Equivariant cohomology of the moduli space of genus three curves with symplectic level two structure via point counts

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Abstract

We determine the cohomology groups of the quartic and hyperelliptic loci inside the moduli space of genus three curves with symplectic level two structure as representations of the symmetric group S_7 together with their mixed Hodge structures by means of making equivariant point counts over finite fields and via purity arguments. This determines the weighted Euler characteristic of the whole moduli space of genus three curves with level two structure.

Keywords Moduli of curves · Cohomology · Point counts · Purity

Mathematics Subject Classification 14H10 · 14H50 · 14F20 · 14F40 · 55R80

1 Introduction

Let n be a positive integer and let C be a curve. A level n structure on C is a basis for the n -torsion of the Jacobian of C . The purpose of this paper is to study the cohomology of the moduli space $\mathcal{M}_3[2]$ of genus 3 curves with symplectic level 2 structure.

A genus 3 curve which is not hyperelliptic is embedded as a plane quartic via its canonical linear system. The corresponding locus in $\mathcal{M}_3[2]$ is called the quartic locus and it is denoted $\mathcal{Q}[2]$. A plane quartic with level 2 structure is specified, up to isomorphism, by an ordered septuple of points in general position in $\mathbb{P}^2$, up to the action of $\mathrm{PGL}(3)$ (see Sect. 3, especially Theorem 3.2). This identification will be the basis for our investigation of $\mathcal{Q}[2]$.

Our main focus will be on $\mathcal{Q}[2]$ but we will also consider its complement in $\mathcal{M}_3[2]$, i.e. the hyperelliptic locus $\mathcal{H}_3[2]$. The spaces $\mathcal{M}_3[2]$, $\mathcal{Q}[2]$ and $\mathcal{H}_3[2]$ are all defined over $\mathbb{Z}[1/2]$, a fact which gives us the flexibility to consider them over the complex numbers as well as over finite fields of characteristic different from 2. In the present

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paper, the latter viewpoint will be the central one and the computations will be carried out via point counts over finite fields (see Sects. 6, 7). By virtue of Lefschetz trace formula (see Sect. 4), such point counts give cohomological information in the form of Euler characteristics. However, we shall see (in Proposition 5.3 resp. Sect. 7) that both $\mathcal{Q}[2]$ and $\mathcal{H}_3[2]$ satisfy certain strong purity conditions which allow us to deduce information about the individual cohomology groups, in the form of Poincaré polynomials, from these Euler characteristics.

The group $\mathrm{Sp}(6, \mathbb{Z}/2\mathbb{Z})$ acts on $\mathcal{M}_3[2]$ as well as on $\mathcal{Q}[2]$ and $\mathcal{H}_3[2]$ by changing level structures. The cohomology groups thus become $\mathrm{Sp}(6, \mathbb{Z}/2\mathbb{Z})$ -representations and our computations will therefore be equivariant. However, the action of $\mathrm{Sp}(6, \mathbb{Z}/2\mathbb{Z})$ is rather subtle on $\mathcal{Q}[2]$ when $\mathcal{Q}[2]$ is identified with the space of septuples of points in general position in $\mathbb{P}^2$. On the other hand, the action of the symmetric group S_7 on seven elements is very clear and we will therefore restrict our attention to this subgroup. The main results are presented in Tables 2 and 5 where we give the cohomology groups of $\mathcal{Q}[2]$ and $\mathcal{H}_3[2]$ as representations of S_7 . In particular, we obtain the following.

Theorem 1.1 *The Poincaré polynomial of $\mathcal{Q}[2]$ is*

$$P_{\mathcal{Q}[2]}(t) = 1 + 35t + 490t^2 + 3485t^3 + 13174t^4 + 24920t^5 + 18375t^6.$$

Theorem 1.2 *The Poincaré polynomial of $\mathcal{H}_3[2]$ is*

$$P_{\mathcal{H}_3[2]}(t) = 36 + 720t + 5580t^2 + 20880t^3 + 37584t^4 + 25920t^5.$$

The present paper may to a large extent be seen as a level 2 analogue of Looijenga [25] and many results, for instance Propositions 1.1 and 1.2, have counterparts in [25] and [19]. Our paper also builds heavily on the work of Dolgachev and Ortland [14], especially the description of $\mathcal{Q}[2]$ in terms of configurations of points in the projective plane, and van Geemen, whose results are unpublished by himself but can be found in [14]. It may also be of interest to compare the present paper to the work of Bergström [3, 4], and Bergström and Tommasi [5] which also investigate cohomological questions about moduli spaces of low genus curves via point counts. Although there are many similarities, a key difference between the works mentioned above and the present paper is that in previous works the most refined information is obtained for compactifications of the various moduli spaces under investigation, and compactness is used in an essential way, whereas in the present paper we obtain the strongest information for “open” moduli spaces. There are also several other works which answer representation theoretic questions about the cohomology groups of various spaces, e.g. [10, 13, 22, 23]. It should also be mentioned that our results are an essential ingredient in the article [7] where further information about the action of $\mathrm{Sp}(6, \mathbb{Z}/2\mathbb{Z})$ on the cohomology $\mathcal{M}_3[2]$ and $\mathcal{Q}[2]$ is obtained via quite different methods.

2 Symplectic level structures

Let K be an algebraically closed field of characteristic not equal to 2 and let C be a smooth and irreducible curve of genus g over K . The 2-torsion part $\text{Jac}(C)[2]$ of the Jacobian of C is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2g}$ as an abelian group and the Weil pairing is a nondegenerate and alternating bilinear form on $\text{Jac}(C)[2]$.

Definition 2.1 A *symplectic level two structure* on a curve C is an ordered basis $(D_1, \dots, D_{2g})$ of $\text{Jac}(C)[2]$ such that the Weil pairing has matrix

$$\begin{pmatrix} 0 & I_g \\ I_g & 0 \end{pmatrix},$$

with respect to this basis. Here, I_g denotes the $g \times g$ identity matrix (note that the matrix of the pairing takes a somewhat simpler form than in general since $\text{Jac}(C)[2]$ is a vector space over a field of characteristic 2).

For more information about the Weil pairing and level structures, see e.g. [2] or [20]. Since we shall only consider symplectic level structures we shall refer to symplectic level structures simply as level structures.

A tuple $(C, D_1, \dots, D_{2g})$ where C is a smooth irreducible curve and $(D_1, \dots, D_{2g})$ is a level 2 structure on C is called a curve with level 2 structure. Let $(C', D'_1, \dots, D'_{2g})$ be another curve with level 2 structure. An isomorphism of curves with level 2 structures is an isomorphism of curves $\phi: C \rightarrow C'$ such that $\phi^*(D'_i) = D_i$ for $i = 1, \dots, 2g$. We denote the moduli space of genus g curves with level 2 structure by $\mathcal{M}_g[2]$. We remark that we shall consider these moduli spaces as coarse spaces and not as stacks. The group $\text{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ acts on $\mathcal{M}_g[2]$ by changing level structures.

A concept closely related to level 2 structures is that of theta characteristics.

Definition 2.2 Let C be a smooth and irreducible curve and let K_C be its canonical class. An element $\theta \in \text{Pic}(C)$ such that $2\theta = K_C$ is called a *theta characteristic*. We denote the set of theta characteristics of C by $\Theta(C)$.

Let C be a curve of genus g . Given two theta characteristics θ_1 and θ_2 on C , we obtain an element $D \in \text{Jac}(C)[2]$ by taking the difference $\theta_1 - \theta_2$. Conversely, given a theta characteristic θ and a 2-torsion element D , we obtain a new theta characteristic as $\theta' = \theta + D$. More precisely we have that $\Theta(C)$ is a $\text{Jac}(C)[2]$ -torsor and the set $\tilde{\Theta}(C) = \Theta(C) \cup \text{Jac}(C)[2]$ is a vector space of dimension $2g + 1$ over the field $\mathbb{Z}/2\mathbb{Z}$ of two elements.

Definition 2.3 An ordered basis $A = (\theta_1, \dots, \theta_{2g+1})$ of theta characteristics of the vector space $\tilde{\Theta}(C)$ is called an *ordered Aronhold basis* if the expression

$$h^0(\theta) \bmod 2,$$

only depends on the number of elements in A that is required to express θ for any theta characteristic θ .

Proposition 2.4 *Let C be a smooth and irreducible curve. There is a bijection between the set of ordered Aronhold bases on C and the set of level 2 structures on C .*

For a proof of Proposition 2.4 as well as a more thorough treatment of theta characteristics and Aronhold bases we refer to [21] and [26, Section 3.3].

Proposition 2.4 provides a more geometric way to think about level 2 structures. In the case of a plane quartic curve, which shall be the case of most importance to us, we point out that each theta characteristic occurring in an Aronhold basis is cut out by a bitangent line. Thus, in the case of plane quartics one can think of ordered Aronhold bases as ordered sets of bitangents (although not every ordered set of bitangents constitute an ordered Aronhold basis).

3 Plane quartics

Let K be an algebraically closed field of characteristic not equal to 2 and let C be a smooth and irreducible curve of genus g over K . If C is not hyperelliptic it is embedded into $\mathbb{P}^{g-1}$ via its canonical linear system. Thus, a non-hyperelliptic curve of genus 3 is embedded into $\mathbb{P}^2$ and by the genus-degree formula we see that the degree of the image is 4. We therefore refer to the corresponding locus in $\mathcal{M}_3$, the moduli space of curves of genus 3, as the *quartic locus* and denote it by $\mathcal{Q}$. It is the complement of the *hyperelliptic locus*, $\mathcal{H}_3$. Similarly, we denote the complement of the hyperelliptic locus in $\mathcal{M}_3[2]$ by $\mathcal{Q}[2]$. Clearly, the action of $\mathrm{Sp}(6, \mathbb{Z}/2\mathbb{Z})$ on $\mathcal{M}_3[2]$ restricts to an action on $\mathcal{Q}[2]$.

The purpose of this section is to give an explicit, combinatorial description of $\mathcal{Q}[2]$. This description will be in terms of points in general position. Intuitively, a set of n points in the projective plane is in general position if there is no “unexpected” curve passing through all of them. In our case, this is made precise by the following definition.

Definition 3.1 Let $(P_1, \dots, P_7)$ be a septuple of points in $\mathbb{P}^2$. We say that the septuple is in *general position* if there is no line passing through any three of the points and no conic passing through any six of them. We denote the moduli space of septuples of points in general position up to projective equivalence by $\mathcal{P}_7^2$.

Let $T = (P_1, \dots, P_7)$ be a septuple of points in general position in the projective plane and let $\mathcal{N}_T$ be the net of cubics passing through T . If we let F_0, F_1 and F_2 be generators for $\mathcal{N}_T$, then the equation

$$\det \left(\frac{\partial F_i}{\partial x_j} \right)_{i,j=0,1,2} = 0,$$

describes a plane sextic curve S_T with double points precisely at $P_1, \dots, P_7$. By the genus-degree formula we see that S_T has geometric genus 3 and it turns out that its smooth model is not hyperelliptic. Moreover, if we let $\rho: C_T \rightarrow S_T$ be a resolution of the singularities, then $D_i = \rho^{-1}(P_i)$ is a theta characteristic and $(D_1, \dots, D_7)$ is an ordered Aronhold basis.

The following result can be found in [14] where it is attributed to van Geemen.

Theorem 3.2 *Sending a septuple $T = (P_1, \dots, P_7)$ of points in general position in the projective plane to $(C_T, D_1, \dots, D_7)$ gives an $\mathrm{Sp}(6, \mathbb{Z}/2\mathbb{Z})$ -equivariant isomorphism*

$$\mathcal{P}_7^2 \rightarrow \mathcal{Q}[2].$$

Remark 3.3 Even though [14] only considers the case $K = \mathbb{C}$, the proof only relies on the theory of geometrically marked Del Pezzo surfaces of degree 2. This theory is the same over any algebraically closed field of characteristic different from 2, see [24, Section III.3].

It should be pointed out that while the action of $\mathrm{Sp}(6, \mathbb{Z}/2\mathbb{Z})$ is clear on $\mathcal{Q}[2]$ its action on $\mathcal{P}_7^2$ is much more subtle. However, we can at least plainly see the symmetric group $S_7 \subset \mathrm{Sp}(6, \mathbb{Z}/2\mathbb{Z})$ acting on $\mathcal{P}_7^2$ by permuting points.

Remark 3.4 Since it is not at all obvious at first glance that the group S_7 embeds into $\mathrm{Sp}(6, \mathbb{Z}/2\mathbb{Z})$ we say a few words about how this can be seen. It is not harder to see the embedding of the larger group S_8 and we begin by explaining this embedding. Recall that S_8 has a presentation given by generators $\sigma_i, i = 1, \dots, 7$, and relations

$$\begin{aligned}\sigma_i^2 &= \mathrm{id}, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i, \quad j \neq i \pm 1, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}.\end{aligned}$$

One should think about σ_i as the transposition of i and $i + 1$.

Let u be a vector in the six-dimensional vector space V over $\mathbb{Z}/2\mathbb{Z}$ on which $\mathrm{Sp}(6, \mathbb{Z}/2\mathbb{Z})$ acts and let b denote the symplectic bilinear form. The transvection

$$T_u: v \mapsto v + b(u, v)u$$

is then an element in $\mathrm{Sp}(6, \mathbb{Z}/2\mathbb{Z})$. Let $x_1, x_2, x_3, y_1, y_2, y_3$ be a symplectic basis of V (i.e. a basis such that $b(x_i, y_j) = \delta_{i,j}$). The embedding of S_8 can now be given explicitly as

$$\begin{aligned}\sigma_1 &\mapsto T_{x_1+y_1}, \\ \sigma_2 &\mapsto T_{x_1+x_2+x_3+y_3}, \\ \sigma_3 &\mapsto T_{x_2+y_2}, \\ \sigma_4 &\mapsto T_{x_1+x_2+x_3+y_1}, \\ \sigma_5 &\mapsto T_{x_3+y_3}, \\ \sigma_6 &\mapsto T_{x_2+y_2+y_3}, \\ \sigma_7 &\mapsto T_{x_1+x_2+x_3+y_1+y_2+y_3}.\end{aligned}$$

An embedding of S_7 into $\mathrm{Sp}(6, \mathbb{Z}/2\mathbb{Z})$ is now given via its natural embedding into S_8 . We also remark that all embedded copies of S_8 and S_7 in $\mathrm{Sp}(6, \mathbb{Z}/2\mathbb{Z})$ are conjugate.

4 Lefschetz trace formula

We are interested in the spaces $\mathcal{M}_3[2]$, $\mathcal{Q}[2]$ and $\mathcal{H}_3[2]$ and in particular we want to know their cohomology. The Lefschetz trace formula provides a way to obtain cohomological information about a space via point counts over finite fields.

Let p be a prime number, let $n \geq 1$ be an integer and let $q = p^n$. Also, let $\mathbb{F}_q$ denote a finite field with q elements, let $\mathbb{F}_{q^m}$ denote a degree m extension of $\mathbb{F}_q$ and let $\overline{\mathbb{F}}_q$ denote an algebraic closure of $\mathbb{F}_q$. Let X be a scheme defined over $\mathbb{F}_q$ and let F denote its geometric Frobenius endomorphism. Finally, let l be a prime number different from p and let $H_{\text{ét},c}^k(X, \mathbb{Q}_l)$ denote the k -th compactly supported étale cohomology group of X with coefficients in $\mathbb{Q}_l$.

Let Γ be a finite group acting on X by rational automorphisms (i.e. automorphisms over $\mathbb{F}_q$). Then each cohomology group $H_{\text{ét},c}^k(X, \mathbb{Q}_l)$ is a Γ -representation. The Lefschetz trace formula allows us to obtain information about these representations by counting the number of fixed points of $F\sigma$ for different $\sigma \in \Gamma$.

Theorem 4.1 (Lefschetz trace formula) *Let X be a separated scheme of finite type over $\mathbb{F}_q$ with Frobenius endomorphism F and let σ be a rational automorphism of X of finite order. Then*

$$|\overline{X}^{F\sigma}| = \sum_{k \geq 0} (-1)^k \cdot \text{Tr}(F\sigma, H_{\text{ét},c}^k(\overline{X}, \mathbb{Q}_l)),$$

where $\overline{X}^{F\sigma}$ denotes the fixed point set of $F\sigma$ in $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$.

For a proof, see [11, Rapport – Théorème 3.2].

Remark 4.2 This theorem is usually only stated in terms of F . To get the above version one simply applies the “usual” theorem to the twist of X by σ , i.e. by descending from $\overline{X}$ via $F\sigma$ instead of F . For more details, see e.g. [12, Section 3].

Remark 4.3 If Γ is a finite group acting on X by rational automorphisms and $\sigma \in \Gamma$, then $|\overline{X}^{F\sigma}|$ will only depend on the conjugacy class of σ in Γ .

Let $R(\Gamma)$ denote the representation ring of Γ and let the compactly supported Γ -equivariant Euler characteristic of $\overline{X}$ be defined as the virtual representation

$$\text{Eul}_{\overline{X},c}^{\Gamma} = \sum_{k \geq 0} (-1)^k \cdot H_{\text{ét},c}^k(\overline{X}, \mathbb{Q}_l) \in R(\Gamma).$$

We also introduce the following notation:

$$\text{Eul}_{\overline{X},c}^{\Gamma}(\sigma) := \text{Tr}(\sigma, \text{Eul}_{\overline{X},c}^{\Gamma}) = \sum_{k \geq 0} (-1)^k \cdot \text{Tr}(\sigma, H_{\text{ét},c}^k(\overline{X}, \mathbb{Q}_l)) \in \mathbb{Z}.$$

Note in particular that $\text{Eul}_{\overline{X},c}^{\Gamma}(\text{id})$ is the ordinary Euler characteristic of $\overline{X}$. By character theory, $\text{Eul}_{\overline{X},c}^{\Gamma}$ is completely determined by computing $\text{Eul}_{\overline{X},c}^{\Gamma}(\sigma)$ for a representative σ of each conjugacy class of Γ . This motivates the following definition.

Definition 4.4 Let X be a separated scheme of finite type over $\mathbb{F}_q$ with Frobenius endomorphism F and let Γ be a finite group acting on X by rational automorphisms. The determination of $|X^{F^\sigma}|$ for all $\sigma \in \Gamma$ is then called a Γ -equivariant point count of X over $\mathbb{F}_q$.

5 Minimal purity

Let X be a separated scheme of finite type over the finite field $\mathbb{F}_q$ and let Γ be a group acting on X by rational automorphisms. We define the compactly supported Γ -equivariant Poincaré polynomial of X as

$$P_{\bar{X},c}^\Gamma(t) = \sum_{k \geq 0} H_{\text{ét},c}^k(\bar{X}, \mathbb{Q}_l) \cdot t^k \in R(\Gamma)[t].$$

As for the Euler characteristic, we introduce the notation

$$P_{\bar{X},c}^\Gamma(\sigma)(t) := \text{Tr}(\sigma, P_{\bar{X},c}^\Gamma(t)) = \sum_{k \geq 0} \text{Tr}(\sigma, H_{\text{ét},c}^k(\bar{X}, \mathbb{Q}_l)) \cdot t^k \in \mathbb{Z}[t]$$

In the previous section we saw that equivariant point counts give equivariant Euler characteristics. Poincaré polynomials contain more information (namely the cohomological grading) and are therefore more desirable to obtain but are typically more complicated to compute. However, if X satisfies a certain purity condition one can recover the Poincaré polynomial from the Euler characteristic. See also [3–5,8] where similar phenomena for compact spaces have been exploited.

Definition 5.1 (Dimca and Lehrer [13]) Let X be an irreducible and separated scheme of finite type over $\mathbb{F}_q$ with Frobenius endomorphism F and let l be a prime not dividing q . The scheme X is called *minimally pure* if F acts on $H_{\text{ét},c}^k(X, \mathbb{Q}_l)$ by multiplication by $q^{k-\dim(X)}$.

A pure dimensional and separated scheme X of finite type over $\mathbb{F}_q$ is minimally pure if for any collection $\{X_1, \dots, X_r\}$ of irreducible components of X , the irreducible scheme $X_1 \setminus (X_2 \cup \dots \cup X_r)$ is minimally pure.

Examples of minimally pure varieties include complements of arrangements of hyperplanes and hypertori, toric varieties and quotients of reductive groups by maximal tori. For more examples, see [13,22,23].

Thus, if X is minimally pure and σ is a rational automorphism of X of finite order, then $\text{Tr}(F \circ \sigma, H_{\text{ét},c}^k(\bar{X}, \mathbb{Q}_l)) = \text{Tr}(\sigma, H_{\text{ét},c}^k(\bar{X}, \mathbb{Q}_l)) q^{k-\dim(X)}$. Therefore, a term $q^{k-\dim(X)}$ in $|\bar{X}^{F \circ \sigma}|$ can only come from $H_{\text{ét},c}^k(\bar{X}, \mathbb{Q}_l)$ and we can determine the Γ -equivariant Poincaré polynomial of X via the relation

$$\text{Eul}_{\bar{X},c}^\Gamma(\sigma) = q^{-2\dim(X)} \cdot P_{\bar{X},c}^\Gamma(\sigma)(-q^2).$$

If we recall that q is a prime power and, in particular, an integer and we see that the above formula indeed yields an integer (as it should). On the other hand we may

view the above expression as a polynomial in q . We then see that the point count is polynomial and that the coefficients of this polynomial are given by the values of the characters of the representations $H_{\text{ét},c}^k(\overline{X}, \mathbb{Q}_l)$.

5.1 Minimal purity of $\mathcal{Q}[2]$

We shall now show that the moduli space $\mathcal{Q}[2]$ is minimally pure. Let $C \subset \mathbb{P}^2$ be a plane quartic, let $P \in C$ be a point and let $T_P C$ denote the tangent line of C at P . We say that P is a *bitangent point* if

$$C \cdot T_P C = 2P + 2Q$$

for some point Q that might coincide with P . If $P \neq Q$ we say that P is a *genuine bitangent point*. We denote the moduli space of plane quartics with level 2 structure marked with a bitangent point by $\mathcal{Q}_{\overline{\text{btg}}}[2]$ and we denote the moduli space of plane quartics with level 2 structure marked with a genuine bitangent point by $\mathcal{Q}_{\text{btg}}[2]$. The space $\mathcal{Q}_{\text{btg}}[2]$ is an open subvariety of $\mathcal{Q}_{\overline{\text{btg}}}[2]$ and its complement $\mathcal{Q}_{\overline{\text{btg}}}[2] \setminus \mathcal{Q}_{\text{btg}}[2]$ is the moduli space of plane quartics with level 2 structure marked with a hyperflex point. We denote the latter space by $\mathcal{Q}_{\text{hfl}}[2]$.

Lemma 5.2 (Looijenga [25, Proposition 1.18, Lemma 3.6]) $\mathcal{Q}_{\overline{\text{btg}}}[2]$ is *minimally pure*.

Proposition 5.3 $\mathcal{Q}[2]$ is *minimally pure*.

