



Recurrence equation and integral representation of Apéry sums

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Abstract

Various methods are used to investigate sums involving a reciprocal central binomial coefficient and a power term. In the first part, new functions are introduced for calculation of sums with a negative exponent in the power term. A recurrence equation for the functions provides an integral representation of the sums using polylogarithm functions. Thus polylogarithms and, in particular, zeta values can be expressed via these functions, too. In the second part, a straightforward recurrence formula is derived for sums having a positive exponent in the power term. Finally, two interesting cases of double sums are presented.

Keywords Apéry sum · Zeta function · Polylogarithm · Central binomial coefficient

Mathematics Subject Classification 11M06 · 11G55 · 11B65

1 Introduction and main results

The subject of this article are infinite sums, A_m^+ and A_m^- , involving a reciprocal central binomial coefficient $1/\binom{2k}{k}$ and a power term k^{-m} :

$$A_m^+ = \sum_{k=1}^{\infty} \frac{1}{k^m \binom{2k}{k}}, \quad A_m^- = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^m \binom{2k}{k}}. \quad (1)$$

In the context of these sums—which we also call the Apéry sums—Roger Apéry proved the irrationality of π^2 and $\zeta(3)$ [1,9,13]. Furthermore, the tight relationship between these sums and values of the zeta function are the topic of many publications [2,6–8,10–14].

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Whereas there exist closed-form expressions for $A_1^+, A_2^+, A_1^-, A_2^-$, and as well A_4^+ , see for instance [10–13], the sums A_m^+ and A_m^- for $m > 4$ can only be expressed by means of polylogarithmic ladders or multiple Clausen values [2,7]. In particular,

$$\begin{aligned} A_1^+ &= \frac{\sqrt{3}\pi}{9}, & A_2^+ &= \frac{\pi^2}{18} = \frac{\zeta(2)}{3}, & A_4^+ &= \frac{17\pi^4}{3240} = \frac{17\zeta(4)}{36}, \\ A_1^- &= \frac{2 \ln \phi}{\sqrt{5}}, & A_2^- &= 2 \ln^2 \phi, & A_3^- &= \frac{2\zeta(3)}{5}, \end{aligned} \tag{2}$$

where $\phi = (\sqrt{5} + 1)/2$, see e.g. [2,11–13].

In this work two approaches are presented to calculate the sums with a negative exponent in the power term, A_m^+ and A_m^- for $m > 0$. First, using a reformulation of the reciprocal central binomial coefficient we define the functions $A_m^+(x)$ and $A_m^-(x)$:

$$A_m^\pm(x) = \sum_{j=1}^\infty \frac{(-x)^j}{j} a_{mj}^\pm, \quad a_{mj}^\pm = \sum_{k=1}^{j-1} \binom{k-1}{j-k-1} \frac{(\mp 1)^{k-1}}{k^{m-1}}. \tag{3}$$

These functions $A_m^+(x)$ and $A_m^-(x)$ together with their derivatives $A_m'^+(x)$ and $A_m'^-(x)$ relate the Apéry sums A_m^+ and A_m^- with the polylogarithm functions $\text{Li}_n(x)$, defined as

$$\text{Li}_n(x) = \sum_{k=1}^\infty \frac{x^k}{k^n}, \quad \text{Li}'_n(x) = \frac{1}{x} \text{Li}_{n-1}(x). \tag{4}$$

Theorem 1.1 $A_m^+ = A_m^+(1), A_m^- = A_m^-(1)$.

Theorem 1.2

$$A_m'^+(x) = \frac{\text{Li}_{m-1}(x - x^2)}{1 - x}, \quad A_m'^-(x) = \frac{\text{Li}_{m-1}(x^2 - x)}{x - 1}. \tag{5}$$

For the proofs of Theorems 1.1 and 1.2 see (11) and Lemma 2.1 in Sect. 2.

Thus, integrating (5) the Apéry sums for $m > 0$ can be calculated as

$$A_m^+ = \int_0^1 \frac{\text{Li}_{m-1}(x - x^2)}{1 - x} dx, \quad A_m^- = \int_0^1 \frac{\text{Li}_{m-1}(x^2 - x)}{x - 1} dx. \tag{6}$$

Furthermore, as shown in Sect. 2, the polylogarithm $\text{Li}_m(z)$, and in particular its value $\text{Li}_m(1) = \zeta(m)$, can be expressed vice versa in terms of $A_m(x)$.

In Sect. 3 we derive an integral representation of the reciprocal central binomial coefficient, which gives us a direct integral formula of the sums, see (6). In Sect. 4 we calculate the integrals for $m = 1, 2, 3$.

