integrating a differential 2-form over an arbitrary 2-dimensional surface in a higher-dimensional space. This variational principle has been proposed as a Lagrangian analogue of the existence of Poisson-commuting Hamilton functions [13,27,28,32]. As in the discrete case, it is not limited to Lagrangian 2-forms describing 2-dimensional PDEs. The corresponding variational principle where a Lagrangian 1-form is integrated over curves applies to integrable ordinary differential equations [22,26,33]. It is conjectured that also for $d > 2$ integrable hierarchies of d-dimensional integrable PDEs can be described by pluri-Lagrangian d-forms.

Thanks to these investigations a quite suggestive scenario has emerged: the pluri-Lagrangian structure is closely related (or even equivalent) to the integrability of the underlying system. This novel characterization of integrability applies to both ordinary differential (or difference) equations and partial differential (or difference) equations.

In the recent paper [22] a connection between the notions of pluri-Lagrangian structures and variational symmetries was proved in the context of classical mechanics. In particular, it was shown that the existence of commuting variational symmetries for a system of variational ordinary differential equations leads to a natural pluri-Lagrangian 1-form, whose multi-time Euler–Lagrange equations consist of the original system and commuting flows corresponding to the variational symmetries. These findings confirmed, in the framework of classical mechanics, that a pluri-Lagrangian structure is hidden behind the existence of a sufficient number of variational symmetries (i.e., of integrals of motion thanks to Noether theorem).

In the present work we extend the above idea to the case of variational 2-dimensional PDEs, thus generalizing the results of [22] to the context of Lagrangian field theory with two independent variables. We consider hierarchies of variational PDEs where the flow of each PDE is a variational symmetry of the Lagrange functions of

all other members of the hierarchy. Under this assumption, we show that there exists a pluri-Lagrangian 2-form for the hierarchy.

The paper is organized as follows. In Sect. 2 we give a short overview of Lagrangian field theory, recalling some classical notions and definitions. In particular we will provide a formulation of the celebrated Noether theorem, which establishes the relation between conservation laws and variational symmetries. In Sect. 3 we review the notion of continuous 2-dimensional pluri-Lagrangian systems. Section 4 is devoted to new results. It will be proved that from a family of variational symmetries one can construct a pluri-Lagrangian structure. The final Sect. 5 contains three examples which illustrate the theoretical results obtained in Sect. 4.

2 A short review of Lagrangian field theory

An exhaustive reference on classical Lagrangian field theory is the book of Olver [20]. The scope of the present section is to recall the main definitions and concepts needed for a self-contained presentation of our results in the next sections.

2.1 Euler–Lagrange equations

Since we will work in a multi-time setting we do not restrict our presentation here to fields depending on only two independent variables. Therefore we start by considering a smooth field $u : \mathbb{R}^N \to \mathbb{R}$ depending on N real independent variables $t_1, \ldots, t_N$.

We will use the multi-index notation for partial derivatives. For any multi-index $I = (i_1, \ldots, i_N) \in \mathbb{N}^N$ we set

$$u_I = \frac{\partial^{|I|} u}{(\partial t_1)^{i_1} \ldots (\partial t_N)^{i_N}},$$

where $|I| = i_1 + \cdots + i_N$ and $u = u(t_1, \ldots, t_N)$. The notations It_k and It_k^{α} will represent the multi-indices $(i_1, \ldots, i_k+1, \ldots, i_N)$ and $(i_1, \ldots, i_k+\alpha, \ldots, i_N)$ respectively. We will write $k \notin I$ if $i_k = 0$ and $k \in I$ if $i_k \neq 0$.

We will denote by D_i the total derivative with respect to the coordinate direction t_i,

$$\mathrm{D}_i = \sum_{I \in \mathbb{N}^N} u_{It_i} \frac{\partial}{\partial u_I}$$

and by $\mathrm{D}_I = \mathrm{D}_1^{i_1} \ldots \mathrm{D}_N^{i_N}$ the corresponding higher order derivatives.

The field u can be considered as a section of the trivial bundle $\mathbb{R}^N \times \mathbb{R}$. The partial derivatives of u of any order span the infinite jet bundle associated with $\mathbb{R}^N \times \mathbb{R}$. We will denote the fiber of the infinite jet bundle by $\mathcal{J}^{\infty}$ and the fiber coordinates by

$$[u] = (u, u_{t_i}, u_{t_i t_j}, \ldots)_{i, j, \ldots \in \{1, \ldots, N\}}.$$

A *variational problem* for a smooth field $u \colon \mathbb{R}^N \to \mathbb{R}$ is described by a *Lagrangian* $L \colon \mathcal{J}^\infty \to \mathbb{R}$ and consists in finding the critical points of the *action functional*

$$S = \int_\Gamma L[u] \, dt_1 \wedge \cdots \wedge dt_N,$$

where $\Gamma \subset \mathbb{R}^N$ is some bounded region. In other words, we look for fields u such that for all fields v such that v and its derivatives vanish at the boundary of Γ, there holds

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_\Gamma L[u + \varepsilon v] \, dt_1 \wedge \cdots \wedge dt_N = 0.$$

Concretely, we will be interested in variational problems for fields $u \colon \mathbb{R}^2 \to \mathbb{R}$. Therefore, let us fix $N = 2$ and write explicitly the variational equations governing the evolution of u. In this case the action functional over some bounded region $\Gamma \subset \mathbb{R}^2$ is

$$S = \int_\Gamma L[u] \, dt_1 \wedge dt_2. \tag{1}$$

The field u is a solution to the variational problem, i.e., a critical point for the action S, if and only if

$$\frac{\delta L}{\delta u} = \sum_{\alpha, \beta \geqslant 0} (-1)^{\alpha+\beta} \, D_1^\alpha \, D_2^\beta \left(\frac{\partial L}{\partial u_{t_1^\alpha t_2^\beta}} \right) = 0, \tag{2}$$

where the left-hand side is called the *variational derivative* of L. Equation (2) gives rise to a variational PDE, called *Euler–Lagrange equation*. Note that if the Lagrangian depends on the n-th order jet, i.e., on derivatives of u up to order n, then the Euler–Lagrange equation depends on the jet of order $2n$. If a given 2-dimensional PDE can be written as in equation (2) for some Lagrangian L, then we say that this PDE has a variational (or Lagrangian) structure.

Of course, the Euler–Lagrange equation (2) admits a straightforward generalization for the case of a field $u \colon \mathbb{R}^N \to \mathbb{R}$ for $N > 2$.

Example 2.1 The Korteweg–de Vries (KdV) equation

$$w_2 = w_{111} + 6 w w_1,$$

where w_i is shorthand notation for the derivative w_{t_i}, can be put into a variational form by introducing the potential $u = w_1$. The corresponding equation is

$$u_{12} = u_{1111} + 6 u_1 u_{11}.$$

Its variational structure comes from the Lagrangian

$$L[u] = \frac{1}{2} u_1 u_2 - u_1^3 - \frac{1}{2} u_1 u_{111}.$$

Indeed, critical points of the action (1) are characterized by the Euler–Lagrange equation

$$
\begin{aligned}
0 = \frac{\delta L}{\delta u} &= -D_1 \frac{\partial L}{\partial u_1} - D_1^3 \frac{\partial L}{\partial u_{111}} - D_2 \frac{\partial L}{\partial u_2} \\
&= \left(-\frac{1}{2} u_{12} + 6u_1 u_{11} + \frac{1}{2} u_{1111} \right) + \frac{1}{2} u_{1111} - \frac{1}{2} u_{12} \\
&= -u_{12} + u_{1111} + 6u_1 u_{11}.
\end{aligned}
$$

2.2 Variational symmetries and Noether's theorem

Let $N = 2$. A vertical generalized vector field on $\mathbb{R}^2 \times \mathbb{R}$ is a vector field of the form $Q\partial_u$, where $Q \colon \mathcal{J}^\infty \to \mathbb{R}$. It is called *vertical* because it does not contain any ∂_{t_i} and *generalized* because Q depends on derivatives of u, not just on u itself. The *prolongation* of $Q\partial_u$ is a vector field on $\mathcal{J}^\infty$ defined as

$$
\mathrm{pr}(Q\partial_u) = \sum_{I \in \mathbb{N}^2} (D_I Q) \frac{\partial}{\partial u_I}.
$$

A vector field $Q\partial_u$ is called a *variational symmetry* of a Lagrangian $L \colon \mathcal{J}^\infty \to \mathbb{R}$ if its prolongation $\mathrm{pr}(Q\partial_u)$ satisfies

$$
\mathrm{pr}(Q\partial_u)L = D_1 F_1 + D_2 F_2 \tag{3}
$$

for some functions $F_1, F_2 \colon \mathcal{J}^\infty \to \mathbb{R}$. The pair (F_1, F_2) is called the *flux* of the variational symmetry.

