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RESEARCH ARTICLE

Non-associative structures of commutative algebras related with quadratic Poisson brackets

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Abstract

There are studied algebraic properties of quadratic Poisson brackets on non-associative non-commutative algebras, compatible with their multiplicative structure. Their relations both with derivations of symmetric tensor algebras and Yang–Baxter structures on the adjacent Lie algebras are demonstrated. Special attention is paid to quadratic Poisson brackets of Lie–Poisson type, examples of Balinsky–Novikov and Leibniz algebras are discussed. The non-associative structures of commutative algebras related with Balinsky–Novikov, Leibniz, Lie, and Zinbiel algebras are studied in detail.

Keywords Balinsky–Novikov algebra · Lie algebra · Leibniz algebra · Zinbiel algebra · Derivation · Pre-Poisson brackets · Lie–Poisson structure

Mathematics Subject Classification 13N15 · 13P25 · 16N60 · 16W10 · 35A30 · 35G25 · 58J70

1 Introduction

Many integrable Hamiltonian systems, discovered during the last decades, were understood [15,16,24,38] owing to the Lie-algebraic properties of their internal hidden symmetry structures. A modern Lie-algebraic approach to describing such systems in many cases allows to represent them as some specially constructed orbits of some hidden group actions on the Poisson manifolds, generated by a set of the suitably con-

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structed Casimir invariants on the adjoint space to the corresponding symmetry Lie algebra. A first formal account of the related Hamiltonian operators and differential-algebraic structures, lying in the background of such integrable systems, was given by Gelfand and Dorfman [19,26] and later was extended by Dubrovin and Novikov [20,21], and also by Balinsky and Novikov [10–13]. There were also devised some new special differential-algebraic techniques [41] for studying the Lax type integrability and the structure of related Hamiltonian operators for a wide class of Riemann type hydrodynamic hierarchies. Just recently a lot of works appeared [7–9,40] being devoted to the finite-dimensional representations of the Novikov algebra. Their importance for constructing integrable multi-component nonlinear Camassa–Holm type dynamical systems on functional manifolds was demonstrated by Strachan and Szablikowski in [47], where there was suggested in part the Lie-algebraic imbedding of the Novikov algebra into the general Lie–Poisson orbits scheme of classification Lax type integrable Hamiltonian systems.

In the work we succeeded in formal differential-algebraic reformulating the classical Lie algebraic scheme and developed an effective approach to classification of algebraic structures lying in the background of integrable multicomponent Hamiltonian systems. We have devised a simple Lie-algebraic algorithm, allowing to construct new algebraic structures within which the corresponding linear and quadratic Hamiltonian operators, generated by the corresponding Lie–Poisson structure on the co-adjoint orbits, exist and describe the related integrable multicomponent dynamical systems. In these cases an interesting problem of describing the Balinsky–Novikov and Leibniz type algebras, whose multiplicative structures satisfy some additional tensor r-structure type relationships naturally arises. We studied also the non-associative structures of commutative algebras related with Balinsky–Novikov, Leibniz, Lie and Zinbiel algebras, having diverse important applications both in theory of integrable dynamical systems and to modern problems of communication technology.

2 Quadratic Poisson brackets: compatibility and related algebraic structures

Let $(A, +, \circ)$ be a finite-dimensional non-associative and non-commutative algebra of dimension $N = \dim A \in \mathbb{Z}_+$ over an algebraically closed field $\mathbb{K}$. To the algebra A one can naturally relate the loop algebra $\widetilde{A}$ of smooth mappings $u : \mathbb{S}^1 \to A$ and endow it with a suitably generalized natural convolution $\langle \cdot, \cdot \rangle$ on $\widetilde{A}^* \times \widetilde{A} \to \mathbb{K}$, where $\widetilde{A}^*$ is the corresponding adjoint to the space $\widetilde{A}$.

First, we shall consider a general scheme of constructing non-trivial ultra-local and local [24] quadratic Poisson structures [11,12,14,45] on the loop space $\widetilde{A}$, compatible with the internal multiplication in the algebra A. Namely, let $\{e_s \in A : s = \overline{1, N}\}$ be a basis of the algebra A and its dual $\{u^s \in \widetilde{A}^* : s = \overline{1, N}\}$ with respect to $\langle \cdot, \cdot \rangle$ on $\widetilde{A}^* \times A$, that is $\langle u^j, e_i \rangle := \delta_i^j$ for all $i, j = \overline{1, N}$, and such that for any

$$u(x) = \sum_{s=1}^{N} u^s(x; u) e_s \in \widetilde{A}, \quad x \in \mathbb{S}^1,$$

the quantities $u^s(x;u) := \langle u^s(x), u\rangle \in \mathbb{K}$ for all $s = \overline{1,N}$, $x \in \mathbb{S}^1$. Denote by $\widetilde{A}^* \wedge \widetilde{A}^* := \mathrm{Skew}(\widetilde{A}^* \otimes \widetilde{A}^*)$ and let $\vartheta^*\colon \widetilde{A}^* \wedge \widetilde{A}^* \to \mathrm{Symm}(\widetilde{A}^* \otimes \widetilde{A}^*)$ be a skew-symmetric bilinear mapping. Then for linear on $\widetilde{A}$ functions $a(u) := \langle a, u\rangle$ and $b(u) := \langle b, u\rangle$ for any $a, b \in \widetilde{A}^*$ the expression

$$\{a(u), b(u)\} := \langle \vartheta^*(a \wedge b), u \otimes u\rangle \tag{1}$$

defines an ultra-local *quadratic* skew-symmetric pre-Poisson bracket on $\widetilde{A}^*$. Since the algebra $\widetilde{A}$ possesses its internal multiplicative structure "$\circ$", the important problem [11,12] arises: *Under what conditions is the pre-Poisson bracket* (1) *Poisson and compatible with this internal structure on* $\widetilde{A}$? To proceed with elucidating this question, we define a co-multiplication $\Delta\colon \widetilde{A}^* \to \widetilde{A}^* \otimes \widetilde{A}^*$ on an arbitrary element $c \in \widetilde{A}^*$ by means of the relationship

$$\langle \Delta c, (w \otimes v)\rangle = \langle c, w \circ v\rangle \tag{2}$$

for arbitrary $w, v \in \widetilde{A}$. Note that the co-multiplication $\Delta\colon \widetilde{A}^* \to \widetilde{A}^* \otimes \widetilde{A}^*$, defined this way, is a *homomorphism* of the tensor algebra $\mathrm{T}^1(\widetilde{A}^*)$ into $\mathrm{T}^2(\widetilde{A}^*)$ and the linear pre-Poisson structure $\{\cdot,\cdot\}$ on $\widetilde{A}^*$ [see (1)] is called *compatible* with the multiplication "$\circ$" on the algebra $\widetilde{A}$, if the following invariance condition:

$$\Delta\{a(u), b(u)\} = \{\Delta a(u), \Delta b(u)\}$$

holds for all $a, b \in \widetilde{A}^*$ and arbitrary $u \in \widetilde{A}$. Now, taking into account that multiplication in the algebra A can be represented for any $i, j = \overline{1,N}$ by means of the relationship

$$e_i \circ e_j := \sum_{s=1}^{N} \sigma_{ij}^s e_s, \tag{3}$$

where the quantities $\sigma_{ij}^s \in \mathbb{K}$ for all i, j and $k = \overline{1,N}$ are constants, the co-multiplication $\Delta\colon A^* \to A^* \otimes A^*$ acts on the basic functionals $u^s \in \widetilde{A}^*$, $s = \overline{1,N}$, as

$$\Delta(u^s) = \sum_{i,j=1}^{N} \sigma_{ij}^s u^i \otimes u^j. \tag{4}$$

Additionally, if the mapping $\vartheta^*\colon \widetilde{A}^* \wedge \widetilde{A}^* \to \mathrm{Symm}(\widetilde{A}^* \otimes \widetilde{A}^*)$ is given, for instance, in the simple linear form

$$\vartheta^*\colon \left(u^i \otimes u^j - u^j \otimes u^i\right) \to \sum_{s,k=1}^{N} \left(c_{sk}^{ij} - c_{ks}^{ji}\right) u^s \otimes u^k, \tag{5}$$

the quantities $c_{sk}^{ij} \in \mathbb{K}$ are constant for all i, j and $s, k = \overline{1, N}$ and chosen to be symmetric in their lower indices, then for the adjoint to (5) mapping $\vartheta : \mathrm{Symm}(\widetilde{A} \otimes \widetilde{A}) \to \widetilde{A} \wedge \widetilde{A}$ one obtains the expression

$$\vartheta : (e_s \otimes e_k + e_k \otimes e_s) \to \sum_{i,j=1}^{N} c_{sk}^{ij} e_i \wedge e_j.$$

Recall that a linear mapping $\vartheta : A \to B$ from an algebra A to the A-bimodule B is called a *derivation* if for any $\lambda, \mu \in A$ there holds the Leibniz property

$$\vartheta(\lambda \cdot \mu) = \vartheta(\lambda)\mu + \lambda\vartheta(\mu). \tag{6}$$

The following theorem [11] gives an effective compatibility criterion for the multiplication in the algebra A.

