European Journal of Mathematics (2018) 4:1278–1292
https://doi.org/10.1007/s40879-018-0272-7

RESEARCH ARTICLE

Equivariant birational geometry of quintic del Pezzo surface

Jonas Wolter[1]

Received: 4 April 2018 / Revised: 1 June 2018 / Accepted: 30 July 2018 / Published online: 13 August 2018

Abstract
We prove that there are exactly two G-minimal surfaces which are G-birational to the quintic del Pezzo surface, where $G \cong C_5 \rtimes C_4$. These surfaces are the quintic del Pezzo surface itself and the surface $\mathbb{P}^1 \times \mathbb{P}^1$.

Keywords G-birational rigidity · del Pezzo surface · Cremona group

Mathematics Subject Classification 14E05 · 14E07 · 14J45

1 Introduction

The study of finite subgroups of the *Cremona group* is classical, but the first serious treatment has been done by Igor V. Dolgachev and Vasily A. Iskovskikh at the beginning of this century, starting with Iskovskikh's paper [11]. In their seminal work [7] all finite subgroups of the Cremona group $\mathrm{Cr}_2(\mathbb{C})$ are classified up to isomorphism. In the section "What is left" in [7] it is stated that not all conjugacy classes of $\mathrm{Cr}_2(\mathbb{C})$ are known and that a finer description of the the conjugacy classes would be desirable.

Let us recall from [11] that two subgroups of the Cremona group given by the biregular actions of a finite group G on two rational surfaces are conjugate if there exists a G-birational map $S_1 \dashrightarrow S_2$. By general theory such a map can be factorised into elementary links [9]. In this paper we will contribute to the open questions from [7] by proving:

Theorem 1.1 *Let S_5 be the smooth del Pezzo surface of degree 5, and let $G_{20} \cong C_5 \rtimes C_4$ be a subgroup of order 20 in $\mathrm{Aut}(S_5)$. Then $\mathrm{Pic}^{G_{20}}(S_5) = \mathbb{Z}$ and*

1. *S_5 is not G_{20}-birational to any conic bundle,*
2. *there exists a unique G-minimal del Pezzo surface which is G_{20}-birational to S_5, that is $\mathbb{P}^1 \times \mathbb{P}^1$,*
3. *the group of G_{20}-birational automorphisms is given by $\mathrm{Bir}^{G_{20}}(S_5) = C_2 \times G_{20}$.*

✉ Jonas Wolter
s1508663@sms.ed.ac.uk

[1] School of Mathematics, The University of Edinburgh, Edinburgh EH9 3FD, UK

Here C_n is a cyclic group of order n. It should be noticed that there are no G-conic fibrations birational to S_5.

In the notation of [1] we can say that S_5 is G_{20}-solid.

Remark 1.2 In the proof of Theorem 1.1 we will also see that the only smooth del Pezzo surfaces G-birational to S_5 are $\mathbb{P}^1 \times \mathbb{P}^1$ and the Clebsch cubic surface. But the latter is not G_{20}-minimal, i.e., $\mathrm{Pic}^{G_{20}}(\widetilde{S}) \neq \mathbb{Z}$. Indeed we will show in Remark 5.2, that its G_{20}-invariant Picard group is $\mathbb{Z}^2$.

Throughout this paper we assume all varieties to be complex and projective. For all notation in birational geometry, such as *G-biregular*, we use the conventions introduced in [7].

2 *G*-Sarkisov links

We will dedicate this section to the introduction of the notion of *G-Sarkisov links* where G is a finite group. For simplicity we will only consider the dimension 2 here. For a more detailed study see [5]. This language will allow us to state Theorem 1.1 in a more precise and technical way. We will firstly define a G-Mori fibre space.

Definition 2.1 A 2-dimensional *G-Mori fibre space* is

DP: a smooth G-minimal del Pezzo surface S, i.e., $\mathrm{Pic}^G(S) = \mathbb{Z}$;
CB: a G-conic bundle, i.e., a G-equivariant morphism $\pi : S \to \mathbb{P}^1$, where S is a smooth surface and the general fibre of π is $\mathbb{P}^1$ such that $\mathrm{Pic}^G(S) = \mathbb{Z}^2$.

The main result about 2-dimensional G-Sarkisov link is the following:

Theorem 2.2 ([5]) *Let S, S' be 2-dimensional G-Mori fibre spaces and let $\chi : S \dashrightarrow S'$ be a non-biregular G-birational map. Then χ is a composition of elementary links known as* G-Sarkisov links.