A *conservation law* for L is a triple of functions $J_1, J_2, Q \colon \mathcal{J}^\infty \to \mathbb{R}$ that satisfy

$$
D_1 J_1 + D_2 J_2 = -Q \frac{\delta L}{\delta u}. \tag{4}
$$

If equation (4) holds true, the pair $J = (J_1, J_2)$ is called the *conserved current* and Q the *characteristic* of the conservation law. On solutions of the Euler–Lagrange equations (2) the conserved current J is divergence-free, hence its name.

The famous Noether's theorem [18] establishes a one-to-one correspondence between conservation laws and variational symmetries.

Theorem 2.2 *Let $Q\partial_u$ be a variational symmetry of L. Then*

$$
J_1[u] = \sum_{I \not\ni t_2} \left((D_I Q) \frac{\delta L}{\delta u_{It_1}} \right) + \frac{1}{2} \sum_I D_2 \left((D_I Q) \frac{\delta L}{\delta u_{It_1 t_2}} \right) - F_1[u], \tag{5}
$$

$$
J_2[u] = \sum_{I \not\ni t_1} \left((D_I Q) \frac{\delta L}{\delta u_{It_2}} \right) + \frac{1}{2} \sum_I D_1 \left((D_I Q) \frac{\delta L}{\delta u_{It_1 t_2}} \right) - F_2[u], \tag{6}
$$

define the components of the conserved current of a conservation law, where the pair of functions (F_1, F_2) is the flux, as in equation (3).

Conversely, given a conserved current (J_1, J_2), equations (5) *and* (6) *define the flux (F_1, F_2) of a variational symmetry.*

Note that equations (5) and (6) contain variational derivatives with respect to partial derivatives of u:

$$\frac{\delta L}{\delta u_I} = \sum_{\alpha, \beta \geqslant 0} (-1)^{\alpha+\beta} \, \mathrm{D}_1^\alpha \, \mathrm{D}_2^\beta \, \frac{\partial L}{\partial u_{I t_1^\alpha t_2^\beta}}.$$

We also observe that J_1 and J_2 can be alternatively written as

$$J_1[u] = \sum_{\alpha \geqslant 0} \left((\mathrm{D}_1^\alpha Q) \, \frac{\delta L}{\delta u_{t_1^{\alpha+1}}} \right) + \frac{1}{2} \sum_{\alpha \geqslant 0} \sum_{\beta \geqslant 0} \mathrm{D}_2 \left((\mathrm{D}_1^\alpha \mathrm{D}_2^\beta Q) \, \frac{\delta L}{\delta u_{t_1^{\alpha+1} t_2^{\beta+1}}} \right) - F_1[u],$$

$$J_2[u] = \sum_{\beta \geqslant 0} \left((\mathrm{D}_2^\beta Q) \, \frac{\delta L}{\delta u_{t_2^{\beta+1}}} \right) + \frac{1}{2} \sum_{\alpha \geqslant 0} \sum_{\beta \geqslant 0} \mathrm{D}_1 \left((\mathrm{D}_1^\alpha \mathrm{D}_2^\beta Q) \, \frac{\delta L}{\delta u_{t_1^{\alpha+1} t_2^{\beta+1}}} \right) - F_2[u].$$

Proof of Theorem 2.2 The key point of the proof consists in the integration by parts of

$$\mathrm{pr}(Q \partial_u) L = \sum_I (\mathrm{D}_I Q) \, \frac{\partial L}{\partial u_I},$$

i.e., to write it in the form

$$\mathrm{pr}(Q \partial_u) L = Q \, \frac{\delta L}{\delta u} + \mathrm{D}_1(\cdots) + \mathrm{D}_2(\cdots).$$

To perform the full calculation, observe that

$$\frac{\partial L}{\partial u_I} = \frac{\delta L}{\delta u_I} + \mathrm{D}_1 \frac{\delta L}{\delta u_{I t_1}} + \mathrm{D}_2 \frac{\delta L}{\delta u_{I t_2}} + \mathrm{D}_1 \mathrm{D}_2 \frac{\delta L}{\delta u_{I t_1 t_2}},$$

hence

$$\mathrm{pr}(Q \partial_u) L = \sum_I (\mathrm{D}_I Q) \left(\frac{\delta L}{\delta u_I} + \mathrm{D}_1 \frac{\delta L}{\delta u_{I t_1}} + \mathrm{D}_2 \frac{\delta L}{\delta u_{I t_2}} + \mathrm{D}_1 \mathrm{D}_2 \frac{\delta L}{\delta u_{I t_1 t_2}} \right)$$

$$= \sum_I \left((\mathrm{D}_{I t_1 t_2} Q) + (\mathrm{D}_{I t_2} Q) \mathrm{D}_1 + (\mathrm{D}_{I t_1} Q) \mathrm{D}_2 + (\mathrm{D}_I Q) \mathrm{D}_1 \mathrm{D}_2 \right) \frac{\delta L}{\delta u_{I t_1 t_2}}$$

$$+ \sum_{I \not\ni t_2} \left((\mathrm{D}_{I t_1} Q) + (\mathrm{D}_I Q) \mathrm{D}_1 \right) \frac{\delta L}{\delta u_{I t_1}}$$

$$+ \sum_{I \not\ni t_1} \left((\mathrm{D}_{I t_2} Q) + (\mathrm{D}_I Q) \mathrm{D}_2 \right) \frac{\delta L}{\delta u_{I t_2}} + Q \, \frac{\delta L}{\delta u},$$

where the last term would be a sum over all $I \not\ni t_1, t_2$, but only the empty multi-index $I = (0, 0)$ satisfies this condition. The above equation can be simplified as

$$\text{pr}(Q\partial_u)L = \sum_I D_1 D_2 \left((D_I Q) \frac{\delta L}{\delta u_{It_1 t_2}} \right)$$

$$+ \sum_{I \not\ni t_2} D_1 \left((D_I Q) \frac{\delta L}{\delta u_{It_1}} \right) + \sum_{I \not\ni t_1} D_2 \left((D_I Q) \frac{\delta L}{\delta u_{It_2}} \right) + Q \frac{\delta L}{\delta u}$$

$$= D_1(J_1 + F_1) + D_2(J_2 + F_2) + Q \frac{\delta L}{\delta u}.$$

It follows that equations (3) and (4) are equivalent. Hence if $Q\partial_u$ is a variational symmetry, then equations (5)–(6) define a conserved current. □

Example 2.3 Consider again the KdV equation

$$u_{12} = u_{1111} + 6u_1 u_{11}$$

and its Lagrangian

$$L[u] = \frac{1}{2} u_1 u_2 - u_1^3 - \frac{1}{2} u_1 u_{111}.$$

As before, indices denote derivatives with respect to the corresponding time variables, e.g. $u_{12} = u_{t_1 t_2}$. We present two variational symmetries of this equation and their associated conservation laws:

(a) The generalized vector field $Q\partial_u$ with $Q[u] = u_1$ corresponds to a translation in the t_1-direction. Indeed,

$$\text{pr}(Q\partial_u)L = u_1 \frac{\partial L}{\partial u} + u_{11} \frac{\partial L}{\partial u_1} + u_{111} \frac{\partial L}{\partial u_{11}} + u_{1111} \frac{\partial L}{\partial u_{111}} + u_{12} \frac{\partial L}{\partial u_2} = D_1 L,$$

hence $Q\partial_u$ is a variational symmetry with flux

$$(F_1[u], F_2[u]) = (L[u], 0).$$

Corresponding to this variational symmetry we find the conservation law

$$- Q[u] \frac{\delta L}{\delta u} = - u_1(- u_{12} + 6u_1 u_{11} + u_{1111}) = D_1 J_1 + D_2 J_2,$$

with

$$J_1[u] = u_1 \frac{\delta L}{\delta u_1} + u_{11} \frac{\delta L}{\delta u_{11}} + u_{111} \frac{\delta L}{\delta u_{111}} - F_1[u] = - 2u_1^3 - u_1 u_{111} + \frac{1}{2} u_{11}^2,$$

$$J_2[u] = u_1 \frac{\delta L}{\delta u_2} - F_2[u] = \frac{1}{2} u_1^2.$$

This in turn implies the conservation of momentum:

$$D_2 \int \frac{1}{2} u_1^2 \, dt_1 = 0.$$

(b) The generalized vector field $Q\partial_u$ with

$$Q[u] = 10u_1^3 + 5u_{11}^2 + 10u_1 u_{111} + u_{11111}.$$

Indeed,

$$\begin{aligned}
&\mathrm{pr}(Q\partial_u)L \\
&= Q \frac{\partial L}{\partial u} + (D_1 Q) \frac{\partial L}{\partial u_1} + (D_1^2 Q) \frac{\partial L}{\partial u_{11}} + (D_1^3 Q) \frac{\partial L}{\partial u_{111}} + (D_2 Q) \frac{\partial L}{\partial u_2} \\
&= D_1 F_1 + D_2 F_2,
\end{aligned}$$

with

$$\begin{aligned}
F_1[u] ={}& -18u_1^5 - 15u_1^2 u_{11}^2 - 45u_1^3 u_{111} + 5u_1^3 u_2 + 4u_{11}^2 u_{111} - 18u_1 u_{111}^2 \\
& - 4u_1 u_{11} u_{1111} - 8u_1^2 u_{11111} - 10u_1 u_{11} u_{12} + \frac{5}{2} u_{11}^2 u_2 + 5u_1 u_{111} u_2 \\
& + \frac{1}{2} u_{1111}^2 - u_{111} u_{11111} + \frac{1}{2} u_{11} u_{111111} - \frac{1}{2} u_1 u_{1111111} + u_{111} u_{112} \\
& - u_{1111} u_{12} + \frac{1}{2} u_{11111} u_2, \\
F_2[u] ={}& \frac{5}{2} u_1^4 + \frac{15}{2} u_1 u_{11}^2 + 5u_1^2 u_{111} - \frac{1}{2} u_{111}^2 + \frac{1}{2} u_1 u_{11111}.
\end{aligned}$$

The corresponding conservation law is

$$-Q[u] \frac{\delta L}{\delta u} = D_1 J_1 + D_2 J_2,$$

with

$$\begin{aligned}
J_1[u] ={}& Q \frac{\delta L}{\delta u_1} + (D_1 Q) \frac{\delta L}{\delta u_{11}} + (D_{11} Q) \frac{\delta L}{\delta u_{111}} - F_1 \\
={}& -12u_1^5 - 15u_1^2 u_{11}^2 - 10u_1^3 u_{111} + u_{11}^2 u_{111} - 2u_1 u_{111}^2 - 6u_1 u_{11} u_{1111} \\
& + 10u_1 u_{11} u_{12} - \frac{1}{2} u_{1111}^2 - u_{111} u_{112} + u_{1111} u_{12}
\end{aligned}$$

and

$$J_2[u] = Q \frac{\delta L}{\delta u_2} - F_2[u] = \frac{5}{2} u_1^4 - 5u_1 u_{11}^2 + \frac{1}{2} u_{111}^2.$$

3 Pluri-Lagrangian field theory

In this section we briefly review the main concepts of pluri-Lagrangian field theory. For further details see [26–28].

3.1 Integrable hierarchies of PDEs

One of the defining features of an integrable PDE is that it possesses an infinite amount of symmetries and, correspondingly, an infinite amount of conservation laws. These symmetries define a family of PDEs that commute with the original one.

Let us illustrate the concept of commuting PDEs on the basis of our leading example.

Example 3.1 In Example 2.3 (b) we proved that the generalized vector field $Q\partial_u$, with

$$Q[u] = 10u_1^3 + 5u_{11}^2 + 10u_1u_{111} + u_{11111},$$

is a variational symmetry of the KdV equation

$$u_{12} = u_{1111} + 6u_1u_{11} = 0.$$

If we introduce a third independent variable t_3, we can define the PDE

$$u_3 = 10u_1^3 + 5u_{11}^2 + 10u_1u_{111} + u_{11111},$$

which commutes with the KdV equation itself. This means that both ways of calculating the mixed derivative u_{123} agree on solutions:

$$
\begin{aligned}
D_3u_{12} &= D_3(u_{1111} + 6u_1u_{11}) \\
&= 540u_1^2u_{11}^2 + 180u_1^3u_{111} + 480u_{11}^2u_{111} + 300u_1u_{111}^2 + 480u_1u_{11}u_{1111} \\
&\quad + 90u_1^2u_{11111} + 70u_{1111}^2 + 110u_{111}u_{11111} + 56u_{11}u_{111111} \\
&\quad + 16u_1u_{1111111} + u_{111111111} \\
&= D_1D_2\big(10u_1^3 + 5u_{11}^2 + 10u_1u_{111} + u_{11111}\big) \\
&= D_1D_2u_3.
\end{aligned}
$$

Since symmetries lead to commuting equations, a natural perspective on an integrable PDE is to consider it as one equation belonging to an infinite *integrable hierarchy*, i.e., an infinite set of integrable PDEs such that any two systems in this set are compatible. Such hierarchies are usually generated by recursion operators or master symmetries [10,17,20].

3.2 Pluri-Lagrangian problems

Let us focus on $(1 + 1)$-dimensional PDEs. A finite number of equations from a hierarchy can be embedded in a higher-dimensional multi-time, where they share a common

space direction, say $t_1 = x$, but each equation has its own time coordinate, $t_2, t_3, \ldots$ Formally, we can embed the whole hierarchy into an infinite-dimensional space in the same way.

In the classical variational description of $(1 + 1)$-dimensional PDEs, we integrate a Lagrange function over (an open subset of) the 2-dimensional space-time. A variational structure of a hierarchy of such PDEs should include the classical variational description of each individual equation, i.e., integration over a 2-dimensional subspace. Therefore, it is natural for the role of a Lagrange function to be played by a differential 2-form.

Let $\mathcal{L} \in \Omega^2(\mathbb{R}^N)$ be a 2-form depending on the infinite jet of a smooth field $u \colon \mathbb{R}^N \to \mathbb{R}$, i.e.,

$$\mathcal{L}[u] = \sum_{i<j} L_{ij}[u] \, dt_i \wedge dt_j, \tag{7}$$

with $L_{ij} \colon \mathcal{J}^\infty \to \mathbb{R}$. We say that u solves the *pluri-Lagrangian problem* for $\mathcal{L}$ if for any 2-dimensional submanifold $\Gamma \subset \mathbb{R}^N$ and for any infinitesimal variation $v(t_1, \ldots, t_N)\partial_u$ of u, where $v \colon \mathbb{R}^N \to \mathbb{R}$ and all its derivatives vanish at the boundary of Γ, we have

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_\Gamma \mathcal{L}[u + \varepsilon v] = 0.$$

This can also be written as

$$\int_\Gamma \mathrm{pr}(v\partial_u)\mathcal{L}[u] = 0,$$

where the vertical vector field $\mathrm{pr}(v\partial_u) = \sum_I v_I \frac{\partial}{\partial u_I}$ acts on the coefficients of $\mathcal{L}[u]$, i.e.,

$$\mathrm{pr}(v\partial_u)\mathcal{L}[u] = \sum_{i<j} \sum_I v_I \frac{\partial L_{ij}[u]}{\partial u_I} \, dt_i \wedge dt_j.$$