Due to the exponential convergence of the reciprocal binomial coefficients even the sums with a positive exponent in the power term, A_m^+ and A_m^- for $m \geq 0$, converge. These sums up to $m = 3$ are studied in various articles [11,12,15]. In this work a straightforward recurrence formula is presented to calculate them.

Theorem 1.3

$$\begin{aligned}
 3A_{-m}^+ &= 1 + 2A_{1-m} + \sum_{j=0}^{m-1} \binom{m}{j} A_{-j}, \\
 5A_{-m}^- &= 1 + 2A_{1-m} - \sum_{j=0}^{m-1} \binom{m}{j} A_{-j}.
 \end{aligned}
 \tag{7}$$

For the proof see (26) in Sect. 5. By reiterative application of (7), the sums A_{-m}^+ and A_{-m}^- are calculated as linear combinations of 1 and A_1^+ , respectively 1 and A_1^- , where $A_1^+ = \sqrt{3}\pi/9$ and $A_1^- = 2 \ln \phi / \sqrt{5}$, see (2).

Theorem 1.4 *The numbers $3^m A_{-m}^+$ and $5^m A_{-m}^-$ are sums of an integer and odd—and therefore non-vanishing—multiple of $2A_1^+$, respectively $2A_1^-$.*

For the proof see Lemma 5.1 in Sect. 5. Since π and (due to the Gelfond–Schneider theorem [3–5]) $\ln \phi$ are transcendental, A_1^+ and A_1^- are transcendental, too. Then, due to Theorem 1.4, A_{-m}^+ and A_{-m}^- for all $m \geq 0$ are transcendental numbers. For instance, this is different from A_3^- which is proven only to be irrational [1].

Although there is up to now—with the exception of A_4^+ —no closed-form expression for A_m^+ and A_m^- for all $m \geq 3$, the sums of $A_m^+ - 1/2$ and $A_m^- - 1/2$ over m are calculated as follows.

Theorem 1.5

$$S^+ = \sum_{m=1}^{\infty} \left(A_m^+ - \frac{1}{2} \right) = \frac{1 - A_1^+}{2}, \quad S^- = \sum_{m=1}^{\infty} \left(A_m^- - \frac{1}{2} \right) = \frac{1 - 3A_1^-}{2}. \tag{8}$$

For the proof see (30) in Sect. 6. Finally, in Sect. 6 the identities, $0 = 6A_0^+ A_2^+ - 4A_1^+ A_2^+ - 3A_1^+ A_1^+$ and $0 = 10A_0^- A_2^- - 4A_1^- A_2^- - 5A_1^- A_1^-$, are presented as vanishing double sums over the products $1 / \left[\binom{2k}{k} \binom{2j}{j} \right]$.

2 Recurrence equation

We show below that non-alternating sums A_m^+ and alternating sums A_m^- subordinate, in general, to the same equations, except for a sign. First, we simplify the notation: $A_m := A_m^\pm$, $a_m := a_m^\pm$, and $A_m(x) := A_m^\pm(x)$. Reiterative application of the recurrence formula for reciprocal binomial coefficients,

$$\binom{n}{m}^{-1} = \frac{m}{m-1} \left[\binom{n-1}{m-1}^{-1} - \binom{n-1}{m}^{-1} \right],$$

to the reciprocal central binomial coefficient $1/\binom{2k}{k}$ gives

$$\binom{2k}{k}^{-1} = k \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{k+j+1} = k \sum_{j=k+1}^{2k} \binom{k-1}{j-k-1} \frac{(-1)^{j-k-1}}{j}.$$

Thus the sums $A_m, (1)$, for $m \geq 1$, can be reformulated via coefficients α_{mjk} as

$$A_m = \sum_{k=1}^{\infty} \frac{(\pm 1)^{k-1}}{k^m \binom{2k}{k}} = \sum_{k=1}^{\infty} \sum_{j=k+1}^{2k} \frac{(-1)^j}{j} \alpha_{mjk}, \tag{9}$$

$$\alpha_{mjk} = \binom{k-1}{j-k-1} \frac{(\mp 1)^{k-1}}{k^{m-1}}.$$

Now, for a more detailed investigation of the sums A_m the functions $A_m(x)$ are introduced, see (3):

$$A_m(x) = \sum_{j=1}^{\infty} \sum_{k=1}^{j-1} \frac{(-x)^j}{j} \alpha_{mjk} = \sum_{j=1}^{\infty} \frac{(-x)^j}{j} a_{mj}, \tag{10}$$

$$a_{mj} = \sum_{k=1}^{j-1} \alpha_{mjk} = \sum_{k=1}^{j-1} \binom{k-1}{j-k-1} \frac{(\mp 1)^{k-1}}{k^{m-1}}.$$