There are five different G-Sarkisov links of dimension 2 which are described below. The first type is given by

$$\begin{tikzcd} & \widehat{S} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ S & & S' \end{tikzcd} \tag{I}$$

where S and S' are G-minimal del Pezzo surfaces and α and β are blow-ups of G-orbits in S and S' respectively. The second type is given by

$$\begin{tikzcd} & \widehat{S} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ S & & \mathbb{P}^1 \end{tikzcd} \tag{II}$$

where S is a G-minimal del Pezzo surface, α is a blow-up of a G-orbit and β is a G-conic bundle. The third type is given by

$$\begin{tikzcd} & \widehat{S} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathbb{P}^1 & & S' \end{tikzcd} \tag{III}$$

where S' is a G-minimal del Pezzo surface, β is a blow-up of a G-orbit and α is a G-conic bundle. We shall notice that this is the inverse link of type (II). The fourth type is given by

$$\begin{tikzcd} & \widehat{S} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{tikzcd} \tag{IV}$$

where α and β are G-conic bundles. Finally, the fifth type is given by

$$\begin{tikzcd} & \widehat{S} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ S \arrow[d, "\pi"'] & & S' \arrow[d, "\pi'"] \\ \mathbb{P}^1 \arrow[rr, equal] & & \mathbb{P}^1 \end{tikzcd} \tag{V}$$

where S and S' are not G-minimal del Pezzo surfaces and α and β are blow-ups of G-orbits in S and S' respectively. Additionally, π and π' are G-conic bundles and we call the whole link an *elementary transformation* of G-conic bundles (see [9]). This diagram commutes.

The notion of G-Sarkisov links is a good way to replace the technical result of the Noether–Fano inequality (see [6,10]).

Remark 2.3 It follows from the definition of G-links that $\widehat{S}$ is a del Pezzo surface if S is a del Pezzo surface. Thus in the links of type (I), (II), (III) and (IV), the surface $\widehat{S}$ is a del Pezzo surface.

Using the notion of G-Sarkisov links we are able to restate Theorem 1.1.

Theorem 2.4 *Let S_5 be the smooth del Pezzo surface of degree* 5*, and let $G_{20} \cong C_5 \rtimes C_4$ be a subgroup of order* 20 *in* $\mathrm{Aut}(S_5)$*. Then* $\mathrm{Pic}^{G_{20}}(S_5) = \mathbb{Z}$ *and the following assertions hold:*

1. *There exists a unique G_{20}-Sarkisov link that starts at S_5. It is given by*

$$\begin{tikzcd} & \widetilde{S} \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\ S_5 & & \mathbb{P}^1 \times \mathbb{P}^1 \end{tikzcd} \tag{1}$$

where π is the blow-up of the unique G_{20}-orbit of length 2 in S_5, σ is a blow-up of one of two G_{20} orbits of length 5 and $\widetilde{S}$ is the Clebsch cubic surface.

2. *Let $\mathbb{P}^1 \times \mathbb{P}^1$ be equipped with the G_{20}-action coming from (1). Then the only G_{20}-Sarkisov links starting from $\mathbb{P}^1 \times \mathbb{P}^1$ are the inverse of (1) and*

$$\begin{tikzcd} & \widetilde{S} \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & \\ \mathbb{P}^1 \times \mathbb{P}^1 & & S_5 \end{tikzcd} \tag{2}$$

where γ is the blow-up of another G_{20}-orbit of length 5, $\widetilde{S}$ is the Clebsch cubic surface and $\delta = \pi$ is the blow-up of the unique G_{20}-orbit of length 2 in S_5.

Combining (1) and (2) yields a non-biregular G_{20}-birational map $S_5 \dashrightarrow S_5$.

These links were constructed and described numerically by Dolgachev and Iskovskikh in [7, Proposition 7.13] but for our purposes we reconstruct them here and will fill in the details for these links.

3 Motivation

In this section, we want to motivate Theorem 1.1. There are various different starting points to investigate conjugacy in the Cremona group. We decided to start our research on del Pezzo surfaces. Those surfaces have been introduced by Pasquale del Pezzo in the late 18th century and since then various ways of studying them have been encountered.

For our purposes we will understand a del Pezzo surface of degree d, denoted by S_d, as the blow-up of $\mathbb{P}^2$ in $9 - d$ points in general position. To start our investigation of conjugacy classes of the Cremona group we need to introduce the notion of G-birational (super-) rigidity.

Definition 3.1 Let S be a smooth del Pezzo surface and $G \subset \mathrm{Aut}(S)$ be a finite group such that $\mathrm{Pic}^G(S) = \mathbb{Z}$. We say S is *G-birationally rigid* whenever

(a) if S is G-birational to any G-minimal del Pezzo surface S', then S' is G-biregular to S, and
(b) S is not G-birational to any G-conic bundle.