The equations that characterize solutions to the pluri-Lagrangian problem are called *multi-time Euler–Lagrange equations*. They were derived in [27] and state that, for all $i, j, k \in \{1, \ldots, N\}$, there holds:

$$\text{for all } I \not\ni t_i, t_j : \quad \frac{\delta_{ij} L_{ij}}{\delta u_I} = 0, \tag{8}$$

$$\text{for all } I \not\ni t_i : \quad \frac{\delta_{ij} L_{ij}}{\delta u_{It_j}} = \frac{\delta_{ik} L_{ik}}{\delta u_{It_k}}, \tag{9}$$

$$\text{for all } I : \quad \frac{\delta_{ij} L_{ij}}{\delta u_{It_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta u_{It_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta u_{It_k t_i}} = 0, \tag{10}$$

where

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = \sum_{\alpha, \beta \geqslant 0} (-1)^{\alpha+\beta} \, \mathrm{D}_i^\alpha \, \mathrm{D}_j^\beta \left(\frac{\partial L_{ij}}{\partial u_{I i^\alpha j^\beta}} \right) \tag{11}$$

is the variational derivative in the (t_i, t_j)-plane. Note that the multi-time Euler–Lagrange equations contain the classical Euler–Lagrange equations in each (t_i, t_j)-plane (8), where derivatives with respect to other times are considered as additional components of the field, plus additional equations (9)–(10) coming from choices of Γ that are not coordinate planes.

In the present work, we will use a different property to recognize solutions to the pluri-Lagrangian problem. There is a remarkable relation between the pluri-Lagrangian problem and the property that the 2-form $\mathcal{L}$ is closed on solutions u to the hierarchy. In fact, this closedness property is often considered to be the fundamental property of the Lagrangian theory of integrable hierarchies [2,13–16,32,33]. When this point of view is taken, the term "Lagrangian multiform" is more commonly used than "pluri-Lagrangian".

Here, we show that a slightly weaker property of the 2-form is a sufficient condition for a solution to the pluri-Lagrangian problem.

Theorem 3.2 *Consider a 2-form $\mathcal{L}$ and a hierarchy of commuting PDEs*

$$u_i = Q_i[u], \quad i = 2, \ldots, N, \tag{12}$$

with $Q_i \colon \mathcal{J}^\infty \to \mathbb{R}$. If the exterior derivative of $\mathcal{L}$ is constant up to a term that attains a double zero on solutions of (12)*, i.e., if*

$$\mathrm{d}\mathcal{L} = \gamma + \sum_{I, J} \sum_{i, j} \omega_{i,j}^{I,J} \, \mathrm{D}_I(u_i - Q_i) \, \mathrm{D}_J(u_j - Q_j)$$

for some $\mathcal{J}^\infty$-dependent 3-forms $\omega_{i,j}^{I,J}$ and a 3-form γ that does not depend on u or its derivatives, then all solutions $u \colon \mathbb{R}^N \to \mathbb{R}$ to the hierarchy (12) *also solve the pluri-Lagrangian problem for $\mathcal{L}$.*

Strictly speaking, the assumption that the PDEs (12) commute can be dropped from this theorem. If they do not commute then there will usually be no non-trivial solutions $u \colon \mathbb{R}^N \to \mathbb{R}$ to all PDEs simultaneously, so in this case the theorem would be of very limited relevance.

Proof of Theorem 3.2 Let u be a solution to the hierarchy and $\Gamma = \partial B$ a surface defined as the boundary of a 3-manifold B. It is sufficient to show that the pluri-Lagrangian property holds on such surfaces. Indeed, without loss of generality we can require variations to be supported on small open subsets and for any sufficiently small open subset Γ' of a given surface, one can find a 3-manifold such that Γ' is contained in its boundary.

As a consequence of the assumption on $\mathcal{L}$ there holds for any variation $v \colon \mathbb{R}^N \to \mathbb{R}$ that

$$\mathrm{pr}\,(v\partial_u)\,\mathrm{d}\mathcal{L}[u] = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \mathrm{d}\mathcal{L}[u + \varepsilon v] = 0.$$

Therefore

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \int_\Gamma \mathcal{L}[u + \varepsilon v] = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \int_B \mathrm{d}\mathcal{L}[u + \varepsilon v] = 0,$$

hence the action integral over any surface Γ is critical with respect to variations of u. $\qquad\square$

There are strong indications that the existence of a pluri-Lagrangian structure is deeply connected to integrability. One such indication comes from within the theory: the multi-time Euler–Lagrange equations are highly overdetermined. Hence if nontrivial solutions exist, then we are dealing with a system with remarkable properties. Other indications are connections to different notions of integrability, including Hamiltonian formulations [26,28] and Lax pairs [24], even though these connections have not yet been studied in full detail.

Despite some recent discoveries, relatively few examples of pluri-Lagrangian hierarchies of PDEs are known. To our knowledge, the list is limited to the potential KdV equation [27] and several related hierarchies obtained as continuum limits from lattice equations [29,30], as well as (a matrix-valued generalization of) the AKNS system [24]. The goal of this paper is to establish a construction of a pluri-Lagrangian 2-form for a given hierarchy of $(1 + 1)$-dimensional PDEs, assuming we know classical Lagrange functions for the individual equations. Furthermore, we will assume that the vector field associated to each of the PDEs is a variational symmetry for the Lagrangians of the rest of the hierarchy. This assumption can be thought of as the Lagrangian analogue to commuting Hamiltonian flows.

4 From variational symmetries to a pluri-Lagrangian 2-form

We will take $t_1 = x$ to be the space coordinate. Then we can take the coefficients L_{1j} of the pluri-Lagrangian 2-form (7) to be classical Lagrangians for the individual equations of the hierarchy. However, the coefficients L_{ij} with $i, j > 1$ do not have an interpretation in a classical variational principle. It is not obvious under which conditions suitable L_{ij} exist, such that the given hierarchy solves the pluri-Lagrangian problem for the 2-form. Below we will give an answer to this question for a large class of Lagrangians.

For a hierarchy of evolutionary equations,

$$u_i = Q_i(u_1, u_{11}, \ldots), \quad i = 2, \ldots, N, \tag{13}$$

it is a reasonable assumption that the corresponding Lagrangians do not contain second or higher derivatives with respect to the time variable. Similarly, we will assume that the Lagrangian does not contain products of time-derivatives. Suppose we have a family of Lagrangians L_{1i} for $i = 2, \ldots, N$ satisfying these assumptions:

$$L_{1i}[u] = p(u, u_1, u_{11}, \ldots) u_i - h_i(u, u_1, u_{11}, \ldots). \tag{14}$$

Here p and h are two arbitrary functions of their arguments. In particular the term $p(u, u_1, u_{11}, \ldots) u_i$ plays the role of a kinetic energy. Note that we are not including mixed derivatives, $u_{1i}, u_{11i}, \ldots$ This does not restrict generality, because if a Lagrangian depends linearly on such derivatives, then we can integrate by parts to get an equivalent Lagrangian of the form (14). Furthermore, note that the factor $p(u, u_1, u_{11}, \ldots)$ in the kinetic term of $L_{1i}[u]$ is the same for all i. This is a direct consequence of the multi-time Euler–Lagrange equations of type (9).

The Euler–Lagrange equations (2) of the Lagrangians (14) will not be evolutionary. Instead we assume that the Euler–Lagrange equations are differential consequences of the hierarchy (13), i.e., equations of the form

$$\mathcal{E}_p(u_i - Q_i(u_1, u_{11}, \ldots)) = 0,$$

where $\mathcal{E}_p$ is some differential operator, depending on the kinetic term of the Lagrangians. In the case of the KdV hierarchy we have $\mathcal{E}_p = D_1$, see Example 2.1.