Proof A plane quartic has 28 bitangents so the morphism

$$\pi : \mathcal{Q}_{\overline{\text{btg}}}[2] \rightarrow \mathcal{Q}[2],$$

forgetting the marked bitangent point, is finite of degree $2 \cdot 28 = 56$. Thus, the map

$$\pi_* \circ \pi^* : H_{\text{ét},c}^k(\mathcal{Q}[2], \mathbb{Q}_l) \rightarrow H_{\text{ét},c}^k(\mathcal{Q}[2], \mathbb{Q}_l)$$

is multiplication with $\deg(\pi) = 56$. In particular, the map

$$\pi_* : H_{\text{ét},c}^k(\mathcal{Q}[2], \mathbb{Q}_l) \rightarrow H_{\text{ét},c}^k(\mathcal{Q}_{\overline{\text{btg}}}[2], \mathbb{Q}_l)$$

is injective. Since F acts on $H_{\text{ét},c}^k(\mathcal{Q}_{\overline{\text{btg}}}[2], \mathbb{Q}_l)$ by multiplication by q^{k-6} , the same is true on the subspace $H_{\text{ét},c}^k(\mathcal{Q}[2], \mathbb{Q}_l)$ and we conclude that $\mathcal{Q}[2]$ is minimally pure. $\square$

Since $\mathcal{Q}[2]$ is isomorphic to $\mathcal{P}_7^2$, we may compute the cohomology of $\mathcal{Q}[2]$ as a representation of S_7 by making S_7 -equivariant point counts of $\mathcal{P}_7^2$.

6 Equivariant point counts

In this section we shall perform an S_7 -equivariant point count of $\mathcal{P}_7^2$. This amounts to the computation of $|(\mathcal{P}_7^2)^{F^\sigma}|$ for one representative σ of each of the fifteen conjugacy

classes of S_7 . The computations will be rather different in the various cases but at least the underlying idea will be the same. Throughout this section we shall work over a finite field $\mathbb{F}_q$ where q is odd.

Let U be a subset of $(\mathbb{P}^2(\overline{\mathbb{F}}_q))^7$ and interpret each point of U as an ordered septuple of points in $\mathbb{P}^2(\overline{\mathbb{F}}_q)$. Define the *discriminant locus* $\Delta \subset U$ as the subset consisting of septuples which are not in general position. If U contains the subset of $(\mathbb{P}^2(\overline{\mathbb{F}}_q))^7$ consisting of all septuples which are in general position, then

$$\mathcal{P}_7^2 = (U \setminus \Delta) / \mathrm{PGL}(3, \overline{\mathbb{F}}_q).$$

An element of $\mathrm{PGL}(3)$ is completely specified by where it takes four points in general position. Therefore, the action of $\mathrm{PGL}(3, \overline{\mathbb{F}}_q)$ on U is free and we have the simple relation

$$|(\mathcal{P}_7^2)^{F\sigma}| = \frac{|U^{F\sigma}| - |\Delta^{F\sigma}|}{|\mathrm{PGL}(3, \overline{\mathbb{F}}_q)|}. \quad (6.1)$$

We shall choose the set U in such a way that counting fixed points of $F\sigma$ in U is easy. We shall therefore focus on the discriminant locus.

The discriminant locus can be decomposed as

$$\Delta = \Delta_l \cup \Delta_c,$$

where Δ_l consists of septuples where at least three points lie on a line and Δ_c consists of septuples where at least six points lie on a conic. The computation of $|\Delta^{F\sigma}|$ will consist of the following three steps:

- the computation of $|\Delta_l^{F\sigma}|$,
- the computation of $|\Delta_c^{F\sigma}|$,
- the computation of $|(\Delta_l \cap \Delta_c)^{F\sigma}|$.

We can then easily determine $|\Delta^{F\sigma}|$ via the principle of inclusion and exclusion.

Lemma 6.1 *Let $C \subset \mathbb{P}^2$ be a smooth conic over a field k and let $P \in \mathbb{P}^2$ be a point such that n tangent lines of C pass through P . Then $n \leq 2$ or $\mathrm{char}(k) = 2$.*

Proof The line $P^\vee$ in the dual projective plane intersects the dual conic $C^\vee$ in n points. The dual conic $C^\vee$ is smooth if the characteristic of k is not 2 and we conclude that n can be at most 2. $\square$

One can, of course, also see this via a direct computation.

Lemma 6.2 *Let $C \in \mathbb{P}^2$ be a smooth, $\mathbb{F}_q$ -rational conic and let P be an $\mathbb{F}_q$ -rational point lying on precisely one tangent line L to C . Then P is a point on C .*

Proof We first observe that L must be $\mathbb{F}_q$ -rational since otherwise L and FL would be two distinct tangent lines passing through P . Let $Q \in C$ be the point of tangency of L and assume $Q \neq P$. If L' is another tangent to C , then L' cannot pass through Q since if that was the case the quadratic curve $L \cup L'$ would intersect C with multiplicity at least 5, contradicting Bezout's theorem.

Fig. 1 A conic C with a point P on the outside of C , a point Q on the inside of C and a point R on C

Now consider the set S of $\mathbb{F}_q$ -rational tangents of C different from L . We have $|S| = q$. By the above observation we have that none of the elements passes through Q and, by assumption, none of them passes through P . Since the number of $\mathbb{F}_q$ -rational points of L different from P and Q is $q - 1$, the pigeon hole principle gives that there must be a point R on L such that two of the elements of S pass through R . But now R is a point with three tangents of C passing through it which is impossible by Lemma 6.1. We conclude that $P = Q$ and that P is a point on C . $\square$

The above results justify the following definition, which will be useful in the analysis of $\Delta_l \cap \Delta_c$. See also Fig. 1 for motivation of the terminology.

Definition 6.3 Let C be a smooth conic over $\mathbb{F}_q$ and let $P \in \mathbb{P}^2(\mathbb{F}_q)$. We then say that

- P is on the $\mathbb{F}_q$ -inside of C if there is no $\mathbb{F}_q$ -tangent to C passing through P ,
- P is on C if there is precisely one $\mathbb{F}_q$ -tangent to C passing through P ,
- P is on the $\mathbb{F}_q$ -outside of C if there are two $\mathbb{F}_q$ -tangents to C passing through P .

Recall that the natural action of S_7 on a septuple $(P_1, \dots, P_7)$ is given by

$$\sigma.(P_1, \dots, P_7) = (P_{\sigma^{-1}(1)}, \dots, P_{\sigma^{-1}(7)}).$$

Thus, a septuple is fixed by $F\sigma$ if and only if $FP_i = P_{\sigma(i)}$ for $i = 1, \dots, 7$. This is the motivation for the following definition.

Definition 6.4 Let X be an $\mathbb{F}_q$ -scheme with Frobenius endomorphism F and let $Z \subset X_{\overline{\mathbb{F}_q}}$ be a subscheme. We say that Z is a *strict* $\mathbb{F}_{q^m}$ -subscheme if Z is an $\mathbb{F}_{q^m}$ -subscheme which is not defined over $\mathbb{F}_{q^n}$ for any $n < m$.

If Z is a strict $\mathbb{F}_{q^m}$ -subscheme, the m -tuple $(Z, \dots, F^{m-1}Z)$ is called a *conjugate m -tuple*. Let r be a positive integer and let $\lambda = [1^{\lambda_1}, \dots, r^{\lambda_r}]$ be a partition of r . An r -tuple $(Z_1, \dots, Z_r)$ of closed subschemes of X is called a *conjugate λ -tuple* if it consists of λ_1 conjugate 1-tuples, λ_2 -conjugate 2-tuples and so on. We denote the set of conjugate λ -tuples of $\overline{\mathbb{F}_q}$ -points of X by $X(\lambda)$.

We shall sometimes drop the adjective “conjugate” and simply write “ λ -tuple”. Since the conjugacy class of an element in S_7 is given by its cycle type, we want to count the number of conjugate λ -tuples in both U and Δ for each partition of 7.

We now recall a number of basic results regarding point counts. We begin by noting that the number of conjugate λ -tuples of hyperplanes in $\mathbb{P}^n$ is equal to the number of conjugate λ -tuples of points in $\mathbb{P}^n$. We also recall that

$$|\mathbb{P}^n(\mathbb{F}_q)| = \sum_{i=0}^n q^i,$$

and that

$$|\mathrm{PGL}(3, \mathbb{F}_q)| = q^3 \cdot (q^3 - 1) \cdot (q^2 - 1).$$

A slightly less elementary result is that the number of smooth conics defined over $\mathbb{F}_q$ is

$$q^5 - q^2.$$

To see this, note that there is a $\mathbb{P}^5$ of conics. Of these there are $q^2 + q + 1$ double $\mathbb{F}_q$ -lines, $\frac{1}{2} \cdot (q^2 + q + 1) \cdot (q^2 + q)$ intersecting pairs of $\mathbb{F}_q$ -lines and $\frac{1}{2} \cdot (q^4 - q)$ conjugate pairs of $\mathbb{F}_{q^2}$ -lines while the remaining conics are smooth. Finally, recall that a smooth conic is rational and thus has $q + 1$ points.

We are now ready for the task of counting the number of conjugate λ -tuples for each element of S_7 .

Remark 6.5 Since $\mathcal{P}_7^2$ is minimally pure, equation (6.1) gives that $|\mathcal{P}_7^2)^{F\sigma}|$ is a monic polynomial in q of degree six so it is in fact enough to make counts for six different finite fields and interpolate. This is however hard to carry out in practice, even with a computer, as soon as λ contains parts of large enough size (where “large enough” means 3 or 4). However, one can always obtain partial information which provides important checks for our computations and for partitions entirely with parts at most 2 we have been able to obtain the entire polynomials also via computer counts. This fact might help to convince the reader of the validity of our results since these cases are by far the hardest to do by hand.

6.1 The case $\lambda = [7]$

For $\lambda = [7]$, a conjugate λ -tuple is a septuple $(P_1, \dots, P_7)$ such that $FP_i = P_{i+1}$ for $i = 1, \dots, 6$ and $FP_7 = P_1$. In this case, we simply take U as the subset of $(\mathbb{P}^2)^7$ of pairwise distinct points. We then have

$$|U^{F\sigma}| = q^{14} + q^7 + 1 - (q^2 + q + 1) = q^{14} + q^7 - q^2 - q.$$

The main observation is the following.

Lemma 6.6 *If $(P_1, \dots, P_7)$ is a λ -tuple with three of its points on a line, then all seven points lie on a line defined over $\mathbb{F}_q$.*

Proof Suppose that the set $S = \{P_i, P_j, P_k\}$ is contained in the line L . Then L is either defined over $\mathbb{F}_q$ or is a strict $\mathbb{F}_{q^7}$ -line. If there is an $1 \leq r \leq 6$ such that $|S \cap F^r S| = 2$ we must have $L = F^r L$ and the result follows. For instance,

- (i) if S is of the form $S = \{P_i, P_{i+1}, P_{i+2}\}$, then $|S \cap F S| = 2$,
- (ii) if S is of the form $S = \{P_i, P_{i+2}, P_{i+4}\}$, then $|S \cap F^2 S| = 2$,
- (iii) if S is of the form $S = \{P_i, P_{i+1}, P_{i+4}\}$, then $|S \cap F^3 S| = 2$.

However, if S is not of the above form we have $|S \cap F^r S| = 1$ for all $1 \leq r \leq 6$ and the above method fails.

We show how to argue in the case $S = P_1, P_2, P_4$, the other cases are analogous. We assume that L is a strict $\mathbb{F}_{q^7}$ -line and derive a contradiction. Since L is a strict $\mathbb{F}_{q^7}$ -line, the points P_1, P_2, P_3 and P_6 must be in general position (otherwise we could find a triple of type (i)–(iii) on a line among these points, which would imply that L is defined over $\mathbb{F}_q$). Let $L_{i,j}$ denote the line between P_i and P_j and define

$$Q_1 = L_{1,2} \cap L_{3,6}, \quad Q_2 = L_{1,3} \cap L_{2,6}, \quad Q_3 = L_{1,6} \cap L_{2,3}$$

Since P_1, P_2, P_3 and P_6 are in general position, the points Q_1, Q_2 and Q_3 do not lie on a line.

Since P_1, P_2 and P_4 lie on a line we have $P_4 \in L_{1,2}$. We have $F^2 P_1 = P_3$ and $F^2 P_4 = P_6$ so $F^2 L_{1,2} = L_{3,6}$. But since $P_2 \in L_{1,2}$ we must have $F^2 P_2 = P_4 \in F^2 L_{1,2} = L_{3,6}$. We thus have $P_4 \in L_{1,2}$ and $P_4 \in L_{3,6}$ so $P_4 = Q_1$. By analogous arguments one shows that $P_5 = Q_3$ and $P_7 = Q_2$. But we now have that $\{Q_1, Q_2, Q_3\} = \{P_4, P_5, P_7\} = F^3 S$ so the points Q_1, Q_2 and Q_3 lie on the line $F^3 L$. This contradiction establishes the claim. $\square$

Lemma 6.7 *If $(P_1, \dots, P_7)$ is a λ -tuple with six of its points on a smooth conic, then all seven points lie on a smooth conic defined over $\mathbb{F}_q$.*

Proof Suppose that the set $S = \{P_{i_1}, \dots, P_{i_6}\}$ lies on a smooth conic C . We have $|FS \cap S| = 5$ and since a conic is defined by any five points on it we have $FC = C$. Hence, we have that C is defined over $\mathbb{F}_q$ and that all seven points lie on C . $\square$

We conclude that $\Delta_l^{F\sigma}$ and $\Delta_c^{F\sigma}$ are disjoint. We obtain $|\Delta_l^{F\sigma}|$ by first choosing an $\mathbb{F}_q$ -line L and then picking a λ -tuple on L . We thus have

$$|\Delta_l^{F\sigma}| = (q^2 + q + 1) \cdot (q^7 - q).$$

To obtain $|\Delta_c|$ we first choose a smooth conic C and then a conjugate λ -tuple on C . We thus have

$$|\Delta_c^{F\sigma}| = (q^5 - q^2) \cdot (q^7 - q).$$

Equation (6.1) now gives

$$|(\mathcal{P}_7^{F\sigma})| = q^6 + q^3.$$

6.2 The case $\lambda = [1, 6]$

For $\lambda = [1, 6]$ we can take a conjugate λ -tuple as a septuple $(P_1, \dots, P_7)$ such that $FP_i = P_{i+1}$ for $i = 1, \dots, 5$, $FP_6 = P_1$ and $FP_7 = P_7$. Also in this case we take U as the subset of $(\mathbb{P}^2)^7$ of pairwise distinct points. We then have

$$\begin{aligned} |U^{F\sigma}| &= ((q^{12} + q^6 + 1) - (q^6 + q^3 + 1) - (q^4 + q^2 + 1) + (q^2 + q + 1))(q^2 + q + 1) \\ &= (q^{12} - q^4 - q^3 + q)(q^2 + q + 1) \end{aligned}$$

The main observation is the following.

Lemma 6.8 *If a λ -tuple has three points on a line, then either*

- (1) *the first six points of the λ -tuple lie on an $\mathbb{F}_q$ -line or,*
- (2) *the first six points lie on two conjugate $\mathbb{F}_{q^2}$ -lines, the $\mathbb{F}_{q^2}$ -lines contain three $\mathbb{F}_{q^6}$ -points each and these triples are interchanged by F , or,*
- (3) *the first six points lie pairwise on three conjugate $\mathbb{F}_{q^3}$ -lines which intersect in P_7 .*

Proof Suppose that $S = \{P_i, P_j, P_k\}$ lie on a line L . Then L is either defined over $\mathbb{F}_q, \mathbb{F}_{q^2}, \mathbb{F}_{q^3}$ or $\mathbb{F}_{q^6}$. One easily checks that for each of the $\binom{7}{3} = 35$ possible choices of S there is an integer $1 \leq r \leq 3$ such that $|F^r S \cap S| \geq 2$ so L is defined over $\mathbb{F}_q, \mathbb{F}_{q^2}$ or $\mathbb{F}_{q^3}$, i.e. we are in one of the three cases above. $\square$

Let $\Delta_{l,i}$ be the subset of Δ_l corresponding to case (i) in Lemma 6.8. The set $\Delta_{l,1}^{F\sigma}$ is clearly disjoint from $\Delta_{l,2}^{F\sigma}$ and $\Delta_{l,3}^{F\sigma}$.

Lemma 6.9 *If six of the points of a λ -tuple $(P_1, \dots, P_7)$ lie on a smooth conic, then $P_1, \dots, P_6$ lie on the conic and the conic is defined over $\mathbb{F}_q$.*

Proof Suppose $S = \{P_1, \dots, P_6\}$ lie on a smooth conic C . Then $|FS \cap S| \geq 5$ so $FC = C$. Let $P \in S$ be an $\mathbb{F}_{q^6}$ -point. Then we have $\{P, FP, \dots, F^5P\} = \{P_1, \dots, P_6\} \subset C$. $\square$

Since a smooth conic does not contain a line, we have that Δ_c only intersects $\Delta_{l,3}$, see Fig. 2.

We compute $|\Delta_{l,1}^{F\sigma}|$ by first choosing an $\mathbb{F}_q$ -line L and then a strict $\mathbb{F}_{q^6}$ point on L . Finally we choose an $\mathbb{F}_q$ -point P_7 anywhere. We thus have

$$|\Delta_{l,1}^{F\sigma}| = (q^2 + q + 1) \cdot (q^6 - q^3 - q^2 + q) \cdot (q^2 + q + 1).$$

To obtain $|\Delta_{l,2}^{F\sigma}|$ we first choose a strict $\mathbb{F}_{q^2}$ -line, L . There are $q^4 - q$ such lines and once a line L is chosen, the other line must be FL . We then choose a strict $\mathbb{F}_{q^6}$ -point P_1 on L . The points $P_2 = FP_1, \dots, P_6 = F^5P_1$ will then be the rest of our conjugate sextuple and there are $q^6 - q^2$ choices. We now have two $\mathbb{F}_{q^2}$ -lines with three of our six $\mathbb{F}_{q^6}$ -points on each so all that remains is to choose an $\mathbb{F}_q$ -point anywhere we want in one of $q^2 + q + 1$ ways. Hence,

$$|\Delta_{l,2}| = (q^4 - q)(q^6 - q^2)(q^2 + q + 1).$$

Fig. 2 An element in $\Delta_l \cap \Delta_c$

To count $|\Delta_{l,3}^{F\sigma}|$ we first choose an $\mathbb{F}_q$ -point P_7 in $q^2 + q + 1$ ways. There is a $\mathbb{P}^1$ of lines through P_7 and we want to choose a strict $\mathbb{F}_{q^3}$ -line L through P . There are $q^3 - q$ choices. Finally, we choose one of the strict $\mathbb{F}_{q^6}$ -points P_1 on L in one of $q^6 - q^3$ possible ways. We thus have

$$|\Delta_{l,3}| = (q^2 + q + 1)(q^3 - q)(q^6 - q^3).$$

In order to finish the computation of Δ_l , we need to compute $|\Delta_{l,2}^{F\sigma} \cap \Delta_{l,3}^{F\sigma}|$. We first choose a pair of conjugate $\mathbb{F}_{q^2}$ -lines in $\frac{1}{2}(q^4 - q)$ ways. These lines intersect in an $\mathbb{F}_q$ -point and we choose P_7 away from this point in one of $q^2 + q$ ways. We then choose a strict $\mathbb{F}_{q^3}$ -line through P_7 in one of $q^3 - q$ ways. This line intersects the two $\mathbb{F}_{q^2}$ -lines in two distinct points which clearly must have $\mathbb{F}_{q^6}$ as their minimal field of definition. We choose one of them to become P_1 in one of two ways. Thus, in total we have

$$|\Delta_{l,2}^{F\sigma} \cap \Delta_{l,3}^{F\sigma}| = (q^4 - q) \cdot (q^2 + q) \cdot (q^3 - q).$$

To compute $|\Delta_c^{F\sigma}|$ we first choose a smooth conic C in $q^5 - q^2$ ways. There are then $q^6 - q^3 - q^2 + q$ ways of choosing a conjugate sextuple on C . Finally, we choose P_7 anywhere we want in $q^2 + q + 1$ ways. We thus see that

$$|\Delta_c^{F\sigma}| = (q^5 - q^2)(q^6 - q^3 - q^2 + q)(q^2 + q + 1).$$

It remains to compute the size of the intersection between $\Delta_l^{F\sigma}$ and $\Delta_c^{F\sigma}$. To do this, we begin by choosing a smooth conic C in $q^5 - q^2$ ways and then an $\mathbb{F}_q$ -point P_7 not on C in $q^2 + q + 1 - (q + 1) = q^2$ ways. There are $q^3 - q$ strict $\mathbb{F}_{q^3}$ -lines passing through P_7 . All of these intersect C in two, not necessarily strict, $\mathbb{F}_{q^6}$ -points since,

by Lemma 6.1, these lines cannot be tangent to C . More precisely, choosing any of the $q^3 - q$ strict $\mathbb{F}_{q^3}$ -points Q of C gives a strict $\mathbb{F}_{q^3}$ -line through Q and P_7 , and since every such line cuts C in exactly two points we conclude that there are precisely $\frac{1}{2}(q^3 - q)$ strict $\mathbb{F}_{q^3}$ -lines through P intersecting C in two strict $\mathbb{F}_{q^3}$ -points. Thus, the remaining

$$q^3 - q - \frac{1}{2}(q^3 - q) = \frac{1}{2}(q^3 - q)$$

strict $\mathbb{F}_{q^3}$ -lines through P_7 will intersect C in two strict $\mathbb{F}_{q^6}$ -points. If we pick such a line and label one of the intersection points as P_1 we obtain an element in $\Delta_l^{F\sigma} \cap \Delta_c^{F\sigma}$. Hence,

$$|\Delta_l^{F\sigma} \cap \Delta_c^{F\sigma}| = (q^5 - q^2)q^2(q^3 - q).$$

We now conclude that

$$|(\mathcal{P}_7^{F\sigma})| = q^6 - 2q^3 + 1.$$

6.3 The case $\lambda = [2, 5]$

For $\lambda = [2, 6]$ we can take a conjugate λ -tuple as a septuple $(P_1, \dots, P_7)$ such that $FP_i = P_{i+1}$ for $i = 1, \dots, 4$, $FP_5 = P_1$, $FP_6 = P_7$ and $FP_7 = P_6$. Also in this case we take U as the subset of $(\mathbb{P}^2)^7$ of pairwise distinct points. We then have

$$\begin{aligned} |U^{F\sigma}| &= ((q^{10} + q^5 + 1) - (q^2 + q + 1))((q^4 + q^2 + 1) - (q^2 + q + 1)) \\ &= (q^{10} + q^5 - q^2 - q)(q^4 - q). \end{aligned}$$

The main observation is the following.