Theorem 2.1 *The pre-Poisson bracket* (2) *is compatible with the multiplication* (3) *if and only if the mapping* $\vartheta : \mathrm{Symm}(\widetilde{A} \otimes \widetilde{A}) \to \widetilde{A} \wedge \widetilde{A}$ *is a derivation of the symmetric algebra* $\mathrm{Symm}(\widetilde{A} \otimes \widetilde{A})$.

Proof The idea of the proof consists in checking the relationships on the corresponding coefficients following both from (2) and (6) for basis elements $\lambda, \mu \in \mathrm{Symm}(\widetilde{A} \otimes \widetilde{A})$. □

Observe now that the pre-Poisson bracket (1) can be equivalently rewritten as

$$\langle a \wedge b, \{u \overset{\otimes}{,} u\}\rangle = \langle a \wedge b, \vartheta(u \otimes u)\rangle,$$

giving rise, owing to the arbitrariness of elements $a, b \in \widetilde{A}^*$, to the following tensor equality:

$$\{u \overset{\otimes}{,} u\} = \vartheta(u \otimes u) \tag{7}$$

with the derivation ϑ. As was remarked in [11,12], the following natural commutator expression:

$$\vartheta(\lambda) := [r, \lambda]$$

for any $\lambda \in \mathrm{Symm}(\widetilde{A} \otimes \widetilde{A})$ and a fixed skew-symmetric constant tensor $r \in \widetilde{A} \otimes \widetilde{A}$ is an inner derivation of the algebra $\mathrm{Symm}(\widetilde{A} \otimes \widetilde{A})$. Thus, one can consider a class of pre-Poisson brackets (7) in the following commutator tensor form:

$$\{u \overset{\otimes}{,} u\} = [r, u \otimes u] \tag{8}$$

and pose a problem of finding conditions on the tensor $r \in \widetilde{A} \otimes \widetilde{A}$ under which the pre-Poisson bracket (8) becomes a Poisson one.

If the algebra $\widetilde{A}$ is non-commutative and associative, the adjacent Lie algebra $\mathcal{L}_{\tilde{A}} \simeq \widetilde{A}$ makes it possible to construct the related formal Lie group $G_{\tilde{A}} := 1 + \widetilde{A}$, whose tangent space at the unity can be identified with the Lie algebra $\mathcal{L}_{\tilde{A}}$ of the right-invariant vector fields on $G_{\tilde{A}}$. For a fixed element $u \in G_{\tilde{A}}$ one can denote by $\rho_u, \xi_u : \mathcal{L}_{\tilde{A}} \to T_u(G_{\tilde{A}})$ the differentials of the right and left shifts on $G_{\tilde{A}}$, respectively. Let $\rho_u^*, \xi_u^* : T_u^*(G_{\tilde{A}}) \to \mathcal{L}_{\tilde{A}}^*$ be dual mappings, respectively. Then the following theorem, stated in [45], holds.

Theorem 2.2 *The following bracket:*

$$\{a(u), b(u)\} = \langle \rho_u^*(b), \mathcal{R}(\rho_u^*(a))\rangle - \langle \xi_u^*(b), \mathcal{R}(\xi_u^*(a))\rangle \tag{9}$$

for any $a, b \in T_u^*(G_{\tilde{A}})$ is Poisson if the homomorphism $\mathcal{R}: \widetilde{A} \to \widetilde{A}$, naturally related with the tensor $r \in \widetilde{A} \otimes \widetilde{A}$, is skew-symmetric and satisfies the modified Yang–Baxter relationship

$$\mathcal{R}([\alpha, \mathcal{R}\beta] + [\mathcal{R}\alpha, \beta]) = [\mathcal{R}\alpha, \mathcal{R}\beta] + [\alpha, \beta] \tag{10}$$

for all $\alpha, \beta \in \mathcal{L}_{\tilde{A}}$ subject to the Lie commutator structure in $\mathcal{L}_{\tilde{A}}$.

If to take into account that in this case there hold the expressions

$$\rho_u^*(c) = \Delta_2 c(u), \quad \xi_u^*(c) = \Delta_1 c(u)$$

for any $c \in \widetilde{A}^*$, where the mappings Δ_1 and Δ_2 stand for convolutions of the co-multiplication $\Delta : \widetilde{A} \to \widetilde{A} \otimes \widetilde{A}$ with the first and second tensor components, respectively, that is

$$\begin{aligned} \langle \Delta c, u \otimes \alpha\rangle &= \langle c, u \circ \alpha\rangle := \langle \Delta_1 c(u), \alpha\rangle, \\ \langle \Delta c, \alpha \otimes u\rangle &= \langle c, \alpha \circ u\rangle := \langle \Delta_2 c(u), \alpha\rangle \end{aligned}$$

for any $\alpha \in \widetilde{A}$, the bracket (9) will become

$$\{a(u), b(u)\} = \langle b, \mathcal{R}(\Delta_2 a(u)) \circ u\rangle - \langle b, u \circ \mathcal{R}(\Delta_2 a(u))\rangle \tag{11}$$

for any $a, b \in T_u^*(G_{\tilde{A}})$, which can be easily enough computed, if to take into account the relationship (4).

The following result [14,45] is a simple consequence of Theorem 2.2 in the case of the associative matrix algebra $\widetilde{A}$ and is almost classical.

Theorem 2.3 *Let the algebra $\widetilde{A}$ be matrix associative with respect to the standard multiplication, and endowed both with the natural commutator Lie structure $[\cdot, \cdot]$ and with the trace-type symmetric scalar product $\langle \cdot, \cdot \rangle := \mathrm{Tr}(\cdot \cdot)$. Define also for the tensor*

$$r := \sum_{i,j=1}^{N} r^{ij} e_i \otimes e_j \in \widetilde{A} \otimes \widetilde{A},$$

the related $\mathcal{R}$-homomorphism

$$\mathcal{R}\alpha := \sum_{i,j=1}^{N} r^{ij} e_i \langle e_j, \alpha\rangle \tag{12}$$

for any $\alpha \in \widetilde{A}$. Then the pre-Poisson bracket (11) is Poisson if the $\mathcal{R}$-homomorphism (12) is skew-symmetric and satisfies the modified Yang–Baxter relationship (10). Moreover, the Poisson bracket (11) can be equivalently rewritten in the following simplified form:

$$\{a(u), b(u)\} = \langle ub, \mathcal{R}(ua)\rangle - \langle bu, \mathcal{R}(au)\rangle$$

for any $a, b \in \widetilde{A}^*$.

Remark 2.4 The Yang–Baxter relationship (10) is basic for finding the corresponding internal multiplication structure of the algebra $\widetilde{A}$, allowing the quadratic Poisson bracket (11). If, for example, to assume that the adjacent loop Lie algebra $\mathcal{L}_{\tilde{A}}$ allows splitting into two subalgebras, $\mathcal{L}_{\tilde{A}} = \mathcal{L}_{\tilde{A}}^{+} \oplus \mathcal{L}_{\tilde{A}}^{-}$, then the homomorphism $\mathcal{R} = \mathcal{P}_{+} - \mathcal{P}_{-}$ solves the relationship (10), where, by definition, the mappings $\mathcal{P}_{\pm} : \mathcal{L}_{\tilde{A}} \to \mathcal{L}_{\tilde{A}}^{\pm} \subset \mathcal{L}_{\tilde{A}}$ are the suitable projections. If to assume, that the adjacent loop Lie algebra $\mathcal{L}_{\tilde{A}}$ is generated by the associative multiplication "$*$" of the Balinsky–Novikov loop algebra $\widetilde{A}$, then the related Lie structure is given by the commutator

$$[\alpha, \beta] := \alpha * \beta - \beta * \alpha \tag{13}$$

for the derivation $D_x = d/dx$ and any $\alpha, \beta \in \widetilde{A}$, giving rise to the ultra-local quadratic Poisson bracket (11). To our regret, we do not know whether the Lie structure

$$[\alpha, \beta] := \alpha \circ D_x \beta - \beta \circ D_x \alpha \tag{14}$$

for any $\alpha, \beta \in \widetilde{A}$ and all $x \in \mathbb{S}^1$, suitably determining the adjacent loop Lie algebra $\mathcal{L}_{\tilde{A}}$, can be generated by some associative multiplication on the loop Balinsky–Novikov algebra, with respect to which the Lie structure (14) could entail the local quadratic Poisson bracket (11).

Problem 2.5 Concerning the algebraic structures discussed above the interesting problem arises: Classify associative Balinsky–Novikov loop algebras $\widetilde{A}$, whose adjacent Lie algebras $\mathcal{L}_{\tilde{A}}$ allow splitting into two non-trivial subalgebras subject to the Lie structure (13).

Remark 2.6 In the case of basic Leibniz loop algebra $\widetilde{A}$, it is well known that the usual commutator structure (13) does not generate the adjacent loop Lie algebra $\mathcal{L}_{\tilde{A}}$, yet the following inverse-derivative Lie structure:

$$[\alpha, \beta] := \alpha \circ D_x^{-1} \beta - \beta \circ D_x^{-1} \alpha, \tag{15}$$

suitably determined for any $\alpha, \beta \in \widetilde{A}$ and all $x \in \mathbb{S}^1$, already does. Yet, we do not know whether the Lie structure (15) can be generated by some associative multiplication "$*$" on the loop Leibniz algebra $\widetilde{A}$.