Assume that the prolonged vector fields $\mathfrak{D}_i = \text{pr}(Q_i \partial_u)$, corresponding to the equations of the hierarchy, commute pairwise and are variational symmetries of the L_{1j}:

$$\mathfrak{D}_i L_{1j} = D_1 A_{ij} + D_j B_{ij} \tag{15}$$

for some functions $A_{ij}, B_{ij} : \mathcal{J}^\infty \to \mathbb{R}$. If we consider only those terms that contain a t_j-derivative, what remains of equation (15) is of the form

$$\mathfrak{D}_i(pu_j) = D_1 \overline{A}_{ij}(u, u_1, u_j, \ldots) + D_j B_{ij}(u, u_1, u_{11}, \ldots)$$

for some function $\overline{A}_{ij} : \mathcal{J}^\infty \to \mathbb{R}$. This is an algebraic identity (as opposed to an equality on solutions), hence we can replace t_j-derivatives by new dependent variables, e.g. u_j by a field denoted by u_t. We find

$$\mathfrak{D}_i(pu_t) = D_1 \overline{A}_{ij}(u, u_1, u_t, \ldots) + D_k B_{ij}(u, u_1, u_{11}, \ldots).$$

Since the left-hand side of this equation is independent of j, we can choose $\overline{A}_{ij}$ and B_{ij} independent of j as well. In particular, we can write $B_{ij} = B_i$ and get

$$\mathfrak{D}_i L_{1j} = D_1 A_{ij} + D_j B_i.$$

Note that $A_{ij}, B_i : \mathcal{J}^\infty \to \mathbb{R}$ are only defined up to a constant, hence we can choose them to be zero on the zero field: $A_{ij}[0] = B_i[0] = 0$.

Lemma 4.1 *For Lagrangians of the form* (14) *with commuting variational symmetries* (15)*, there exist functions* $F_{ij} \colon \mathfrak{J}^\infty \to \mathbb{R} \colon [u] \mapsto F_{ij}(u, u_1, u_{11}, \ldots)$*, that do not depend on any time-derivatives, such that*

$$\mathrm{D}_1 F_{ij} = \mathrm{D}_i L_{1j} - \mathrm{D}_j L_{1i}$$

on solutions of the hierarchy (13)*.*

Proof Since the variational symmetries $\mathfrak{D}_i = \mathrm{pr}(Q_i \partial_u)$ commute, we have for any $k \neq i, j$,

$$
\begin{aligned}
0 &= [\mathfrak{D}_i, \mathfrak{D}_j] L_{1k} \\
&= \mathrm{D}_1(\mathfrak{D}_i A_{jk} - \mathfrak{D}_j A_{ik}) + \mathrm{D}_k(\mathfrak{D}_i B_j - \mathfrak{D}_j B_i).
\end{aligned}
$$

Now let u be an arbitrary compactly supported smooth field. Then

$$
\begin{aligned}
0 &= \int_{-\infty}^{\infty} \mathrm{D}_1(\mathfrak{D}_i A_{jk} - \mathfrak{D}_j A_{ik}) + \mathrm{D}_k(\mathfrak{D}_i B_j - \mathfrak{D}_j B_i) \, \mathrm{d}t_1 \\
&= \int_{-\infty}^{\infty} \mathrm{D}_k(\mathfrak{D}_i B_j - \mathfrak{D}_j B_i) \, \mathrm{d}t_1 \\
&= \mathrm{D}_k \int_{-\infty}^{\infty} (\mathfrak{D}_i B_j - \mathfrak{D}_j B_i) \, \mathrm{d}t_1.
\end{aligned}
$$

Since u and in particular its t_k-derivatives are arbitrary, it follows that $\mathfrak{D}_i B_j - \mathfrak{D}_j B_i$ is a null Lagrangian. This implies (see e.g. [20, Theorem 4.7]) that there exists a function $G_{ij} \colon \mathfrak{J}^\infty \to \mathbb{R}$ such that

$$\mathfrak{D}_i B_j - \mathfrak{D}_j B_i = \mathrm{D}_1(G_{ij}).$$

Hence with $F_{ij} = G_{ij} + A_{ij} - A_{ji}$ we find that, on solutions of the hierarchy (13),

$$
\begin{aligned}
\mathfrak{D}_i L_{1j} - \mathfrak{D}_j L_{1i} &= \mathrm{D}_1 A_{ij} + \mathrm{D}_j B_i - \mathrm{D}_1 A_{ji} - \mathrm{D}_i B_j \\
&= \mathrm{D}_1 A_{ij} + \mathfrak{D}_j B_i - \mathrm{D}_1 A_{ji} - \mathfrak{D}_i B_j \\
&= \mathrm{D}_1 F_{ij}.
\end{aligned}
$$

Since we are working on solutions of the equations the hierarchy, we can use those equations to eliminate time-derivatives from F_{ij}, hence we can assume it depends on the jet as $F_{ij}(u, u_1, u_{11}, \ldots)$. $\qquad \square$

We now present our main result, which is the analogue in 2-dimensional field theory of [22, Theorem 10].

Theorem 4.2 *Assume we have Lagrangians of the form* (14) *with commuting variational symmetries* (15). *Let*

$$
L_{ij}[u] = \sum_{\alpha \geqslant 0} \frac{\delta_{1j} L_{1j}}{\delta u_{t_1^{\alpha+1}}} \, \mathrm{D}_1^\alpha (u_i - Q_i) - \sum_{\alpha \geqslant 0} \frac{\delta_{1i} L_{1i}}{\delta u_{t_1^{\alpha+1}}} \, \mathrm{D}_1^\alpha (u_j - Q_j)
$$
$$
+ F_{ij}(u, u_1, u_{11}, \ldots), \tag{16}
$$

where $F_{ij} \colon \mathcal{J}^\infty \to \mathbb{R}$ *is as in Lemma* 4.1 *and the operator* $\frac{\delta_{ij}}{\delta}$ *is the variational derivative from equation* (11). *Then every solution of the hierarchy* (13) *is a critical point of*

$$
\mathcal{L}[u] = \sum_{i<j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j
$$

in the pluri-Lagrangian sense.

Proof We show that $\mathcal{L}$ is almost-closed in the sense of Theorem 3.2. We start by calculating $\mathrm{D}_1 L_{ij}$. We have

$$
\mathrm{D}_1 \left(\sum_{\alpha \geqslant 0} \frac{\delta_{1j} L_{1j}}{\delta u_{t_1^{\alpha+1}}} \, \mathrm{D}_1^\alpha (u_i - Q_i) \right)
$$
$$
= \sum_{\alpha \geqslant 0} \mathrm{D}_1 \frac{\delta_{1j} L_{1j}}{\delta u_{t_1^{\alpha+1}}} \, \mathrm{D}_1^\alpha (u_i - Q_i) + \sum_{\alpha \geqslant 0} \frac{\delta_{1j} L_{1j}}{\delta u_{t_1^{\alpha+1}}} \, \mathrm{D}_1^{\alpha+1} (u_i - Q_i)
$$
$$
= \sum_{\alpha \geqslant 0} \left(\mathrm{D}_1 \frac{\delta_{1j} L_{1j}}{\delta u_{t_1^{\alpha+1}}} + \frac{\delta_{1j} L_{1j}}{\delta u_{t_1^{\alpha}}} \right) \mathrm{D}_1^\alpha (u_i - Q_i) - \frac{\delta_{1j} L_{1j}}{\delta u} \, \mathrm{D}_1 (u_i - Q_i)
$$
$$
= \sum_{\alpha \geqslant 0} \left(\frac{\partial L_{1j}}{\partial u_{t_1^\alpha}} - \mathrm{D}_j \frac{\delta_{1j} L_{1j}}{\delta u_{t_1^\alpha t_j}} - \mathrm{D}_1 \mathrm{D}_j \frac{\delta_{1j} L_{1j}}{\delta u_{t_1^{\alpha+1} t_j}} \right) \mathrm{D}_1^\alpha (u_i - Q_i)
$$
$$
- \frac{\delta_{1j} L_{1j}}{\delta u} \, \mathrm{D}_1 (u_i - Q_i).
$$