Lemma 6.10 *If $(P_1, \dots, P_7)$ is a λ -tuple with three of its points on a line, then all five $\mathbb{F}_{q^5}$ -points lie on a line defined over $\mathbb{F}_q$. If six of the points lie on a smooth conic C , then all seven points lie on C and C is defined over $\mathbb{F}_q$.*

Proof The proof is very similar to the proofs of Lemmas 6.6 and 6.7 and is therefore omitted. $\square$

There are $q^{10} + q^5 - q^2 - q$ conjugate quintuples whereof $(q^2 + q + 1)(q^5 - q)$ lie on a line. We may thus choose a conjugate quintuple whose points do not lie on a line in $q^{10} - q^7 - q^6 + q^3$ ways. This quintuple defines a smooth conic C . By Lemma 6.10, it is enough to choose a conjugate pair outside C in order to obtain an element of $U^{F\sigma} \setminus \Delta^{F\sigma}$ of the desired type. Since there are $q^4 - q$ conjugate pairs of which $q^2 - q$ lie on C there are $q^4 - q^2$ remaining choices. We thus obtain

$$|(\mathcal{P}_7^{F\sigma})| = q^6 - q^2.$$

6.4 The case $\lambda = [1^2, 5]$

The computation in this case is very similar to that of the case $\lambda = [2, 5]$ and we therefore simply state the result:

$$|(\mathcal{P}_7^{F\sigma})| = q^6 - q^2.$$

6.5 The case $\lambda = [3^1, 4^1]$

For $\lambda = [3, 4]$ we can take a conjugate λ -tuple as a septuple $(P_1, \dots, P_7)$ such that F permutes the tuples (P_1, P_2, P_3, P_4) and (P_5, P_6, P_7) cyclically. Also in this case we take U as the subset of $(\mathbb{P}^2)^7$ of pairwise distinct points. We then have

$$\begin{aligned} |U^{F\sigma}| &= ((q^8 + q^4 + 1) - (q^4 + q^2 + 1))((q^6 + q^3 + 1) - (q^2 + q + 1)) \\ &= (q^8 - q^2)(q^6 + q^3 - q^2 - q). \end{aligned}$$

The main observation is the following.

Lemma 6.11 *If a conjugate λ -tuple has three points on a line, then either*

- (1) *the four $\mathbb{F}_{q^4}$ -points lie on an $\mathbb{F}_q$ -line, or*
- (2) *the three $\mathbb{F}_{q^3}$ -points lie on an $\mathbb{F}_q$ -line.*

Proof It is easy to see that if three $\mathbb{F}_{q^4}$ -points lie on a line, then all four $\mathbb{F}_{q^4}$ -points lie on that line and even easier to see the corresponding result for three $\mathbb{F}_{q^3}$ -points.

Suppose that two $\mathbb{F}_{q^4}$ -points P_i and P_j and an $\mathbb{F}_{q^3}$ -point P lie on a line L . Since $F^4 P_i = P_i$ and $F^4 P_j = P_j$ we see that $F^4 L = L$. Thus, $F^4 P = FP \neq P$ lies on L . Repeating this argument again, with FP in the place of P , shows that also $F^2 P$ lies on L . We are thus in case (2).

If we assume that two $\mathbb{F}_{q^3}$ -points and an $\mathbb{F}_{q^4}$ -point lie on a line, then a completely analogous argument shows that all four $\mathbb{F}_{q^4}$ -points lie on that line. $\square$

We decompose $\Delta_l^{F\sigma}$ as

$$\Delta_l^{F\sigma} = \Delta_{l,1} \cup \Delta_{l,2},$$

where $\Delta_{l,1}$ consists of tuples with the four $\mathbb{F}_{q^4}$ -points on a line and $\Delta_{l,2}$ consists of tuples with the three $\mathbb{F}_{q^3}$ -points on a line. The computations of $|\Delta_{l,1}|$, $|\Delta_{l,2}|$ and $|\Delta_{l,1} \cap \Delta_{l,2}|$ are straightforward and we get

$$\begin{aligned} |\Delta_{l,1}| &= (q^2 + q + 1)(q^4 - q^2)(q^6 + q^3 - q^2 - q), \\ |\Delta_{l,2}| &= (q^2 + q + 1)(q^3 - q)(q^8 - q^2), \\ |\Delta_{l,1} \cap \Delta_{l,2}| &= (q^2 + q + 1)(q^3 - q)(q^4 - q^2), \end{aligned}$$

so

$$|\Delta_l^{F\sigma}| = q^{13} + 2q^{12} - 3q^{10} - 2q^9 + q^8 + q^7 - q^6 - q^5 + q^4 + q^3.$$

To compute $|\Delta_c^{F\sigma}|$ we start by noting that if six of the points of a λ -tuple lie on a smooth conic C , then all seven points lie on C and C is defined over $\mathbb{F}_q$. Thus, the problem consists of choosing a smooth conic C over $\mathbb{F}_q$ and then picking a λ -tuple on C . We thus have

$$|\Delta_c^{F\sigma}| = (q^5 - q^2)(q^4 - q^2)(q^3 - q).$$

Since no three points on a smooth conic lie on a line, we conclude that the intersection $\Delta_l^{F\sigma} \cap \Delta_c^{F\sigma}$ is empty. We thus obtain

$$|(\mathcal{P}_7^2)^{F\sigma}| = q^6 - q^5 - 2q^4 + q^3 + q^2.$$

6.6 The case $\lambda = [1, 2, 4]$

For $\lambda = [1, 2, 4]$ we can take a conjugate λ -tuple as a septuple $(P_1, \dots, P_7)$ such that F permutes the tuple (P_1, P_2, P_3, P_4) cyclically, transposes the points P_5 and P_6 and fixes the point P_7 . Also in this case we take U as the subset of $(\mathbb{P}^2)^7$ of pairwise distinct points. We then have

$$|U^{F\sigma}| = (q^8 - q^2)(q^4 - q)(q^2 + q + 1).$$

We have the following trivial decomposition of $\Delta_l^{F\sigma}$:

$$\Delta_l^{F\sigma} = \bigcup_{i=1}^6 \Delta_{l,i},$$

where

- $\Delta_{l,1}$ consists of λ -tuples with three strict $\mathbb{F}_{q^4}$ -points lying on a line,
- $\Delta_{l,2}$ consists of λ -tuples with two strict $\mathbb{F}_{q^4}$ -points and a strict $\mathbb{F}_{q^2}$ -point lying on a line,
- $\Delta_{l,3}$ consists of λ -tuples with two strict $\mathbb{F}_{q^4}$ -points and the $\mathbb{F}_q$ -point lying on a line,
- $\Delta_{l,4}$ consists of λ -tuples with a strict $\mathbb{F}_{q^4}$ -point and two strict $\mathbb{F}_{q^2}$ -points lying on a line,
- $\Delta_{l,5}$ consists of λ -tuples with a strict $\mathbb{F}_{q^4}$ -point, a strict $\mathbb{F}_{q^2}$ -point and an $\mathbb{F}_q$ -point lying on a line, and,
- $\Delta_{l,6}$ consists of λ -tuples with the two strict $\mathbb{F}_{q^2}$ -points and the $\mathbb{F}_q$ -point lying on a line.

This decomposition is of course naive and is not very nice to work with since none of the possible intersections are empty and the reader can surely think of many other decompositions which, a priori, look more promising. However, the more “clever” approaches we have tried have turned out to be quite hard to work with in practice. The positive thing about the above decomposition is that most intersections are rather easily handled and that quadruple intersections (and higher) all consist of tuples where all seven points lie on an $\mathbb{F}_q$ -line.

The two slightly more complicated sets in the above list are $\Delta_{l,2}$ and $\Delta_{l,3}$. We shall therefore comment a bit about the computations involving them.

The set $\Delta_{l,2}$ splits into three disjoint subsets

$$\Delta_{l,2} = \Delta_{l,2}^1 \cup \Delta_{l,2}^2 \cup \Delta_{l,2}^3,$$

where

- $\Delta_{l,2}^1$ consists of λ -tuples such that the four strict $\mathbb{F}_{q^4}$ -points and the two strict $\mathbb{F}_{q^2}$ -points lie on an $\mathbb{F}_q$ -line, or,
- $\Delta_{l,2}^2$ consists of λ -tuples such that the two strict $\mathbb{F}_{q^4}$ -points and the strict $\mathbb{F}_{q^2}$ -point lie on a strict $\mathbb{F}_{q^2}$ -line L (and the other two strict $\mathbb{F}_{q^4}$ -points and the second strict $\mathbb{F}_{q^2}$ -point lie on FL), or,
- $\Delta_{l,2}^3$ consists of λ -tuples such that the four strict $\mathbb{F}_{q^4}$ -points and the two strict $\mathbb{F}_{q^2}$ -points are intersection points of four conjugate $\mathbb{F}_{q^4}$ -lines.

The sets $\Delta_{l,2}^2$ and $\Delta_{l,2}^3$ are illustrated in Fig. 3. The cardinality of $\Delta_{l,2}^1$ is easily computed to be $(q^2 + q + 1)^2(q^4 - q^2)(q^2 - q)$. To get the cardinality of $\Delta_{l,2}^2$, we first choose a strict $\mathbb{F}_{q^2}$ -line L in one of $q^4 - q$ ways and then a strict $\mathbb{F}_{q^4}$ -point P_1 on L in one of $q^4 - q^2$ ways. This determines all the four $\mathbb{F}_{q^4}$ points since they must be $P_2 = FP_1$, $P_3 = F^2P_1$ and $P_4 = F^3P_1$. We must now decide if P_5 should lie on L or FL . We then choose a strict $\mathbb{F}_{q^2}$ -point on the chosen line. The lines L and FL both contain $q^2 + 1$ points defined over $\mathbb{F}_{q^2}$ of which precisely one is defined over $\mathbb{F}_q$ (namely the point $L \cap FL$). Hence, there are q^2 choices for P_5 . It now only remains to choose P_7 in one of $q^2 + q + 1$ ways. We thus have

$$|\Delta_{l,2}^2| = 2(q^4 - q)(q^4 - q^2)q^2(q^2 + q + 1).$$

It remains to compute $|\Delta_{l,2}^3|$. We first choose an $\mathbb{F}_{q^2}$ -point P_5 not defined over $\mathbb{F}_q$ in one of $q^4 - q$ ways. There are $q^4 - q^2$ lines L strictly defined over $\mathbb{F}_{q^4}$ through P_5 and we choose one of them. We thus get four $\mathbb{F}_{q^4}$ -lines which intersect in the two $\mathbb{F}_{q^2}$ -points P_5 and P_6 as well as in four strict $\mathbb{F}_{q^4}$ -points. We choose one of these to become P_1 and the labels of the other three points are then given. However, we could as well have chosen the line F^2L and ended up with the same four $\mathbb{F}_{q^4}$ -points. We therefore must divide by 2. Finally, we choose any of the $q^2 + q + 1$ $\mathbb{F}_{q^4}$ -points to become P_7 . We thus have

$$|\Delta_{l,2}^3| = 2(q^4 - q)(q^4 - q^2)(q^2 + q + 1).$$

The set $\Delta_{l,3}^{F\sigma}$ splits into two disjoint subsets

$$\Delta_{l,3}^{F\sigma} = \Delta_{l,3}^1 \cup \Delta_{l,3}^2,$$

Fig. 3 Illustration of elements of the sets $\Delta_{l,2}^2$ and $\Delta_{l,2}^3$

where

- $\Delta_{l,3}^1$ consists of λ -tuples such that the four strict $\mathbb{F}_{q^4}$ -points and the $\mathbb{F}_q$ -point lie on an $\mathbb{F}_q$ -line, and,
- $\Delta_{l,3}^2$ consists of λ -tuples such that there are two conjugate $\mathbb{F}_{q^2}$ -lines intersecting in the $\mathbb{F}_q$ -point, each $\mathbb{F}_{q^2}$ -line containing two of the strict $\mathbb{F}_{q^4}$ -points.

To compute $|\Delta_{l,3}^1|$ we first choose an $\mathbb{F}_q$ -line L , then a conjugate quadruple and an $\mathbb{F}_q$ -point on L and finally a conjugate pair of $\mathbb{F}_{q^2}$ -points anywhere. Hence

$$|\Delta_{l,3}^1| = (q^2 + q + 1)(q^4 - q^2)(q + 1)(q^4 - q).$$

To compute $|\Delta_{l,3}^2|$ we first choose an $\mathbb{F}_{q^2}$ -line L not defined over $\mathbb{F}_q$, then an $\mathbb{F}_{q^4}$ -point P_4 not defined over $\mathbb{F}_{q^2}$ on L and finally a pair of conjugate $\mathbb{F}_{q^2}$ -points anywhere. We thus have

$$|\Delta_{l,3}^2| = (q^4 - q)^2(q^4 - q^2).$$

We now consider the intersection $\Delta_{l,2} \cap \Delta_{l,3}$. The decompositions above yield a decomposition

$$\Delta_{l,2} \cap \Delta_{l,3} = \bigcup_{i,j} \Delta_{l,2}^i \cap \Delta_{l,3}^j.$$

The intersection $\Delta_{l,2}^1 \cap \Delta_{l,3}^1$ consists of configurations where all seven points lie on an $\mathbb{F}_q$ -line. There are

$$(q^2 + q + 1)(q^4 - q^2)(q^2 - q)(q + 1)$$

such λ -tuples. Both the intersections $\Delta_{l,2}^1 \cap \Delta_{l,3}^2$ and $\Delta_{l,2}^2 \cap \Delta_{l,3}^1$ are empty.

To compute the cardinality of $\Delta_{l,2}^2 \cap \Delta_{l,3}^2$ we first choose a strict $\mathbb{F}_{q^2}$ -line L in one of $q^4 - q$ ways and then a strict $\mathbb{F}_{q^4}$ -point P_1 on L in one of $q^4 - q^2$ ways. We must now decide if P_5 should lie on L or FL . We then choose a strict $\mathbb{F}_{q^2}$ -point on the chosen line in one of q^2 ways. We are now sure to have a tuple in $\Delta_{l,2}^2$. To make sure that the tuple also lies in $\Delta_{l,3}^2$ we have no choice but to put P_7 at the intersection of L and FL . There are thus

$$2(q^4 - q)(q^4 - q^2)q^2$$

elements in the intersection $\Delta_{l,2}^2 \cap \Delta_{l,3}^2$.

The intersection $\Delta_{l,2}^3 \cap \Delta_{l,3}^1$ is empty so there is only one intersection remaining. As explained in the computation of $|\Delta_{l,2}^3|$, there are $2(q^4 - q)(q^4 - q^2)$ ways to obtain four strict $\mathbb{F}_{q^4}$ -points and two strict $\mathbb{F}_{q^2}$ -points which are the intersection points of four conjugate $\mathbb{F}_{q^4}$ -lines. We now note that there are precisely six lines through pairs of points among the four strict $\mathbb{F}_{q^4}$ -points. Four of these lines are of course the four $\mathbb{F}_{q^4}$ -lines. The remaining two lines are defined over $\mathbb{F}_{q^2}$ and therefore intersect in an $\mathbb{F}_q$ -point. To obtain a tuple in $\Delta_{l,3}^2$ we have no choice but to choose P_7 as this intersection point. Therefore, there are

$$2(q^4 - q)(q^4 - q^2)$$

elements in the intersection $\Delta_{l,2}^3 \cap \Delta_{l,3}^2$.

The remaining computations are rather straightforward and one obtains

$$|\Delta_l^{F\sigma}| = q^{13} + 5q^{12} - 4q^{10} - 5q^9 - 3q^8 + 2q^7 + q^6 + 3q^5 + q^4 - q^3.$$

We now turn to Δ_c . We have that if six points of a conjugate λ -tuple lie on a smooth conic C , then the four $\mathbb{F}_{q^4}$ -points and the two $\mathbb{F}_{q^2}$ -points lie on C and C is defined over $\mathbb{F}_q$. Thus, the computation of $|\Delta_c|$ consists of choosing a smooth conic C defined over $\mathbb{F}_q$, choosing a strict $\mathbb{F}_{q^4}$ -point and a strict $\mathbb{F}_{q^2}$ -point on C and finally choosing an $\mathbb{F}_q$ -point anywhere. We thus have

$$|\Delta_c^{F\sigma}| = (q^5 - q^2)(q^4 - q^2)(q^2 - q)(q^2 + q + 1).$$

It remains to investigate the intersection $\Delta_l \cap \Delta_c$. Since a smooth conic cannot contain three collinear points, we only have nonempty intersection between $\Delta_c^{F\sigma}$ and the sets $\Delta_{l,3}^2$ and $\Delta_{l,6}$.

To compute $|\Delta_{l,3}^2 \cap \Delta_c^{F\sigma}|$ we first choose a smooth conic C , then a conjugate quadruple on C and finally a pair of conjugate $\mathbb{F}_{q^2}$ -points on C . The $\mathbb{F}_q$ -point is then uniquely defined as the intersection point of the two $\mathbb{F}_{q^2}$ -lines through pairs of the four strict $\mathbb{F}_{q^4}$ -points. We thus have

$$|\Delta_{l,3}^2 \cap \Delta_c| = (q^5 - q^2)(q^4 - q^2)(q^2 - q).$$

The same construction as above works for the intersection $\Delta_{l,6} \cap \Delta_c^{F\sigma}$ if we remember that we now have some choice for the $\mathbb{F}_q$ -point since it can lie anywhere on the line through the two $\mathbb{F}_{q^2}$ -points. We thus have

$$|\Delta_{l,6} \cap \Delta_c^{F\sigma}| = (q^5 - q^2)(q^4 - q^2)(q^2 - q)(q + 1).$$

The only thing that remains to compute is the cardinality of the triple intersection $\Delta_{l,3}^2 \cap \Delta_{l,6} \cap \Delta_c^{F\sigma}$. We first assume that the $\mathbb{F}_q$ -point is on the $\mathbb{F}_q$ -outside of a smooth conic C containing the other six points. We first choose C in $q^5 - q^2$ ways. There are $\frac{1}{2}(q + 1)q$ ways to choose two $\mathbb{F}_q$ -points P and Q on C . Intersecting the tangents

$T_P C$ and $T_Q C$ gives an $\mathbb{F}_q$ -point P_7 which will clearly lie on the $\mathbb{F}_q$ -outside of C . Hence, there are precisely $\frac{1}{2}(q+1)q$ ways to choose an $\mathbb{F}_q$ -point on the $\mathbb{F}_q$ -outside of C .

We now want to choose an $\mathbb{F}_q$ -line through P_7 intersecting C in two $\mathbb{F}_{q^2}$ -points. There are $q+1$ $\mathbb{F}_q$ -lines through P_7 of which two are tangent to C . These tangent lines contain an $\mathbb{F}_q$ -point of C each so there are $q-1$ remaining $\mathbb{F}_q$ -points on C . Picking such a point gives a line through this point, P_7 and one further point on C . We thus see that exactly $\frac{1}{2}(q-1)$ of the $\mathbb{F}_q$ -lines through P_7 intersect C in two conjugate $\mathbb{F}_{q^2}$ -points. We label one of them as P_5 .

We shall now choose a conjugate pair of $\mathbb{F}_{q^2}$ -lines through P_7 intersecting C in four $\mathbb{F}_{q^4}$ -points. There are $q^2 - q$ conjugate pairs of $\mathbb{F}_{q^2}$ -lines through P_7 . No $\mathbb{F}_{q^2}$ -line through P_7 is tangent to C so each $\mathbb{F}_{q^2}$ -line through P_7 will intersect C in two points. The conic C contains $q^2 - q$ points which are defined over $\mathbb{F}_{q^2}$ but not $\mathbb{F}_q$. Picking such a point gives a line through this point and P_7 as well as one further $\mathbb{F}_{q^2}$ -point not defined over $\mathbb{F}_q$. Thus, there are $\frac{1}{2}(q^2 - q)$ lines obtained in this way. Typically, such a line will be defined over $\mathbb{F}_{q^2}$ but not $\mathbb{F}_q$. We saw above that the number of such lines which are defined over $\mathbb{F}_q$ is precisely $\frac{1}{2}(q-1)$. Thus, there are precisely

$$\frac{1}{2}(q^2 - q) - \frac{1}{2}(q - 1) = \frac{1}{2}(q^2 - 2q + 1)$$

$\mathbb{F}_{q^2}$ -lines, not defined over $\mathbb{F}_q$, which intersect C in two strict $\mathbb{F}_{q^2}$ -points. Thus, the remaining

$$q^2 - q - \frac{1}{2}(q^2 - 2q + 1) = \frac{1}{2}(q^2 - 1) \quad (6.2)$$

$\mathbb{F}_{q^2}$ -lines must intersect C in two strict $\mathbb{F}_{q^4}$ -points. Picking such a line and labeling one of the points P_1 gives a configuration belonging to $\Delta_{1,3}^2 \cap \Delta_{1,6} \cap \Delta_c^{F\sigma}$ and we thus see that there are

$$\frac{1}{2}(q^5 - q^2)q(q+1)(q-1)(q^2 - 1)$$

such configurations with P_7 on the $\mathbb{F}_q$ -outside of C .

We now assume that P_7 is on the $\mathbb{F}_q$ -inside of C . We first choose C in one of $q^5 - q^2$ ways. Since the number of $\mathbb{F}_q$ -points is $q^2 + q + 1$ and $q+1$ of these lie on C the number of $\mathbb{F}_q$ -points not on C is precisely q^2 . We just saw that $\frac{1}{2}(q+1)q$ of these lie on the $\mathbb{F}_q$ -outside of C so there must be

$$q^2 - \frac{1}{2}(q+1)q = \frac{1}{2}(q^2 - q)$$

$\mathbb{F}_q$ -points which lie on the $\mathbb{F}_q$ -inside of C .

Since P_7 now lies on the $\mathbb{F}_q$ -inside of C , every $\mathbb{F}_q$ -line through P_7 will intersect C in two points. Exactly $\frac{1}{2}(q+1)$ will intersect C in two $\mathbb{F}_q$ -points so the remaining $\frac{1}{2}(q+1)$ will intersect C in two conjugate $\mathbb{F}_{q^2}$ -points. We pick such a pair of points and label one of them P_5 .

We now choose a conjugate pair of $\mathbb{F}_{q^2}$ -lines through P_7 intersecting C in a conjugate quadruple of $\mathbb{F}_{q^4}$ -points. The number of $\mathbb{F}_{q^2}$ -lines, not defined over $\mathbb{F}_q$, through

P_7 is $q^2 - q$. Two of these are tangent to C so, using ideas similar to those above, we see that

$$\frac{1}{2}(q^2 - q - 2) - \frac{1}{2}(q + 1) + 2 = \frac{1}{2}(q^2 - 2q + 1)$$

of these lines intersect C in points strictly defined over $\mathbb{F}_{q^2}$. Hence, the remaining

$$q^2 - q - \frac{1}{2}(q^2 - 2q + 1) = \frac{1}{2}(q^2 - 1) \quad (6.3)$$

lines intersect C in two $\mathbb{F}_{q^4}$ -points which are not defined over $\mathbb{F}_{q^2}$. If we pick one of these points to become P_1 we end up with a configuration in $\Delta_{l,3}^2 \cap \Delta_{l,6} \cap \Delta_c^{F\sigma}$. We thus have

$$\frac{1}{2}(q^5 - q^2)(q^2 - q)(q + 1)(q^2 - 1)$$

such configurations with P_7 on the inside of C . One may note that the expression above actually is the same as the expression when P_7 was on the outside, but this will not always be the case in this type of decomposition. We thus have

$$|\Delta_l^{F\sigma} \cap \Delta_c^{F\sigma}| = q^{12} + q^{11} - 4q^{10} - 2q^9 + 3q^8 + 4q^7 - 4q^5 + q^3,$$

and finally

$$|(\mathcal{P}_7^2)^{F\sigma}| = q^6 - q^5 - 2q^4 + q^3 - 2q^2 + 3.$$

6.7 The case $\lambda = [1^3, 4]$

For $\lambda = [1^3, 4]$ we can take a conjugate λ -tuple as a septuple $(P_1, \dots, P_7)$ such that F permutes the tuple (P_1, P_2, P_3, P_4) cyclically and fixes each of the points P_5, P_6 and P_7 . In this case we take U as the subset of $(\mathbb{P}^2)^7$ consisting of pairwise distinct points such that the last three of the points are not collinear. Thus, $U^{F\sigma}$ consists of a conjugate quadruple and three $\mathbb{F}_q$ -points not on a line. We then have

$$|U^{F\sigma}| = (q^8 - q^2)(q^2 + q + 1)(q^2 + q)q^2$$

We can decompose $\Delta_l^{F\sigma}$ into a disjoint union as

$$\Delta_l^{F\sigma} = \Delta_{l,1} \cup \Delta_{l,2},$$

where $\Delta_{l,1}$ consists of septuples such that all four $\mathbb{F}_{q^4}$ -points lie on a $\mathbb{F}_q$ -line and $\Delta_{l,2}$ consists of septuples such that the $\mathbb{F}_{q^2}$ -line through P_1 and P_3 intersects the $\mathbb{F}_{q^2}$ -line through P_2 and P_4 in P_5, P_6 or P_7 .