Condition (a) is equivalent to saying that for any birational G-map $\chi\colon S \dashrightarrow S'$, where S' is a G-minimal del Pezzo surface, there exists a G-birational automorphism $\theta\colon S \dashrightarrow S$ such that $\chi \circ \theta$ is a G-isomorphism.

Definition 3.1 means, that the only G-Sarkisov links starting in S are of the form

$$\begin{tikzcd} & \widehat{S} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ S \arrow[rr, dashed, "\phi"'] & & S \end{tikzcd} \qquad (\bigstar)$$

where α and β are blow-ups of G-orbits.

Definition 3.2 Let S be a smooth del Pezzo surface and $G \subset \mathrm{Aut}(S)$ be a finite group such that $\mathrm{Pic}^G(S) = \mathbb{Z}$. The surface S is *G-birationally superrigid* if it is G-birationally rigid and $\mathrm{Bir}^G(S) = \mathrm{Aut}^G(S)$.

Definition 3.2 means that there are no G-Sarkisov links starting at S. With these definitions in hand we are able to state.

Theorem 3.3 ([7]) *Let S be a smooth del Pezzo surface of degree d, that is $K_S^2 = d$, and let $G \subseteq \mathrm{Aut}(S)$ be such that $\mathrm{Pic}^G(S) = \mathbb{Z}$. Then the following assertions hold:*

(a) *If S does not contain a G-orbit of length less then d, then S is G-birationally superrigid.*
(b) *If S does not contain a G-orbit of length less then $d-2$, then S is G-birationally rigid.*

Sketched Proof Assume $\widehat{S}$ is a smooth del Pezzo surface. Then by Remark 2.3, $\alpha\colon S \to \widehat{S}$ is a blow-up of a G-orbit of length less then d, because $K_{\widehat{S}} > 0$. This proves (a).

If there is a G-orbit of length $d-1$, the blow-up of this orbit is $\widehat{S} = S_1$, the del Pezzo surface of degree 1, so we can use the Bertini involution there. Similarly, if there exists a G-orbit of length $d-2$, the del Pezzo surface of degree 2, we can blow up this orbit to obtain $\widehat{S} = S_2$, and we can use the Geiser involution. This proves (b). □

From Theorem 3.3 we can immediately deduce the following corollary.

Corollary 3.4 ([7]) *Let S be a smooth del Pezzo surface of degree $d < 3$, and let $G \subseteq \mathrm{Aut}(S)$ be a finite group such that $\mathrm{Pic}^G(S) = \mathbb{Z}$. If S is of degree 1, then S is G-birationally superrigid. If S is of degree 2 or 3, then S is G-birationally rigid.*

This result is known for quite some time and was implicitly proven by Segre in 1943 and Manin in 1962. For proofs of Theorem 3.3 and Corollary 3.4 see [7, Section 7.1]. The proof of Theorem 3.3 easily implies

Theorem 3.5 ([7]) *Let S be a smooth del Pezzo surface of degree 4, and let $G \subset \mathrm{Aut}(S)$ be a finite group such that $\mathrm{Pic}^G(S) = \mathbb{Z}$. Then*

1. *if there are no G-fixed points, then S is G-birational rigid,*
2. *if there exists a G-fixed point, then there exists a G-Sarkisov link*

$$\begin{tikzcd} & \widetilde{S} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ S_4 \arrow[rr, dashed] & & \mathbb{P}^1 \end{tikzcd}$$

where α is the blow-up of a G-orbit, $\widetilde{S}$ is a smooth cubic surface and β is a conic bundle.

In this paper, we are mostly interested in G-birational rigid del Pezzo surfaces or those which are close to them. By *close* we mean that these are del Pezzo surfaces which are not G-birational to any conic bundle (in the language of [1] these are G-solid del Pezzo surfaces).

Following Corollary 3.4 and Theorem 3.5 we will investigate links starting from the smooth del Pezzo surface of degree 5, which we will call S_5, in this paper. It is well known that

$$\mathrm{Aut}(S_5) \cong \mathfrak{S}_5,$$

the symmetric group of five elements. A proof is provided in [4] . If we want S_5 to be a G-minimal surface (i.e., $\mathrm{Pic}^G(S_5) = \mathbb{Z}$), we require G to be one of the following (see [7, Theorem 6.4]):

- the symmetric group $\mathfrak{S}_5$ of five elements of order 120;
- the alternating group $\mathfrak{A}_5$ of five elements of order 60;
- the semidirect product $G_{20} \cong C_5 \rtimes C_4$ of order 20;
- the dihedral group D_{10} of order 10;
- the cyclic group C_5 of order 5.