Since L_{1j} does not depend on any mixed derivatives $u_{t_1^{\alpha+1} t_j}$, this simplifies to

$$
\mathrm{D}_1 \left(\sum_{\alpha \geqslant 0} \frac{\delta_{1j} L_{1j}}{\delta u_{t_1^{\alpha+1}}} \, \mathrm{D}_1^\alpha (u_i - Q_i) \right)
$$
$$
= \sum_{\alpha \geqslant 0} \frac{\partial L_{1j}}{\partial u_{t_1^\alpha}} \, \mathrm{D}_1^\alpha (u_i - Q_i) - \mathrm{D}_j \frac{\delta_{1j} L_{1j}}{\delta u_{t_j}} \, (u_i - Q_i) - \frac{\delta_{1j} L_{1j}}{\delta u} \, \mathrm{D}_1 (u_i - Q_i)
$$
$$
\equiv \sum_{\alpha \geqslant 0} \frac{\partial L_{1j}}{\partial u_{t_1^\alpha}} \, \mathrm{D}_1^\alpha (u_i - Q_i) - (\mathrm{D}_j p)(u_i - Q_i),
$$

where $\equiv$ denotes equality modulo double zeros. Similarly, there holds

$$D_1 \left(\sum_{\alpha \geqslant 0} \frac{\delta_{1i} L_{1i}}{\delta u_{t_1^{\alpha+1}}} D_1^\alpha (u_j - Q_j) \right) \equiv \sum_{\alpha \geqslant 0} \frac{\partial L_{1i}}{\partial u_{t_1^\alpha}} D_1^\alpha (u_j - Q_j) - (D_i p)(u_j - Q_j).$$

Hence

$$D_1 L_{ij} \equiv \sum_{\alpha \geqslant 0} \frac{\partial L_{1j}}{\partial u_{t_1^\alpha}} D_1^\alpha (u_i - Q_i) - \sum_{\alpha \geqslant 0} \frac{\partial L_{1i}}{\partial u_{t_1^\alpha}} D_1^\alpha (u_j - Q_j)$$
$$- (D_j p)(u_i - Q_i) + (D_i p)(u_j - Q_j) + D_1 F_{ij}.$$

Using the assumption that the Lagrangians L_{1i} and L_{1j} are of the form (14), we can write

$$D_i L_{1j} - \mathfrak{D}_i L_{1j} = p D_j (u_i - Q_i) + \sum_{\alpha \geqslant 0} \frac{\partial L_{1j}}{\partial u_{t_1^\alpha}} D_1^\alpha (u_i - Q_i),$$

$$D_j L_{1i} - \mathfrak{D}_j L_{1i} = p D_i (u_j - Q_j) + \sum_{\alpha \geqslant 0} \frac{\partial L_{1i}}{\partial u_{t_1^\alpha}} D_1^\alpha (u_j - Q_j),$$

where $\mathfrak{D}_i = \mathrm{pr}(Q_i \partial_u)$ and $\mathfrak{D}_j = \mathrm{pr}(Q_j \partial_u)$. Hence

$$D_1 L_{ij} - D_i L_{1j} + D_j L_{1i} \equiv -\mathfrak{D}_i L_{1j} + \mathfrak{D}_j L_{1i} - (D_j p)(u_i - Q_i)$$
$$- p D_j (u_i - Q_i) + (D_i p)(u_j - Q_j) \quad (17)$$
$$+ p D_i (u_j - Q_j) + D_1 F_{ij}.$$

By definition of F_{ij} we have that $D_1 F_{ij} - \mathfrak{D}_i L_{1j} + \mathfrak{D}_j L_{1i} = 0$ on solutions of (13). Furthermore, the only time derivatives in this expression come from the kinetic parts $p u_i$ and $p u_j$ of the Lagrangians. Therefore,

$$D_1 F_{ij} - \mathfrak{D}_i L_{1j} + \mathfrak{D}_j L_{1i}$$
$$= -\mathfrak{D}_i (p u_j - p Q_j) + \mathfrak{D}_j (p u_i - p Q_i)$$
$$= -p \mathfrak{D}_i (u_j - Q_j) - (\mathfrak{D}_i p)(u_j - Q_j)$$
$$+ p \mathfrak{D}_j (u_i - Q_i) + (\mathfrak{D}_j p)(u_i - Q_i) \quad (18)$$
$$\equiv -p D_i (u_j - Q_j) - (D_i p)(u_j - Q_j)$$
$$+ p D_j (u_i - Q_i) + (D_j p)(u_i - Q_i).$$

Combining equations (17) and (18) gives

$$D_1 L_{ij} - D_i L_{1j} + D_j L_{1i} \equiv 0. \quad (19)$$

Consider three copies of equation (19), each with an additional differentiation:

$$D_k(D_1 L_{ij} - D_i L_{1j} + D_j L_{1i}) \equiv 0,$$
$$D_j(D_1 L_{ik} - D_i L_{1k} + D_k L_{1i}) \equiv 0,$$
$$D_i(D_1 L_{jk} - D_j L_{1k} + D_k L_{1j}) \equiv 0.$$

A linear combination of these three equations gives us

$$D_1(D_k L_{ij} - D_j L_{ij} + D_i L_{jk}) \equiv 0.$$

Since all coefficients are autonomous, this implies that

$$D_k L_{ij} - D_j L_{ij} + D_i L_{jk} \equiv \text{const.} \tag{20}$$

Equations (19) and (20) together imply that $\mathcal{L}$ fulfills the conditions of Theorem 3.2, hence every solution of the hierarchy (13) is a critical point of the pluri-Lagrangian problem for $\mathcal{L}$. □

Theorem 4.2 and its proof are formulated for scalar systems, but they can be extended to the case of multi-component systems. If $u = (u^1, \ldots, u^\ell)$ satisfies the equations $u_i^k = Q_i^k$, we construct the Lagrangian coefficients by

$$L_{ij}[u] = \sum_{k=1}^{\ell} \sum_{\alpha \geqslant 0} \frac{\delta_{1j} L_{1j}}{\delta u_{t_1^{\alpha+1}}^k} D_1^\alpha(u_i^k - Q_i^k) - \sum_{k=1}^{\ell} \sum_{\alpha \geqslant 0} \frac{\delta_{1i} L_{1i}}{\delta u_{t_1^{\alpha+1}}^k} D_1^\alpha(u_j^k - Q_j^k)$$
$$+ F_{ij}(u, u_1, u_{11}, \ldots).$$

5 Examples

In this last section we discuss three examples. For the first one, the potential Korteweg–de Vries hierarchy, a pluri-Lagrangian structure is known in the literature [28]. Our discussion illustrates that this structure can be obtained using Theorem 4.2. The second example is the Nonlinear Schrödinger (NLS) hierarchy. Its pluri-Lagrangian structure can be considered as a special case of the one for the AKNS hierarchy obtained in [24]. The final example is the system consisting of the sine-Gordon and modified KdV equations, which indicates that the construction of Theorem 4.2 can be adapted to non-evolutionary equations.

The calculations in this section were performed in the SageMath software system [23]. The code is available at [31].

5.1 Potential KdV hierarchy

We start with our running example of the Korteweg–de Vries equation. The potential KdV hierarchy was the first complete hierarchy of PDEs for which a pluri-Lagrangian

structure was found [27]. Here we show that this structure can also be derived using Theorem 4.2. We present only a minimal example consisting of just the first two equations in the hierarchy,

$$u_2 = 3u_1^2 + u_{111}, \tag{21}$$

$$u_3 = 10u_1^3 + 5u_{11}^2 + 10u_1u_{111} + u_{11111}. \tag{22}$$

The corresponding Lagrangians are

$$L_{12}[u] = \frac{1}{2}u_1u_2 - u_1^3 - \frac{1}{2}u_1u_{111},$$

$$L_{13}[u] = \frac{1}{2}u_1u_3 - \frac{5}{2}u_1^4 + 5u_1u_{11}^2 - \frac{1}{2}u_{111}^2,$$

and have as their Euler–Lagrange equations

$$D_1(u_2 - (3u_1^2 + u_{111})) = 0, \tag{23}$$

$$D_1(u_3 - (10u_1^3 + 5u_{11}^2 + 10u_1u_{111} + u_{11111})) = 0. \tag{24}$$

On solutions of the evolutionary equations, there holds

$$
\begin{aligned}
D_2L_{13} - D_3L_{12} = {} & -10u_1^3u_{12} + 10u_1u_{11}u_{112} + 5u_{11}^2u_{12} + 3u_1^2u_{13} - uu_{111}u_{1112} \\
& + \frac{1}{2}u_1u_{1113} + \frac{1}{2}u_{111}u_{13} - \frac{1}{2}u_{13}u_2 + \frac{1}{2}u_{12}u_3 \\
= {} & 15u_1^4u_{11} + 135u_1u_{11}^3 + 210u_1^2u_{11}u_{111} + 25u_1^3u_{1111} \\
& - 18u_{11}u_{111}^2 + \frac{15}{2}u_{11}^2u_{1111} + 34u_1u_{111}u_{1111} \\
& + 33u_1u_{11}u_{11111} + \frac{13}{2}u_1^2u_{111111} + \frac{1}{2}u_{1111}u_{11111} \\
& - u_{111}u_{111111} + \frac{1}{2}u_1u_{11111111}.
\end{aligned}
$$