3 Quadratic Poisson structures: the Lie–Poisson type generalization

Assume as above that $(A, +, \circ)$ is a finite-dimensional algebra of the dimension $N = \dim A \in Z_+$ (in general non-associative and non-commutative) over an algebraically closed field $\mathbb{K}$. Based on the algebra A one can construct the related loop algebra $\widetilde{A}$ of smooth mappings $u\colon \mathbb{S}^1 \to A$ and endow it with the suitably generalized natural convolution $\langle\cdot,\cdot\rangle$ on $\widetilde{A}^* \times \widetilde{A} \to \mathbb{K}$, where $\widetilde{A}^*$ is the corresponding adjoint space to $\widetilde{A}$.

First, we will consider a general scheme of constructing non-trivial ultra-local and local Poisson structures on the adjoint space $\widetilde{A}^*$ [24] compatible with the internal multiplication in the loop algebra $\widetilde{A}$. Consider a basis $\{e_s \in A : s = \overline{1,N}\}$ of the algebra A and its dual $\{e^s \in A^* : s = \overline{1,N}\}$ with respect to the natural convolution $\langle\cdot,\cdot\rangle$ on $A^* \times A$, that is $\langle e^j, e_i\rangle := \delta^j_i$, $i, j = \overline{1,N}$, and such that for any

$$u(x) = \sum_{s=1}^{N} u_s(x) e^s \in \widetilde{A}^*, \quad x \in \mathbb{S}^1,$$

the quantities $u_s(x) := \langle u(x), e_s\rangle \in \mathbb{K}$ for all $s = \overline{1,N}$, $x \in \mathbb{S}^1$. Denote by $\widetilde{A} \wedge \widetilde{A} := \mathrm{Skew}(\widetilde{A} \otimes \widetilde{A})$ and let $\vartheta^*\colon \widetilde{A} \wedge \widetilde{A} \to \mathrm{Symm}(\widetilde{A})$ be a skew-symmetric bilinear mapping. Then the expression

$$\{u(a), u(b)\} := \langle u(x), \vartheta^*(a \wedge b)\rangle \tag{16}$$

defines for any $a, b \in \widetilde{A}$ an ultra-local *linear* skew-symmetric pre-Poisson bracket on $\widetilde{A}^*$. If the mapping $\vartheta^*\colon \widetilde{A} \wedge \widetilde{A} \to \mathrm{Symm}(\widetilde{A})$ is given, for instance, in the simple linear form

$$\vartheta^*\colon (e_i \otimes e_j - e_j \otimes e_i) \to \sum_{s=1}^{N} \left(c^s_{ij} - c^s_{ji}\right) e_s, \tag{17}$$

where quantities $c^s_{ij} \in \mathbb{K}$ are constant for all i, j and $s = \overline{1,N}$, then for the adjoint to (17) mapping $\vartheta\colon \widetilde{A}^* \to \widetilde{A}^* \wedge \widetilde{A}^*$ one obtains the expression

$$\vartheta : e^s \to \sum_{i,j=1}^{N} \left(c^s_{ij} - c^s_{ji}\right) e^i \otimes e^j. \tag{18}$$

For the pre-Poisson bracket to be a Poisson bracket on $\widetilde{A}^*$, it should satisfy additionally the Jacobi identity. To find the corresponding additional constraints on the internal multiplication "$\circ$" on the algebra $\widetilde{A}$, define for any $u(x) \in \widetilde{A}^*$ the skew-symmetric linear mapping

$$\vartheta(u) \colon \widetilde{A} \to \widetilde{A}^*, \tag{19}$$

called by the *Hamiltonian operator* [26], via the identity

$$\langle \vartheta(u)a, b\rangle := \langle \vartheta u(x), a \wedge b\rangle$$

for any $a, b \in \widetilde{A}$, where the mapping $\vartheta \colon \widetilde{A}^* \to \widetilde{A}^* \wedge \widetilde{A}^*$ is determined by the expression (18), being adjoint to it. Then it is well known [26] that the pre-Poisson bracket (16) is a Poisson one if and only if the Hamiltonian operator (19) satisfies the Schouten–Nijenhuis condition

$$[[\vartheta(u), \vartheta(u)]] = 0 \tag{20}$$

for any $u(x) \in \widetilde{A}^*$. Since

$$\vartheta(u)e_i = \sum_{s,k=1}^{N} \left(c_{ik}^s - c_{ki}^s\right) u_s(x) e^k \tag{21}$$

holds for any basis element $e_i \in A$, $i = \overline{1, N}$, the resulting pre-Poisson bracket (16) is equal to

$$\begin{aligned} \{u(a), u(b)\} = \langle \vartheta(u)a, b\rangle &= \sum_{s=1}^{N} \sum_{i,j=1}^{N} \left(c_{ij}^s - c_{ji}^s\right) a^i b^j u_s(x) \\ &= \left\langle u(x), \sum_{i,j=1}^{N} \left(c_{ij}^s - c_{ji}^s\right) a^i b^j e_s \right\rangle \end{aligned} \tag{22}$$

for any $u(x) \in \widetilde{A}^*$ and all $a, b \in \widetilde{A}$. If now to define on the algebra A the natural adjacent Lie algebra structure to the algebra A

$$[e_i, e_j] = e_i \circ e_j - e_j \circ e_i := \sum_{s=1}^{N} \left(c_{ij}^s - c_{ji}^s\right) e_s \tag{23}$$

for any basis elements $e_i, e_j \in A$, $i, j = \overline{1, N}$, the expression (22) yields for all $a, b \in \widetilde{A}$ the well-known [1,4] *classical Lie–Poisson bracket*

$$\{u(a), u(b)\} = \langle u, [a, b]\rangle. \tag{24}$$

Concerning the adjacent Lie algebra structure condition (23), it can be easily rewritten as the set of relationships

$$\sigma_{ij}^{s}-\sigma_{ji}^{s}=c_{ij}^{s}-c_{ji}^{s},$$

whose obvious solution is

$$c_{ij}^{s}=\sigma_{ij}^{s} \tag{25}$$

for any $i, j, s=\overline{1, N}$. As the bracket (24) is of the classical Lie–Poisson type, for the Hamiltonian operator (21) to satisfy the Schouten–Nijenhuis condition (20) it is enough to check only the weak Jacobi identity for the loop Lie algebra $\mathcal{L}_{\tilde{A}}$, adjacent to the algebra $\widetilde{A}$ via imposing the Lie structure (23), taking into account the relationships (25). For instance, if the commutator of the adjacent loop Lie algebra $\mathcal{L}_{\tilde{A}}$ is given by the expression

$$[a, b]=a\circ D_x b-b\circ D_x a, \tag{26}$$

the corresponding algebra A coincides with the well-known *Balinsky–Novikov algebra*, determined by means of the following relationships:

$$[L_a, L_b]=L_{[a,b]}, \quad [R_{a,} R_b]=0, \tag{27}$$

where, by definition, $R_a b=b\circ a=L_b a$ for any $a, b\in\widetilde{A}$. If, for instance, the commutator of the adjacent loop Lie algebra $\mathcal{L}_{\tilde{A}}$ is given by the expression

$$[a, b]=a\circ D_x^{-1}b-b\circ D_x^{-1}a \tag{28}$$

for a suitably determined inverse-derivation mapping $D_x^{-1}\colon \widetilde{A}\rightarrow\widetilde{A}$, the corresponding algebra A coincides with the well-known *right Leibniz algebra*, described by the relationships

$$[R_a, R_b]=R_{a\circ b}, \quad [R_{a,} R_b]=0 \tag{29}$$

for any $a, b\in\widetilde{A}$. As a consequence of reasonings above one can formulate the following general theorem.

Theorem 3.1 *The linear pre-Poisson bracket* (24) *on* $\widetilde{A}^*$ *is Lie–Poisson on the adjoint space* $\mathcal{L}_{\tilde{A}}^*$ *if and only if the internal multiplicative structure of the algebra* A *is compatible with the weak Lie algebra structure on the adjacent loop Lie algebra* $\mathcal{L}_{\tilde{A}}$.