To compute $|\Delta_{l,1}|$ we simply choose a $\mathbb{F}_q$ -line L , a conjugate quadruple on L and finally place the three $\mathbb{F}_q$ -points in such a way that they do not lie on a line. This can be done in

$$|\Delta_{l,1}| = (q^2 + q + 1)^2(q^4 - q^2)(q^2 + q)q^2$$

ways. To compute $|\Delta_{l,2}|$ we first choose P_5 , P_6 or P_7 then a $\mathbb{F}_{q^2}$ -line L not defined over $\mathbb{F}_q$ through this point. Finally, we choose a $\mathbb{F}_{q^4}$ -point P_1 which is not defined over $\mathbb{F}_{q^2}$ on L and make sure that the final two $\mathbb{F}_q$ -points are not collinear with the first. This can be done in

$$|\Delta_{l,2}| = 3(q^2 + q + 1)(q^2 - q)(q^4 - q^2)(q^2 + q)q^2$$

ways. This gives that

$$|\Delta_l^{F\sigma}| = 4q^{12} + 6q^{11} + q^{10} - 4q^9 - 5q^8 - q^7 - q^5.$$

We now turn to investigate Δ_c . It is not hard to see that if six of the points lie on a smooth conic C , then the four $\mathbb{F}_{q^4}$ -points must lie on that conic and C must be defined over $\mathbb{F}_q$. We thus choose a smooth conic C over $\mathbb{F}_q$ and a conjugate quadruple on C . Then we choose P_5 , P_6 or P_7 to possibly not lie on C and place the other two on C . Finally, we place the final $\mathbb{F}_q$ -point anywhere except on the line through the other two $\mathbb{F}_q$ -points. This gives the number

$$3(q^5 - q^2)(q^4 - q^2)(q + 1)q \cdot q^2.$$

However, we have now counted the configurations where all seven points lie on a smooth conic three times. There are

$$(q^5 - q^2)(q^4 - q^2)(q + 1)q(q - 1)$$

such configurations and it thus follows that

$$|\Delta_c^{F\sigma}| = 3q^{13} + q^{12} - 3q^{11} - 2q^{10} - q^9 + q^8 - q^7 + 2q^5.$$

We now turn to the intersection $\Delta_l^{F\sigma} \cap \Delta_c^{F\sigma} = \Delta_{l,2} \cap \Delta_c^{F\sigma}$. We begin by choosing one of the $\mathbb{F}_q$ -points P_5 , P_6 and P_7 to not lie on the conic C . We call the chosen point P and the remaining two points P_i and P_j where $i < j$. We now have three disjoint possibilities:

- (i) the point P may lie on the $\mathbb{F}_q$ -outside of C with one of the tangents through P also passing through P_i ,
- (ii) the point P may lie on the $\mathbb{F}_q$ -outside of C with none of the tangents through P passing through P_i ,
- (iii) the point P may lie on the $\mathbb{F}_q$ -inside of C .

We consider the three cases (i)–(iii) separately.

(i). We begin by choosing a smooth conic C over $\mathbb{F}_q$ in one of $q^5 - q^2$ ways and a $\mathbb{F}_q$ -point P on the $\mathbb{F}_q$ -outside of C in one of $\frac{1}{2}(q + 1)q$ ways. By equation (6.2), there are now $q^2 - 1$ ways to choose P_1 . Since we require P_i to lie on a tangent to C which

passes through P , we only have 2 choices for P_i . Finally, we may choose P_j as any of the q remaining points on C . We thus have

$$3(q^5 - q^2)(q + 1)q(q^2 - 1)q$$

possibilities in this case.

(ii). Again, we begin by choosing a smooth conic C over $\mathbb{F}_q$ in one of $q^5 - q^2$ ways and a $\mathbb{F}_q$ -point P on the $\mathbb{F}_q$ -outside of C in one of $\frac{1}{2}(q + 1)q$ ways. We then choose P_1 on C in one of $q^2 - 1$ ways. The point P_i should now not lie on a tangent to C which passes through P so we have $q - 1$ choices. Since the line between P and P_i is not a tangent to C , there is one further intersection point between this line and C . Since P , P_i and P_j should not be collinear, we must take P_j away from this point and P_i and we thus have $q - 1$ possible choices. Hence, we have

$$\frac{3}{2}(q^5 - q^2)(q + 1)q(q^2 - 1)(q - 1)^2$$

possibilities in this case.

(iii). We start by choosing a smooth conic C over $\mathbb{F}_q$ in $q^5 - q^2$ ways and then a point P on the $\mathbb{F}_q$ -inside of C in $\frac{1}{2}(q^2 - q)$ ways. We continue by using equation (6.3) to see that we have $q^2 - 1$ choices for P_1 . We now choose P_i as any of the $\mathbb{F}_q$ -points on C and thus have $q + 1$ choices. Finally, we may choose P_j as any $\mathbb{F}_q$ -point on C , except in the intersection of C with the line through P_i and P . Hence, we have $q - 1$ choices. We thus have

$$\frac{3}{2}(q^5 - q^2)(q^2 - q)(q^2 - 1)(q + 1)(q - 1)$$

possibilities in this case. We conclude that

$$|\Delta_l^{F\Sigma} \cap \Delta_c^{F\sigma}| = 3q^{11} - 3q^9 - 3q^5 + 3q^3,$$

and, finally,

$$|(\mathcal{P}_7^{F\sigma})| = q^6 - q^5 - 2q^4 + q^3 - 2q^2 + 3.$$

6.8 The case $\lambda = [1, 3^2]$

For $\lambda = [1, 3^2]$ we can take a conjugate λ -tuple as a septuple $(P_1, \dots, P_7)$ such that F permutes the tuples (P_1, P_2, P_3) and (P_4, P_5, P_6) cyclically and fixes the point P_7 . In this case we take U as the subset of $(\mathbb{P}^2)^7$ consisting of pairwise distinct points. We then have

$$|U^{F\sigma}| = (q^6 + q^3 - q^2 - q)(q^6 + q^3 - q^2 - q - 3)(q^2 + q + 1).$$

We shall also denote the second triple of strict $\mathbb{F}_{q^3}$ -points by (Q_1, Q_2, Q_3) and the $\mathbb{F}_q$ -point by R .

Fig. 4 Illustration of the decomposition of Δ_l

We decompose $\Delta_l^{F\sigma}$ as

$$\Delta_l^{F\sigma} = \bigcup_{i=1}^5 \Delta_{l,i},$$

where

- $\Delta_{l,1}$ consists of λ -tuples such that the points P_1 , P_2 and P_3 lie on an $\mathbb{F}_q$ -line,
- $\Delta_{l,2}$ consists of λ -tuples such that the points P_1 , P_2 and P_3 are the intersection points of a conjugate triple of $\mathbb{F}_{q^3}$ -lines with each of the lines containing one of the points Q_1 , Q_2 and Q_3 ,
- $\Delta_{l,3}$ consists of λ -tuples such that the points Q_1 , Q_2 and Q_3 are the intersection points of a conjugate triple of $\mathbb{F}_{q^3}$ -lines with each of the lines containing one of the points P_1 , P_2 and P_3 ,
- $\Delta_{l,4}$ consists of λ -tuples such that the points Q_1 , Q_2 and Q_3 lie on an $\mathbb{F}_q$ -line, and
- $\Delta_{l,5}$ consists of λ -tuples such that the point R is the intersection of three conjugate $\mathbb{F}_{q^3}$ -lines with each of the $\mathbb{F}_{q^3}$ -lines containing one of the points P_1 , P_2 and P_3 and one of the points Q_1 , Q_2 and Q_3 (Fig. 4).

We investigate each of the cases separately.

6.8.1 $\Delta_{l,1}$ and $\Delta_{l,4}$

The cardinalities of $\Delta_{l,1}$ and $\Delta_{l,4}$ are obviously the same so we only make the computation for $\Delta_{l,1}$. We thus choose an $\mathbb{F}_q$ -line L , a conjugate $\mathbb{F}_{q^3}$ -tuple P_1, P_2, P_3 on L , a conjugate $\mathbb{F}_{q^3}$ -tuple Q_1, Q_2, Q_3 anywhere except equal to the other $\mathbb{F}_{q^3}$ -tuple and, finally, an $\mathbb{F}_q$ -point anywhere. We thus get

$$|\Delta_{l,1}| = |\Delta_{l,4}| = (q^2 + q + 1)^2 (q^3 - q)(q^6 + q^3 - q^2 - q - 3).$$

6.8.2 $\Delta_{l,2}$ and $\Delta_{l,3}$

The cardinalities of $\Delta_{l,2}$ and $\Delta_{l,3}$ are of course also the same. To compute $|\Delta_{l,2}|$ we first choose a conjugate triple of lines, L, FL, F^2L , which do not intersect in a point. There are $q^6 + q^3 - q^2 - q$ conjugate triples of lines of which $(q^2 + q + 1)(q^3 - q)$ intersect in a point. There are thus $q^6 - q^5 - q^4 + q^3$ possible triples. We give the label P_1 to the point $L \cap FL$ which determines the labels of the other two intersection points. We must now choose if Q_1 should lie on L, FL or F^2L and then place Q_1 on

the chosen line. There are $3(q^3 - 1)$ ways to do this. Finally, we choose any $\mathbb{F}_q$ -point. We thus have

$$|\Delta_{l,2}| = |\Delta_{l,3}| = 3(q^6 - q^5 - q^4 + q^3)(q^3 - 1)(q^2 + q + 1).$$

6.8.3 $\Delta_{l,5}$

We first choose an $\mathbb{F}_q$ -point R anywhere and then a conjugate triple of lines, L, FL, F^2L through R . We then choose a point P_1 somewhere on L in q^3 ways. We must now decide if Q_1 should lie on L, FL or F^2L and then pick a point on the chosen line in one of $q^3 - 1$ ways. We thus see that

$$|\Delta_{l,5}| = 3(q^2 + q + 1)(q^3 - q)q^3(q^3 - 1).$$

We must now compute the cardinalities of the different intersections. Firstly, note that the intersection between $\Delta_{l,1}$ and $\Delta_{l,2}$ is empty. Secondly, the size of the intersection of $\Delta_{l,1}$ and $\Delta_{l,3}$ is equal to that of the intersection of $\Delta_{l,2}$ and $\Delta_{l,4}$. To obtain $|\Delta_{l,1} \cap \Delta_{l,3}|$ we first choose a conjugate triple of lines, L, FL, F^2L , which do not intersect in a point and label the intersection $L \cap FL$ by Q_1 . We then choose an $\mathbb{F}_q$ -line L' and thus get three strict $\mathbb{F}_{q^3}$ -points $L' \cap L, L' \cap FL$ and $L' \cap F^2L$. We label one of these by P_1 . We may now choose any $\mathbb{F}_q$ -point to become R . We thus see that that

$$|\Delta_{l,1} \cap \Delta_{l,3}| = |\Delta_{l,2} \cap \Delta_{l,4}| = 3(q^6 - q^5 - q^4 + q^3)(q^2 + q + 1)^2.$$

When we consider the intersection between $\Delta_{l,1}$ and $\Delta_{l,4}$ we must distinguish between the cases where the two triples of strict $\mathbb{F}_{q^3}$ -points lie on the same line and when they do not. A simple computation then gives

$$|\Delta_{l,1} \cap \Delta_{l,4}| = (q^2 + q + 1)^2(q^3 - q)(q^3 - q - 3) + (q^2 + q + 1)^2(q^2 + q)(q^3 - q)^2.$$

We continue by observing that $|\Delta_{l,1} \cap \Delta_{l,5}| = |\Delta_{l,4} \cap \Delta_{l,5}|$. To compute $|\Delta_{l,1} \cap \Delta_{l,5}|$ we first choose an $\mathbb{F}_q$ -point R and then a conjugate $\mathbb{F}_{q^3}$ -tuple of lines L, FL, F^2L through R . We continue by choosing an $\mathbb{F}_q$ -line L' not through R in one of q^2 ways and then label $L' \cap L, L' \cap FL$ or $L' \cap F^2L$ by Q_1 . Finally, we choose one of the remaining $q^3 - 1$ points of L to become P_1 . Hence,

$$|\Delta_{l,1} \cap \Delta_{l,5}| = |\Delta_{l,4} \cap \Delta_{l,5}| = 3(q^2 + q + 1)(q^3 - q)q^2(q^3 - 1).$$

The sets $\Delta_{l,2}$ and $\Delta_{l,3}$ do not intersect and neither do the sets $\Delta_{l,3}$ and $\Delta_{l,4}$. Hence, there are only two intersections left, namely the one between $\Delta_{l,2}$ and $\Delta_{l,5}$ and the one between $\Delta_{l,3}$ and $\Delta_{l,5}$. These have equal cardinalities. To compute $|\Delta_{l,2} \cap \Delta_{l,5}|$ we first choose a conjugate triple of lines, L, FL, F^2L , which do not intersect in a point and label the intersection $L \cap FL$ by Q_1 . We then choose an $\mathbb{F}_q$ -point R . The lines between R and the points Q_1, Q_2 and Q_3 intersect the lines L, FL and F^2L in three points and we label one of them by P_1 . We thus have

$$|\Delta_{l,2} \cap \Delta_{l,5}| = |\Delta_{l,3} \cap \Delta_{l,5}| = 3(q^6 - q^5 - q^4 + q^3)(q^2 + q + 1).$$

There is only one triple intersection, namely between $\Delta_{l,1}$, $\Delta_{l,4}$ and $\Delta_{l,5}$. To compute the size of this intersection we first choose an $\mathbb{F}_q$ -point R and then a conjugate triple of lines, L , FL , F^2L through R . We then choose an $\mathbb{F}_q$ -line L' not through R and label the intersection $L \cap L'$ by P_1 . Finally, we choose another $\mathbb{F}_q$ -line L'' and label one of the intersections $L'' \cap L$, $L'' \cap FL$ and $L'' \cap F^2L$ by Q_1 . This shows that

$$|\Delta_{l,1} \cap \Delta_{l,4} \cap \Delta_{l,5}| = 3(q^2 + q + 1)(q^3 - q)q^2(q^2 - 1).$$

We now turn to the computation of $|\Delta_c|$. If six points of a λ -tuple lie on a smooth conic C , then both of the conjugate triples must lie on C and C must be defined over $\mathbb{F}_q$. Hence, to obtain $|\Delta_c|$ we only have to choose a smooth $\mathbb{F}_q$ -conic C , two conjugate triples on C and an $\mathbb{F}_q$ -point anywhere. We thus have that

$$|\Delta_c| = (q^5 - q^2)(q^3 - q)(q^3 - q - 3)(q^2 + q + 1).$$

Since the sets $\Delta_{l,1}$, $\Delta_{l,2}$, $\Delta_{l,3}$ and $\Delta_{l,4}$ all require three of the $\mathbb{F}_{q^3}$ -points to lie on a line, they will have empty intersection with Δ_c . This is however not true for the set $\Delta_{l,5}$. To obtain such a configuration we first choose a smooth conic C and an $\mathbb{F}_q$ -point R . Now choose an $\mathbb{F}_{q^3}$ -point P_1 on C which is not defined over $\mathbb{F}_q$ in one of $q^3 - q$ ways. Since both C and R are defined over $\mathbb{F}_q$ we know that any tangent to C which passes through R must be defined over $\mathbb{F}_{q^2}$. Hence, the line through R and P_1 will intersect C in P_1 and one point more. We label this point with Q_1 , Q_2 or Q_3 . We thus have

$$|\Delta_l \cap \Delta_c| = 3(q^5 - q^2)q^2(q^3 - q).$$

We now obtain

$$|(\mathcal{P}_7^2)^{F\sigma}| = q^6 - 2q^5 - 2q^4 - 8q^3 + 16q^2 + 10q + 21.$$

6.9 The case $\lambda = [2^2, 3]$

For $\lambda = [2^2, 3]$ we can take a conjugate λ -tuple as a septuple $(P_1, \dots, P_7)$ such that F permutes the tuple (P_1, P_2, P_3) cyclically and transposes the pairs (P_4, P_5) and (P_6, P_7) . In this case we take U as the subset of $(\mathbb{P}^2)^7$ consisting of pairwise distinct points. We then have

$$|U^{F\sigma}| = (q^6 + q^3 - q^2 - q)(q^4 - q)(q^4 - q - 2).$$

We can decompose $\Delta_l^{F\sigma}$ as

$$\Delta_l^{F\sigma} = \Delta_{l,1} \cup \Delta_{l,2},$$

where $\Delta_{l,1}$ consists of septuples such that the three strict $\mathbb{F}_{q^3}$ -points lie on a line and $\Delta_{l,2}$ consists of septuples such that the four strict $\mathbb{F}_{q^2}$ -points lie on a line.

We have

$$|\Delta_{l,1}| = (q^2 + q + 1)(q^3 - q)(q^4 - q)(q^4 - q - 2),$$

and

$$|\Delta_{l,2}| = (q^2 + q + 1)(q^2 - q)(q^2 - q - 2)(q^6 + q^3 - q^2 - q).$$

The cardinality of the intersection is easily computed to be

$$|\Delta_{l,1} \cap \Delta_{l,2}| = (q^2 + q + 1)^2 (q^3 - q)(q^2 - q)(q^2 - q - 2).$$

This allows us to compute $|\Delta_l^{F\sigma}|$.

We have that if six of the points of a λ -tuple lie on a smooth conic C , then all seven points lie on C and C is defined over $\mathbb{F}_q$. We thus have that $\Delta_c^{F\sigma}$ is disjoint from $\Delta_l^{F\sigma}$. We also see that

$$|\Delta_c| = (q^5 - q^2)(q^3 - q)(q^2 - q)(q^2 - q - 2),$$

so,

$$|(\mathcal{P}_7^{F\sigma})| = q^6 - q^5 - 2q^4 + 3q^3 + q^2 - 2q.$$

6.10 The case $\lambda = [1^2, 2, 3]$

For $\lambda = [1^2, 2, 3]$ we can take a conjugate λ -tuple as a septuple $(P_1, \dots, P_7)$ such that F permutes the tuple (P_1, P_2, P_3) cyclically, transposes the pair (P_4, P_5) and fixes P_6 and P_7 . In this case we take U as the subset of $(\mathbb{P}^2)^7$ consisting of pairwise distinct points. We then have

$$|U^{F\sigma}| = (q^6 + q^3 - q^2 - q)(q^4 - q)(q^2 + q + 1)(q^2 + q).$$

We shall use the notation P_1, P_2, P_3 for the conjugate triple, Q_1, Q_2 for the conjugate pair and use R_6 and R_7 to denote the two $\mathbb{F}_q$ -points.

We decompose $\Delta_l^{F\sigma}$ as

$$\Delta_l^{F\sigma} = \Delta_{l,1} \cup \Delta_{l,2},$$

where $\Delta_{l,1}$ consists of λ -tuples such that the three strict $\mathbb{F}_{q^3}$ -points lie on an $\mathbb{F}_q$ -line and $\Delta_{l,2}$ consists of λ -tuples such that the two conjugate $\mathbb{F}_{q^2}$ -points and one of the $\mathbb{F}_q$ -points lie on an $\mathbb{F}_q$ -line.

We have

$$|\Delta_{l,1}| = (q^2 + q + 1)^2 (q^3 - q)(q^4 - q)(q^2 + q).$$

We decompose $\Delta_{l,2}$ as

$$\Delta_{l,2} = \Delta_{l,2}^6 \cup \Delta_{l,2}^7,$$

where $\Delta_{l,2}^i$ consists of tuples such that the line through the two $\mathbb{F}_{q^2}$ -points passes through R_i . We have

$$|\Delta_{l,2}^6| = |\Delta_{l,2}^7| = (q^2 + q + 1)(q^2 - q)(q + 1)(q^6 + q^3 - q^2 - q)(q^2 + q).$$

We now turn to the double intersections. We have

$$|\Delta_{l,1} \cap \Delta_{l,2}^6| = |\Delta_{l,1} \cap \Delta_{l,2}^7| = (q^2 + q + 1)^2 (q^3 - q)(q^2 - q)(q + 1)(q^2 + q),$$

and

$$|\Delta_{l,2}^6 \cap \Delta_{l,2}^7| = (q^2 + q + 1)(q^2 - q)(q + 1)q(q^6 + q^3 - q^2 - q).$$

Finally, we compute the cardinality of the intersection of all three sets

$$|\Delta_{l,1} \cap \Delta_{l,2}^6 \cap \Delta_{l,2}^7| = (q^2 + q + 1)^2 (q^3 - q)(q^2 - q)(q + 1)q.$$

This allows us to compute Δ_l .

If six points of a λ -tuple lie on a smooth conic C , then the three $\mathbb{F}_{q^3}$ -points and the two $\mathbb{F}_{q^2}$ -points lie on C and C is defined over $\mathbb{F}_q$. Thus, to compute $|\Delta_c^{F\sigma}|$ we begin by choosing a smooth conic C over $\mathbb{F}_q$. We then choose a conjugate triple and a conjugate pair of $\mathbb{F}_{q^2}$ -points on C . Then, we choose either R_6 or R_7 and place the chosen point on C . Finally, we place the remaining point anywhere we want. We thus obtain the number

$$2(q^5 - q^2)(q^3 - q)(q^2 - q)(q + 1)(q^2 + q).$$

However, in the above we have counted the configurations where all seven points lie on the conic twice. We thus have to take away

$$(q^5 - q^2)(q^3 - q)(q^2 - q)(q + 1)q,$$

in order to obtain $|\Delta_c^{F\sigma}|$.

It only remains to compute the cardinality of the intersection $\Delta_l^{F\sigma} \cap \Delta_c^{F\sigma}$. We only have nonempty intersection between the set $\Delta_c^{F\sigma}$ and the set $\Delta_{l,2}$. To compute the cardinality of this intersection, we only have to make sure to choose the point R_6 (resp. R_7) on the line through the two $\mathbb{F}_{q^2}$ -points. Hence, we have

$$|\Delta_{l,2}^6 \cap \Delta_c| = |\Delta_{l,2}^7 \cap \Delta_c| = (q^5 - q^2)(q^3 - q)(q^2 - q)(q + 1)^2,$$

and, therefore,

$$|\Delta_l^{F\sigma} \cap \Delta_c^{F\sigma}| = 2(q^5 - q^2)(q^3 - q)(q^2 - q)(q + 1)^2.$$

This gives us

$$|(\mathcal{P}_7^{F\sigma})| = q^6 - 3q^5 - q^4 + 5q^3 - 2q.$$

6.11 The case $\lambda = [1^4, 3^1]$

For $\lambda = [1^4, 3]$ we can take a conjugate λ -tuple as a septuple $(P_1, \dots, P_7)$ such that F permutes the tuple (P_1, P_2, P_3) cyclically and fixes the points P_4, P_5, P_6 and P_7 . In

this case we take U as the subset of $(\mathbb{P}^2)^7$ consisting of pairwise distinct points such that the last four points lie in general position. We then have

$$|U^{F\sigma}| = (q^6 + q^3 - q^2 - q)(q^2 + q + 1)(q^2 + q)q^2(q^2 - 2q + 1).$$

We shall denote the four $\mathbb{F}_q$ -points by Q_1, Q_2, Q_3 and Q_4 .