For $\mathfrak{S}_5$ and $\mathfrak{A}_5$ the quintic del Pezzo surface is G-birationally superrigid (see [2]). For C_5 there exists a G-birational map from S_5 to $\mathbb{P}^2$ (see [3]) such that C_5 has a fixed point there (see Lemma 4.1). The construction of this map can be generalised for D_{10} which is done in Corollary 4.2. Hence these groups are better addressed when studying the G-equivariant birational geometry of $\mathbb{P}^2$. This has been done in [12]. We shall also notice that S_5 is not G-solid in this case.

In this paper we will therefore focus on the group $G_{20} \cong C_5 \rtimes C_4$ as a subgroup of $\mathrm{Aut}(S_5) \cong \mathfrak{S}_5$, which is also known as the general affine group of degree 1 over the field with five elements, denoted by $\mathrm{GA}(1, 5)$.

4 The quintic del Pezzo surface

In the proof of Theorem 2.4 we will investigate the existence of G_{20}-equivariant birational maps between quintic del Pezzo surface, denoted by S_5, and the surface $\mathbb{P}^1 \times \mathbb{P}^1$. First we need to understand the action of G_{20} on S_5. To do this we use the following result from [3].

Lemma 4.1 ([3]) *There is a C_5-birational map ϕ (i.e., a C_5-Sarkisov link) between S_5 and $\mathbb{P}^2$ given by the C_5-commutative diagram*

$$\begin{tikzcd} & S_4 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ S_5 \arrow[rr, dashed, "\phi"'] & & \mathbb{P}^2. \end{tikzcd} \qquad (\blacktriangle)$$

Here α is the blow-up of a C_5-fixed point in S_5, and β is the blow-up of five points in $\mathbb{P}^2$ which form a C_5-orbit. S_4 is a quartic del Pezzo surface.

Proof For the proof we will start with $\mathbb{P}^2$ and invert the link ($\blacktriangle$). Consider C_5 as a subgroup of $\mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}_2(C)$. There exists a C_5-equivariant Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ which defines a faithful action of C_5 on $\mathbb{P}^2$ such that there exists a C_5-invariant conic $K \subseteq \mathbb{P}^2$ (that is the image of $\mathbb{P}^1$). Thus we can blow up the C_5-orbit of length 5 on this conic to obtain the quartic del Pezzo surface, denoted by S_4.

If we contract the the proper transform of K there we get the unique quintic del Pezzo surface. Since $C_5 \subseteq \mathfrak{S} \cong \mathrm{Aut}(S_5)$ is unique up to conjugation the composition of the two described maps yields the desired link ϕ.

In more elementary terms, we may say that five points $P_1, \ldots, P_5$ in general position in $\mathbb{P}^2$ always lie on a unique conic K. Then the group C_5 fixes two points A_1, A_2 on a conic [4] (i.e., the line through these two points is C_5-invariant). Additionally, it fixes a point $B \in \mathbb{P}^2$ which does not lie on the conic. The blow-up α of $P_1, \ldots, P_5$ does not affect B, neither does the contraction β. Thus there is a point

$$Q_2 = \phi^{-1}(B) \in S_5,$$

which is fixed by C_5. We know that $\alpha^{-1}(K)$ is a β-exceptional curve in S_4. After the contraction β, we have

$$\phi^{-1}(A_1) = \phi^{-1}(A_2) = Q_1,$$

which is another fixed point of C_5 in S_5. Thus we know that for $C_5 \subseteq \mathrm{Aut}(S_5)$ there exist two C_5-fixed points Q_1 and Q_2. We shall mention that all other orbits are of length 5. $\square$

From the proof of Lemma 4.1 we can easily deduce.

Corollary 4.2 *There is a D_{10}-birational map ϕ (i.e., a D_{10}-Sarkisov link) between S_5 and $\mathbb{P}^2$ corresponding to the D_{10}-commutative diagram ($\blacktriangle$).*

Proof In the same way as in the proof of Lemma 4.1 we can construct the inverse link of ($\blacktriangle$). Furthermore the action of $D_{10} \subseteq \mathrm{Aut}(\mathbb{P}^1)$ lifts to an action on $\mathbb{P}^2$. Then we can use the same argument as before.

In the notation of the proof of Lemma 4.1 we may say that the action of D_{10} on $\mathbb{P}^2$ interchanges the points A_1 and A_2 but fixes the point B. Thus we can use the same link ϕ as in Lemma 4.1 and by the same argument as above D_{10} fixes points Q_1 and Q_2 in S_5. □

We are now in the position to investigate orbits of small length $r < 5$ of the G_{20}-action on S_5.