Similarly, one can consider a simple ultra-local quadratic pre-Poisson bracket on $\widetilde{A}^*$ in the form

$$\{u(a), u(b)\}:=\langle u(x)\otimes u(x), \vartheta^*(a\wedge b)\rangle \tag{30}$$

for any $a, b \in \widetilde{A}$, where the skew-symmetric mapping $\vartheta^*\colon \widetilde{A} \wedge \widetilde{A} \to \mathrm{Symm}(\widetilde{A} \otimes \widetilde{A})$ is given for any $i, j = \overline{1, N}$ in the quadratic form

$$\vartheta^*(e_i \otimes e_j - e_j \otimes e_i) := \sum_{k,s=1}^{N} \left(c_{ij}^{ks} - c_{ji}^{ks}\right)(e_k \otimes e_s + e_s \otimes e_k).$$

In particular, if to assume that the coefficients $c_{ij}^{ks} = \sigma_{ij}^k \alpha^s$ for some constant numbers σ_{ij}^k and $\alpha^s \in \mathbb{K}$ for all i, j and $k, s = \overline{1, N}$, where, by definition, the multiplications

$$e_k \circ e_s := \sum_{k=1}^{N} \sigma_{ij}^k e_k$$

coincides with that of the algebra A, then the pre-Poissson bracket (30) yields for any $a, b \in A$ a very compact form

$$\{u(a), u(b)\} := \langle u(x) \otimes u(x), \alpha \otimes [a, b] + [a, b] \otimes \alpha \rangle, \tag{31}$$

generalizing the classical Lie–Poisson expression (24) and parametrically depending on the constant vector

$$\alpha := \sum_{s=1}^{N} \alpha^s e_s \in A.$$

Thus, for the pre-Lie–Poisson bracket (31) one can formulate suitable constraints on the algebraic structure of A. For instance, if the weak algebraic structure on the adjacent Lie algebra $\mathcal{L}_{\tilde{A}}$ is given, respectively, either by the Lie commutator (26) or by (28), then the corresponding multiplicative structures of the algebra A are generated, respectively, by the Balinsky–Novikov (27) and Leibniz (29) algebras relationships, augmented with the following common tensor multiplicative constraint:

$$R_a \otimes R_\alpha = 0 = R_\alpha \otimes R_a, \tag{32}$$

which holds for any $a \in A$ and arbitrary but fixed element $\alpha \in A$. So, one can formulate the following theorem.

Theorem 3.2 *The quadratic pre-Lie–Poisson bracket* (31) *on* $\widetilde{A}^*$ *is Poisson if and only if the internal multiplicative structure of the algebra* A *is compatible both with the weak Lie algebra structure on the adjacent loop Lie algebra* $\mathcal{L}_{\tilde{A}}$ *and with the tensor multiplicative relationships* (32).

In these cases there arises an interesting problem of describing the Balinsky–Novikov and Leibniz algebras, whose multiplicative structures additionally satisfy tensor relationships (32). Such and related algebraic structure problems are planned to be studied in detail elsewhere. In the next section we proceed to study general algebraic structures related both with generalized Balinsky–Novikov and Leibniz algebras, and so-called Zinbiel algebras, having diverse important applications in communications technology.

4 Algebraic structures preliminaries

Let A be an associative commutative algebra over a field $\mathbb{K}$ of any finite or infinite dimension (with the addition "$+$" and multiplication "$\cdot$") not necessary with identity and δ its derivation, i.e., $\delta\colon A \to A$ is a $\mathbb{K}$-linear map satisfying the Leibniz rule. Then

$$A^{\delta,\xi} = (A, +, *)$$

is a *Balinsky–Novikov algebra* (the so-called δ-*adjancent* or δ-*associated Balinsky–Novikov algebra* of A) with respect to "$*$" defined by the rule

$$a * b = a \cdot \delta(b) + \xi \cdot a \cdot b$$

(where ξ is a fixed element of A) and so

$$(a * b) * c = (a * c) * b$$

and

$$(a * b) * c - a * (b * c) = (b * a) * c - b * (a * c)$$

for all $a, b, c \in A$. In particular, $A^{\delta,0} := A^{\delta,\xi}$ with $\xi = 0$. Balinsky–Novikov algebras were introduced in connection with the so-called Hamiltonian operators [26] and Poisson brackets of hydrodynamic type [13]. Note here, that the term "*Novikov algebra*" was suggested by Osborn in [40]. Moreover,

$$A^{\delta,L} = (A, +, [-, -])$$

is a Lie algebra (the so-called δ-*adjancent* or δ-*associated Lie algebra* of A) with respect to the Lie bracket "$[-, -]$" defined by the rule

$$[a, b] = a * b - b * a$$

for any $a, b \in A$, see [29,30,37], [42, p. 285] and [43, p. 245].

Let $(D, +, \diamond)$ be a (Lie, Balinsky–Novikov, Zinbiel or associative) algebra with the derivation algebra Der D, $\varnothing \neq \Delta \subseteq$ Der D and $\theta \in$ Der D. Then ann $W := \{a \in D : a \diamond W = 0 = W \diamond a\}$ is the annihilator of $W \subseteq D$ and $Z(D) := \{z \in D : z \diamond a = a \diamond z$ for all $a \in D\}$ is the center of D. If I is an ideal of D and $\theta(I) \subseteq I$ (respectively, for any $\theta \in \Delta$), then we say that I is a θ-*ideal* (respectively, Δ-*ideal*) of D. Recall that D is called:

- Δ-*simple* if $D \diamond D \neq 0$ and any Δ-ideal I of D is 0 or D,
- Δ-*prime* if, for any Δ-ideals B, C of D, the condition $B \diamond C = 0$ implies that $B = 0$ or $C = 0$,
- Δ-*semiprime* if, for any Δ-ideal B of D, the condition $B \diamond B = 0$ implies that $B = 0$.

Every Δ-prime algebra is Δ-semiprime and every Δ-simple algebra is Δ-prime. If $\Delta = \{\theta\}$ and D is a Δ-simple (respectively, Δ-prime or Δ-semiprime), then we say that D is θ-simple (respectively, θ-prime or θ-semiprime). Moreover, if $\Delta = \{0\}$, then a Δ-simple (respectively, Δ-prime or Δ-semiprime) algebra is simple (respectively, prime or semiprime). Any unexplained terminology is standard as in [27,28,31,34].

The purpose of this paper is also to study relationships between associative commutative algebras A, their δ-associated Balinsky–Novikov algebras $A^{\delta,\xi}$ and δ-associated Lie algebras $A^{\delta,L}$. Connections between properties of an associative commutative algebra A and its δ-associated algebra $A^{\delta,L}$ have been investigated by Ribenboim [43], Jordan, Jordan [29,30], and Nowicki [37]. Xu [49] found some classes of infinite-dimensional simple Balinsky–Novikov algebras of type $A^{\delta,\xi}$. Bai and Meng [6] proved that, if A is a finite-dimensional associative commutative algebra and $0 \neq \delta \in \operatorname{Der} A$, then $A^{\delta,0}$ is transitive (i.e., $r_a \colon A \ni x \mapsto x * a = x \cdot \delta(a) \in A$ is a nilpotent right transformation operator of $A^{\delta,0}$ for any $a \in A$) and $A^{\delta,L}$ is a solvable Lie algebra [32]. In [48, Proposition 2.8] it is proved that the Balinsky–Novikov algebra $A^{\delta,\xi}$ is simple if and only if an associative commutative ring A is δ-simple. As noted in [5], there is a conjecture: *the Balinsky–Novikov algebras N can be realized as the algebras $A^{\delta,0}$, where A is a suitable associative commutative algebras, and their* (*compatible*) *linear deformation.* Recall that a binary operation $G_1 \colon N \times N \to N$ of a Balinsky–Novikov algebra $(N, +, *)$ is called its *linear deformation* if algebras $(N, +, g_t)$, where $g_t(a, b) = a * b + tG_1(a, b)$, are Balinsky–Novikov algebras for every t. If G_1 is commutative, then it is called *compatible*.

As noted in [5], a "good" structure theory for algebraic systems means an existence of a well-defined radical and the quotient by the radical is semisimple. Our result in this direction is the following.

Theorem 4.1 *Let A be an associative commutative algebra with* 1*,* char $\mathbb{K} \neq 2$*,* $0 \neq \delta \in \operatorname{Der} A$ *and $\xi \in A$. Then the following conditions are equivalent:*

(i) *A is a δ-semiprime* (*respectively, δ-prime or δ-simple*) *algebra,*
(ii) *$A^{\delta,\xi}$ is a semiprime* (*respectively, prime or simple*) *Balinsky–Novikov algebra,*
(iii) *$A^{\delta,L}$ is a semiprime* (*respectively, prime or simple*) *Lie algebra.*

A triple $(Z, +, \circ)$ is called a *Zinbiel algebra* (or a *dual Leibniz algebra*) if

- $(Z, +)$ is an abelian group,
- $(x \circ y) \circ z = x \circ (y \circ z) + x \circ (z \circ y)$,
- $(x + y) \circ z = (x \circ z) + (y \circ z)$ and $x \circ (y + z) = (x \circ y) + (x \circ z)$

for all $x, y, z \in Z$. As a consequence, $(x \circ y) \circ z = (x \circ z) \circ y$. If $x \odot y = x \circ y + y \circ x$ for all $x, y \in Z$ (see Lemma 5.1), then $(Z, +, \odot)$ is an associative commutative algebra (the so-called *adjacent* or *associated associative algebra* Z^A of a Zinbiel algebra Z). Zinbiel algebras were introduced by Loday in [33,34] and are very popular in the control theory (in context of "chronological" algebras, see e.g. [2,3,31,44]) and in the theory of Leibniz cohomology [35]. Zinbiel rings can be defined by analogy.

Some interesting properties of Zinbiel algebras were obtained by Dzhumadil'daev, Tulenbaev [22,23], and Omirov [39]. In particular, Dzhumadil'daev [22] proved that any finite-dimensional Zinbiel algebra over the complex numbers field is nilpotent. We prove the next result.

Proposition 4.2 *Let Z be a Zinbiel algebra over a field $\mathbb{K}$. If the characteristic* $\operatorname{char}\mathbb{K} = p > 0$ *is prime, then the associated associative algebra Z^A is nil of bounded degree p.*

5 An associative commutative structure of a Zinbiel algebra

Recall that a (Zinbiel or associative) algebra $(A, +, \diamond)$ is called *reduced* if the implication $a \diamond a = 0 \Rightarrow a = 0$ is true for any $a \in A$.