Integrating this gives us

$$
\begin{aligned}
F_{23}(u, u_1, u_{11}, \ldots) = {} & 3u_1^5 + \frac{135}{2}u_1^2u_{11}^2 + 25u_1^3u_{111} - \frac{25}{2}u_{11}^2u_{111} + 7u_1u_{111}^2 \\
& + 20u_1u_{11}u_{1111} + \frac{13}{2}u_1^2u_{11111} + \frac{1}{2}u_{1111}^2 - \frac{1}{2}u_{111}u_{11111} \\
& - \frac{1}{2}u_{11}u_{111111} + \frac{1}{2}u_1u_{1111111}.
\end{aligned}
$$

Let Q_2 and Q_3 be the right-hand sides of equations (21) and (22) . Then the remaining terms in equation (16) are

$$
\frac{\delta_{13}L_{13}}{\delta u_1}(u_2 - Q_2) = \left(\frac{1}{2}u_3 - 10u_1^3 - 5u_{11}^2 - 10u_1u_{111} - u_{11111}\right)
$$
$$
\cdot (u_2 - 3u_1^2 - u_{111}),
$$
$$
\frac{\delta_{13}L_{13}}{\delta u_{11}}D_1(u_2 - Q_2) = (10u_1u_{11} - u_{1111})(u_{12} - 6u_1u_{11} - u_{1111}),
$$
$$
\frac{\delta_{13}L_{13}}{\delta u_{111}}D_{11}(u_2 - Q_2) = u_{111}(u_{112} - 6u_1u_{111} - 6u_{11}^2 - u_{11111}),
$$

and

$$
-\frac{\delta_{12}L_{12}}{\delta u_1}(u_3 - Q_3) = -\left(\frac{1}{2}u_2 - 3u_1^2 - u_{111}\right)
$$
$$
\cdot (u_3 - 10u_1^3 - 5u_{11}^2 - 10u_1u_{111} - u_{11111}),
$$
$$
-\frac{\delta_{12}L_{12}}{\delta u_{11}}D_1(u_3 - Q_3) = -\frac{1}{2}u_{11}(u_{13} - 30u_1^2u_{11} - 20u_{11}u_{111}
$$
$$
- 10u_1u_{1111} - u_{111111}),
$$
$$
-\frac{\delta_{12}L_{12}}{\delta u_{111}}D_{11}(u_3 - Q_3) = \frac{1}{2}u_1(u_{113} - 60u_1u_{11}^2 - 30u_1^2u_{111} - 20u_{111}^2
$$
$$
- 30u_{11}u_{1111} - 10u_1u_{11111} - u_{1111111}).
$$

Adding everything together, as in equation (16) of Theorem 4.2, we find

$$
L_{23}[u] = 3u_1^5 - \frac{15}{2}u_1^2u_{11}^2 + 10u_1^3u_{111} - 5u_1^3u_2 + \frac{7}{2}u_{11}^2u_{111} + 3u_1u_{111}^2
$$
$$
- 6u_1u_{11}u_{1111} + \frac{3}{2}u_1^2u_{11111} + 10u_1u_{11}u_{12} - \frac{5}{2}u_{11}^2u_2 - 5u_1u_{111}u_2
$$
$$
+ \frac{3}{2}u_1^2u_3 - \frac{1}{2}u_{1111}^2 + \frac{1}{2}u_{111}u_{11111} - u_{111}u_{112} + \frac{1}{2}u_1u_{113}
$$
$$
+ u_{1111}u_{12} - \frac{1}{2}u_{11}u_{13} - \frac{1}{2}u_{11111}u_2 + \frac{1}{2}u_{111}u_3.
$$

Note that the classical Euler–Lagrange equations

$$
\frac{\delta_{12}L_{12}}{\delta u} = 0 \quad \text{and} \quad \frac{\delta_{13}L_{13}}{\delta u} = 0
$$

yield equations (23)–(24), which are the t_1-derivatives of the potential KdV equations (21)–(22). However, the multi-time Euler–Lagrange equations also contain the potential KdV equations themselves:

$$\frac{\delta_{12}L_{12}}{\delta u_1} = -\frac{\delta_{23}L_{23}}{\delta u_3} \implies \frac{1}{2}u_2 - 3u_1^2 - u_{111} = -\frac{3}{2}u_1^2 - \frac{1}{2}u_{111},$$

$$\frac{\delta_{13}L_{13}}{\delta u_1} = \frac{\delta_{23}L_{23}}{\delta u_2} \implies \frac{1}{2}u_3 - 10u_1^3 - 5u_{11}^2 - 10u_1u_{111} - u_{11111}$$

$$= -5u_1^3 - \frac{5}{2}u_{11}^2 - 5u_1u_{111} - \frac{1}{2}u_{11111}.$$

5.2 Nonlinear Schrödinger hierarchy

The nonlinear Schrödinger equation is one of the most prominent integrable PDEs [11,12]. The corresponding hierarchy is discussed for example in [1,21]. It is a special case of the AKNS hierarchy, the pluri-Lagrangian structure of which is studied in [24]. Here we construct a pluri-Lagrangian structure for the NLS hierarchy using Theorem 4.2.

In this example we consider a complex field $u\colon \mathbb{R}^N \to \mathbb{C}$. The first two equations of the hierarchy are the nonlinear Schrödinger equation itself and the complex modified KdV equation,

$$u_2 = iu_{11} - 2i|u|^2u, \tag{25}$$

$$u_3 = u_{111} - 6|u|^2u_1. \tag{26}$$

Fields u that solve both these equations and their complex conjugates are critical fields for the Lagrangians (see e.g. [3])

$$L_{12}[u] = \frac{i}{2}(u_2\bar{u} - u\bar{u}_2) - |u_1|^2 - |u|^4,$$

$$L_{13}[u] = \frac{i}{2}(u_3\bar{u} - u\bar{u}_3) + \frac{i}{2}(u_{11}\bar{u}_1 - u_1\bar{u}_{11}) + \frac{3i}{2}|u|^2(u_1\bar{u} - u\bar{u}_1).$$

For these Lagrangians Lemma 4.1 gives us the function

$$F_{23}(u, u_1, u_{11}, \ldots) = 2|u|^6 - \frac{3}{2}|u|^2(u_{11}\bar{u} - u\bar{u}_{11}) - 6|uu_1|^2$$
$$+ \frac{1}{2}(u_{111}\bar{u}_1 + u_1\bar{u}_{111}) + |u_{11}|^2$$

and Theorem 4.2 provides the coefficient

$$L_{23}[u] = -4|u|^6 - u_1^2\bar{u}_1^2 - u^2\bar{u}_1^2 + 2|uu_1|^2 + 2|u|^2(u_{11}\bar{u} + u\bar{u}_{11})$$
$$+ \frac{3}{2}i|u|^2(u_2\bar{u} - u\bar{u}_2) + \frac{i}{2}(u_{12}\bar{u}_1 - u_1\bar{u}_{12})$$
$$+ u_3\bar{u}_1 + u_1\bar{u}_3 - |u_{11}|^2 + i(u_{11}\bar{u}_2 - u_2\bar{u}_{11})$$

of a pluri-Lagrangian 2-form

$$\mathcal{L}[u] = L_{12}[u]\,\mathrm{d}t_1 \wedge \mathrm{d}t_2 + L_{13}[u]\,\mathrm{d}t_1 \wedge \mathrm{d}t_3 + L_{23}[u]\,\mathrm{d}t_2 \wedge \mathrm{d}t_3.$$

Interestingly, in this example the classical Euler–Lagrange equations

$$\frac{\delta_{12}L_{12}}{\delta u} = 0 \quad \text{and} \quad \frac{\delta_{13}L_{13}}{\delta u} = 0$$

already yield the evolutionary form of the NLS equations (25)–(26). All other multi-time Euler–Lagrange equations, in particular those of the form

$$\frac{\delta_{12}L_{12}}{\delta u_1} = -\frac{\delta_{23}L_{23}}{\delta u_3} \quad \text{and} \quad \frac{\delta_{13}L_{13}}{\delta u_1} = \frac{\delta_{23}L_{23}}{\delta u_2}$$

are trivially satisfied.