In the end of this section the coefficients a_{1j}^+ and a_{1j}^- are calculated. Convergence of the sums $A_m^+(x)$ and $A_m^-(x)$ is guaranteed for $|x| < 1$, respectively $|x| < 1/\phi$, only. However, by the process of analytic continuation the order of summation in (10) can be switched and a comparison with (9) shows

$$A_m(x) = \sum_{j=1}^{\infty} \sum_{k=1}^{j-1} \frac{(-x)^j}{j} \alpha_{mjk} = \sum_{k=1}^{\infty} \sum_{j=k+1}^{2k} \frac{(-x)^j}{j} \alpha_{mjk}, \quad A_m(1) = A_m. \tag{11}$$

The derivatives $A'_m(x)$ and $A''_m(x)$ are given by

$$A'_m(x) = - \sum_{j=1}^{\infty} (-x)^{j-1} a_{mj}, \quad A''_m(x) = \sum_{j=1}^{\infty} (j-1)(-x)^{j-2} a_{mj}. \tag{12}$$

For $m = 1$ the derivative $A'_1(x)$ —as the analytic continuation of (12)—is calculated as

$$\begin{aligned}
 A'_1(x) &= - \sum_{j=1}^{\infty} (-x)^{j-1} \sum_{k=1}^{j-1} (\mp 1)^{k-1} \binom{k-1}{j-k-1} \\
 &= - \sum_{k=1}^{\infty} \sum_{j=1}^k (\mp 1)^{k-1} (-x)^{k+j-1} \binom{k-1}{j-1} \\
 &= x \sum_{k=1}^{\infty} (\pm x)^{k-1} (1-x)^{k-1} = \frac{x}{1 \pm (x^2-x)}.
 \end{aligned}
 \tag{13}$$

In fact, the coefficients a_{mj} from (10) satisfy the identity

$$\begin{aligned}
 &(j-1)a_{mj} - a_{m-1,j} + (j-1)a_{m,j-1} - 2a_{m-1,j-1} \\
 &= \sum_{k=1}^{j-1} \frac{(\mp 1)^{k-1}}{k^{m-1}} \left((j-1-k) \binom{k-1}{j-k-1} + (j-1-2k) \binom{k-1}{j-k-2} \right) = 0.
 \end{aligned}$$

Thus, for the functions $A_m(x)$ from (10) the following differential equation is obtained:

$$\begin{aligned}
 &(x^2-x)A''_m(x) + xA'_m(x) - (2x-1)A'_{m-1}(x) \\
 &= \sum_{j=1}^{\infty} (j-1)a_{mj}(x^2-x)(-x)^{j-2} - \sum_{j=1}^{\infty} [a_{mj}x - a_{m-1,j}(2x-1)](-x)^{j-1} \\
 &= \sum_{j=1}^{\infty} [(j-1)a_{mj} - a_{m-1,j}](-x)^{j-1} + \sum_{j=1}^{\infty} [ja_{mj} - 2a_{m-1,j}](-x)^j \\
 &= \sum_{j=1}^{\infty} [(j-1)a_{mj} - a_{m-1,j} + (j-1)a_{m,j-1} - 2a_{m-1,j-1}](-x)^{j-1} = 0.
 \end{aligned}$$

Therefore, the derivatives $A'_m(x)$ can be calculated recurrently:

$$\frac{2x-1}{x} A'_{m-1}(x) = (x-1)A''_m(x) + A'_m(x) = [(x-1)A'_m(x)]'. \tag{14}$$

Since $A'_{m-1}(0) = a_{m1} = 0$, by integration $A'_m(x)$ can be written as

$$A'_m(x) = \frac{1}{x-1} \int_0^x \frac{2t-1}{t} A'_{m-1}(t) dt.$$

Lemma 2.1 *The function $(x - 1)A'_m(x)$ and the polylogarithm $\mp\text{Li}_{m-1}(\mp(x^2 - x))$ are identical:*

$$(x - 1)A'_m(x) = \mp\text{Li}_{m-1}(\mp(x^2 - x)). \tag{15}$$

Proof (i) For $m = 1$ both sides are identical and (ii) for $m > 1$ they are described by the same recurrence equation.