Lemma 5.1 [18, Theorem 3.4] *If $(Z, +, \circ)$ is a Zinbiel algebra, then $(Z, +, \odot)$ is an associative commutative algebra, where "$\odot$" is defined by the rule $a \odot b = a \circ b + b \circ a$ for any $a, b \in Z$.*

An additive subgroup I of a Zinbiel ring (respectively, algebra) Z is said to be an *associative ideal* of Z if $I \odot Z \subseteq I$. It is easy to see that I is an associative ideal of Z if and only if it is an ideal of Z^A. A Zinbiel ring Z is called 2*-torsion-free* if, for any $x \in Z$, $2x = 0$ implies that $x = 0$.

Lemma 5.2 *Let Z be a Zinbiel ring (respectively, algebra over a field $\mathbb{K}$), $\varnothing \neq \Delta \subseteq$* $\operatorname{Der} Z$ *and $a \in Z$. Then the following properties hold:*

(i) *$a \circ Z := \{a \circ z : z \in Z\}$ is a right ideal of Z,*
(ii) *if X is a non-empty subset of Z, then the right annihilator* $\operatorname{rann} X := \{t \in Z : X \circ t = 0\}$ *of X is an associative ideal of Z and* $\operatorname{rann} X \subseteq \operatorname{rann}(X \circ Z)$,
(iii) *if I is a right Δ-ideal of Z, then $Z \circ I$ and $I + (Z \circ I)$ are Δ-ideals of Z,*
(iv) *if X is an associative Δ-ideal of Z, then the left annihilator* $\operatorname{lann} X := \{t \in Z : t \circ X = 0\}$ *is a right Δ-ideal of Z,*
(v) *if Z is 2-torsion-free (respectively,* $\operatorname{char}\mathbb{K} \neq 2$*), then Z is reduced if and only if Z^A is reduced,*
(vi) *if Z is 2-torsion-free (respectively,* $\operatorname{char}\mathbb{K} \neq 2$*) and I, J are commutating ideals of Z (i.e., $i \circ j = j \circ i$ for $i \in I$ and $j \in J$), then $I \circ J \subseteq$* $\operatorname{rann} Z$,
(vii) *if K is an additive Δ-closed subgroup of Z, then $S(K) := \{a \in K : a \circ Z \subseteq K\}$ is a right Δ-ideal of Z,*
(viii) *if I, J are Δ-ideals of Z, then $I \circ J$ is a right Δ-ideal of Z,*
(ix) *if K is an associative ideal of Z, then $K \circ K \subseteq S(K)$,*
(x) *if $e = e^2 \in Z^A$, then $e = 0$.*

Proof Let $z, t \in Z$.

(i) Clearly $a \circ Z$ is a subgroup of the additive group $(Z, +)$ and $(a \circ z) \circ t = a \circ (z \circ t) + a \circ (t \circ z) \in a \circ Z$. Hence $a \circ Z$ is a right ideal of Z.

(ii) If $x \in X$, $u \in \operatorname{rann} X$, then $0 = (x \circ u) \circ z = x \circ (u \odot z)$ and so $u \odot z \in \operatorname{rann} X$. Moreover, $(x \circ z) \circ u = x \circ (z \odot u) \in X \circ \operatorname{rann} X = 0$.

(iii) Assume that $i, j \in I$. Since $(z \circ t) \circ i, z \circ (i \circ t), t \circ ((i \circ z), (t \circ z) \circ i \in Z \circ I$, $(z \circ i) \circ t = (z \circ t) \circ i = z \circ (t \circ i) + z \circ (i \circ t)$ and $(t \circ z) \circ i = t \circ (z \circ i) + t \circ (i \circ z)$, we deduce that $z \circ (t \circ i), t \circ (z \circ i) \in Z \circ I$. We also see that

$$t \circ (i + z \circ j) = t \circ i + t \circ (z \circ j) = t \circ i + (t \circ z) \circ j - t \circ (j \circ z) \in (Z \circ I) + I$$

and

$$(i + z \circ j) \circ t = i \circ t + (z \circ j) \circ t = i \circ t + (z \circ t) \circ j \in (Z \circ I) + I.$$

(iv) If $a \in \operatorname{lann} X, i \in X$, then $(a \circ t) \circ i = a \circ (t \odot i) = 0$.

(v) It follows from $z \circ z = 0 \Leftrightarrow z \odot z = 0$.

(vi) Assume that $i \in I$ and $j \in J$. Then

$$\begin{aligned} j \circ (i \circ z) = (i \circ z) \circ j = i \circ (z \circ j) + i \circ (j \circ z) &= (z \circ j) \circ i + (j \circ z) \circ i \\ &= z \circ (i \circ j) + z \circ (i \circ j) + j \circ (z \circ i) + j \circ (i \circ z) \end{aligned}$$

and from this

$$2(z \circ (i \circ j)) = -(j \circ (z \circ i)). \tag{33}$$

On the other hand,

$$j \circ (z \circ i) = (z \circ i) \circ j = z \circ (i \circ j) + z \circ (j \circ i) = 2(z \circ (i \circ j)). \tag{34}$$

Then (33) and (34) imply $z \circ (i \circ j) = 0$.

(vii) If $u \in S(K)$, then

$$(u \circ t) \circ z = u \circ (t \circ z) + u \circ (z \circ t) = u \circ (t \odot z) \in u \circ Z \subseteq K.$$

(viii) Straightforward.

(ix) If $a, b \in K$, then $(a \circ b) \circ z = a \circ (b \odot z) \in K \circ K \subseteq K$ and the assertion holds.

(x) Since $e = e \circ e = (e \circ e) \circ e = e \circ (e \circ e) + e \circ (e \circ e) = e \circ e + e \circ e = 2e$, we conclude that $e = 0$. □

Lemma 5.3 *If Z is a Zinbiel ring (respectively, algebra) and $\varnothing \neq \Delta \subseteq \operatorname{Der} Z$, then the following properties hold:*

(i) *if K is an associative Δ-ideal of Z, then $S_0(K) = S(K) + (Z \circ S(K))$ is a Δ-ideal of Z,*
(ii) *if I, J are associative ideals of Z such that $I \odot J = 0$, then $S_0(I) \circ S_0(J) = 0$.*

Proof (i) By Lemma 5.2 (vii), (ix) and (iii), $S_0(K)$ is a Δ-ideal of A.

(ii) Let $a \in S(I)$, $b \in S(J)$ and $z, t \in Z$. Then $0 = b \odot a = b \circ a + a \circ b$ and we have $a \circ b = -b \circ a$. Since $(z \circ b) \circ a = z \circ (b \odot a) = 0$,

$$\begin{aligned} a \circ (z \circ b) &= (a \circ z) \circ b - a \circ (b \circ z) \\ &= (a \circ z) \circ b + (b \circ z) \circ a = (a \circ b) \circ z + (b \circ a) \circ z = (a \odot b) \circ z = 0 \end{aligned}$$

and

$$\begin{aligned} (t \circ a) \circ (z \circ b) &= t \circ (a \circ (z \circ b)) + t \circ ((z \circ b) \circ a) \\ &= t \circ (a \circ (z \circ b)) + t \circ (z \circ (b \odot a)) = t \circ (a \circ (z \circ b)) = 0, \end{aligned}$$

we conclude that $S_0(I) \circ S_0(J) = 0$. □

Corollary 5.4 *Let Z be a Zinbiel ring (respectively, algebra) and $\varnothing \neq \Delta \subseteq \mathrm{Der}\, Z$. If Z^A is Δ-simple (respectively, Δ-prime or Δ-semiprime), then Z is the same.*

Proof If $\Delta \subseteq \mathrm{Der}\, Z$, then $\Delta \subseteq \mathrm{Der}\, Z^A$.

Simplicity. Since every Δ-ideal of Z is a Δ-ideal of Z^A, the simplicity of Z^A implies that Z is simple.

Primeness. Let Z^A be a Δ-prime ring (respectively, algebra) and I, J be Δ-ideals of Z such that $I \circ J = 0$. Then I, J, and $J \cap I$ are Δ-ideals of Z^A and

$$(J \cap I) \circ (J \cap I) \subseteq I \circ J = 0.$$

Since $(J \cap I) \odot (J \cap I) = 0$, we conclude that $J \circ I \subseteq J \cap I = 0$. But then $I \odot J = 0$ and consequently $I = 0$ or $J = 0$.

Semiprimeness. By analogy as in the prime case. □

Proof of Proposition 4.2 Let $a \in Z$. By $a^{\odot n}$ we denote the n-th power of a in the algebra Z^A (n is a positive integer). We have

- $a^{\odot 2} = a \odot a = 2!\,(a \circ a)$,
- $a^{\odot 3} = (2!\,(a \circ a)) \odot a = 3!\,(a \circ (a \circ a))$.