We want to proceed in a similar way as in [11] which means that we need to classify all G-orbits of length $r < \deg(S_5) = \mathrm{K}_{S_5}^2 = 5$. Then we will concentrate on those orbits of which the points are in general position, because this is a necessary condition for the existence of links starting from the surface S_5.

Remark 4.3 We say that points of an orbit are *in general position* if the blow-up of S_5 in this orbit is a del Pezzo surface again.

Lemma 4.4 *There is a unique G_{20}-orbit of length $r < 5$ on S_5. It is the orbit of length $r = 2$ consisting of the points Q_1 and Q_2.*

Proof Let us consider all possible lengths for orbits.

$r = 1$: Such an orbit does not exist. Assume it does. By Lemma 4.1 and Corollary 4.2, this point can only be Q_1, because if all of G_{20} fixes it, the normal subgroups C_5 and D_{10} fix it in particular. Hence the link (▲) yields G_{20}-equivariant link from S_5 to $\mathbb{P}^2$. This means that G_{20} acts on $\mathbb{P}^2$ and preserves the conic K. This implies that G_{20} acts faithfully on $K \cong \mathbb{P}^1$, but this is clearly a contradiction. Hence no orbit of length $r = 1$ exists.

$r = 2$: $\{Q_1, Q_2\}$ is such an orbit. We know that G_{20} has D_{10} as a normal subgroup. If we consider the action of D_{10} on S_5, then Corollary 4.2 tells us that there is indeed a unique orbit of length 2 which is the orbit $\{Q_1, Q_2\}$.

$r = 3$: Such an orbit does not exist because $3 \nmid 20 = |G_{20}|$, which is required by the orbit-stabilizer theorem.

$r = 4$: Such an orbit does not exist. If there were such an orbit the stabilizer would satisfy $\mathrm{Stab}_G = C_5$ but we know that C_5 actually fixes the same points as D_{10} by Theorem 4.2 and hence the stabilizer would actually be D_{10} which cannot give an orbit of length 4. □

Lemma 4.4 implies that the only possible G_{20}-Sarkisov link starting from S_5 consists of a blow-up of the described orbit of length $r = 2$.

Lemma 4.5 *The blow-up of Q_1 and Q_2 in S_5 yields a smooth del Pezzo surface $\widetilde{S}$.*

Proof We need to prove that $-K_{\widetilde{S}}$ is ample. This is equivalent to saying that Q_1 and Q_2 neither lie on the (-1)-curves nor in an exceptional conic in S_5. We prove this by contradiction. For this we will consider different cases.

- We first prove that there are no (-1)-curves containing Q_1 or Q_2. Assume Q_1 lies on one of the 10 exceptional curves in S_5. Clearly Q_2 needs to lie on such a curve as well. If they lie on two different exceptional curves these two are interchanged by the group action of G_{20}. This contradicts the fact that $\mathrm{Pic}^{G_{20}}(S_5) = \mathbb{Z}$.
  Similarly, we may assume that Q_1 lies on one of the intersections of two exceptional

curves. Again this contradicts $\mathrm{Pic}^{G_{20}}(S_5) = \mathbb{Z}$. So indeed Q_1 and Q_2 do not lie on the (-1)-curves in S_5 which proves that the blow-up of these two points yields another del Pezzo surface.

- It remains to show that Q_1 and Q_2 are not contained in an exceptional conic S_5. There are five classes of conic in S_5 and each of them has self-intersection $C^2 = 0$. Going through all these cases in detail one can show that Q_1 and Q_2 either lie on one line which we ruled out previously or they cannot lie on one conic. Due to heavy computational work we omit the different cases at this point. □

The resulting surface of this blow-up will have degree $5 - 2 = 3$, so it is a cubic surface. The only smooth cubic surface with a G_{20}-action is the *Clebsch cubic surface* (this was proved in [8]) which we will investigate in the next section.

5 The Clebsch cubic surface

Theorem 2.4 states that the only G-Sarkisov links starting from the quintic del Pezzo surface S_5 are of the form

$$\begin{tikzcd} & \widetilde{S} \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\ S_5 \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^1 \times \mathbb{P}^1. \end{tikzcd} \qquad (\blacklozenge)$$

From Lemma 4.4 we know that π is the blow-up of the unique G_{20}-orbit $\{Q_1, Q_2\}$ of length 2. Hence $\widetilde{S}$ is the Clebsch cubic surface, which is defined as follows:

Definition 5.1 The *Clebsch cubic surface*, denoted by $\widetilde{S}$, is a cubic given by two defining equations in $\mathbb{P}^4$:

$$\begin{cases} x_0 + x_1 + x_2 + x_3 + x_4 = 0; \\ x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0. \end{cases}$$

Remark 5.2 In [7] it is shown that $\mathrm{Pic}^{G_{20}}(\widetilde{S}) \neq \mathbb{Z}$. The link ($\blacklozenge$) proves that in fact, $\mathrm{Pic}^{G_{20}}(\widetilde{S}) = \mathbb{Z}^2$.