(i) In the case $m = 1$, it follows from (13) and the definition $\text{Li}_0(z) = z/(1 - z)$:

$$(x - 1)A'_1(x) = \mp\text{Li}_0(\mp(x^2 - x)) = \frac{x^2 - x}{1 \pm (x^2 - x)}.$$

(ii) Applying (15) to the recurrence equation of polylogarithm, $\text{Li}'_{m-1}(z) = \text{Li}_{m-2}(z)/z$ with $z = \mp(x^2 - x)$, results in (14), again:

$$\begin{aligned} [\mp\text{Li}_{m-1}(\mp(x^2 - x))]' &= \mp(2x - 1) \frac{\text{Li}_{m-2}(\mp(x^2 - x))}{x^2 - x} \\ &= [(x - 1)A'_m(x)]' = \frac{2x - 1}{x} A'_{m-1}(x). \end{aligned} \tag{16}$$

□

Due to the symmetry of (15),

$$xA'_m(1 - x) = (1 - x)A'_m(x), \tag{17}$$

and the transformation, $x \mapsto \frac{1}{2} - x$, the Apéry sums $A_m = A_m(1)$ are calculated as

$$A_m = \int_0^1 A'_m(x) dx = \int_0^1 \frac{\text{Li}_{m-1}(\mp(x^2 - x))}{\mp(x - 1)} dx = \int_0^{\frac{1}{2}} \frac{\text{Li}_{m-1}(\pm(\frac{1}{4} - x^2))}{\pm(\frac{1}{4} - x^2)} dx. \tag{18}$$

Integrating (16), the polylogarithm $\mp\text{Li}_m(\mp(x - x^2))$ can be described by the function $A_m(x)$:

$$\begin{aligned} \mp\text{Li}_m(\mp(x^2 - x)) &= \int_0^x \frac{2t - 1}{t} A'_m(t) dt \\ &= \int_0^x (A'_m(t) - A'_m(1 - t)) dt = A_m(x) + A_m(1 - x) - A_m(1), \end{aligned}$$

in which $\text{Li}_m(0) = A_m(0) = 0$ and the symmetry of (17) are used.

The values of the zeta function are given by $\zeta(m) = \text{Li}_m(1)$. Using the definition of the coefficients a_{mj}^+ from (9), and writing $\varphi = \exp(i\pi/3)$ with $\varphi - \varphi^2 = 1$ and $1 - \varphi = \varphi^{-1}$, one obtains

$$\begin{aligned} \zeta(m) &= \text{Li}_m(1) = \text{Li}_m(\varphi - \varphi^2) = A_m^+(1) - A_m^+(\varphi) - A_m^+(\varphi^{-1}) \\ &= \sum_{j=1}^{\infty} \frac{(-1)^j}{j} a_{mj}^+(1 - \varphi^j - \varphi^{-j}) = \sum_{j=1}^{\infty} \frac{(-1)^j}{j} a_{mj}^+ \left(1 - 2 \cos \frac{j\pi}{3}\right). \end{aligned}$$

Thus, the functions $A_m^+(x)$, or respectively the coefficients a_{mj}^+ , enable a formulation for both the sums A_m^+ and the values $\zeta(m)$.

Next, we want to focus on the coefficients a_{mj} from (9) in the case $m = 1$. Here, the coefficients a_{1j}^+ of the non-alternating sum $A_1^+(x)$ fulfil the equation

$$a_{1j}^+ = \sum_{k=1}^{j-1} (-1)^{k-1} \binom{k-1}{j-k-1} = -a_{1,j-1}^+ - a_{1,j-2}^+.$$

With $a_{11}^+ = 0$ and $a_{12}^+ = 1$ one gets recurrently,

$$a_{1,3j+1}^+ = 0, \quad a_{1,3j+2}^+ = 1, \quad a_{1,3j+3}^+ = -1, \quad j \geq 0.$$

Hence, comparing the sum $A_1^+ = A_1^+(1)$ from (10) with (2) gives

$$A_1^+ = \sum_{j=1}^{\infty} \frac{(-1)^j a_{1j}^+}{j} = \sum_{j=0}^{\infty} \frac{(-1)^j (6j + 5)}{(3j + 2)(3j + 3)} = \frac{\pi\sqrt{3}}{9}.$$

The coefficients a_{1j}^- of the alternating sum $A_1^-(x)$ can be written as

$$a_{1j}^- = \sum_{k=1}^{j-1} \binom{k-1}{j-k-1} = a_{1,j-1}^- + a_{1,j-2}^-.$$

With $a_{11}^- = 0$ and $a_{12}^- = 1$ the coefficients $a_{1,j+1}^-$ are given by the Fibonacci numbers F_j :

$$a_{1,j+1}^- = F_j = \frac{\phi^j - (1 - \phi)^j}{\sqrt{5}}, \quad j \geq 0, \quad \phi = \frac{\sqrt{5} + 1}{2}. \tag{19}$$

Since Fibonacci numbers grow exponentially with $\lim_{j \rightarrow \infty} F_j/F_{j-1} = \phi$, at $x = 1$ the sum $A_1^-(1) = \sum_{j=1}^{\infty} (-1)^j a_{1j}^-/j$ does not converge.