A septuple in $\Delta_l^{F\sigma}$ will have the three strict $\mathbb{F}_{q^3}$ -points on an $\mathbb{F}_q$ -line. Thus, to compute the size of $\Delta_l^{F\sigma}$ we only need to place the four $\mathbb{F}_q$ -points in general position, choose $\mathbb{F}_q$ -line L and place the conjugate $\mathbb{F}_{q^3}$ -tuple on L . We thus have,

$$|\Delta_l^{F\sigma}| = (q^2 + q + 1)^2(q^2 + q)q^2(q^2 - 2q + 1)(q^3 - q).$$

A septuple in $\Delta_c^{F\sigma}$ will have the three strict $\mathbb{F}_{q^3}$ -points on a smooth conic C defined over $\mathbb{F}_q$. Thus, to compute $|\Delta_c^{F\sigma}|$, we first choose a smooth conic C defined over $\mathbb{F}_q$ and then a conjugate triple on C . We then choose one of the points Q_1, Q_2, Q_3 and Q_4 to possibly not lie on C . Call this point Q . We then place the other three points on C . These three points define three lines which, in total, contain $(q+1)+q+(q-1) = 3q$ points. As long as we choose Q away from these points, the four $\mathbb{F}_q$ -points will be in general position. We thus obtain

$$4(q^5 - q^2)(q^3 - q)(q+1)q(q-1)(q^2 - 2q + 1).$$

However, we have counted the septuples with all seven points on a smooth conic four times. We thus need to take away

$$3(q^5 - q^2)(q^3 - q)(q+1)q(q-1)(q-2).$$

Since $\Delta_l^{F\sigma}$ and $\Delta_c^{F\sigma}$ are disjoint we are done and conclude that

$$|(\mathcal{P}_7^{F\sigma})| = q^6 - 5q^5 + 10q^4 - 5q^3 - 11q^2 + 10q.$$

6.12 The case $\lambda = [1, 2^3]$

For $\lambda = [1, 2^3]$ we can take a conjugate λ -tuple as a septuple $(P_1, \dots, P_7)$ such that F transposes the pairs (P_1, P_2) , (P_3, P_4) and (P_5, P_6) and fixes the point P_7 . In this case we take U as the subset of $(\mathbb{P}^2)^7$ consisting of pairwise distinct points such that the first six points have no three points on a line. We then have

$$|U^{F\sigma}| = (q^4 - q)(q^4 - q^2)(q^4 - 6q^2 + q + 8)(q^2 + q + 1).$$

We shall denote the three conjugate pairs of $\mathbb{F}_{q^2}$ -points by P_1, P_2, Q_1, Q_2 and R_1, R_2 and the $\mathbb{F}_q$ -point by O .

We decompose $\Delta_l^{F\sigma}$ as

$$\Delta_l^{F\sigma} = \Delta_{l,1} \cup \Delta_{l,2},$$

where $\Delta_{l,1}$ consists of those septuples where two conjugate $\mathbb{F}_{q^2}$ -points and the $\mathbb{F}_q$ -point lie on an $\mathbb{F}_q$ -line and $\Delta_{l,2}$ consists of those septuples where two conjugate $\mathbb{F}_{q^2}$ -lines, containing two $\mathbb{F}_{q^2}$ -points each, intersect in the point defined over $\mathbb{F}_q$.

The set $\Delta_{l,1}$ naturally decomposes into three equally large, but not disjoint, subsets:

- the set $\Delta_{l,1}^a$ where P_1, P_2 and O lie on an $\mathbb{F}_q$ -line,
- the set $\Delta_{l,1}^b$ where Q_1, Q_2 and O lie on an $\mathbb{F}_q$ -line, and
- the set $\Delta_{l,1}^c$ where R_1, R_2 and O lie on an $\mathbb{F}_q$ -line.

Similarly, the set $\Delta_{l,2}$ decomposes into six disjoint and equally large subsets:

- the two sets $\Delta_{l,2}^{P_1, Q_i}$ where the line through the points P_1 and Q_i also passes through the point O ,
- the two sets $\Delta_{l,2}^{P_1, R_i}$ where the line through the points P_1 and R_i also passes through the point O , and
- the two sets $\Delta_{l,2}^{Q_1, R_i}$ where the line through the points Q_1 and R_i also passes through the point O .

The cardinalities of these sets are easily computed to be

$$|\Delta_{l,1}^a| = |\Delta_{l,1}^b| = |\Delta_{l,1}^c| = (q^4 - q)(q^4 - q^2)(q^4 - 6q^2 + q + 8)(q + 1),$$

and

$$|\Delta_{l,2}^{P_1, Q_i}| = |\Delta_{l,2}^{P_1, R_i}| = |\Delta_{l,2}^{Q_1, R_i}| = (q^4 - q)(q^4 - q^2)(q^4 - 6q^2 + q + 8).$$

To compute $|\Delta_{l,1}^a \cap \Delta_{l,1}^b|$ we note that if we place the three pairs of conjugate $\mathbb{F}_{q^2}$ -points so that no three lie on a line, then the line through P_1 and P_2 and the line through Q_1 and Q_2 will intersect in an $\mathbb{F}_q$ -point. By choosing this point as O we obtain an element of $\Delta_{l,1}^a \cap \Delta_{l,1}^b$. We now see that

$$|\Delta_{l,1}^a \cap \Delta_{l,1}^b| = |\Delta_{l,1}^a \cap \Delta_{l,1}^c| = |\Delta_{l,1}^b \cap \Delta_{l,1}^c| = (q^4 - q)(q^4 - q^2)(q^4 - 6q^2 + q + 8).$$

To compute $|\Delta_{l,1}^a \cap \Delta_{l,2}^{Q_1, R_1}|$ we first choose a conjugate pair Q_1, Q_2 and then a conjugate pair of $\mathbb{F}_{q^2}$ -points R_1, R_2 which do not lie on the line through Q_1 and Q_2 . We now only have one choice for O . We choose an $\mathbb{F}_q$ -line L through O . There are two possibilities: either L will pass through the intersection point P of the line through Q_1 and R_2 and the line through Q_2 and R_1 or it will not. If L passes through P , then we have $q^2 - q$ possible choices for P_1 and P_2 on L . Otherwise, we only have $q^2 - q - 2$ choices. Hence

$$\begin{aligned} |\Delta_{l,1}^a \cap \Delta_{l,2}^{Q_1, R_1}| &= |\Delta_{l,1}^b \cap \Delta_{l,2}^{P_1, R_1}| = |\Delta_{l,1}^c \cap \Delta_{l,2}^{P_1, Q_1}| \\ &= (q^4 - q)(q^4 - q^2)(q^2 - q) + (q^4 - q)(q^4 - q^2)q(q^2 - q - 2). \end{aligned}$$

The only nonempty triple intersection is $\Delta_{l,1}^a \cap \Delta_{l,1}^b \cap \Delta_{l,1}^c$. A computation very similar to the one for $|\Delta_{l,1}^a \cap \Delta_{l,2}^{Q_1, R_1}|$ gives

$$|\Delta_{l,1}^a \cap \Delta_{l,1}^b \cap \Delta_{l,1}^c| = 2(q^4 - q)(q^4 - q^2)(q^2 - q - 2) \\ + (q^4 - q)(q^4 - q^2)(q - 3)(q^2 - q - 4).$$

This finishes the investigation of $\Delta_l^{F\sigma}$.

We now turn to $\Delta_c^{F\sigma}$. If six points of a λ -tuple lie on a smooth conic C , then the six $\mathbb{F}_{q^2}$ -points lie on C and C is defined over $\mathbb{F}_q$. Since no three points of a smooth conic can lie on a line, we shall obtain an element of $\Delta_c^{F\sigma}$ simply by choosing a smooth conic C , three conjugate pairs on C and, finally, an $\mathbb{F}_q$ -point anywhere. We thus have

$$|\Delta_c| = (q^5 - q^2)(q^2 - q)(q^2 - q - 2)(q^2 - q - 4)(q^2 + q + 1).$$

We shall now compute the cardinality of the intersection between Δ_l and Δ_c . The intersections with the cases $\Delta_{l,1}^a$, $\Delta_{l,1}^b$, and $\Delta_{l,1}^c$ are easily handled — we simply choose a smooth conic with three conjugate pairs on it and then place O on the line through the right conjugate pair. We thus get

$$|\Delta_{l,1}^a \cap \Delta_c| = |\Delta_{l,1}^b \cap \Delta_c| = |\Delta_{l,1}^c \cap \Delta_c| \\ = (q^5 - q^2)(q^2 - q)(q^2 - q - 2)(q^2 - q - 4)(q + 1).$$

The intersections with the sets $\Delta_{l,2}^{a_1, Q_i}$, $\Delta_{l,2}^{P_1, R_i}$ and $\Delta_{l,2}^{Q_1, R_i}$ are just as simple. Once we have chosen our conic C and our conjugate pairs we have only one choice for O . Hence,

$$|\Delta_{l,2}^{P_1, Q_i} \cap \Delta_c| = |\Delta_{l,2}^{P_1, R_i} \cap \Delta_c| = |\Delta_{l,2}^{Q_1, R_i} \cap \Delta_c| \\ = (q^5 - q^2)(q^2 - q)(q^2 - q - 2)(q^2 - q - 4).$$

An analogous argument shows that

$$|\Delta_{l,1}^a \cap \Delta_{l,1}^b \cap \Delta_c| = |\Delta_{l,1}^a \cap \Delta_{l,1}^c \cap \Delta_c| = |\Delta_{l,1}^b \cap \Delta_{l,1}^c \cap \Delta_c| \\ = (q^5 - q^2)(q^2 - q)(q^2 - q - 2)(q^2 - q - 4).$$

The remaining intersections are quite a bit harder than the previous ones. We begin by considering the intersection $\Delta_{l,1}^a \cap \Delta_{l,2}^{Q_1, R_1} \cap \Delta_c$, but the other intersections of this type are completely analogous and have the same size.

We first consider the case when O is on the $\mathbb{F}_q$ -outside of C . There are $q + 1$ lines through O . Of these, precisely two are tangents and $\frac{1}{2}(q - 1)$ intersect C in $\mathbb{F}_q$ -points. Thus, the remaining $\frac{1}{2}(q - 1)$ lines will intersect C in two conjugate $\mathbb{F}_{q^2}$ -points. We thus pick one of these lines and label one of the intersection points by P_1 .

Picking an $\mathbb{F}_{q^2}$ -point not defined over $\mathbb{F}_q$ on C will typically define an $\mathbb{F}_{q^2}$ -line through O which is not defined over $\mathbb{F}_q$. However, some of these choices will give

$\mathbb{F}_q$ -lines and we saw above that the number of such $\mathbb{F}_q$ -lines is $\frac{1}{2}(q-1)$. Thus, the number of $\mathbb{F}_{q^2}$ -lines, not defined over $\mathbb{F}_q$, intersecting C in two strict $\mathbb{F}_{q^2}$ -points is

$$\frac{1}{2}(q^2 - q) - \frac{1}{2}(q - 1) = \frac{1}{2}(q^2 - 2q + 1).$$

We pick one such line, label one of the intersection points Q_1 and the other intersection point R_1 . This gives us a configuration of the desired type. Hence, the number of tuples in $\Delta_{l,1}^a \cap \Delta_{l,2}^{Q_1, R_1} \cap \Delta_c$ with O on the $\mathbb{F}_q$ -outside of C is

$$\frac{1}{2}(q^5 - q^2)(q+1)q(q-1)(q^2 - 2q + 1).$$

We now turn to the case when O is on the $\mathbb{F}_q$ -inside of C . Of the $q+1$ lines defined over $\mathbb{F}_q$ which pass through O , $\frac{1}{2}(q+1)$ will now intersect C in $\mathbb{F}_q$ -points and equally many in conjugate $\mathbb{F}_{q^2}$ -points. We thus pick a line that intersects C in two conjugate $\mathbb{F}_{q^2}$ -points and label one of them by P_1 .

We now want to pick an $\mathbb{F}_{q^2}$ -line through O which is not defined over $\mathbb{F}_q$ and which intersects C in two $\mathbb{F}_{q^2}$ -points that are not defined over $\mathbb{F}_q$. To obtain such a line we pick an $\mathbb{F}_{q^2}$ -point which is not defined over $\mathbb{F}_q$ on C . However, two such points define tangents to C which pass through O and $\frac{1}{2}(q+1)$ of the lines obtained in this way are actually defined over $\mathbb{F}_q$. We thus have

$$\frac{1}{2}(q^2 - q - 2) - \frac{1}{2}(q + 1) = \frac{1}{2}(q^2 - 2q - 3)$$

choices. We pick such a line and label the intersection points by Q_1 and R_1 . Hence, the number of tuples in $\Delta_{l,1}^a \cap \Delta_{l,2}^{Q_1, R_1} \cap \Delta_c$ with O on the $\mathbb{F}_q$ -inside of C is

$$\frac{1}{2}(q^5 - q^2)(q^2 - q)(q+1)(q^2 - 2q - 3).$$

This finishes the computation of $|\Delta_{l,1}^a \cap \Delta_{l,2}^{Q_1, R_1} \cap \Delta_c|$, $|\Delta_{l,1}^b \cap \Delta_{l,2}^{P_1, R_1} \cap \Delta_c|$ and $|\Delta_{l,1}^c \cap \Delta_{l,2}^{P_1, Q_1} \cap \Delta_c|$.

The only remaining intersection is $\Delta_{l,1}^a \cap \Delta_{l,1}^b \cap \Delta_{l,1}^c \cap \Delta_c$ which we shall handle in a way similar to that above. Fortunately, much of the work has already been done. To start, if O is on the $\mathbb{F}_q$ -outside of C , then there are $\frac{1}{2}(q-1)$ lines through O which are defined over $\mathbb{F}_q$ and intersect C in conjugate pairs of $\mathbb{F}_{q^2}$ -points. Thus, there are

$$(q-1)(q-3)(q-5)$$

ways to pick three lines and label the intersection points with P_1 and P_2 , Q_1 and Q_2 and R_1 and R_2 . Hence, the number of λ -tuples in $\Delta_{l,1}^a \cap \Delta_{l,1}^b \cap \Delta_{l,1}^c \cap \Delta_c$ with O on the outside of C is

$$\frac{1}{2}(q^5 - q^2)(q+1)q(q-1)(q-3)(q-5).$$

Similarly, if O lies on the $\mathbb{F}_q$ -inside of C we have seen that there are $\frac{1}{2}(q+1)$ lines through O which are defined over $\mathbb{F}_q$ and which intersect C in a pair of conjugate $\mathbb{F}_{q^2}$ -points. Thus, there are

$$(q+1)(q-1)(q-3)$$

ways to pick three lines and label the intersection points with P_1 and P_2 , Q_1 and Q_2 and R_1 and R_2 . Hence, the number of λ -tuples in $\Delta_{l,1}^a \cap \Delta_{l,1}^b \cap \Delta_{l,1}^c \cap \Delta_c$ with O on the inside of C is

$$\frac{1}{2}(q^5 - q^2)(q+1)q(q+1)(q-1)(q-3).$$

We finally obtain

$$|(\mathcal{P}_7^{F\sigma})| = q^6 - 3q^5 - 6q^4 + 19q^3 + 6q^2 - 24q + 7.$$

6.13 The case $\lambda = [1^3, 2^2]$

For $\lambda = [1^3, 2^2]$ we can take a conjugate λ -tuple as a septuple $(P_1, \dots, P_7)$ such that F fixes the points P_1 , P_2 and P_3 and transposes the pairs (P_4, P_5) and (P_6, P_7) . In this case we take U as the subset of $(\mathbb{P}^2)^7$ consisting of pairwise distinct points such that the first five points lie in general position. We then have

$$|U^{F\sigma}| = (q^2 + q + 1)(q^2 + q)q^2(q^4 - 3q^3 + 3q^2 - q)(q^4 - q - 2).$$

We shall denote the $\mathbb{F}_q$ -points by P_1 , P_2 and P_3 and the two conjugate pairs of $\mathbb{F}_{q^2}$ -points by Q_1 , Q_2 and R_1 , R_2 .

The set $\Delta_l^{F\sigma}$ can be decomposed as

$$\Delta_l^{F\sigma} = \Delta_{l,1} \cup \Delta_{l,2} \cup \Delta_{l,3},$$

where

- $\Delta_{l,1}$ consists of tuples such that the line through R_1 and R_2 also passes through P_1 , P_2 or P_3 ,
- $\Delta_{l,2}$ consists of tuples such that the points R_1 and R_2 lie on the line through Q_1 and Q_2 , and
- $\Delta_{l,3}$ consists of tuples such that a line through Q_1 and one of the points P_1 , P_2 and P_3 also contains R_1 or R_2 .

The set $\Delta_{l,1}$ decomposes as a union of the sets $\Delta_{l,1}^1$, $\Delta_{l,1}^2$ and $\Delta_{l,1}^3$ consisting of tuples with the line through R_1 and R_2 passing through P_1 , P_2 and P_3 , respectively. Similarly, the set $\Delta_{l,3}$ is the union of the six sets $\Delta_{l,3}^{i,j}$, $i = 1, 2$, $j = 1, 2, 3$, where $\Delta_{l,3}^{i,j}$ contains all tuples such that Q_1 , R_i and P_j lie on a line.

The cardinalities of the above sets are easily computed to be

$$|\Delta_{l,1}^i| = (q^2 + q + 1)(q^2 + q)q^2(q^4 - 3q^3 + 3q^2 - q)(q+1)(q^2 - q),$$

$$|\Delta_{l,2}| = (q^2 + q + 1)(q^2 + q)q^2(q^4 - 3q^3 + 3q^2 - q)(q^2 - q - 2),$$

$$|\Delta_{l,3}^{i,j}| = (q^2 + q + 1)(q^2 + q)q^2(q^4 - 3q^3 + 3q^2 - q)(q^2 - 1).$$

The cardinality of $\Delta_{l,1}^i \cap \Delta_{l,1}^j$, $i \neq j$, is also easily computed

$$|\Delta_{l,1}^i \cap \Delta_{l,1}^j| = (q^2 + q + 1)(q^2 + q)q^2(q^4 - 3q^3 + 3q^2 - q)(q^2 - q).$$

There is only nonempty intersection between the set $\Delta_{l,1}^i$ and the set $\Delta_{l,3}^{j,k}$ if $k \neq i$. We then place the first five points in general position and choose an $\mathbb{F}_q$ -line through P_k which does not pass through P_i in one of q ways. This gives a tuple of the desired form. We thus see that

$$|\Delta_{l,1}^i \cap \Delta_{l,3}^{j,k}| = (q^2 + q + 1)(q^2 + q)q^2(q^4 - 3q^3 + 3q^2 - q)q.$$

We also have nonempty intersection between the sets $\Delta_{l,3}^{1,i}$ and the set $\Delta_{l,3}^{2,j}$ where $i \neq j$. Such a configuration is actually given by specifying the first five points in general position since we must then take R_1 as the intersection point of the line between Q_1 and P_i and the line between Q_2 and P_j and similarly for R_2 . Hence,

$$|\Delta_{l,3}^{1,i} \cap \Delta_{l,3}^{2,j}| = (q^2 + q + 1)(q^2 + q)q^2(q^4 - 3q^3 + 3q^2 - q).$$

The set $\Delta_{l,2}$ cannot intersect any of the other sets since this would require Q_1 and Q_2 to lie on a line through one of the $\mathbb{F}_q$ -points. It is thus time to consider the triple intersections.

Since P_1 , P_2 and P_3 do not lie on a line we have that the intersection of $\Delta_{l,1}^1$, $\Delta_{l,1}^2$ and $\Delta_{l,1}^3$ is empty. We thus only have two types of triple intersections, namely $\Delta_{l,1}^i \cap \Delta_{l,1}^j \cap \Delta_{l,3}^{r,s}$ and $\Delta_{l,1}^i \cap \Delta_{l,3}^{1,j} \cap \Delta_{l,3}^{2,k}$ where, of course, i , j and k are assumed to be distinct.

An element of $\Delta_{l,1}^i \cap \Delta_{l,1}^j \cap \Delta_{l,3}^{r,s}$ is specified by choosing the first five points in general position. The point R_r must then be chosen as the intersection point of the line between P_i and P_j and the line between Q_1 and P_s and similarly for FR_r . We thus have

$$|\Delta_{l,1}^i \cap \Delta_{l,1}^j \cap \Delta_{l,3}^{r,s}| = (q^2 + q + 1)(q^2 + q)q^2(q^4 - 3q^3 + 3q^2 - q).$$

To compute the cardinality of the intersection $\Delta_{l,1}^i \cap \Delta_{l,3}^{1,j} \cap \Delta_{l,3}^{2,k}$ we first choose two $\mathbb{F}_q$ -points P_j and P_k . We then choose a conjugate pair of $\mathbb{F}_{q^2}$ -lines through each of these points. The intersections of these lines give four $\mathbb{F}_{q^2}$ -points which we only have one way to label with Q_1 , Q_2 , R_1 and R_2 . We must now place the point P_i somewhere on the line L through R_1 and R_2 . The line through P_i and P_k intersects L in one $\mathbb{F}_q$ -point and the line through Q_1 and Q_2 intersects L in another. Thus, we have $q - 1$ choices for P_i . We thus see that

$$|\Delta_{l,1}^i \cap \Delta_{l,3}^{1,j} \cap \Delta_{l,3}^{2,s}| = (q^2 + q + 1)(q^2 + q)(q^2 - q)^2(q - 1).$$

This completes the investigation of $\Delta_l^{F\sigma}$.

If a smooth conic C contains six of the points, then C contains both the conjugate $\mathbb{F}_{q^2}$ -pairs and C is defined over $\mathbb{F}_q$. Thus, to compute $|\Delta_c^{F\sigma}|$ we first choose a smooth conic C over $\mathbb{F}_q$ and then pick one of the points P_1 , P_2 and P_3 to possibly lie outside C . We call the chosen point P . We then place the other two points and the two $\mathbb{F}_{q^2}$ -pairs on C . Finally, we must place P somewhere to make P_1 , P_2 , P_3 , Q_1 and Q_2 lie in general position. Hence, we must choose P away from the line through the two other $\mathbb{F}_q$ -points and away from the line through Q_1 and Q_2 . This gives us

$$(q^5 - q^2)(q + 1)q(q^2 - q)(q^2 - q - 2)(q^2 - q).$$

However, in the above we have counted the configurations where all seven points lie on C three times. We must therefore take away

$$2 \cdot (q^5 - q^2)(q + 1)q(q - 1)(q^2 - q)(q^2 - q - 2)$$

in order to obtain $|\Delta_c^{F\sigma}|$.