Now it is well known that the automorphism group of the Clebsch cubic surface is $\mathfrak{S}_5$. Thus the action of G_{20} can be described very explicitly by understanding G_{20} as a subgroup of $\mathfrak{S}_5$ acting by permutation on the coordinates of this surface.

We know that all representations of G_{20} are conjugate to each other and thus we will use a generation by $\sigma_{(12345)}$ and $\sigma_{(2354)}$, where we use the notation introduced in [4]. Considering orbits of length 4 on $\widetilde{S}$ we obtain

Lemma 5.3 *There is a unique orbit of length* 4 *of the* G_{20}*-action on the Clebsch cubic surface given by the points*

$$\mathcal{O} = \left\{(1:\zeta:\zeta^2:\zeta^3:\zeta^4), (1:\zeta^2:\zeta^4:\zeta:\zeta^3), (1:\zeta^3:\zeta:\zeta^4:\zeta^2), (1:\zeta^4:\zeta^3:\zeta^2:\zeta)\right\},$$

with ζ being a primitive fifth root of unity.

Proof An orbit of length 4 has the stabilizer $\mathrm{Stab}_G = K \cong C_5$ which is isomorphic to the group generated by $\sigma_{(12345)}$, which is the unique subgroup of G_{20} isomorphic to C_5. It has exactly the fixed points as stated in Lemma 5.3, which are obtained by straightforward calculations. It is easy to verify that these four points lie indeed on $\widetilde{S}$ and form an orbit of length 4. □

The orbits of length 5 are a bit more sophisticated.

Lemma 5.4 *There are three orbits of length* 5 *of the G_{20}-action on the Clebsch cubic surface given by:*

$$\begin{aligned}
\mathcal{O}_1 = \big\{&V_1 = (0:-1:1:1:-1), V_2 = (-1:0:-1:1:1), V_3 = (1:-1:0:-1:1),\\
&V_4 = (1:1:-1:0:-1), V_5 = (-1:1:1:-1:0)\big\},\\
\mathcal{O}_2 = \big\{&U_1 = (0:-i:-1:1:i), U_2 = (i:0:-i:-1:1), U_3 = (1:i:0:-i:-1),\\
&U_4 = (-1:1:i:0:-i), U_5 = (-i:-1:1:i:0)\big\},\\
\mathcal{O}_3 = \big\{&W_1 = (0:i:-1:1:-i), W_2 = (-i:0:i:-1:1), W_3 = (1:-i:0:i:-1),\\
&W_4 = (-1:1:-i:0:i), W_5 = (i:-1:1:-i:0)\big\}.
\end{aligned}$$

Proof An orbit of length 5 has the stabilizer $\mathrm{Stab}_G = H \cong C_4$ in G_{20}. There are five subgroups of G_{20} which are isomorphic to C_4. Let $H \cong C_4$ be the subgroup generated by $\sigma_{(2354)}$. Then H fixes four points in $\mathbb{P}^4$ with $\sum_{i=1}^{5} x_i = 0$ which are

$$\begin{aligned}
R_1 &= (0:-1:1:1:-1), & R_2 &= (0:-i:-1:1:i),\\
R_3 &= (0:i:-1:1:-i), & R_4 &= (-4:1:1:1:1),
\end{aligned}$$

whereas the R_4 does not lie on $\widetilde{S}$ because the cubes of the coordinates do not sum to zero. Again it is easy to verify that the points $(R_1, \dots, R_4)$ are indeed fixed points. Acting by an element of order 5, we obtain fixed points corresponding to the action of $\sigma_{(12345)}$ on the coordinates of R_i. Thus we deduce, that there are three orbits of length 5 on $\widetilde{S}$ as stated in Lemma 5.4. □

We shall notice that R_2 and R_3 lie on the line $x_1 + x_4 = x_2 + x_3 = 0$. Generalising this we make the following important observation.

Corollary 5.5 *The points $U_i \in \mathcal{O}_2$ and $W_i \in \mathcal{O}_3$ respectively lie on one of the* 27 *real lines on the Clebsch cubic surfaces. These five resulting lines in the link are:*

(i) $L_1 : x_1 + x_4 = x_2 + x_3 = 0$ *through U_1 and W_1.*
(ii) $L_2 : x_0 + x_2 = x_3 + x_4 = 0$ *through U_2 and W_2.*
(iii) $L_3 : x_0 + x_4 = x_1 + x_3 = 0$ *through U_3 and W_3.*
(iv) $L_4 : x_0 + x_1 = x_2 + x_4 = 0$ *through U_4 and W_4.*
(v) $L_5 : x_0 + x_3 = x_1 + x_2 = 0$ *through U_5 and W_5.*

It is easy to see that these five lines are disjoint.