Having $|a_{1j}^+| \leq 1$ for all j , the convergence radius of $A_1^+(x)$, (10), is $R_1^+ = 1$. The convergence radius of $A_1^-(x)$ is given by $R_1^- = \lim_{j \rightarrow \infty} |a_{1,j-1}^-/a_{1j}^-| = 1/\phi$. Furthermore, with $|a_{mj}^+| \leq |a_{m-1,j}^+|$ and $|a_{mj}^-| \leq |a_{m-1,j}^-|$ it follows $R_m^+ \geq 1$ and $R_m^- \geq 1/\phi$.

3 Integral representation

The integrals A_m can be obtained directly by the integral representation of the reciprocal central binomial coefficients $1/\binom{2k}{k}$, see e.g. [15]. Looking at the integrals $\int_0^{1/2} (\frac{1}{4} - x^2)^{k-1} dx$, partial integration results in

$$\int_0^{\frac{1}{2}} \left(\frac{1}{4} - x^2\right)^{k-1} dx = \frac{4k + 2}{k} \int_0^{\frac{1}{2}} \left(\frac{1}{4} - x^2\right)^k dx.$$

Thus, starting with $\int_0^{1/2} (\frac{1}{4} - x^2)^0 dx = \frac{1}{2}$, we arrive to the integral by induction:

$$\int_0^{\frac{1}{2}} \left(\frac{1}{4} - x^2\right)^k dx = \frac{k!(k + 1)!}{(2k + 2)!}.$$

So, the reciprocal central binomial coefficients $1/\binom{2k}{k}$ are

$$\binom{2k}{k}^{-1} = \frac{k!^2}{(2k)!} = k \int_0^{\frac{1}{2}} \left(\frac{1}{4} - x^2\right)^{k-1} dx. \tag{20}$$

With (8) and the definition of the polylogarithm function, $\text{Li}_m(z) = \sum_{k=1}^{\infty} k^{-m} z^k$, the sums A_m can be calculated as

$$A_m = \sum_{k=1}^{\infty} \frac{(\pm 1)^{k-1}}{k^m \binom{2k}{k}} = \int_0^{\frac{1}{2}} \sum_{k=1}^{\infty} \frac{(\pm(\frac{1}{4} - x^2))^{k-1}}{k^{m-1}} dx = \int_0^{\frac{1}{2}} \frac{\text{Li}_{m-1}(\pm(\frac{1}{4} - x^2))}{\pm(\frac{1}{4} - x^2)} dx,$$

in agreement with (18). A similar result was derived by Taylor [15].

Furthermore, calculating the integral formula for the coefficients $1/\binom{2k}{k}$ in (20) one obtains the identity

$$\binom{2k}{k}^{-1} = k \sum_{h=0}^{k-1} \binom{k-1}{h} \int_0^{\frac{1}{2}} \frac{(-x^2)^h}{4^{k-1-h}} dx = \frac{2k}{4^k} \sum_{h=0}^{k-1} \binom{k-1}{h} \frac{(-1)^h}{2h+1}.$$

4 Integrals for A_1, A_2 , and A_3

The calculation of the sums A_m for $m > 0$, and in particular A_1 and A_2 , see (2), has been considered in many publications, see e.g. [11–13]. In this section we employ the integrals from (18) to calculate A_1 and A_2 and to give an integral formula of A_3 . For $m = 1$ the integral A_1 from (18) is given by $\text{Li}_0(z) = z/(1 - z)$:

$$A_1 = \int_0^{\frac{1}{2}} \frac{\text{Li}_0(\pm(\frac{1}{4} - x^2))}{\pm(\frac{1}{4} - x^2)} dx = \int_0^{\frac{1}{2}} \frac{dx}{1 \pm (x^2 - \frac{1}{4})}.$$

In particular, A_1^+ and A_1^- are calculated as

$$A_1^+ = \int_0^{\frac{1}{2}} \frac{dx}{\frac{3}{4} + x^2} = \left[\frac{2}{\sqrt{3}} \arctan \frac{2x}{\sqrt{3}} \right]_0^{\frac{1}{2}} = \frac{2}{\sqrt{3}} \frac{\pi}{6} = \frac{\pi\sqrt{3}}{9}, \tag{21}$$

$$A_1^- = \int_0^{\frac{1}{2}} \frac{dx}{\frac{5}{4} - x^2} = \left[\frac{1}{\sqrt{5}} \ln \frac{\sqrt{5} + 2x}{\sqrt{5} - 2x} \right]_0^{\frac{1}{2}} = \frac{2}{\sqrt{5}} \ln \frac{\sqrt{5} + 1}{2}. \tag{22}$$