The intersection $\Delta_{l,2} \cap \Delta_c^{F\sigma}$ is empty but the intersections of $\Delta_c^{F\sigma}$ with the other sets in the decomposition of $\Delta_l^{F\sigma}$ are not. To compute $|\Delta_{l,1}^i \cap \Delta_c^{F\sigma}|$ we shall first assume that P_i lies on the $\mathbb{F}_q$ -outside of C . Of the $q + 1$ lines through P_i which are defined over $\mathbb{F}_q$ we have that two are tangent to C and $\frac{1}{2}(q - 1)$ intersect C in two $\mathbb{F}_q$ -points. Thus, there are $\frac{1}{2}(q - 1)$ lines left which must intersect C in a pair of conjugate $\mathbb{F}_{q^2}$ -points. We pick such a line and label the intersection points by R_1 and R_2 in one of two ways. We shall now place the other two $\mathbb{F}_q$ -points on C . There are $\frac{1}{2}(q + 1)q$ ways to choose two $\mathbb{F}_q$ -points on C of which $\frac{1}{2}(q - 1)$ pairs lie on an $\mathbb{F}_q$ -line through P_i . There are thus $\frac{1}{2}(q^2 + 1)$ pairs which do not lie on a line through P_i and, since there are two ways to label each pair, we thus have $q^2 + 1$ choices for the two $\mathbb{F}_q$ -points. Finally, we shall place Q_1 and Q_2 somewhere on C but we have to make sure that the points P_1 , P_2 , P_3 , Q_1 and Q_2 are in general position. Since the lines between P_i and the other two $\mathbb{F}_q$ -points intersect C only in $\mathbb{F}_q$ -points, the only thing that might go wrong when choosing Q_1 and Q_2 is that the line through Q_1 and Q_2 might also go through P_i . As seen above, there are exactly $q - 1$ choices for Q_1 and Q_2 for which this happens, so the remaining $q^2 - q - (q - 1) = q^2 - 2q + 1$ choices will give a configuration of the desired type. We thus have that the number of elements in $\Delta_{l,1}^i \cap \Delta_c^{F\sigma}$ such that P_i lies on the $\mathbb{F}_q$ -outside of C is

$$\frac{1}{2}(q^5 - q^2)(q + 1)q(q - 1)(q^2 + 1)(q^2 - 2q + 1).$$

We now assume that P_i lies on the $\mathbb{F}_q$ -inside of C . We proceed similarly to the above. First we observe that the number of $\mathbb{F}_q$ -lines through P_i is $q + 1$ of which half intersect C in two $\mathbb{F}_q$ -points and half intersect C in conjugate pairs of $\mathbb{F}_{q^2}$ -points. We choose a line which intersects C in two conjugate $\mathbb{F}_{q^2}$ -points and label the intersection points by R_1 and R_2 . We now choose an $\mathbb{F}_q$ -point P_j on C in one of $q + 1$ ways. The line through P_i and P_j intersects C in another $\mathbb{F}_q$ -point and we choose the final $\mathbb{F}_q$ -point away from this intersection point and P_j . Finally, we shall place the points Q_1 and Q_2 on C in a way so that the points P_1 , P_2 , P_3 , Q_1 and Q_2 are in general position. As

above, the only thing that might go wrong is that the line through Q_1 and Q_2 might go through P_i and there are precisely $q + 1$ choices for Q_1 and Q_2 for which this happens. Thus, there are $q^2 - q - (q + 1) = q^2 - 2q - 1$ valid choices for Q_1 and Q_2 . Hence, there are

$$\frac{1}{2} (q^5 - q^2)(q^2 - q)(q + 1)(q + 1)(q - 1)(q^2 - 2q - 1)$$

elements in $\Delta_{l,1}^i \cap \Delta_c^{F\sigma}$ such that P_i lies on the $\mathbb{F}_q$ -inside of C .

To compute the intersection $\Delta_{l,3}^{i,j} \cap \Delta_c^{F\sigma}$ we note that if we place P_j on the $\mathbb{F}_q$ -outside of C and then choose two $\mathbb{F}_q$ -points on C and two conjugate $\mathbb{F}_{q^2}$ -points Q_1 and Q_2 on C such that P_1, P_2, P_3, Q_1 and Q_2 are in general position, then we must choose R_i as the other intersection point of C with the line through Q_1 and P_j . We may thus use constructions analogous to those above to see that there are

$$\frac{1}{2} (q^5 - q^2)(q + 1)q(q^2 + 1)(q^2 - 2q + 1)$$

elements in $\Delta_{l,1}^{i,j} \cap \Delta_c^{F\sigma}$ with P_j on the $\mathbb{F}_q$ -outside of C and

$$\frac{1}{2} (q^5 - q^2)(q^2 - q)(q + 1)(q - 1)(q^2 - 2q - 3)$$

elements with P_j on the $\mathbb{F}_q$ -inside of C .

We may now put all the pieces together to obtain

$$|(\mathcal{P}_7^{F\sigma})| = q^6 - 7q^5 + 10q^4 + 15q^3 - 26q^2 - 8q + 15.$$

6.14 The case $\lambda = [1^5, 2]$

For $\lambda = [1^3, 2^2]$ we can take a conjugate λ -tuple as a septuple $(P_1, \dots, P_7)$ such that F fixes the points P_1, P_2, P_3, P_4 and P_5 and transposes the pair (P_6, P_7) . In this case we take U as the subset of $(\mathbb{P}^2)^7$ consisting of pairwise distinct points such that the first five points lie in general position. We then have

$$|U^{F\sigma}| = (q^2 + q + 1)(q^2 + q)q^2(q^2 - 2q + 1)(q^2 - 5q + 6)(q^4 - q).$$

We shall denote the conjugate pair of $\mathbb{F}_{q^2}$ -points by Q_1, Q_2 .

If three points of a conjugate λ -tuple in $U(\lambda)$ lie on a line, then Q_1 and Q_2 lie on a line passing through one of the $\mathbb{F}_q$ -points. There are

$$(q + 1) + q + (q - 1) + (q - 2) + (q - 3) = 5q - 5,$$

$\mathbb{F}_q$ -lines passing through P_1, P_2, P_3, P_4 or P_5 (or possibly two of them). Each of these lines contain $q^2 - q$ conjugate pairs of $\mathbb{F}_{q^2}$ -points and no conjugate pair lies on two such lines. We thus have

$$|\Delta_l^{F\sigma}| = (q^2 + q + 1)(q^2 + q)q^2(q^2 - 2q + 1)(q^2 - 5q + 6)(5q - 5)(q^2 - q).$$

If six of the points of a conjugate λ -tuple lie on a smooth conic C , then C is defined over $\mathbb{F}_q$ and contains Q_1 and Q_2 . Therefore, to compute the cardinality of $\Delta_c^{F\sigma}$ we first choose a smooth conic C defined over $\mathbb{F}_q$ and one of the points P_1, P_2, P_3, P_4 or P_5 to possibly lie outside C . We call the chosen point P . Then, we choose four $\mathbb{F}_q$ -points and a conjugate pair on C . Finally, we choose P away from the six lines through pairs of the other four $\mathbb{F}_q$ -points. We thus get

$$5(q^5 - q^2)(q + 1)q(q - 1)(q - 2)(q^2 - q)(q^2 - 5q + 6).$$

In the above we have counted the λ -tuples with all seven points on a conic five times. We therefore must take away

$$4(q^5 - q^2)(q + 1)q(q - 1)(q - 2)(q - 3)(q^2 - q),$$

in order to obtain $|\Delta_c^{F\sigma}|$.

To compute the size of the intersection $\Delta_l^{F\sigma} \cap \Delta_c^{F\sigma}$ we shall decompose this set into a disjoint union of five subsets A_i , $i = 1, \dots, 5$, where A_i consists of those tuples where P_i does not lie on the conic C through the other six points. Each of the sets A_i is then decomposed further into a union of the sets A_i^{out} and A_i^{in} where A_i^{out} consists of those tuples with P_i on the $\mathbb{F}_q$ -outside of C and A_i^{in} consists of those with P_i on the $\mathbb{F}_q$ -inside of C . Finally, we shall decompose A_i^{out} into a union of the three disjoint subsets:

- the set $A_{i,0}^{\text{out}}$ consisting of λ -tuples such that the tangent lines to C passing through P_i do not pass through any of the other points of the λ -tuple,
- the set $A_{i,1}^{\text{out}}$ consisting of λ -tuples such that exactly one of the tangent lines to C passing through P_i pass through one of the other points of the λ -tuple,
- the set $A_{i,2}^{\text{out}}$ consisting of λ -tuples such that both the tangent lines to C passing through P_i passes through another point of the λ -tuple.

To compute $|A_i^{\text{out}}|$, we first choose a smooth conic C defined over $\mathbb{F}_q$ in $q^5 - q^2$ ways and then a point P_i outside C in $\frac{1}{2}(q + 1)q$ ways. As seen many times before, there are exactly $\frac{1}{2}(q - 1)$ lines through P_i which are defined over $\mathbb{F}_q$ and which intersect C in a conjugate pair of points. We pick such a line and label the points Q_1 and Q_2 in one of two ways. From this point on, the computations are a little bit different for the three subsets of A_i^{out} .

The subset $A_{i,0}^{\text{out}}$. We shall now pick the other four $\mathbb{F}_q$ -points of the λ -tuple. Since we should not pick points whose tangents pass through P_i , we have $q - 1$ choices for the first point. For the second point, we should stay away from the tangent points, the first point and the other intersection point of C and the line through P_i and the first point. Hence, we have $q - 3$ choices. In a similar way, we see that we have $q - 5$ choices for the third point and $q - 7$ for the fourth. Hence,

$$|A_{i,0}^{\text{out}}| = \frac{1}{2}(q^5 - q^2)(q + 1)q(q - 1)^2(q - 3)(q - 5)(q - 7).$$

The subset $A_{i,1}^{\text{out}}$. We begin by choosing one of the four $\mathbb{F}_q$ -points to lie on a tangent to C passing through P_i and then we pick the tangent it should lie on. For the first of the remaining three points we now have $q - 1$ choices and, similarly to the above case, we have $q - 3$ choices for the second and $q - 5$ for the third. Thus,

$$|A_{i,1}^{\text{out}}| = 4 \cdot 2 \cdot \frac{1}{2} (q^5 - q^2)(q + 1)q(q - 1)^2(q - 3)(q - 5).$$

The subset $A_{i,2}^{\text{out}}$. We begin by choosing two of the four $\mathbb{F}_q$ -points to lie on tangents to C passing through P_i and then we pick which point should lie on which tangent. For the first of the remaining two points we now have $q - 1$ choices and we then have $q - 3$ choices for the second. Thus,

$$|A_{i,2}^{\text{out}}| = \binom{4}{2} \cdot 2 \cdot \frac{1}{2} (q^5 - q^2)(q + 1)q(q - 1)^2(q - 3).$$

It remains to compute $|A_i^{\text{in}}|$. We first choose a smooth conic C defined over $\mathbb{F}_q$ in $q^5 - q^2$ ways and then a point P_i on the inside of C in $\frac{1}{2}(q^2 - q)$ ways. We have already seen that there now are $\frac{1}{2}(q + 1)$ lines passing through P_i which are defined over $\mathbb{F}_q$ and which intersect C in a conjugate pair of points. We thus pick such a line and label the intersection points by Q_1 and Q_2 . Since any $\mathbb{F}_q$ -line through P_i will intersect C in precisely two points, we have $(q + 1)(q - 1)(q - 3)(q - 5)$ choices for the remaining four $\mathbb{F}_q$ -points of the λ -tuple. We thus see that

$$|A_i^{\text{in}}| = \frac{1}{2} (q^5 - q^2)(q^2 - q)(q + 1)(q - 1)(q - 3)(q - 5).$$

We now conclude that

$$|(\mathcal{P}_7^2)^{F\sigma}| = q^6 - 15q^5 + 90q^4 - 265q^3 + 374q^2 - 200q + 15.$$

6.15 The case $\lambda = [1^7]$

Since we shall almost exclusively be interested in objects defined over $\mathbb{F}_q$, we shall often omit the decoration “ $\mathbb{F}_q$ ”. For instance, we shall simply write “point” to mean “ $\mathbb{F}_q$ -point”. Let $U \subset (\mathbb{P}^2)^7$ be the subset consisting of septuples of pairwise distinct points such that the first four points lie in general position. We thus have

$$|U^{F\sigma}| = (q^2 + q + 1)(q^2 + q)q^2(q^2 - 2q + 1)(q^2 + q - 3)(q^2 + q - 4)(q^2 + q - 5).$$

The following notation will be quite convenient.

Definition 6.12 If P and Q are two points in $\mathbb{P}^2$, then the line through P and Q shall be denoted PQ .

Fig. 5 A typical element of $\Delta_{l,1}(\{i\})$

Since we shall often want to stay away from lines through two of the first four points we define

$$\mathcal{S} = \bigcup_{1 \leq i < j \leq 4} P_i P_j.$$

We note that $\mathcal{S}^{F\sigma}$ contains

$$|\mathcal{S}^{F\sigma}| = 6q - 5$$

points.

6.15.1 The set Δ_l

The set Δ_l decomposes into a disjoint union of three sets

$$\Delta_l = \Delta_{l,1} \cup \Delta_{l,2} \cup \Delta_{l,3},$$

where

- the points of $\Delta_{l,1}$ are such that at least one of the points P_5, P_6 or P_7 lies in $\mathcal{S}$,
- the points of $\Delta_{l,2}$ are such that one of the lines $P_i P_j$, $5 \leq i < j \leq 7$, contains one of the points P_1, P_2, P_3 and P_4 , but $\{P_5, P_6, P_7\} \cap \mathcal{S} = \emptyset$, and
- the points of $\Delta_{l,3}$ are such that the three points P_5, P_6 and P_7 lie on a line which does not pass through P_1, P_2, P_3 or P_4 .

We shall consider the three subsets separately.

The set $\Delta_{l,1}$. For each subset $I \subset \{5, 6, 7\}$, let $\Delta_{l,1}(I)$ denote the set of points in $\Delta_{l,1}$ such that $P_i \in \mathcal{S}$ for all $i \in I$. We can then decompose $\Delta_{l,1}$ further as

$$\Delta_{l,1} = \Delta_{l,1}(\{5\}) \cup \Delta_{l,1}(\{6\}) \cup \Delta_{l,1}(\{7\}).$$

Clearly, $\Delta_{l,1}(\{i\}) \cap \Delta_{l,1}(\{j\}) = \Delta_{l,1}(\{i, j\})$.

A typical element of $\Delta_{l,1}(\{i\})$ is illustrated in Fig. 5. To compute $|\Delta_{l,1}(\{i\})|$ we first place the first four points in general position, then choose P_i as any point in $\mathcal{S}$ and finally place the remaining two points anywhere. Hence

$$|\Delta_{l,1}(\{i\})| = \underbrace{(q^2 + q + 1)(q^2 + q)q^2(q^2 - 2q + 1)(6q - 9)(q^2 + q - 4)(q^2 + q - 5)}_{|\mathrm{PGL}(3, \mathbb{F}_q)|}.$$

Fig. 6 A typical element of $\Delta_{l,2}^r(\{i, j\})$

Similarly, we have

$$|\Delta_{l,1}(\{i, j\})| = |\mathrm{PGL}(3, \mathbb{F}_q)| \cdot (6q - 9)(6q - 10)(q^2 + q - 5),$$

and

$$|\Delta_{l,1}(\{5, 6, 7\})| = |\mathrm{PGL}(3, \mathbb{F}_q)| \cdot (6q - 9)(6q - 10)(6q - 11).$$

This allows us to compute $|\Delta_{l,1}|$ as

$$|\Delta_{l,1}| = |\mathrm{PGL}(3, \mathbb{F}_q)| \cdot (18q^5 - 99q^4 + 252q^3 - 414q^2 + 417q - 180).$$

The set $\Delta_{l,2}$. Let $\{i, j\} \in \{5, 6, 7\}$, $r \in \{1, 2, 3, 4\}$ and let $\Delta_{l,2}^r(\{i, j\})$ be the subset of points in $\Delta_{l,2}$ such that $P_i P_j \cap \{P_1, P_2, P_3, P_4\} = \{P_r\}$. We also define

$$\Delta_{l,2}(\{i, j\}) = \bigcup_{r=1}^4 \Delta_{l,2}^r(\{i, j\}).$$

A typical element of $\Delta_{l,2}^r(\{i, j\})$ is illustrated in Fig. 6. To obtain an element of $\Delta_{l,2}^r(\{i, j\})$ we first place P_1, P_2, P_3 or P_4 in general position. There are $q + 1$ lines through P_r of which three are contained in $\mathcal{S}$. We choose $P_i P_j$ as one of the remaining $q - 2$ lines. Note that $P_i P_j$ will not pass through any of the points

$$Q_1 = P_1 P_4 \cap P_2 P_3, \quad Q_2 = P_2 P_4 \cap P_1 P_3, \quad Q_3 = P_3 P_4 \cap P_1 P_2. \quad (6.4)$$

Hence, $P_i P_j$ will intersect $\mathcal{S}$ in P_r and three further points. There are thus $q - 3$ ways to choose P_i and then $q - 4$ ways to choose P_j . Finally, there are

$$|\mathbb{P}^2(\mathbb{F}_q) \setminus \mathcal{S}^{F\sigma}| - 2 = q^2 + q + 1 - (6q - 5) - 2 = q^2 - 5q + 4,$$

choices for the seventh point. We thus have

$$|\Delta_{l,2}^r(\{i, j\})| = |\mathrm{PGL}(3, \mathbb{F}_q)| \cdot (q - 2)(q - 3)(q - 4)(q^2 - 5q + 4).$$

Fig. 7 A typical element of $\Delta_{l,2}^r(\{i, j\}) \cap \Delta_{l,2}^s(\{i, k\})$

Fig. 8 A typical element of $\Delta_{l,2}^r(\{5, 6\}) \cap \Delta_{l,2}^s(\{5, 7\}) \cap \Delta_{l,2}^t(\{6, 7\})$

We have counted some tuples several times. To begin with, the points of

$$\Delta_{l,2}^r(\{5, 6\}) \cap \Delta_{l,2}^r(\{5, 7\}) \cap \Delta_{l,2}^r(\{6, 7\})$$

have been counted three times. There are

$$|\mathrm{PGL}(3, \mathbb{F}_q)| \cdot (q-2)(q-3)(q-4)(q-5),$$

of these. Further, the sets $\Delta_{l,2}^r(\{i, j\})$ and $\Delta_{l,2}^s(\{i, k\})$ will intersect if $r \neq s$ and $j \neq k$. A typical element is illustrated in Fig. 7.

To compute $|\Delta_{l,2}^r(\{i, j\}) \cap \Delta_{l,2}^s(\{i, k\})|$ we begin by choosing P_1, P_2, P_3 and P_4 in general position and continue by choosing P_i outside $\mathcal{S}$ in one of $q^2 - 5q + 6$ ways. This gives us two lines $P_i P_r$ and $P_i P_s$ which intersect $\mathcal{S}$ in four points each. We choose P_j on $P_i P_r$ away from P_i and $\mathcal{S}$ in $q-4$ ways and similarly for P_k . This gives

$$|\Delta_{l,2}^r(\{i, j\}) \cap \Delta_{l,2}^s(\{i, k\})| = |\mathrm{PGL}(3, \mathbb{F}_q)| \cdot (q^2 - 5q + 6)(q-4)^2.$$

Finally, we must compute the cardinality of the triple intersection

$$\Delta_{l,2}^r(\{5, 6\}) \cap \Delta_{l,2}^s(\{5, 7\}) \cap \Delta_{l,2}^t(\{6, 7\}),$$

where r, s and t are distinct. A typical element of the intersection is illustrated in Fig. 8.

Fig. 9 A typical element of $T^{r,s,t}$

This is where we have to pay for the awkward requirement that P_5, P_6 and P_7 should not be in $\mathcal{S}$. We shall view $\Delta_{l,2}^r(\{5, 6\}) \cap \Delta_{l,2}^s(\{5, 7\}) \cap \Delta_{l,2}^t(\{6, 7\})$ as an open subset of the set $T^{r,s,t}$ consisting of tuples such that

- the line $P_5 P_6$ passes through P_r , $P_5 P_7$ passes through P_s and $P_6 P_7$ passes through P_t but,
- we allow P_5, P_6 and P_7 to lie in $\mathcal{S}$, but,
- we do not allow the lines $P_i P_j$, $5 \leq i < j \leq 7$ to be contained in $\mathcal{S}$.

The complement of $\Delta_{l,2}^r(\{5, 6\}) \cap \Delta_{l,2}^s(\{5, 7\}) \cap \Delta_{l,2}^t(\{6, 7\})$ in $T^{r,s,t}$ can be decomposed into a union of three subsets $T_i^{r,s,t}$, $i = 5, 6, 7$, consisting of those tuples with P_i in $\mathcal{S}$.

We begin with the computation of $|T^{r,s,t}|$. To obtain such a tuple, we begin by choosing a line L_r through P_r in $q - 2$ ways. We shall then choose a line L_s through P_s . There are however two cases that may occur. Typically, the intersection point $P_5 = L_r \cap L_s$ will lie outside $\mathcal{S}$ but for one choice of L_s it will lie in $\mathcal{S}$. The situation is illustrated in Fig. 9.

There are $q - 3$ ways to choose L_s so that $L_r \cap L_s$ lies outside $\mathcal{S}$. When we choose the line L_t through P_t we must make sure that L_t is not contained in $\mathcal{S}$ and that L_t does not pass through $L_r \cap L_s$, since we want to end up with three distinct intersection points. We thus have $q - 3$ choices. On the other hand, if we choose L_s as the one line making the intersection point $L_r \cap L_s$ lie in $\mathcal{S}$ we only need to make sure that L_t is not contained in $\mathcal{S}$ and we thus have $q - 2$ choices. Hence, we see that

$$|T^{r,s,t}| = |\mathrm{PGL}(3, \mathbb{F}_q)| \cdot ((q - 2)(q - 3)^2 + (q - 2)^2).$$

We now turn to the computation of $|T_i^{r,s,t}|$, $i = 5, 6, 7$. We then begin by choosing a line L_r through P_r in $q - 2$ ways. The line L_s through P_s is then completely determined since we must have $P_i \in \mathcal{S}$. This gives us $q - 2$ choices for the final line L_t through P_t . Hence,

$$|T_i^{r,s,t}| = |\mathrm{PGL}(3, \mathbb{F}_q)| \cdot (q - 2)^2.$$

We now turn to the computation of $|T_i^{r,s,t} \cap T_j^{r,s,t}|$, $5 \leq i < j \leq 7$. As above, we begin by choosing a line L_r through P_r in $q - 2$ ways. Since P_i must lie in $\mathcal{S}$ we have only one choice for L_s . Since $P_j = L_s \cap L_t$ we see that we now have precisely one

Fig. 10 The only element in $T_5^{r,s,t} \cap T_6^{r,s,t} \cap T_7^{r,s,t}$

choice for L_t also. Hence,

$$|T_i^{r,s,t} \cap T_j^{r,s,t}| = |\mathrm{PGL}(3, \mathbb{F}_q)| \cdot (q - 2).$$

We now consider $T_5^{r,s,t} \cap T_6^{r,s,t} \cap T_7^{r,s,t}$. It turns out that once the four points P_1, P_2, P_3 and P_4 have been placed in general position, there is precisely one such tuple. The situation is illustrated in Fig. 10.