Proof This is an easy exercise of calculating the lines through each pair of points and comparing it with the lines on the Clebsch cubic, which are well known. □

Lemmas 5.3 and 5.4 allow us to state the main result for this section.

Proposition 5.6 *Let $\widetilde{S}$ be the Clebsch cubic surface. Then the G_{20}-orbits of length $r < 8$ on $\widetilde{S}$ are:*

(a) *The unique orbit $\mathcal{O}$ described in Lemma 5.3 of length* 4.
(b) *The three orbits $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ described in Lemma 5.4 of length* 5.

Proof The orbit-stabilizer theorem tells us immediately that orbits of length $r = 6$ or $r = 7$ cannot exist. It remains to show that there are no orbits of length 1 or 2 on $\widetilde{S}$. This follows directly from our description of the orbits but we include computational explanation, too. An orbit of length 1 would have the whole group G_{20} as its stabilizer. We see immediately that this is not possible because the subgroups K and H generated by $\sigma_{(12345)}$ and $\sigma_{(2354)}$ have completely different fixed points.

By a similar argument, there cannot be any orbits of length 2. These would have the subgroup $F \cong D_{10}$ generated by $\sigma_{(12345)}$ and $\sigma_{(25)(34)}$ as its stabilizer. Again it is easy to verify that F has $K \cong C_5$ as a subgroup. On the other hand F has the group generated isomorphic to C_2 generated by $\sigma_{(25)(34)}$ which is a subgroup of H as a subgroup.

But we have seen that H and K do not have any common fixed points. Hence F cannot have fixed points which means that there does not exist an orbit of length 2. □

Remark 5.7 Proposition 5.6 supports the statement of Lemma 4.4. For the unique orbit $\mathcal{O}$ of length 4 each pair of points lies on one of 27 real lines on the Clebsch cubic. Hence after contracting two of them to obtain S_5, we are left with an orbit of length 2.

An orbit of length 4 in S_5 would lift to a different orbit of length 4 in $\widetilde{S}$, but for the given reason this cannot be $\mathcal{O}$, which means that there do not exist orbits of length 4 in the quintic del Pezzo surface.

Given Corollary 5.5, we may consider the contraction of these five lines.

Proposition 5.8 *The contraction of the five lines $L_1, \ldots, L_5$ described in Corollary 5.5 yields the surface $\mathbb{P}^1 \times \mathbb{P}^1$ and this is the only other contraction that can be conducted apart from the inverse of the blow-up from S_5.*

Proof We know that $\widetilde{S}$ is a del Pezzo surface, so $-K_{\widetilde{S}}$ is ample. Remark 2.3 tells us that the resulting surface of the described contraction will be a del Pezzo surface of degree $3 + 5 = 8$, so it can only be $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$, but $\mathrm{Pic}^{G_{20}}(\mathbb{F}_1) \neq \mathbb{Z}$, which we require.

In Remark 5.2 we have seen that $\mathrm{Pic}^{G_{20}}(\widetilde{S}) = \mathbb{Z}^2$. From this we conclude that there are two external rays in the Mori cone. We have shown that one consists of two lines and the other one of five. These are the only possible contraction of $\widetilde{S}$. □

Proposition 5.8 allows us to state the following lemma about the link (♦) which we introduced at the beginning of this section.

Lemma 5.9 *Considering the desired link (♦) from S_5 to $\mathbb{P}^1 \times \mathbb{P}^1$, we know*

1. *π is the contraction of two disjoint lines E_1, E_2 in the Clebsch cubic surface (respectively the blow-up of Q_1 and Q_2 in S_5).*
2. *σ is the contraction of five disjoint lines $F_1, \dots, F_5$ in the Clebsch cubic surface (respectively the blow-up of five points in $\mathbb{P}^1 \times \mathbb{P}^1$).*
3. *The following equations hold for the exceptional divisors:*

$$\sigma^*(H) = 2\pi^*(-K_{S_5}) - 3(E_1 + E_2),$$
$$\sum_{i=1}^{5} F_i = 3\pi^*(-K_{S_5}) - 5(E_1 + E_2),$$

where $-K_{S_5}$ is the anticanonical divisor of S_5, $E_1 + E_2$ are the two (-1)-curves of the blow-up of Q_1 and Q_2, H is a divisor of bidegree $(1, 1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and $\sum_{i=1}^{5} F_i$ are the (-1)-curves of the blow-up of five points in $\mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 5.9 implies that $\pi(F_i)$ is a smooth twisted cubic curve in S_5 and E_1 and E_2 are smooth twisted cubics of bidegree $(2, 1)$ and $(1, 2)$ respectively.