For $m = 2$ the integral A_2 from (18) can be calculated via $\text{Li}_1(z) = -\ln(1 - z)$:

$$A_2 = \pm \int_0^1 \frac{\text{Li}_1(\mp(x^2 - x))}{x} dx = \mp \int_0^1 \frac{\ln(1 \pm (x^2 - x))}{x} dx.$$

Having the factorizations,

$$1 - x + x^2 = (\varphi - x)(\varphi^{-1} - x), \quad \varphi = e^{i\pi/3}, \quad \varphi^{-1} = e^{-i\pi/3}, \quad \varphi + \varphi^{-1} = 1,$$

$$1 + x - x^2 = (\phi - x)(\phi^{-1} + x), \quad \phi = \frac{\sqrt{5} + 1}{2}, \quad \phi^{-1} = \frac{\sqrt{5} - 1}{2}, \quad \phi - \phi^{-1} = 1,$$

the integral $\int \ln(a \pm x)/x dx = \ln x \ln a - \text{Li}_2(\mp x/a)$, the dilogarithm $\text{Li}_2(0) = 0$, and the Landen identities,

$$\text{Li}_2(\varphi^{-1}) + \text{Li}_2(\varphi) = \text{Li}_2(1 - \varphi) + \text{Li}_2(1 - \varphi^{-1}) = -\frac{\ln^2 \varphi}{2} = \frac{\pi^2}{18},$$

$$\text{Li}_2(\phi^{-1}) + \text{Li}_2(-\phi) = \text{Li}_2(1 - \phi^{-2}) + \text{Li}_2(1 - \phi^2) = -\frac{\ln^2 \phi^2}{2} = -2 \ln^2 \phi,$$

A_2^+ and A_2^- are obtained as

$$A_2^+ = - \int_0^1 \frac{\ln(1 - x + x^2)}{x} dx = \left[\text{Li}_2 \frac{x}{\varphi} + \text{Li}_2 \frac{x}{\varphi^{-1}} \right]_0^1 = \frac{\pi^2}{18}, \tag{23}$$

$$A_2^- = \int_0^1 \frac{\ln(1 + x - x^2)}{x} dx = \left[-\text{Li}_2 \frac{x}{\phi} - \text{Li}_2 \frac{-x}{\phi^{-1}} \right]_0^1 = 2 \ln^2 \phi. \tag{24}$$

The integral A_3 from (18) is calculated by integration of the dilogarithm Li_2 . Integrating by parts one finds

$$A_3 = \int_0^{\frac{1}{2}} \frac{\text{Li}_2(\pm(\frac{1}{4} - x^2))}{\pm(\frac{1}{4} - x^2)} dx = \int_0^{\frac{1}{2}} \frac{\pm 2x \ln(1 \pm (x^2 - \frac{1}{4}))}{x^2 - \frac{1}{4}} \ln \frac{1 + 2x}{1 - 2x} dx.$$

Thus the sums A_3^+ and $A_3^- = \frac{2}{5}\zeta(3)$, see (2), are given by the following integrals:

$$A_3^+ = \int_0^{\frac{1}{2}} \frac{\text{Li}_2\left(\frac{1}{4} - x^2\right)}{\frac{1}{4} - x^2} dx = \int_0^{\frac{1}{2}} \frac{2x \ln\left(\frac{3}{4} + x^2\right)}{x^2 - \frac{1}{4}} \ln \frac{1 + 2x}{1 - 2x} dx,$$

$$A_3^- = \int_0^{\frac{1}{2}} \frac{\text{Li}_2\left(x^2 - \frac{1}{4}\right)}{x^2 - \frac{1}{4}} dx = \int_0^{\frac{1}{2}} \frac{2x \ln\left(\frac{5}{4} - x^2\right)}{\frac{1}{4} - x^2} \ln \frac{1 + 2x}{1 - 2x} dx = \frac{2\zeta(3)}{5},$$

leading to another integral form for $\zeta(3)$.

5 Apéry sums for positive powers

The sums for positive powers, A_{-m} for $m \geq 0$, have also been studied in [11,12,15]. Calculation of A_0, A_{-1}, A_{-2} , and A_{-3} was done by Lehmer [12]. In this section, a general recurrence formula for A_{-m} is derived. The sums A_{-m} converge, since $1/\binom{2k}{k} < \frac{k}{4^k}$ and

$$|A_{-m}| = \left| \sum_{k=1}^{\infty} \frac{(\pm 1)^k k^m}{\binom{2k}{k}} \right| < \sum_{k=1}^{\infty} \frac{k^{m+1}}{4^k} = \text{Li}_{-(m+1)}\left(\frac{1}{4}\right).$$