Now assume that

$$a^{\odot (p-1)} = (p-1)!\,(\underbrace{a \circ (a \circ \cdots \circ (a \circ a) \cdots)}_{p-1 \text{ times}})$$

and compute

$$
\begin{aligned}
a^{\odot p} &= a^{\odot(p-1)} \odot a = a \circ a^{\odot(p-1)} + a^{\odot(p-1)} \circ a \\
&= (p-1)!\,(\underbrace{a \circ (a \circ \cdots \circ (a \circ a) \cdots)}_{p \text{ times}}) + (p-1)!\,((a \circ (\underbrace{a \circ \cdots \circ (a \circ a) \cdots}_{p-2 \text{ times}})) \circ a) \\
&= (p-1)!\,(\underbrace{a \circ (a \circ \cdots \circ (a \circ a) \cdots)}_{p \text{ times}}) + (p-1)!\,((a \circ (\underbrace{(a \circ \cdots \circ (a \circ a) \cdots)}_{p-2 \text{ times}} \circ a)) \\
&\qquad + (a \circ (\underbrace{(a \circ \cdots \circ (a \circ a) \cdots)}_{p-2\text{times}} \circ a))) \\
&= \cdots = p!(\underbrace{a \circ (a \circ \cdots \circ (a \circ a) \cdots)}_{p \text{ times}}) = 0.
\end{aligned}
$$

□

6 Balinsky–Novikov properties

Lemma 6.1 ([25,26,43], [48, Lemma 2.3] and [50, Proposition 2.4]) *If A is an associative commutative algebra, $\delta \in \operatorname{Der} A$, and $\xi \in A$, then $A^{\delta,\xi}$ is a Balinsky–Novikov algebra.*

Lemma 6.2 *Let A be an associative commutative algebra, $\delta \in \operatorname{Der} A$, and $\xi \in A$. Then we have:*

(i) *$d \in \operatorname{Der} A^{\delta,\xi}$ if and only if $[d, \delta](b) + d(\xi) \cdot b \in \operatorname{ann} A$ for all $b \in A$,*
(ii) *if $1 \in A$, then $d \in \operatorname{Der} A^{\delta,\xi}$ if and only if $[d, \delta](b) + d(\xi) \cdot b = 0$ for all $b \in A$,*
(iii) *$d \in \operatorname{Der} A^{\delta,0}$ if and only if $A \cdot [d, \delta] = 0$,*
(iv) *if $1 \in A$, then $d \in \operatorname{Der} A^{\delta,0}$ if and only if $[d, \delta] = 0$.*

Proof (i) For any $a, b \in A$ and $d \in \operatorname{Der} A^{\delta,\xi}$ we have

$$
\begin{aligned}
& d(a) \cdot \delta(b) + a \cdot d(\delta(b)) + d(\xi) \cdot a \cdot b + \xi \cdot d(a) \cdot b + \xi \cdot a \cdot d(b) \\
&\quad = d(a \cdot \delta(b) + \xi \cdot a \cdot b) = d(a * b) = d(a) * b + a * d(b) \\
&\quad = d(a) \cdot \delta(b) + \xi \cdot d(a) \cdot b + a \cdot \delta(d(b)) + \xi \cdot a \cdot d(b)
\end{aligned}
$$

if and only if $a \cdot [d, \delta](b) + d(\xi) \cdot a \cdot b = 0$.

(ii)–(iv) The rest follows from part (i). □

Lemma 6.3 *Let δ be a surjective derivation of an associative commutative algebra A with 1. If I is a right ideal of a Balinsky–Novikov algebra $A^{\delta,\xi}$, then I is an ideal of A.*

Proof Indeed, if $i \in I$ and $a \in A$, then $I \ni i * a = i \cdot \delta(a) + \xi \cdot i \cdot a$ and therefore $\xi \cdot i = i * 1 \in I$. Since δ is surjective, we have $i \cdot \delta(a) = i * a - \xi \cdot i \cdot a \in I$ and so $i \cdot A \subseteq I$. □

It is easy to see that $e * e = 0$ for any idempotent $e^2 = e \in A$.

Lemma 6.4 *Let A be an associative commutative algebra, $\delta \in \operatorname{Der} A$, and $\xi \in A$. Then the following properties hold:*

(i) [37, Lemma 3.1] *if* $\operatorname{char} \mathbb{K} \neq 2$ *and* U *is an ideal of the Lie algebra* $A^{\delta,L}$*, then* $[U, U] = 0$ *or* U *contains a non-zero* δ*-ideal of* A*,*
(ii) *if* I *is a* δ*-ideal of* A*, then* I *is an ideal of the Balinsky–Novikov algebra* $A^{\delta,\xi}$*,*
(iii) *if* K *is an additive subgroup of a Balinsky–Novikov algebra* $A^{\delta,\xi}$ *and* $\delta(K) \subseteq K$*, then* $I_A(K) := \{k \in A : k \cdot A \subseteq K\}$ *is a* δ*-ideal of* A*,*
(iv) *if* $1 \in A$ *and* B *is an ideal of* $A^{\delta,\xi}$*, then* $\xi \cdot B, \delta(B) \subseteq B$*,*
(v) *if* C *is a left ideal of* $A^{\delta,0}$*, then* $\delta(C) \subseteq I_A(C)$*,*
(vi) *if* I *is a* δ*-ideal of* A*, then* I *is an ideal of* $A^{\delta,L}$*,*
(vii) *if* e *is an idempotent of* A*, then* $e \in \operatorname{rann} A^{\delta,0}$*,*
(viii) *the kernel* $\ker \delta := \{a \in A^{\delta,0} : \delta(a) = 0\}$ *of* δ *is a left ideal of* $A^{\delta,0}$*,*
(ix) *if* $\delta(a) \in a \cdot A$*, then* $a \cdot A$ *is an ideal of* $A^{\delta,L}$*,*
(x) *if* B *is an ideal of* $A^{\delta,\xi}$*, then* B *is an ideal of the Lie algebra* $A^{\delta,L}$*,*
(xi) *if* S *is an ideal of* $A^{\delta,\xi}$*, then* $T_A(S) := \{s \in S : s * A \subseteq S\}$ *is an ideal of* $A^{\delta,L}$ *and* $T_A(S) \subseteq S$*,*
(xii) *if* $1 \in A$ *and* I *is an ideal* $A^{\delta,L}$*, then* $\delta(I) \subseteq I$ *and* $I \cdot A$ *is a* δ*-ideal of* A*,*
(xiii) *if* W *is an ideal of* $A^{\delta,\xi}$*, then* $\delta(w) - \xi \cdot w \in I_A(W)$ *for any* $w \in W$*,*
(xiv) *if* $\operatorname{char} \mathbb{K} \neq 2$ *and* $a \cdot a = 0$*, then* $a \cdot A$ *is a right ideal of* $A^{\delta,\xi}$ *such that* $(a \cdot A) * (a \cdot A) = 0$.

Proof (i) For the proof see [37].

(ii) Indeed, $i * a = i \cdot \delta(a) + \xi \cdot i \cdot a \in I$ and $a * i = a \cdot \delta(i) + \xi \cdot i \cdot a \in I$ for any $i \in I$ and $a \in A$.

(iii) Assume that $k \in I_A(K)$ and $x \in A$. Then $(x \cdot k) \cdot A = k \cdot (x \cdot A) \subseteq k \cdot A \subseteq K$, what implies that $I_A(K)$ is an ideal of A. Since $\delta(k) \cdot A + k \cdot \delta(A) = \delta(k \cdot A) \subseteq \delta(K) \subseteq K$ and $k \cdot \delta(A) \subseteq K$, we conclude that $\delta(k) \cdot A \subseteq K$. Hence $\delta(I_A(K)) \subseteq I_A(K)$.

(iv) It is easy to see that $B \ni b * 1 = b \cdot \delta(1) + \xi \cdot b = \xi \cdot b$ and $B \ni 1 * b = \delta(b) + \xi \cdot b$ for any $b \in B$. Consequently, $\delta(B), \xi \cdot B \subseteq B$.

(v) For any $a \in A$ and $c \in C$ we see that $C \ni a * c = a \cdot \delta(c)$, whence $\delta(C) \subseteq I_A(C)$.

(vi) If $i \in I$ and $a \in A$, then $[i, a] = i \cdot \delta(a) - a \cdot \delta(i) \in I$, the claim follows.

(vii) Since $\delta(e) = 0$, we obtain $A * e = A \cdot \delta(e) = 0$.

(viii) If $u \in \ker \delta$ and $a \in A$, then $\delta(a * u) = \delta(a \cdot \delta(u)) = 0$. Hence $a * u \in \ker \delta$.

(ix) For any $t, b \in A$

$$\begin{aligned}[a \cdot t, b] = (a \cdot t) * b - b * (a \cdot t) &= a \cdot t \cdot \delta(b) - b \cdot \delta(a) \cdot t - b \cdot a \cdot \delta(t) \\ &= a \cdot [t, b] - b \cdot \delta(a) \cdot t \in a \cdot A.\end{aligned}$$

(x) Since $B * A \subseteq B$, $A * B \subseteq B$, we deduce that $[B, A] \subseteq B$.

(xi) Let $a, x \in A$ and $s \in T_A(S)$. Then

$$(s * x) * a = s * (x * a) - x * (s * a) + (x * s) * a$$

and therefore $[s, x] * a = s * (x * a) - x * (s * a) \in S$.