This finally allows us to compute

$$\Delta_{l,2} = |\mathrm{PGL}(3, \mathbb{F}_q)| \cdot (12q^5 - 212q^4 + 1504q^3 - 5320q^2 + 9296q - 6360).$$

The set $\Delta_{l,3}$. Recall the definition of the three points Q_1, Q_2 and Q_3 from equation (6.4). Using these three points we may decompose $\Delta_{l,3}$ into a disjoint union of the following subsets:

- $\Delta_{l,3}(\{Q_r, Q_s\})$ consisting of those tuples of $\Delta_{l,3}$ where P_5, P_6 and P_7 lie on the line $Q_r Q_s$, $1 \leq r < s \leq 3$, and,
- $\Delta_{l,3}(\{Q_r\})$ consisting of those tuples of $\Delta_{l,3}$ with P_5, P_6 and P_7 on a line through Q_r , $1 \leq r \leq 3$, which does not pass through any of the other Q_i , and
- $\Delta_{l,3}(\emptyset)$ consisting of those tuples of $\Delta_{l,3}$ with P_5, P_6 and P_7 on a line which does not pass through Q_1, Q_2 or Q_3 .

We begin by considering $\Delta_{l,3}(\{Q_r, Q_s\})$. The line $Q_r Q_s$ contains $q + 1$ points of which four lie in $\mathcal{S}$. There are thus $q - 3$ choices for P_5 , $q - 4$ choices for P_6 and $q - 5$ choices for P_7 . Hence,

$$|\Delta_{l,3}(\{Q_r, Q_s\})| = |\mathrm{PGL}(3, \mathbb{F}_q)| \cdot (q - 3)(q - 4)(q - 5).$$

We continue with $|\Delta_{l,3}(\{Q_r\})|$. There are $q + 1$ lines through Q_r of which two are contained in $\mathcal{S}$ and two are the lines through the other two Q_i . Hence, there are $q - 3$ choices for a line L though Q_r . The line L intersects $\mathcal{S}$ in five points so we have $q - 4$ choices for P_5 , $q - 5$ choices for P_6 and $q - 6$ choices for P_7 . We conclude that

$$|\Delta_{l,3}(\{Q_r\})| = |\mathrm{PGL}(3, \mathbb{F}_q)| \cdot (q - 3)(q - 4)(q - 5)(q - 6).$$

To compute $|\Delta_{l,3}(\emptyset)|$ we begin by choosing a line L which does not pass through any of the points $P_1, P_2, P_3, P_4, Q_1, Q_2$ and Q_3 . There are $q^2 + q + 1$ lines in $\mathbb{P}^2$, of which $q + 1$ passes through $P_i, i = 1, 2, 3, 4$. There is exactly one line through each pair of these points so there are

$$q^2 + q + 1 - 4(q + 1) + 6 = q^2 - 3q + 3$$

lines which do not pass through P_1, P_2, P_3 and P_4 . Of the $q + 1$ lines through $Q_i, i = 1, 2, 3$, precisely two have been removed above and the line $Q_i Q_j$ passes through both Q_i and Q_j . Hence, we have

$$q^2 - 3q + 3 - 3(q - 1) + 3 = q^2 - 6q + 9$$

choices for L .

The line L intersects $\mathcal{S}$ in six points. We therefore have $q - 5$ choices for $P_5, q - 6$ choices for P_6 and $q - 7$ choices for P_7 . Hence,

$$|\Delta_{l,3}(\emptyset)| = |\mathrm{PGL}(3, \mathbb{F}_q)| \cdot (q^2 - 6q + 9)(q - 5)(q - 6)(q - 7).$$

We now add everything together to obtain

$$|\Delta_{l,3}| = |\mathrm{PGL}(3, \mathbb{F}_q)| \cdot (q^5 - 21q^4 + 173q^3 - 693q^2 + 1338q - 990)$$

and, finally,

$$|\Delta_l^{F\sigma}| = |\mathrm{PGL}(3, \mathbb{F}_q)| \cdot (31q^5 - 332q^4 + 1929q^3 - 6427q^2 + 11051q - 7530).$$

6.15.2 The set Δ_c

We decompose $\Delta_c^{F\sigma}$ as

$$\Delta_c^{F\sigma} = \Delta_{c,1} \cup \Delta_{c,2},$$

where $\Delta_{c,1}$ consists of tuples where six points lie on a smooth conic C with one of the points P_1, P_2, P_3 or P_4 possibly outside C and $\Delta_{c,2}$ consists of tuples where six points lie on a smooth conic C with one of the points P_5, P_6 or P_7 possibly outside the conic C .

To obtain an element of $\Delta_{c,1}$ we first choose one of the points P_1, P_2, P_3 and P_4 and call it P . Then we choose a smooth conic C in $q^5 - q^2$ ways and place all of the seven points except P on C in

$$(q + 1)q(q - 1)(q - 2)(q - 3)(q - 4),$$

ways. There are three lines through pairs of points in $\{P_1, P_2, P_3, P_4\} \setminus \{P\}$ which together contain $3q$ points. These lines do not contain P_5, P_6 and P_7 so we have

$$q^2 + q + 1 - 3q - 3 = q^2 - 2q - 2,$$

choices for P . Multiplying everything together we obtain

$$N_1 := 4(q^5 - q^2)(q + 1)q(q - 1)(q - 2)(q - 3)(q - 4)(q^2 - 2q - 2),$$

which is almost $|\Delta_{c,1}|$ except that we have counted the tuples where all seven points lie on C four times.

To obtain an element of $\Delta_{c,2}$ we first choose P_5, P_6 and P_7 and call the chosen point P . We then choose a smooth conic C and place all but the chosen points on C . Finally, we place P anywhere in $\mathbb{P}^2$ except at the six chosen points. In this way we obtain the number

$$N_2 := 3(q^5 - q^2)(q + 1)q(q - 1)(q - 2)(q - 3)(q - 4)(q^2 + q - 5),$$

which is almost equal to $|\Delta_{c,2}|$ except that we have counted the tuples with all seven points on C three times.

We now want to compute the number of tuples with all seven points on a smooth conic C . We thus choose a smooth conic C and place all seven points on it in

$$N_7 := (q^5 - q^2)(q + 1)q(q - 1)(q - 2)(q - 3)(q - 4)(q - 5),$$

ways. We thus have

$$|\Delta_c^{F\sigma}| = |\mathrm{PGL}(3, \mathbb{F}_q)| \cdot (7q^5 - 74q^4 + 288q^3 - 517q^2 + 446q - 168).$$

6.15.3 The set $\Delta_l \cap \Delta_c$

We introduce the filtration $\mathcal{F}_3 \subset \mathcal{F}_2 \subset \mathcal{F}_1 = \Delta_l^{F\sigma} \cap \Delta_c^{F\sigma}$ where

- the set $\mathcal{F}_1$ consists of tuples such that at least one line contains three points of the tuple,
- the set $\mathcal{F}_2$ consists of tuples such that at least two lines contain three points of the tuple,
- the set $\mathcal{F}_3$ consists of tuples such that at least three lines contain three points of the tuple.

The strategy will be to compute the numbers:

$$N_1 = |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3|,$$

$$N_2 = |\mathcal{F}_2| + 2|\mathcal{F}_3|,$$

$$N_3 = |\mathcal{F}_3|,$$

and thereby obtain the desired cardinality.

Since the points P_1, P_2, P_3 and P_4 are assumed to constitute a frame, we must do things a little bit differently depending on whether the point not on the conic is one of these four or not. We therefore make further subdivisions.

The subsets with P_5, P_6 or P_7 not on the conic. We shall denote the subsets in question by $\mathcal{F}_i^{5,6,7}$ and, similarly

$$\begin{aligned} N_1^{5,6,7} &= |\mathcal{F}_1^{5,6,7}| + |\mathcal{F}_2^{5,6,7}| + |\mathcal{F}_3^{5,6,7}|, \\ N_2^{5,6,7} &= |\mathcal{F}_2^{5,6,7}| + 2|\mathcal{F}_3^{5,6,7}|, \\ N_3^{5,6,7} &= |\mathcal{F}_3^{5,6,7}|. \end{aligned}$$

To compute $N_1^{5,6,7}$, we first choose one of the points P_5, P_6 or P_7 to be the point P not on the smooth conic C and call the remaining two points P_i and P_j . We then choose C in $q^5 - q^2$ ways and choose two points among $\{P_1, P_2, P_3, P_4, P_i, P_j\}$ and call them R_1 and R_2 . There are $(q+1)q$ ways to place R_1 and R_2 on C and there are then $q-1$ ways to place P on the line $R_1 R_2$. Finally, we place the remaining four points on C in $(q-1)(q-2)(q-3)(q-4)$ ways. Multiplying everything together we obtain

$$N_1^{5,6,7} := 3 \cdot \binom{6}{2} \cdot (q^5 - q^2)(q+1)q(q-1)^2(q-2)(q-3)(q-4).$$

In order to compute $N_2^{5,6,7}$, we first choose one of the points P_5, P_6 or P_7 to be the point P not on the smooth conic C and call the remaining two points P_i and P_j . We then choose C in $q^5 - q^2$ ways and choose two unordered pairs of unordered points among $\{P_1, P_2, P_3, P_4, P_i, P_j\}$. This can be done in $\frac{1}{2} \cdot \binom{6}{4} \cdot \binom{4}{2}$ ways. We call the points of the first pair R_1 and R_2 and those of the second O_1 and O_2 . There are $(q+1)q(q-1)(q-2)(q-3)(q-4)$ ways to place $\{P_1, P_2, P_3, P_4, P_i, P_j\}$ on C and the point P is then completely determined as $P = R_1 R_2 \cap O_1 O_2$. Thus

$$N_2^{5,6,7} = 3 \cdot \frac{1}{2} \cdot \binom{6}{4} \cdot \binom{4}{2} \cdot (q^5 - q^2)(q+1)q(q-1)(q-2)(q-3)(q-4).$$

The computation of $N_3^{5,6,7}$ is slightly more complicated since we need to subdivide into two subcases depending on if P is on the outside or on the inside of C . We call the two corresponding numbers $N_{3,\text{out}}^{5,6,7}$ and $N_{3,\text{in}}^{5,6,7}$.

To compute $N_{3,\text{out}}^{5,6,7}$ we first choose one of the points P_5, P_6 or P_7 to be the point P not on the smooth conic C . We proceed by choosing the smooth conic C in $q^5 - q^2$ ways and then the point P on the outside of C in $\frac{1}{2}(q+1)q$ ways. We now place P_1 at one of the $q-1$ points of C whose tangent does not pass through P and choose one of the remaining five points as the other intersection point in $C \cap P_1 P$. There are now four remaining points P_i, P_j, P_k and P_l to place on C . We place P_i at one of the $q-3$ remaining points of C whose tangent does not pass through P and choose one of the remaining three points as the other intersection point in $C \cap P_i P$. There are now two points P_r and P_s to place on C . We place P_r at one of the $q-5$ possible points and the point P_s is then determined. We thus have

$$N_{3,\text{out}}^{5,6,7} = 3 \cdot (q^5 - q^2) \cdot \frac{1}{2} (q+1)q \cdot (q-1) \cdot 5 \cdot (q-3) \cdot 3 \cdot (q-5).$$

We proceed by computing $N_{3,\text{in}}^{5,6,7}$. We first choose one of the points P_5, P_6 or P_7 to be the point P not on the smooth conic C . We proceed by choosing the smooth conic C in $q^5 - q^2$ ways and then the point P on the outside of C in $\frac{1}{2}(q-1)q$ ways.

We now place P_1 at one of the $q+1$ points of C whose tangent does not pass through P and choose one of the remaining five points as the other intersection point in $C \cap P_1P$. There are now four remaining points P_i, P_j, P_k and P_l to place on C . We place P_i at one of the $q-1$ remaining points of C and choose one of the remaining three points as the other intersection point in $C \cap P_iP$. There are now two points P_r and P_s to place on C . We place P_r at one of the $q-3$ possible points and the point P_s is then determined. We now see that

$$N_{3,\text{in}}^{5,6,7} = 3 \cdot (q^5 - q^2) \cdot \frac{1}{2}(q-1)q \cdot (q+1) \cdot 5 \cdot (q-1) \cdot 3 \cdot (q-3).$$

The subsets with P_1, P_2, P_3 or P_4 not on the conic. We shall denote the subsets in question by $\mathcal{F}_i^{1,2,3,4}$ and, similarly

$$\begin{aligned} N_1^{1,2,3,4} &= |\mathcal{F}_1^{1,2,3,4}| + |\mathcal{F}_2^{1,2,3,4}| + |\mathcal{F}_3^{1,2,3,4}|, \\ N_2^{1,2,3,4} &= |\mathcal{F}_2^{1,2,3,4}| + 2|\mathcal{F}_3^{1,2,3,4}|, \\ N_3^{1,2,3,4} &= |\mathcal{F}_3^{1,2,3,4}|. \end{aligned}$$

In order to compute $N_1^{1,2,3,4}$, we first choose one of the points P_1, P_2, P_3 or P_4 to be the point P not on the smooth conic C and call the remaining three points P_r, P_s and P_t . We continue by choosing a smooth conic C in $q^5 - q^2$ ways.

We first assume that P lies on a line R_1R_2 where $\{R_1, R_2\} \subset \{P_5, P_6, P_7\}$. We therefore choose the two points in three ways and call the remaining point P_i . We then place R_1 and R_2 on C in $(q+1)q$ ways. We continue by choosing the three points P_r, P_s and P_t on C in $(q-1)(q-2)(q-3)$ ways. The lines P_rP_s, P_rP_t and P_sP_t intersect the line R_1R_2 in three distinct points so there are $q-4$ ways to choose the point P on R_1R_2 but away from these three points and R_1 and R_2 . Finally, we place P_i at one of the $q-4$ remaining points of C . Multiplying everything together we get

$$4 \cdot 3 \cdot (q^5 - q^2)(q+1)q(q-1)(q-2)(q-3)(q-4)^2.$$

We now assume that P lies on a line ab with $a \in \{P_1, P_2, P_3, P_4\}$ and $b \in \{P_5, P_6, P_7\}$. We thus first choose a as one of the points in $\{P_r, P_s, P_t\}$ and the point b as one of the points $\{P_5, P_6, P_7\}$ and place a and b on C in one of $(q+1)q$ ways. We then place the remaining two points, c and d , of $\{P_1, P_2, P_3, P_4\}$ on C in $(q-1)(q-2)$ ways. The line cd intersects ab in a point outside of C so there are $q-2$ ways to choose P on ab but away from this intersection point and a and b . Finally, we place the remaining two points of $\{P_5, P_6, P_7\}$ on C in one of $(q-3)(q-4)$ ways. Multiplying everything together we obtain

$$4 \cdot 3 \cdot 3 \cdot (q^5 - q^2)(q+1)q(q-1)(q-2)^2(q-3)(q-4).$$

We now add the two answers above together to get

$$N_1^{1,2,3,4} = 24q^3(q-2)(q-3)(q-4)(2q-5)(q+1)(q^2+q+1)(q-1)^2.$$

To compute $N_2^{1,2,3,4}$, we first choose one of the points P_1, P_2, P_3 or P_4 to be the point P not on the smooth conic C and call the remaining three points P_r, P_s and P_t . We continue by choosing a smooth conic C in $q^5 - q^2$ ways.

We first assume that P lies on two lines R_1R_2 and O_1O_2 where $\{R_1, O_1\} \subset \{P_5, P_6, P_7\}$ and $\{R_2, O_2\} \subset \{P_r, P_s, P_t\}$. We now choose two points among $\{P_5, P_6, P_7\}$ in three ways and choose two points among $\{P_r, P_s, P_t\}$ in three ways and rename the remaining two points to P_u and P_v . There are now two possible ways to label the four chosen points R_1, R_2, O_1 and O_2 in such a way that $\{R_1, O_1\} \subset \{P_5, P_6, P_7\}$ and $\{R_2, O_2\} \subset \{P_r, P_s, P_t\}$ and we choose one of them. We then place the four points R_1, R_2, O_1 and O_2 on the conic C in $(q+1)q(q-1)(q-2)$ ways. The point P is now given as $P = R_1R_2 \cap O_1O_2$ and no matter how we place P_u and P_v , the three lines P_rP_s, P_rP_t and P_sP_t will not go through P . We can now multiply everything together to obtain

$$4 \cdot 3 \cdot 3 \cdot 2 \cdot (q^5 - q^2)(q+1)q(q-1)(q-2)(q-3)(q-4).$$

The other possibility is that P lies on two lines R_1R_2 and R_3b where $\{R_1, R_2, R_3\}$ is the set $\{P_5, P_6, P_7\}$ and $b \in \{P_r, P_s, P_t\}$. We thus choose b in three ways and rename the remaining two points in $\{P_r, P_s, P_t\}$ to P_u and P_v . From now on, we must differentiate between when P is on the outside and on the inside of C .

First, we choose P on the outside of C in $\frac{1}{2}(q+1)q$ ways. We then choose b as a point on C whose tangent does not pass through P in $q-1$ ways. We then choose one of the points P_5, P_6 and P_7 to become the second intersection point in $C \cap bP$. Then, we place the remaining two points among $\{P_5, P_6, P_7\}$ on C such that the line through them passes through P in $q-3$ ways. There are now $q-5$ ways to choose P_u and P_v such that the line P_uP_v will pass through P . Thus, the remaining $(q-3)(q-4) - (q-5) = q^2 - 8q + 17$ choices must give P_u and P_v such that none of the lines P_rP_s, P_rP_t and P_sP_t will contain P . We may now multiply everything together to obtain

$$4 \cdot (q^5 - q^2) \cdot 3 \cdot \frac{1}{2}(q+1)q \cdot (q-1) \cdot 3 \cdot (q-3) \cdot (q^2 - 8q + 17).$$

Now we choose P on the inside of C in one of $\frac{1}{2}(q-1)q$ ways. We then choose b as a point on C whose tangent does not pass through P in $q+1$ ways. We then choose one of the points P_5, P_6 and P_7 to become the second intersection point in $C \cap bP$. Then, we place the remaining two points among $\{P_5, P_6, P_7\}$ on C such that the line through them passes through P in $q-1$ ways. There are now $q-3$ ways to choose P_u and P_v such that the line P_uP_v will pass through P . Thus, the remaining $(q-3)(q-4) - (q-3) = (q-3)(q-5)$ choices must give P_u and P_v such that none of the lines P_rP_s, P_rP_t and P_sP_t will contain P . We may now multiply everything together to obtain

$$4 \cdot (q^5 - q^2) \cdot 3 \cdot \frac{1}{2} (q - 1) q \cdot (q + 1) \cdot 3 \cdot (q - 1) \cdot (q - 3)(q - 5).$$

We may now add everything together to get

$$N_2^{1,2,3,4} = 36q^3(q + 1)(q^2 + q + 1)(5q^3 - 37q^2 + 82q - 60)(q - 1)^2.$$

Finally, we need to compute $N_3^{1,2,3,4}$. We begin by choosing one of the points P_1, P_2, P_3 or P_4 to be the point P not on the smooth conic C and call the remaining three points P_r, P_s and P_t . We continue by choosing a smooth conic C in $q^5 - q^2$ ways.

Here, we only have the possibility that P lies on three lines $R_1 O_1, R_2 O_2$ and $R_3 O_3$ where $\{R_1, R_2, R_3\} = \{P_5, P_6, P_7\}$ and $\{O_1, O_2, O_3\} = \{P_r, P_s, P_t\}$. However, we must take care of the case that P is on the outside of C and the case that P is on the inside of C separately. We call the corresponding numbers $N_{3,\text{out}}^{1,2,3,4}$ and $N_{3,\text{in}}^{1,2,3,4}$.

We begin by computing $N_{3,\text{out}}^{1,2,3,4}$. We thus choose the point P as a point on the outside of C in $\frac{1}{2}(q + 1)q$ ways. We begin by placing P_5 at one of the points of C whose tangent does not pass through P in $q - 1$ ways. We label the second intersection point of $C \cap P_5 P$ with P_r, P_s or P_t and call the remaining two points P_u and P_v . We then place P_6 at one of the $q - 3$ remaining points of C whose tangent does not pass through P and then choose one of the points P_u and P_v to become the other intersection point of $C \cap P_6 P$. Finally, we place P_7 at one of the remaining $q - 5$ points and label the other point of $C \cap P_7 P$ in the only possible way. We thus have

$$N_{3,\text{out}}^{1,2,3,4} = 4 \cdot (q^5 - q^2) \cdot \frac{1}{2} (q + 1) q \cdot (q - 1) \cdot 3 \cdot (q - 3) \cdot 2 \cdot (q - 5).$$

We now turn to computing $N_{3,\text{in}}^{1,2,3,4}$. We thus choose the point P as a point on the inside of C in $\frac{1}{2}(q - 1)q$ ways. We begin by placing P_5 at one of the points of C whose tangent does not pass through P in $q + 1$ ways. We label the second intersection point of $C \cap P_5 P$ with P_r, P_s or P_t and call the remaining two points P_u and P_v . We then place P_6 at one of the $q - 1$ remaining points of C and then choose one of the points P_u and P_v to become the other intersection point of $C \cap P_6 P$. Finally, we place P_7 at one of the remaining $q - 3$ points and label the other point of $C \cap P_7 P$ in the only possible way. We now see that

$$N_{3,\text{in}}^{1,2,3,4} = 4 \cdot (q^5 - q^2) \cdot \frac{1}{2} (q - 1) q \cdot (q + 1) \cdot 3 \cdot (q - 1) \cdot 2 \cdot (q - 3),$$

and we get

$$N_3^{1,2,3,4} = 192q^3(q + 1)(q^2 + q + 1)(q^2 - 3q + 3)(q - 1)^2.$$

We now obtain

$$|\Delta_l^{F\sigma} \cap \Delta_c^{F\sigma}| = |\text{PGL}(3, \mathbb{F}_q)| \cdot (93q^4 - 1245q^3 + 6195q^2 - 13470q + 10737),$$

and, finally,

$$|(\mathcal{P}_7^2)^{F\sigma}| = q^6 - 35q^5 + 490q^4 - 3485q^3 + 13174q^2 - 24920q + 18375.$$

This concludes the equivariant point count of $\mathcal{Q}[2]$. In Sect. 8 we provide a summary of the results of the computations.

7 The hyperelliptic locus

Up to this point we have almost exclusively discussed plane quartics. We shall now briefly turn our attention to the other type of genus 3 curves — the hyperelliptic curves. There are many possible ways to approach the computation of the cohomology of $\mathcal{H}_3[2]$. Our choice is by means of equivariant point counts as in the previous section.

Recall that a hyperelliptic curve C of genus g is determined, up to isomorphism, by $2g + 2$ distinct points on $\mathbb{P}^1$, up to projective equivalence and that any such collection S of $2g + 2$ points determines a double cover $\pi : C \rightarrow \mathbb{P}^1$ branched precisely over S (and C is thus a hyperelliptic curve). Moreover, if we pick $2g + 2$ ordered points $P_1, \dots, P_{2g+2}$ on $\mathbb{P}^1$, the curve C also attains a level 2 structure. In the genus 3 case, we get eight points $Q_i = \pi^{-1}(P_i)$ which determine $\binom{8}{2} = 28$ odd theta characteristics $Q_i + Q_j$, $i < j$ and $\{Q_1 + Q_8, \dots, Q_7 + Q_8\}$ is an ordered Aronhold basis, see [2, Appendix B.32–33] and [21], and an ordered Aronhold basis determines a level 2 structure.

However, not all level 2 structures on the hyperelliptic curve C arise from different orderings of the points. Nevertheless, there is an intimate relationship between the moduli space $\mathcal{H}_g[2]$ of hyperelliptic curves with level 2 structure and the moduli space $\mathcal{M}_{0,2g+2}$ of $2g + 2$ ordered points on $\mathbb{P}^1$ given by the following theorem which can be found in [14, Theorem VIII.1].