With (1) the expression $4A_m - 2A_{m+1}$ can be calculated by

$$4A_m - 2A_{m+1} = \sum_{k=1}^{\infty} \frac{(4k - 2)(\pm 1)^{k-1}}{k^{m+1} \binom{2k}{k}} \tag{25}$$

$$= \sum_{k=1}^{\infty} \frac{(\pm 1)^{k-1}}{k^m \binom{2k-2}{k-1}} = 1 \pm \sum_{k=1}^{\infty} \frac{(\pm 1)^{k-1}}{(k + 1)^m \binom{2k}{k}}.$$

Thus, we obtain in the case of positive powers, with $m \geq 0$, that

$$4A_{-m} - 2A_{1-m} = 1 \pm \sum_{k=1}^{\infty} (k + 1)^m \frac{(\pm 1)^{k-1}}{\binom{2k}{k}} = 1 \pm \sum_{j=0}^m \binom{m}{j} A_{-j}.$$

This directly leads to the recurrence equation for the sums A_{-m} :

$$A_0 = \frac{1 + 2A_1}{4 \mp 1}, \quad A_{-m} = \frac{1}{4 \mp 1} \left(1 + 2A_{1-m} \pm \sum_{j=0}^{m-1} \binom{m}{j} A_{-j} \right). \tag{26}$$

Here, the sums A_{-m} can be calculated recurrently starting with A_1 . For $0 \leq m \leq 4$ the sums A_{-m}^+ and A_{-m}^- are given by:

$$\begin{aligned}
 A_1^+ &= \frac{\pi\sqrt{3}}{9}, \\
 A_0^+ &= \frac{1 + 2A_1^+}{3} = \frac{1}{3} + \frac{2A_1^+}{3}, \\
 A_{-1}^+ &= \frac{1 + 3A_0^+}{3} = \frac{2}{3} + \frac{2A_1^+}{3}, \\
 A_{-2}^+ &= \frac{1 + 4A_{-1}^+ + 1A_0^+}{3} = \frac{4}{3} + \frac{10A_1^+}{9}, \\
 A_{-3}^+ &= \frac{1 + 5A_{-2}^+ + 3A_{-1}^+ + 1A_0^+}{3} = \frac{10}{3} + \frac{74A_1^+}{27}, \\
 A_{-4}^+ &= \frac{1 + 6A_{-3}^+ + 6A_{-2}^+ + 4A_{-1}^+ + 1A_0^+}{3} = \frac{32}{3} + \frac{238A_1^+}{27},
 \end{aligned}
 \tag{27}$$

$$\begin{aligned}
 A_1^- &= \frac{2 \ln \phi}{\sqrt{5}}, \\
 A_0^- &= \frac{1 + 2A_1^-}{5} = \frac{1}{5} + \frac{2A_1^-}{5}, \\
 A_{-1}^- &= \frac{1 + 1A_0^-}{5} = \frac{6}{25} + \frac{2A_1^-}{25}, \\
 A_{-2}^- &= \frac{1 + 0A_{-1}^- - 1A_0^-}{5} = \frac{4}{25} - \frac{2A_1^-}{25}, \\
 A_{-3}^- &= \frac{1 - 1A_{-2}^- - 3A_{-1}^- - 1A_0^-}{5} = -\frac{2}{125} - \frac{14A_1^-}{125}, \\
 A_{-4}^- &= \frac{1 - 2A_{-3}^- - 6A_{-2}^- - 4A_{-1}^- - 1A_0^-}{5} = -\frac{136}{625} - \frac{2A_1^-}{625}.
 \end{aligned}
 \tag{28}$$

Using the substitution $\tilde{A}_{-m} = (4 \mp 1)^{m+1} A_{-m}$, (26) gives

$$\begin{aligned}
 \tilde{A}_0 &= 1 + 2A_1, \quad \tilde{A}_{-m} = (4 \mp 1)^m + \sum_{j=0}^{m-1} w_{mj} \tilde{A}_{-j}, \\
 w_{mj} &= 2\delta_{m-1,j} \pm \binom{m}{j} (4 \mp 1)^{m-1-j}.
 \end{aligned}$$

Here, w_{mj} are the weights of \tilde{A}_{-j} contributing to \tilde{A}_{-m} .