(xii) For any $i \in I$,

$$I \supseteq [I, A] \ni [i, 1] = i * 1 - 1 * i = i \cdot \delta(1) - 1 \cdot \delta(i) = -\delta(i).$$

(xiii) By (x), W is an ideal of the Lie algebra $A^{\delta,L}$ and so $W \ni w * a = w \cdot \delta(a) + \xi \cdot w \cdot a$ and $W \ni [w, a] = w * a - a * w = w \cdot \delta(a) - a \cdot \delta(w)$ for any $w \in W$ and $a \in A$. Then $W \ni w * a - [w, a] = (\xi \cdot w + \delta(w)) \cdot a$. Hence $\delta(w) + \xi \cdot w \in I_A(W)$.

(xiv) We compute that $0 = \delta(a \cdot a) = 2a \cdot \delta(a)$, $a * a = a \cdot \delta(a) + \xi \cdot a \cdot a = 0$ and

$$\begin{aligned}(a \cdot A) * (a \cdot A) &\ni a \cdot b \cdot \delta(a \cdot c) + \xi \cdot a \cdot b \cdot a \cdot c \\ &= a \cdot b \cdot \delta(a) \cdot c + a \cdot b \cdot a \cdot \delta(c) = 0\end{aligned}$$

for any $b, c \in A$. □

If $x \in A$, then

$$l_x \colon A^{\delta,\xi} \ni a \mapsto x * a \in A^{\delta,\xi}$$

is called a *left transformation operator* of the Balinsky–Novikov algebra $A^{\delta,\xi}$.

Lemma 6.5 *Let A be an associative commutative algebra, $\delta \in \mathrm{Der}\, A$, and $x, \xi \in A$. Then the following properties hold:*

- (i) *if $\delta \in Z(\mathrm{Der}\, A) := \{\mu \in \mathrm{Der}\, A : \mu\theta = \theta\mu$ for any $\theta \in \mathrm{Der}\, A\}$, then $\mathrm{Der}\, A \subseteq \mathrm{Der}\, A^{\delta,0}$,*
- (ii) *$r_x \in \mathrm{Der}\, A^{\delta,0}$ if and only if $A * (A * x) = 0$,*
- (iii) *$l_x \in \mathrm{Der}\, A^{\delta,0}$ if and only if $(A * A) * x = 0$,*
- (iv) *$[r_a, r_b] = 0$ for any $a, b \in A^{\delta,\xi}$,*
- (v) *$[l_a, l_b] = l_{[a,b]}$ for any $a, b \in A^{\delta,\xi}$,*
- (vi) *$L(A^{\delta,\xi}) := \{l_a : a \in A\}$ is a Lie algebra.*

Proof (i) If $\delta \in \mathrm{Der}\, A$, then

$$\begin{aligned}d(a * b) &= d(a \cdot \delta(b)) = d(a) \cdot \delta(b) + a \cdot d(\delta(b)) \\ &= d(a) \cdot \delta(b) + a \cdot \delta(d(b)) = d(a) * b + a * d(b)\end{aligned}$$

for any $a, b \in A^{\delta,0}$. Therefore, $d \in \mathrm{Der}\, A^{\delta,0}$.

(ii) If $r_x \in \mathrm{Der}\, A^{\delta,0}$, then

$$\begin{aligned}a \cdot \delta(b) \cdot \delta(x) &= (a * b) * x = r_x(a * b) = r_x(a) * b + a * r_x(b) \\ &= a \cdot \delta(x) \cdot \delta(b) + a \cdot \delta(b) \cdot \delta(x) + a \cdot b \cdot \delta^2(x)\end{aligned}$$

and so $a \cdot (\delta(b) \cdot \delta(x) + b \cdot \delta^2(x)) = 0$ for any $a, b \in A^{\delta,0}$. This is equivalent to $a \cdot \delta(b \cdot \delta(x)) = 0$. Hence $a * (b * x) = 0$.

(iii) By the same argument as in (ii).

(iv)–(vi) Obvious. □

Zhelyabin and Tikhov [51] asked: *Is true that an associative commutative algebra* $(A, +, \cdot)$ *with a derivation* δ *is* δ*-simple in the usual sense if and only if its corresponding Balinsky–Novikov algebra* $(A, +, *)$ *is simple*?

Lemma 6.6 *Let* A *be an associative commutative algebra,* $\delta \in \operatorname{Der} A$*, and* $\xi \in A$*. Then* A *is a* δ*-simple algebra if and only if* $A^{\delta,\xi}$ *is a simple Balinsky–Novikov algebra.*

Proof For the proof see [48, Proposition 2.8]. □

Corollary 6.7 *If* A *is a field,* $\delta \in \operatorname{Der} A$*, and* $\xi \in A$*, then* $A^{\delta,\xi}$ *is a simple Balinsky–Novikov algebra.*

Further we shall need the following result.

Lemma 6.8 *Let* A *be an associative commutative* δ*-semiprime algebra with* 1*,* $\operatorname{char} \mathbb{K} \neq 2$*, and* $\delta \in \operatorname{Der} A$*. If* I *is a* δ*-ideal of* A *and* $\delta^2(I) = 0$*, then* $\delta(I) = 0$ *and* $I \cdot \delta(A) = 0$*.*

Proof If $i \in I$, then

$$0 = \delta^2(i \cdot i) = \delta(2i \cdot \delta(i)) = 2\delta(i) \cdot \delta(i) + 2i \cdot \delta^2(i) = 2\delta(i) \cdot \delta(i)$$

and therefore $\delta(i) \cdot \delta(i) = 0$. Then $(\delta(i) \cdot A)^2 = 0$ and so $\delta(i) = 0$. Moreover,

$$0 = \delta(I) = \delta(I \cdot A) = \delta(I) \cdot A + I \cdot \delta(A) = I \cdot \delta(A).$$

□

Lemma 6.9 *Let* A *be an associative commutative algebra with* 1*,* $0 \neq \delta \in \operatorname{Der} A$*, and* $\xi \in A$*. Then* A *is a* δ*-prime algebra if and only if* $A^{\delta,\xi}$ *is a prime Balinsky–Novikov algebra.*

Proof $(\Rightarrow)$ Let I and J be ideals of $A^{\delta,\xi}$ such that $I * J = 0$. This means that $i \cdot \delta(j) + \xi \cdot i \cdot j = 0$ for all $i \in I$ and $j \in J$. By Lemma 6.4 (iv), $\xi \cdot I, \delta(I) \subseteq I$ and $\xi \cdot J, \delta(J) \subseteq J$. Moreover, $\operatorname{ann} I$ and $\operatorname{ann}(\operatorname{ann} I)$ are δ-ideals of A, $I \subseteq \operatorname{ann}(\operatorname{ann} I)$ and

$$\xi \cdot j + \delta(j) \in \operatorname{ann} I. \tag{35}$$

Assume that $I \neq 0$. Then $\operatorname{ann} I = 0$ and so $\delta(j) = -\xi \cdot j$ for any $j \in J$. Then

$$-\xi \cdot j \cdot k = \delta(j \cdot k) = \delta(j) \cdot k + j \cdot \delta(k) = -2\xi \cdot j \cdot k$$

for any $k \in J$ and, as a consequence, $\xi \cdot J \cdot J = 0$. Since $J \cdot A$ is a δ-ideal of A, we conclude that $J \cdot J \neq 0$. Then $\xi = 0$ and, in view of (35),

$$\delta(J) = 0. \tag{36}$$

Since $J \cdot \delta(I) \cdot \delta(I) \cdot \delta(J) + J \cdot J \cdot \delta(I) \cdot \delta^2(I) \subseteq (J * I) * (J * I) \subseteq I * J = 0$, we obtain that $J \cdot J \cdot \delta(I) \cdot \delta^2(I) = 0$ by (35) and (36).

If $\mathrm{ann}(J\cdot J)\neq 0$, then $J\cdot J\subseteq \mathrm{ann}(\mathrm{ann}(J\cdot J))=0$, a contradiction. Hence $\mathrm{ann}(J\cdot J)=0$. Then $(\delta(I)\cdot A)\cdot(\delta^2(I)\cdot A)=0$ and $\delta(I)=0$ by Lemma 6.8. As a consequence, $I\cdot\delta(A)=0$. This means that $\delta(A)\subseteq \mathrm{ann}(I\cdot A)$, what forces that $\delta(A)=0$, a contradiction.