5.3 Sine-Gordon equation and modified KdV hierarchy

Consider the sine-Gordon equation

$$u_{12} = \sin u$$

and the (potential) modified KdV hierarchy

$$u_3 = u_{111} + \frac{1}{2} u_1^3,$$
$$u_4 = \frac{3}{8} u_1^5 + \frac{5}{2} u_1 u_{11}^2 + \frac{5}{2} u_1^2 u_{111} + u_{11111},$$
$$\vdots$$

This hierarchy consists of symmetries of the sine-Gordon equation (see, e.g. [19] or [17, Section 5k]). The corresponding Lagrangians are

$$L_{12}[u] = \frac{1}{2} u_1 u_2 - \cos u,$$
$$L_{13}[u] = \frac{1}{2} u_1 u_3 - \frac{1}{8} u_1^4 + \frac{1}{2} u_{11}^2,$$
$$L_{14}[u] = \frac{1}{2} u_1 u_4 - \frac{1}{16} u_1^6 - \frac{5}{12} u_1^3 u_{111} - \frac{1}{2} u_{111}^2,$$
$$\vdots$$

Since the sine-Gordon equation is not evolutionary, Theorem 4.2 does not apply to this hierarchy. Surprisingly, a naive adaptation of the construction leads to a suitable 2-form, at least for the first few equations of the hierarchy.

We start the construction of a pluri-Lagrangian 2-form in three dimensions, considering only t_1, t_2 and t_3. Let $Q_3 = u_{111} + \frac{1}{2}u_1^3$. Then on solutions of the equations, there holds

$$D_2 L_{13} - D_3 L_{12} = \frac{1}{2} u_{12} u_3 - \frac{1}{2} u_1^3 u_{12} + u_{11} u_{112} - \frac{1}{2} u_{13} u_2 - u_3 \sin u$$

$$= -\frac{1}{2} u_{12} Q_3 - \frac{1}{2} u_2 D_1 Q_3 - \frac{1}{2} u_1^3 \sin u + u_{11} u_1 \cos u$$

$$= D_1 F_{23}$$

for

$$F_{23}[u] = -\frac{1}{2} u_2 \left(u_{111} + \frac{1}{2} u_1^3 \right) + \frac{1}{2} u_1^2 \cos u.$$

Since there is no evolutionary equation for u_2, we tolerate the dependence of F_{23} on this derivative. For the same reason, the term $\frac{\delta_{12} L_{12}}{\delta u_{1\alpha+1}} D_1^\alpha (u_2 - Q_2)$ in equation (16) only makes sense for $\alpha > 0$. For $\alpha = 0$ we just remove it. We are left with

$$L_{23}[u] = \frac{\delta_{13} L_{13}}{\delta u_{11}} (u_{12} - \sin u) - \frac{\delta_{12} L_{12}}{\delta u_1} \left(u_3 - u_{111} - \frac{1}{2} u_1^3 \right) + F_{23}[u]$$

$$= u_{11}(u_{12} - \sin u) - \frac{1}{2} u_2 \left(u_3 - u_{111} - \frac{1}{2} u_1^3 \right)$$

$$- \frac{1}{2} u_2 \left(u_{111} + \frac{1}{2} u_1^3 \right) + \frac{1}{2} u_1^2 \cos u$$

$$= u_{11}(u_{12} - \sin u) - \frac{1}{2} u_2 u_3 + \frac{1}{2} u_1^2 \cos u.$$

This pluri-Lagrangian structure in $\mathbb{R}^3$ was first found in [27], but a pluri-Lagrangian structure incorporating more equations of the hierarchy has not been given previously. With the method presented here, such an extension is obtained by a straightforward (but long) calculation. For example, we can calculate F_{24} and F_{34} analogously to F_{23} above. This in turn allow us to calculate the coefficients of the Lagrangian 2-form,

$$L_{24}[u] = \frac{3}{8} u_1^4 \cos u - \frac{5}{12} u_1^3 u_{112} + \frac{5}{4} u_1^2 u_{11} u_{12} - \frac{3}{2} u_1^2 u_{11} \sin u - \frac{1}{2} u_{11}^2 \cos u$$

$$+ u_1 u_{111} \cos u - u_{111} u_{112} + u_{1111} u_{12} - \frac{1}{2} u_2 u_4 - u_{1111} \sin u$$

and

$$L_{34}[u] = \frac{3}{128} u_1^8 - \frac{5}{16} u_1^4 u_{11}^2 + \frac{7}{16} u_1^5 u_{111} - \frac{3}{16} u_1^5 u_3 - \frac{1}{8} u_{11}^4 + \frac{7}{4} u_1 u_{11}^2 u_{111}$$

$$+ \frac{3}{4} u_1^2 u_{111}^2 - \frac{3}{2} u_1^2 u_{11} u_{1111} + \frac{1}{4} u_1^3 u_{11111} - \frac{5}{12} u_1^3 u_{113} + \frac{5}{4} u_1^2 u_{11} u_{13}$$

$$- \frac{5}{4} u_1 u_{11}^2 u_3 - \frac{5}{4} u_1^2 u_{111} u_3 + \frac{1}{4} u_1^3 u_4 - \frac{1}{2} u_{1111}^2 + \frac{1}{2} u_{111} u_{11111}$$

$$- u_{111} u_{113} + u_{1111} u_{13} - u_{11} u_{14} - \frac{1}{2} u_{11111} u_3 + \frac{1}{2} u_{111} u_4.$$

The presented hierarchy can be extended to a doubly-infinite hierarchy, where the sine-Gordon equation connects two copies of the modified KdV hierarchy, one as stated above and one where t_2 is used as space variable. The calculations presented here can be easily extended to cover both sides of the hierarchy. A pluri-Lagrangian structure of this double hierarchy was previously obtained using a carefully chosen continuum limit [29].

In this example, a straightforward adaptation of equation (16) gives us suitable coefficients L_{ij}. However, there does not seem to be a simple generalization of the proof we gave for Theorem 4.2 to cover this case. In this example we have verified by direct calculation that the multi-time Euler–Lagrange equations consist of the Sine-Gordon and modified KdV equations and differential consequences thereof. Showing the validity of our construction in a more general setting, ideally with a more conceptual proof, is a goal for future research.

6 Conclusions

We have shown that a hierarchy of 2-dimensional variational PDEs, that are variational symmetries of each other, possesses a pluri-Lagrangian structure. This extends the results of [22], where a similar result was obtained for variational ODEs. The existence of a hierarchy of variational symmetries for a PDE is closely related to its integrability. Hence our result contributes significantly to the evidence that pluri-Lagrangian structures are a fundamental feature of integrability. Furthermore, our construction can be used to obtain new examples of pluri-Lagrangian 2-forms, as we illustrated in the context of the nonlinear Schrödinger hierarchy.

As illustrated by the example of the sine-Gordon and mKdV equations, our construction applies more generally than the proof we provided. More research is needed to determine the most general form of the ideas presented here. Relevant to this line of investigation is the paper [25], which deals with the same topics as the present work (and appeared on the arXiv one day after it).

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