Lemma 5.1 *The numbers \tilde{A}_{-m} are sums of an integer and an odd multiple of $2A_1$.*

Proof First we prove \tilde{A}_0 is a sum of an integer and an odd multiple of $2A_1$. All weights w_{mj} are integer numbers and the sum of weights,

$$\sum_{j=0}^{m-1} w_{mj} = 2 \pm \sum_{j=0}^{m-1} (4 \mp 1)^{m-1-j} \binom{m}{j} = 2 \pm \frac{(5 \mp 1)^m - 1}{4 \mp 1},$$

is an odd number. Then, by induction, each number \tilde{A}_{-m} is a sum of an integer and an odd multiple of $2A_1$. □

6 Further relations on Apéry sums

In the same way as the zeta function $\zeta(m) = \sum_{k=1}^{\infty} k^{-m}$ and eta function $\eta(m) = \sum_{k=1}^{\infty} (-1)^{k-1} k^{-m}$ converge to 1 for $m \rightarrow \infty$, the sums $A_m = \sum_{k=1}^{\infty} (\pm 1)^{k-1} k^{-m} / \binom{2k}{k}$ converge to $\frac{1}{2}$, since all terms for $k \geq 2$ vanish when $m \rightarrow \infty$, and the term for $k = 1$ establishes the limit. The sum of Apéry sums, $S = \sum_{m=1}^{\infty} (A_m - \frac{1}{2})$, can be written as

$$S = \sum_{m=1}^{\infty} \left(A_m - \frac{1}{2} \right) = \sum_{m=1}^{\infty} \sum_{k=2}^{\infty} \frac{(\pm 1)^{k-1}}{k^m \binom{2k}{k}} = \sum_{k=2}^{\infty} \frac{(\pm 1)^{k-1}}{(k-1) \binom{2k}{k}}.$$

To calculate S , first we construct the sum of $4A_m - 2A_{m+1} - 1$, (25), over m :

$$\begin{aligned} \sum_{m=1}^{\infty} (4A_m - 2A_{m+1} - 1) &= \pm \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{(\pm 1)^{k-1}}{(k+1)^m \binom{2k}{k}} \\ &= \pm \sum_{k=1}^{\infty} \frac{(\pm 1)^{k-1}}{k \binom{2k}{k}} = \pm A_1. \end{aligned} \tag{29}$$

Regrouping (29) leads to

$$\sum_{m=1}^{\infty} (4A_m - 2A_{m+1} - 1) = 4 \sum_{m=1}^{\infty} \left(A_m - \frac{1}{2} \right) - 2 \sum_{m=2}^{\infty} \left(A_m - \frac{1}{2} \right) = 2S + 2A_1 - 1.$$

Comparing one gets

$$2S + (2 \mp 1)A_1 - 1 = 0. \tag{30}$$

Thus, the sum S^+ of the non-alternating sums A_m^+ and the sum S^- of the alternating sums A_m^- are given by

$$S^+ = \sum_{m=1}^{\infty} \left(A_m^+ - \frac{1}{2} \right) = \sum_{k=2}^{\infty} \frac{1}{(k-1) \binom{2k}{k}} = \frac{1}{2} - \frac{1}{2} A_1^+ = \frac{1}{2} - \frac{\pi\sqrt{3}}{18},$$

$$S^- = \sum_{m=1}^{\infty} \left(A_m^- - \frac{1}{2} \right) = \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{(k-1) \binom{2k}{k}} = \frac{1}{2} - \frac{3}{2} A_1^- = \frac{1}{2} - \frac{3}{\sqrt{5}} \ln \phi.$$

Again, these equations resemble similar equations for the zeta and eta functions, namely $\sum_{m=2}^{\infty} (\zeta(m) - 1) = 1$ and $\sum_{m=2}^{\infty} (\eta(m) - 1) = 1 - 2\eta(1) = 1 - 2 \ln 2$.

Finally, the sums A_1^+ and A_2^+ from (21) and (23), and also A_1^- and A_2^- of (22) and (24), are related by

$$2A_2^+ = 3(A_1^+)^2 = \frac{\pi^2}{9}, \quad 2A_2^- = 5(A_1^-)^2 = 4 \ln^2 \phi. \tag{31}$$

In addition, the relations between A_0^+ and A_1^+ , and also between A_0^- and A_1^- , are given by (27) and (28):

$$3A_0^+ - 2A_1^+ = 1, \quad 5A_0^- - 2A_1^- = 1. \tag{32}$$

Thus, combining (31) and (32), the following double sums are shown to be 0:

$$0 = 6A_0^+ A_2^+ - 4A_1^+ A_2^+ - 3A_1^+ A_1^+ = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{6k^2 - 4k - 3kj}{k^2 j^2 \binom{2k}{k} \binom{2j}{j}},$$

$$0 = 10A_0^- A_2^- - 4A_1^- A_2^- - 5A_1^- A_1^- = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (-1)^{k+j} \frac{10k^2 - 4k - 5kj}{k^2 j^2 \binom{2k}{k} \binom{2j}{j}}.$$

Here, the question remains open whether these equations could be proved in rational terms, i.e., without an explicit calculation of the irrational sums A_1 and A_2 .

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