Theorem 7.1 *Each irreducible component of $\mathcal{H}_g[2]$ is isomorphic to the moduli space $\mathcal{M}_{0,2g+2}$ of $2g + 2$ ordered points on the projective line.*

Dolgachev and Ortland [14] pose the question whether the irreducible components of $\mathcal{H}_g[2]$ also are the connected components or, in other words, if $\mathcal{H}_g[2]$ is smooth. In the complex case, the question was answered positively by Tsuyumine in [29] and later, by a shorter argument, by Runge in [28]. Using the results of [1], the argument of Runge carries over word for word to an algebraically closed field of positive characteristic different from 2.

Theorem 7.2 *If $g \geq 2$, then each irreducible component of $\mathcal{H}_g[2]$ is also a connected component.*

We have a natural action of S_{2g+2} on the space $\mathcal{M}_{0,2g+2}$. Since different orderings of the points correspond to different symplectic level 2 structures, S_{2g+2} sits naturally inside $\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ and, in fact, for $g = 3$ and for even g it is a maximal subgroup, see [15]. With Theorems 7.1 and 7.2 at hand, the following slight generalization of a corollary in [14, p. 145] is clear.

Corollary 7.3 *Let $g \geq 2$ and let $X_{[\tau]} = \mathcal{M}_{0,2g+2}$ for each left coset $[\tau] \in \mathcal{T} := \mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})/S_{2g+2}$. Then*

$$\mathcal{H}_g[2] \cong \coprod_{[\tau] \in \mathcal{T}} X_{[\tau]},$$

and the group $\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ acts transitively on the set of connected components $X_{[\tau]}$ of $\mathcal{H}_g[2]$. In particular, there are

$$\frac{|\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})|}{|S_{2g+2}|} = \frac{2^{g^2} (2^{2g} - 1)(2^{2g-2} - 1) \cdots (2^2 - 1)}{(2g + 2)!},$$

connected components of $\mathcal{H}_g[2]$.

Remark 7.4 As pointed out in [28], the argument to prove the corollary stated in [14] is not quite correct in full generality as it is given there. However, it is enough to prove the result for $g = 3$ and for even g , and in [28] it is explained how to obtain the full result.

Let us now, once and for all, choose a set T of representatives of $\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})/S_{2g+2}$. If we denote the elements of $X_{[\mathrm{id}]}$ by x , then any element in $X_{[\tau]}$ can be written as τx for some $x \in X_{[\mathrm{id}]}$. Let α be any element of $\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$. Then

$$\alpha\tau = \tau'\sigma,$$

for some $\sigma \in S_{2g+2}$ and some $\tau' \in T$. Since the Frobenius commutes with the action of $\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ we have that

$$F\alpha(\tau x) = \tau x$$

if and only if

$$F(\tau'\sigma x) = \tau'(F\sigma x) = \tau x.$$

But the Frobenius acts on each of the components of $\mathcal{H}_g[2]$ so we see that $F\alpha(\tau x) = \tau x$ if and only if $\tau' = \tau$ and $F\sigma x = x$.

We now translate the above observation into more standard representation theoretic vocabulary. Define a class function ψ on S_{2g+2} by

$$\psi(\sigma) = |X_{\mathrm{id}}^{F\sigma}|,$$

and define a class function $\hat{\psi}$ on $\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ by setting

$$\hat{\psi}(\alpha) = |\mathcal{H}_g[2]^{F\alpha}|,$$

for any $\alpha \in \mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$. By the above observation we have that

$$\hat{\psi}(\alpha) = \sum_{\tau \in T} \tilde{\psi}(\tau^{-1}\alpha\tau),$$

where

$$\tilde{\psi}(\beta) = \begin{cases} \psi(\beta) & \text{if } \beta \in S_{2g+2}, \\ 0, & \text{otherwise.} \end{cases}$$

In other words, $\widehat{\psi}$ is the class function ψ induced from S_{2g+2} up to $\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$. Thus, to make an S_{2g+2} -equivariant point count of $\mathcal{H}_g[2]$ we can make an S_{2g+2} -equivariant point count of $\mathcal{M}_{0,2g+2}$ and then use the representation theory of S_{2g+2} and $\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ in order to first induce the class function up to $\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ and then restrict it down again to S_{2g+2} . Once this is done, we can obtain the S_{2g+1} -equivariant point count by restricting from S_{2g+2} to S_{2g+1} .

Compared to $Q[2]$, the S_8 -equivariant point count of $\mathcal{H}_3[2]$ is very easy. We first compute the number of λ -tuples of $\mathbb{P}^1$ for each partition of λ of 8 and then divide by $|\mathrm{PGL}(2)|$ in order to obtain $|\mathcal{M}_{0,8}^{F\sigma}|$, where σ is a permutation in S_8 of cycle type λ . The result is given in Table 3. Once this is done, we induce up to $\mathrm{Sp}(6, \mathbb{Z}/2\mathbb{Z})$ in order to obtain the $\mathrm{Sp}(6, \mathbb{Z}/2\mathbb{Z})$ -equivariant cohomology of $\mathcal{H}_3[2]$. The results are given in Table 4. Finally, we restrict to S_7 to get the results of Tables 5 and 6. The computations present no difficulties whatsoever.

It is not very hard to see that $\mathcal{M}_{0,n}$ is isomorphic to the complement of a hyperplane arrangement. One way to see this is to start by placing the first three points at 0, 1 and ∞ . Then $\mathcal{M}_{0,n}$ is isomorphic to $(\mathbb{A}^1 \setminus \{0, 1\})^{2g-1} \setminus \Delta$, where $\Delta \subset (\mathbb{A}^1 \setminus \{0, 1\})^{n-3}$ is the subset of points where at least two coordinates are equal. Thus, $\mathcal{H}_3[2]$ is minimally pure and by the results of Sect. 5 we can deduce the cohomology of $\mathcal{H}_3[2]$ from the equivariant point counts. In Sect. 8 we provide a summary of the results of the computations. We also mention that the equivariant cohomology of $\mathcal{M}_{0,n}$ and $\overline{\mathcal{M}}_{0,n}$ have been computed for all $n \geq 3$ in [18] and later by many others, e.g. Gaiffi [17], Kisin and Lehrer [22] and Robinson and Whitehouse [27]. Thus, using Corollary 7.3, the cohomology of $\mathcal{H}_g[2]$ is in principle known (up to our understanding of the representation theory of $\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$). For instance, the Poincaré polynomial of $\mathcal{H}_g[2]$ is

$$P_{\mathcal{H}_g[2]}(t) = \frac{\prod_{i=1}^g (2^{2i} - 1) 2^{2i-1}}{(2g+2)!} \cdot \prod_{i=1}^{2g-1} (1 + (i+1)t).$$

8 Summary of computations

We summarize the computations related to $Q[2]$ in Table 1. Also recall that the Poincaré polynomial of $Q[2]$ is given in Theorem 1.1. In Table 2 we give the cohomology of $Q[2]$ as a representation of S_7 . The rows correspond to the cohomology groups and the columns correspond to the irreducible representations of S_7 . The symbol s_λ denotes the irreducible representation of S_7 corresponding to the partition λ and a number n in row H^k and column s_λ means that s_λ occurs in H^k with multiplicity n .

Tables 3, 4 and 5 give equivariant point counts for various spaces and groups related to the equivariant point count of $\mathcal{H}_3[2]$ and in Table 6 we give the cohomology groups of $\mathcal{H}_3[2]$ as representations of S_7 . Recall that the Poincaré polynomial of $\mathcal{H}_3[2]$ is given in Theorem 1.2.

Table 1 The S_7 -equivariant point count of $\mathcal{Q}[2]$

λ	$ \mathcal{Q}[2]^{F \cdot \sigma_\lambda} $
[7]	$q^6 + q^3$
[6,1]	$q^6 - 2q^3 + 1$
[5,2]	$q^6 - q^2$
[5, 1 ²]	$q^6 - q^2$
[4,3]	$q^6 - q^5 - 2q^4 + q^3 + q^2$
[4,2,1]	$q^6 - q^5 - 2q^4 + q^3 - 2q^2 + 3$
[4, 1 ³]	$q^6 - q^5 - 2q^4 + q^3 - 2q^2 + 3$
[3 ² , 1]	$q^6 - 2q^5 - 2q^4 - 8q^3 + 16q^2 + 10q + 21$
[3, 2 ²]	$q^6 - q^5 - 2q^4 + 3q^3 + q^2 - 2q$
[3, 2, 1 ²]	$q^6 - 3q^5 + 5q^3 - q^2 - 2q$
[3, 1 ⁴]	$q^6 - 5q^5 + 10q^4 - 5q^3 - 11q^2 + 10q$
[2 ³ , 1]	$q^6 - 3q^5 - 6q^4 + 19q^3 + 6q^2 - 24q + 7$
[2 ² , 1 ³]	$q^6 - 7q^5 + 10q^4 + 15q^3 - 26q^2 - 8q + 15$
[2, 1 ⁵]	$q^6 - 15q^5 + 90q^4 - 265q^3 + 374q^2 - 200q + 15$
[1 ⁷]	$q^6 - 35q^5 + 490q^4 - 3485q^3 + 13174q^2 - 24920q + 18375$

We use σ_λ to denote any permutation in S_7 of cycle type λ

Table 2 The cohomology of $\mathcal{Q}[2]$ as a representation of S_7

	s_7	$s_{6,1}$	$s_{5,2}$	$s_{5,1^2}$	$s_{4,3}$	$s_{4,2,1}$	$s_{4,1^3}$	$s_{3^2,1}$	$s_{3,2^2}$	$s_{3,2,1^2}$
H^0	1	0	0	0	0	0	0	0	0	0
H^1	1	1	1	0	1	0	0	0	0	0
H^2	0	3	4	4	3	5	1	3	1	1
H^3	1	8	14	18	14	30	16	16	12	18
H^4	4	20	44	47	44	99	56	56	54	83
H^5	6	33	76	76	72	178	97	104	105	169
H^6	6	23	51	54	54	127	74	76	77	126
	$s_{3,1^4}$			$s_{2^3,1}$		$s_{2^2,1^3}$		$s_{2,1^5}$		s_{1^7}
H^0	0			0		0		0		0
H^1	0			0		0		0		0
H^2	0			0		0		0		0
H^3	4			6		3		0		0
H^4	32			31		25		6		1
H^5	71			65		64		26		3
H^6	54			54		50		22		5

Table 3 The S_8 -equivariant point count of $\mathcal{M}_{0,8}$

λ	$ \mathcal{M}_{0,8}^{F,\sigma_\lambda} $
[8]	$(q^2 + 1)q^3$
[7, 1]	$(q + 1)(q^2 + q + 1)(q^2 - q + 1)$
[6, 2]	$q(q - 1)(q^3 + q - 1)$
[6, 1 ²]	$q(q + 1)(q^3 + q - 1)$
[5, 3]	$q(q - 1)(q + 1)(q^2 + 1)$
[5, 2, 1]	$q(q - 1)(q + 1)(q^2 + 1)$
[5, 1 ³]	$q(q - 1)(q + 1)(q^2 + 1)$
[4 ²]	$q(q^4 - q^2 - 4)$
[4, 3, 1]	$(q - 1)q^2(q + 1)^2$
[4, 2 ²]	$(q - 1)(q - 2)(q + 1)q^2$
[4, 2, 1 ²]	$(q - 1)(q + 1)q^3$
[4, 1 ⁴]	$(q - 1)(q - 2)(q + 1)q^2$
[3 ² , 2]	$q(q - 1)(q^3 - q - 3)$
[3 ² , 1 ²]	$q(q + 1)(q^3 - q - 3)$
[3, 2 ² , 1]	$q(q - 1)(q - 2)(q + 1)^2$
[3, 2, 1 ³]	$(q + 1)q^2(q - 1)^2$
[3, 1 ⁵]	$q(q - 1)(q - 2)(q - 3)(q + 1)$
[2 ⁴]	$(q - 2)(q - 3)(q + 2)(q^2 - q - 4)$
[2 ³ , 1 ²]	$q(q - 2)(q + 1)(q^2 - q - 4)$
[2 ² , 1 ⁴]	$q(q - 1)(q + 1)(q - 2)^2$
[2, 1 ⁶]	$q(q - 1)(q - 2)(q - 3)(q - 4)$
[1 ⁸]	$(q - 2)(q - 3)(q - 4)(q - 5)(q - 6)$

We use σ_λ to denote any permutation in S_8 of cycle type λ

We mention that since both $H^k(\mathcal{Q}[2], \mathbb{C})$ and $H^k(\mathcal{H}_3[2], \mathbb{C})$ are of pure Tate type (k, k) , see [7], we have that their full mixed Hodge structure can be determined from Theorems 1.1 and 1.2. More precisely, replacing t by uv^2 in the polynomials gives the corresponding Poincaré–Serre polynomials. Here u encodes the cohomological degree and v encodes the Hodge weight.

9 The total moduli space

By adding the results of $\mathcal{Q}[2]$ and $\mathcal{H}_3[2]$ we find the S_7 -equivariant point count of $\mathcal{M}_3[2]$.

Table 4 The S_8 -equivariant point count of $\mathcal{H}_3[2]$

λ	$ \mathcal{H}_3[2]^{F \cdot \sigma_\lambda} $
[8]	$2q^5 + 2q^3$
[7,1]	$q^5 + q^4 + q^3 + q^2 + q + 1$
[6,2]	$3q^5 + 3q^3 - 6q^2 - 3q^4 + 3q$
[6, 1 ²]	$q^5 + q^4 + q^3 - q$
[5,3]	$q^5 - q$
[5,2,1]	$q^5 - q$
[5, 1 ³]	$q^5 - q$
[4 ²]	$4q^5 - 16q - 4q^3$
[4,3,1]	$2q^5 + 2q^4 - 2q^3 - 2q^2$
[4, 2 ²]	$6q^5 + 12q^2 - 12q^4 - 6q^3$
[4, 2, 1 ²]	$2q^5 - 2q^3$
[4, 1 ⁴]	$2q^5 - 4q^4 - 2q^3 + 4q^2$
[3 ² , 2]	$q^5 - q^4 - q^3 - 2q^2 + 3q$
[3 ² , 1 ²]	$3q^5 + 3q^4 - 3q^3 - 12q^2 - 9q$
[3, 2 ² , 1]	$2q^5 - 2q^4 - 6q^3 + 2q^2 + 4q$
[3, 2, 1 ³]	$4q^5 - 4q^4 - 4q^3 + 4q^2$
[3, 1 ⁵]	$6q^5 - 30q^4 + 30q^3 + 30q^2 - 36q$
[2 ⁴]	$12q^5 + 48q - 60q^3 + 336q^2 - 48q^4 - 576$
[2 ³ , 1 ²]	$4q^5 - 8q^4 - 20q^3 + 24q^2 + 32q$
[2 ² , 1 ⁴]	$8q^5 - 32q^4 + 24q^3 + 32q^2 - 32q$
[2, 1 ⁶]	$16q^5 - 160q^4 + 560q^3 - 800q^2 + 384q$
[1 ⁸]	$36q^5 - 720q^4 + 5580q^3 - 20880q^2 + 37584q - 25920$

We use σ_λ to denote any permutation in S_8 of cycle type λ

Theorem 9.1 *The S_7 -equivariant point count of $\mathcal{M}_3[2]$ is given by*

$$\begin{aligned}
& s_7 q^6 + s_7 q^5 \\
& - (2s_7 + 4s_{6,1} + 5s_{5,2} + s_{5,1^2} + 2s_{4,3} + 2s_{4,2,1} + s_{3,2^2}) q^4 \\
& + (2s_7 + 10s_{6,1} + 16s_{5,2} + 13s_{5,1^2} + 11s_{4,3} + 20s_{4,2,1} + 4s_{4,1^3} \\
& \quad + 10s_{3^2,1} + 7s_{3,2^2} + 8s_{3,2,1^2} + s_{3,1^4} + s_{2^3,1} + s_{2^2,1^3}) q^3 \\
& - (2s_7 + 15s_{6,1} + 30s_{5,2} + 33s_{5,1^2} + 28s_{4,3} + 63s_{4,2,1} + 30s_{4,1^3} + 36s_{3^2,1} \\
& \quad + 29s_{3,2^2} + 46s_{3,2,1^2} + 11s_{3,1^4} + 14s_{2^3,1} + 11s_{2^2,1^3} + 4s_{2,1^5}) q^2 \\
& + (2s_7 + 15s_{6,1} + 38s_{5,2} + 41s_{5,1^2} + 37s_{4,3} + 93s_{4,2,1} + 53s_{4,1^3} + 53s_{3^2,1} \\
& \quad + 53s_{3,2^2} + 85s_{3,2,1^2} + 34s_{3,1^4} + 31s_{2^3,1} + 28s_{2^2,1^3} + 9s_{2,1^5} + s_{1^7}) q \\
& + s_7 - 8s_{6,1} - 21s_{5,2} - 23s_{5,1^2} - 18s_{4,3} - 53s_{4,2,1} - 29s_{4,1^3} - 32s_{3^2,1} \\
& \quad - 31s_{3,2^2} - 54s_{3,2,1^2} - 23s_{3,1^4} - 18s_{2^3,1} - 22s_{2^2,1^3} - 9s_{2,1^5}.
\end{aligned}$$

Table 5 The S_7 -equivariant point count of $\mathcal{H}_3[2]$

λ	$ \mathcal{H}_3[2]^{F \cdot \sigma_\lambda} $
[7]	$q^5 + q^4 + q^3 + q^2 + q + 1$
[6,1]	$q^5 + q^4 + q^3 - q$
[5,2]	$q^5 - q$
[5, 1 ²]	$q^5 - q$
[4,3]	$2q^5 + 2q^4 - 2q^3 - 2q^2$
[4,2,1]	$2q^5 - 2q^3$
[4, 1 ³]	$2q^5 - 4q^4 - 2q^3 + 4q^2$
[3 ² , 1]	$3q^5 + 3q^4 - 3q^3 - 12q^2 - 9q$
[3, 2 ²]	$2q^5 - 2q^4 - 6q^3 + 2q^2 + 4q$
[3, 2, 1 ²]	$4q^5 - 4q^4 - 4q^3 + 4q^2$
[3, 1 ⁴]	$6q^5 - 30q^4 + 30q^3 + 30q^2 - 36q$
[2 ³ , 1]	$4q^5 - 8q^4 - 20q^3 + 24q^2 + 32q$
[2 ² , 1 ³]	$8q^5 - 32q^4 + 24q^3 + 32q^2 - 32q$
[2, 1 ⁵]	$16q^5 - 160q^4 + 560q^3 - 800q^2 + 384q$
[1 ⁷]	$36q^5 - 720q^4 + 5580q^3 - 20880q^2 + 37584q - 25920$

We use σ_λ to denote any permutation in S_7 of cycle type λ

Table 6 The cohomology of $\mathcal{H}_3[2]$ as a representation of S_7

	s_7	$s_{6,1}$	$s_{5,2}$	$s_{5,1^2}$	$s_{4,3}$	$s_{4,2,1}$	$s_{4,1^3}$	$s_{3^2,1}$	$s_{3,2^2}$	$s_{3,2,1^2}$
H^0	2	1	1	0	1	0	0	0	0	0
H^1	2	7	9	5	5	7	1	3	2	1
H^2	3	18	30	31	25	50	20	26	19	26
H^3	6	35	74	80	72	162	86	92	83	129
H^4	8	48	114	117	109	271	150	157	158	254
H^5	5	31	72	77	72	180	103	108	108	180
	$s_{3,1^4}$		$s_{2^3,1}$		$s_{2^2,1^3}$		$s_{2,1^5}$		s_{1^7}	
H^0	0		0		0		0		0	
H^1	0		0		0		0		0	
H^2	5		7		4		0		0	
H^3	43		45		36		10		1	
H^4	105		96		92		35		4	
H^5	77		72		72		31		5	

Note in particular that the representations in the coefficients of each power of q occur with the same sign, except in the constant term where s_7 occurs with coefficient 1 while all others occur with negative coefficients. Thus, in contrast with $\mathcal{Q}[2]$ and $\mathcal{H}_3[2]$, we cannot have that $H^k(\mathcal{M}_3[2], \mathbb{C})$ is pure of Tate type (k, k) . Moreover, Looijenga [25]

shows that $H^k(\mathcal{M}_3, \mathbb{C})$ is pure of Tate type (i, i) for some i which is sometimes different from k . In our case, neither this weaker purity holds.

To go a bit further in the investigation of the cohomology of $\mathcal{M}_3[2]$, we recall the following result.

Lemma 9.2 (Looijenga [25]) *Let X be a rational homology manifold (see [9]) of pure dimension and let $Y \subset X$ be a hypersurface. Then there is a Gysin exact sequence of mixed Hodge structures*

$$\cdots \rightarrow H^{k-2}(Y, \mathbb{C})(-1) \rightarrow H^k(X, \mathbb{C}) \rightarrow H^k(X \setminus Y, \mathbb{C}) \rightarrow H^{k-1}(Y, \mathbb{C})(-1) \rightarrow \cdots$$

Recall that $H^k(\mathcal{Q}[2], \mathbb{C})$ and $H^k(\mathcal{H}_3[2], \mathbb{C})$ are both pure of Tate type (k, k) . Thus, if we apply Lemma 9.2 to $X = \mathcal{M}_3[2]$, $Y = \mathcal{H}_3[2]$ and $X \setminus Y = \mathcal{Q}[2]$ we have that the long exact sequence splits into four term sequences

$$\begin{aligned} 0 \rightarrow \mathrm{gr}_{2k}^W H^k(X, \mathbb{C}) &\rightarrow H^k(X \setminus Y, \mathbb{C}) \\ &\rightarrow H^{k-1}(Y, \mathbb{C})(-1) \rightarrow \mathrm{gr}_{2k}^W H^{k+1}(X, \mathbb{C}) \rightarrow 0, \end{aligned}$$

where $\mathrm{gr}_{2i}^W H^k(X, \mathbb{C})$ denotes the weight $2i$ graded quotient of $H^k(X, \mathbb{C})$ (which in this case is of Tate type (i, i)). We may thus conclude the following.

Proposition 9.3 *The cohomology group $H^k(\mathcal{M}_3[2], \mathbb{C})$ consists of one part of Tate type $(k-1, k-1)$ and one part of Tate type (k, k) .*

Moreover, let $m_X^k(\lambda)$ denote the multiplicity of s_λ in $H^k(X)$ and let $n^k(\lambda) = m_{\mathcal{Q}[2]}^k(\lambda) - m_{\mathcal{H}_3[2]}^{k-2}(\lambda)$. If $n^k(\lambda) \geq 0$, then s_λ occurs with multiplicity at least $n^k(\lambda)$ in $W_k H^k(\mathcal{M}_3[2])$ and if $n^k(\lambda) \leq 0$, then s_λ occurs with multiplicity at least $-n^k(\lambda)$ in $W_k H^{k+1}(\mathcal{M}_3[2])$. Thus, Tables 2 and 6 provide explicit bounds for the cohomology groups of $\mathcal{M}_3[2]$. We note that a sharper bound for the dimension of $H^7(\mathcal{M}_3[2], \mathbb{C})$ than that obtained here can be found in recent work of Fullarton and Putman [16]. However, the observation above gives bounds for all cohomology groups.

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