($\Leftarrow$) Let $A^{\delta,\xi}$ be a δ-prime Balinsky–Novikov algebra. Assume that X and Y are δ-ideals of A such that $X\cdot Y=0$. By Lemma 6.4 (ii), X and Y are ideals of $A^{\delta,\xi}$ and $X*Y=0$. Thus $X=0$ or $Y=0$. □

Lemma 6.10 *Let A be an associative commutative algebra with* 1*,* $0\neq\delta\in\mathrm{Der}\,A$*, and* $\xi\in A$*. Then A is a δ-semiprime algebra if and only if $A^{\delta,\xi}$ is a semiprime Balinsky–Novikov algebra.*

Proof By the same argument as in the proof of Lemma 6.9. □

Lemma 6.11 [17] *Let $(N,+,*)$ be a Balinsky–Novikov algebra. Then $Z(N)$ and $[N,N]$ are ideals of N and $Z(N)*[N,N]=0$.*

Lemma 6.12 *Let A be an associative commutative algebra with* 1*,* $\mathrm{char}\,\mathbb{K}\neq 2$*,* $0\neq\delta\in\mathrm{Der}\,A$*, and $\xi\in A$. If A is a δ-prime algebra, then $Z(A^{\delta,\xi})=0$.*

Proof By Lemma 6.11, $Z(A^{\delta,\xi})*[A^{\delta,\xi},A^{\delta,\xi}]=0$. If $[A^{\delta,\xi},A^{\delta,\xi}]=0$, then $a\cdot\delta(b)=b\cdot\delta(a)$ for all $a,b\in A$. Therefore,

$$a\cdot\delta(a)\cdot b+a\cdot a\cdot\delta(b)=a\cdot\delta(a\cdot b)=a\cdot b\cdot\delta(a).$$

This gives that $a\cdot a\cdot\delta(b)=0$ and so $a\cdot a\in\mathrm{ann}(\delta(A)\cdot A)$. Since $\mathrm{ann}(\delta(A)\cdot A)$ is a δ-ideal and $\delta(A)\neq 0$, we obtain that $a\cdot a=0$. Then $(a+b)\cdot(a+b)=0$ for any $a,b\in A$ and $a\cdot b=-b\cdot a$. This yields that $A\cdot A=0$, a contradiction. Consequently, $[A^{\delta,\xi},A^{\delta,\xi}]\neq 0$ and thus $Z(A^{\delta,\xi})=0$. □

7 Lie properties

Lemma 7.1 *Let A be an associative commutative algebra with* 1*, $\xi\in A$, and* $\delta\in\mathrm{Der}\,A$*. If $\delta(A)\nsubseteq P$ for any minimal δ-prime ideal P of A, then:*

(i) *every abelian ideal I of the Lie algebra $A^{\delta,L}$ is contained in the δ-prime radical* $\mathbb{P}_\delta(A):=\bigcap\{P : P \text{ is a } \delta\text{-prime ideal of } A\}$*,*

(ii) *the Lie algebra $A^{\delta,L}$ is not solvable.*

Proof (i) Let I be a non-zero abelian ideal of the Lie algebra $A^{\delta,L}$. If $I\nsubseteq\mathbb{P}_\delta(A)$, then there exists a minimal δ-prime ideal of P of A such that $I\nsubseteq P$. Obviously,

$$\Delta\colon A/P\ni a+P\mapsto\delta(a)+P\in A/P$$

is a non-zero derivation of the quotient algebra A/P. Since A/P is a Δ-prime algebra, $(A/P)^{\Delta,\eta}$ is a prime Lie algebra, where $\eta=\xi+P$. Hence $(I+P)/P$ is zero, a contradiction.

(ii) It follows from (i). □

Lemma 7.2 [37, Theorem 3.3] *Let A be an associative commutative algebra with* 1 *and* $0 \neq \delta \in \operatorname{Der} A$. *Then A is a δ-simple algebra if and only if $A^{\delta,L}$ is a simple Lie algebra.*

Proof By the same argument as in the proof of Lemma 6.9. □

Lemma 7.3 *Let A be an associative commutative algebra with* 1, $\operatorname{char}\mathbb{K} \neq 2$, *and* $0 \neq \delta \in \operatorname{Der} A$. *If I is an abelian Lie ideal of a semiprime Balinsky–Novikov algebra $A^{\delta,\xi}$, then $\delta(I) = 0$. If, moreover, $A^{\delta,L}$ is prime, then $I = 0$.*

Proof (a) Let I be an ideal of $A^{\delta,L}$ such that $[I, I] = 0$. Then $0 = [u, v] = u \cdot \delta(v) - v \cdot \delta(u)$ for any $u, v \in I$. If $x \in A$, then

$$\begin{aligned}
0 &= [u, [v, x]] = u \cdot \delta([v, x]) - [v, x] \cdot \delta(u) \\
&= u \cdot [\delta(v), x] + u \cdot [v, \delta(x)] - [v, x] \cdot \delta(u) \\
&= u \cdot (\delta(v) * x - x * \delta(v)) + u \cdot (v * \delta(x) - \delta(x) * v) - (v * x - x * v) \cdot \delta(u) \\
&= (u \cdot \delta(v) - v \cdot \delta(u)) \cdot \delta(x) + x \cdot (\delta(v) \cdot \delta(u) - u \cdot \delta^2(v)) \\
&\qquad\qquad\qquad\qquad + u \cdot (v \cdot \delta^2(x) - \delta(x) \cdot \delta(v)) \\
&= u \cdot [v, \delta(x)].
\end{aligned}$$

This means that

$$[I, \delta(A)] \subseteq I \cap \operatorname{ann} I = 0 \tag{37}$$

because $\operatorname{ann} I$ is a δ-ideal of A. If $y \in A$, then

$$\begin{aligned}
0 &= [u, \delta(x \cdot y)] = [u, \delta(x) \cdot y + x \cdot \delta(y)] \\
&= u \cdot \delta^2(x) \cdot y + u \cdot \delta(x) \cdot \delta(y) + u \cdot \delta(x) \cdot \delta(y) \\
&\qquad\qquad + u \cdot x \cdot \delta^2(y) - \delta(x) \cdot y \cdot \delta(u) - x \cdot \delta(y) \cdot \delta(u) \\
&= y \cdot [u, \delta(x)] + x \cdot [u, \delta(y)] + 2u \cdot \delta(x) \cdot \delta(y) = 2u \cdot \delta(x) \cdot \delta(y).
\end{aligned}$$

Hence $\delta(A) \cdot \delta(A) \subseteq \operatorname{ann} I$. Then $\delta(I) \cdot \delta(I) \subseteq (\operatorname{ann} I) \cap I = 0$. Since $\operatorname{ann} \delta(I)$ is a δ-ideal of A and $\delta(I) \subseteq \operatorname{ann} \delta(I)$, we conclude that $\delta(I) = 0$.

(b) Now assume that $A^{\delta,\xi}$ is prime. In view of (37), $0 = [I, \delta(A)] = I \cdot \delta^2(A)$. If $I \neq 0$, then $\delta^2(A) = 0$ and so $\delta = 0$ by Lemma 6.8, a contradiction. □

Lemma 7.4 *Let A be an associative commutative algebra with* 1, $\operatorname{char}\mathbb{K} \neq 2$, $0 \neq \delta \in \operatorname{Der} A$, *and $\xi \in A$. Then A is a δ-prime algebra if and only if $A^{\delta,L}$ is a prime Lie algebra.*

Proof ($\Leftarrow$) Suppose that B and C are δ-ideals of A such that $B \cdot C = 0$. Then $(C \cdot B)^2 = 0$ and therefore $C \cdot B = 0$ by the δ-primeness of A. Since B and C are ideals of $A^{\delta,L}$ by Lemma 6.4 (vi) and $[B, C] = 0$, we deduce that $B = 0$ or $C = 0$.

($\Rightarrow$) By Lemma 6.9, $A^{\delta,\xi}$ is a prime Balinsky–Novikov algebra. Assume that I and J are non-zero ideals of $A^{\delta,L}$ such that $[I, J] = 0$. Then I (respectively, J) is a Lie ideal of $A^{\delta,\xi}$. If $[I, I] = 0$ (respectively, $[J, J] = 0$), then $\delta(I) = 0$ (respectively, $\delta(J) = 0$) by Lemma 7.3 and so $I * J = 0$ (respectively, $J * I = 0$), what implies that $I = 0$ or $J = 0$, a contradiction. Therefore, $[I, I] \neq 0$, $[J, J] \neq 0$ and therefore I (respectively, J) contains a non-zero δ-ideal I_0 (respectively, J_0) of A by Lemma 6.4 (i) such that $[I_0, J_0] = 0$. Since $[I_0 \cap J_0, I_0 \cap J_0] = 0$ and $I_0 \cap J_0$ is an ideal of $A^{\delta,\xi}$ by Lemma 6.4 (ii), we obtain that $\delta(I_0 \cap J_0) = 0$ by Lemma 7.3. Then $(I_0 \cap J_0) * (I_0 \cap J_0) = 0$ and, consequently, $I_0 \cdot J_0 \subseteq I_0 \cap J_0 = 0$, which leads to a contradiction. □

Lemma 7.5 *Let A be an associative commutative algebra with* 1, $\operatorname{char} \mathbb{K} \neq 2$, $0 \neq \delta \in \operatorname{Der} A$, *and* $\xi \in A$. *Then A is a δ-semiprime algebra if and only if $A^{\delta,L}$ is a semiprime Lie algebra.*

Proof By the same argument as in the proof of Lemma 7.4. □

Proof of Theorem 4.1 It follows from Lemmas 6.6, 6.9, 7.2, 7.4 and 7.5. □

We specified some of interesting properties of an associative commutative algebra A and its δ-associated algebra $A^{\delta,L}$, which earlier were investigated by Ribenboim [43], C.R. Jordan, D.A. Jordan [29,30], and Nowicki [37]. Moreover, as follows from the results stated above, there exist deep and very interesting relationships between associative commutative algebras A, their δ-associated Balinsky–Novikov algebras $A^{\delta,\xi}$ and δ-associated Lie algebras $A^{\delta,L}$.

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