6 The surface $\mathbb{P}^1 \times \mathbb{P}^1$

The G_{20}-action on $\mathbb{P}^1 \times \mathbb{P}^1$ cannot be understood in a way which is as simple as in Sect. 4 or Sect. 5. For that reason, we will use our previous observations to analyse the G_{20}-orbits on $\mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 6.1 *There is a unique G_{20}-orbit $\mathcal{K}$ of length 4 in $\mathbb{P}^1 \times \mathbb{P}^1$. The four points are given by the intersections $F_{11} \cap F_{21}$, $F_{11} \cap F_{22}$, $F_{12} \cap F_{21}$ and $F_{12} \cap F_{22}$ of the four rulings F_{11}, F_{12}, F_{21} and F_{22}.*

Proof The orbit of length 4 described in Lemma 5.3 lies away from the lines of contraction, thus it has an embedding in $\mathbb{P}^1 \times \mathbb{P}^1$.

Generally, four points need eight different lines to describe them. A G_{20}-orbit of length 4 has the stabilizer C_5. But we know that $D_{10} \subset G_{20}$ acts on the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ which are copies of $\mathbb{P}^1$. Hence the C_5-action cannot split over each of the four points (i.e., interchanging the two lines) but fixes the rulings.

For this reason, the four points on the G_{20}-orbit need to lie on the four intersections of four copies of $\mathbb{P}^1$ (i.e., the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$), because otherwise it would not be an orbit of length 4. Hence all four points in this orbit lie on two rulings in $\mathbb{P}^1 \times \mathbb{P}^1$. □

Remark 6.2 We could prove Lemma 6.1 in a different way by considering the orbit of length 4 in the Clebsch cubic surface and considering their configuration there. We can show merely computationally that there exist four conics each passing through exactly one of the four points and not intersecting the (-1)-curves. Considering the blow-up σ we obtain Lemma 6.1 for the four points on $\mathbb{P}^1 \times \mathbb{P}^1$.

As in Sect. 5 the orbits of length 5 are a bit more difficult.

Lemma 6.3 *There are exactly two G_{20}-orbits $\mathcal{K}_1$ and $\mathcal{K}_2$ of length* 5 *in* $\mathbb{P}^1 \times \mathbb{P}^1$.

Proof We will now identify $\mathbb{P}^1 \times \mathbb{P}^1$ as a quadric $\mathcal{Q}$ in $\mathbb{P}^3$. We know that $\mathbb{P}^1 \times \mathbb{P}^1$ has a natural embedding (Segre) into $\mathbb{P}^3$. Similar to the Clebsch cubic we can understand $\mathbb{P}^3$ as a hyperplane in $\mathbb{P}^4$ with $\sum_{i=0}^4 x_i = 0$. Now let $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ be the quadric given by

$$\mathcal{Q}\colon \sum_{i=0}^{4} x_i = \sum_{i=0}^{4} x_i^2 = 0.$$

G_{20} acts on $\mathcal{Q}$ by permutations of coordinates in a similar way as described in the previous section for the Clebsch cubic.

We found the orbits of length 5 explicitly on $\widetilde{S}$. Observe that the orbits $\mathcal{O}_2$ and $\mathcal{O}_3$ lie in $\mathcal{Q}$, whereas $\mathcal{O}_1$ does not. Hence we may assume $\mathcal{K}_1 = \mathcal{O}_2$ and $\mathcal{K}_2 = \mathcal{O}_3$.

Additionally, we can check computationally that the points of each of the orbits lie in general position (i.e., no two on a line in $\mathcal{Q}$ and no four on a plane). Hence we can indeed consider the blow-up of each of these orbits which will yield the Clebsch cubic surface and together with Lemma 5.9 shows that this blow-up is indeed the inverse link of the described contraction.

Furthermore, we see that the orbits $\mathcal{O}_2$ and $\mathcal{O}_3$ are essentially the same orbits, only permuted by complex conjugation. In fact these two orbits are interchanged by an automorphism of a quadric which is proven in Theorem 7.3. □

Now we can finally state the last proposition we need for the proof of Theorem 1.1.

Proposition 6.4 *The only G_{20}-orbits of length $r < 8$ on $\mathbb{P}^1 \times \mathbb{P}^1$ are:*

(a) *The unique orbit $\mathcal{K}$ described in Lemma 6.1 of length* 4.
(b) *The two orbits $\mathcal{K}_1$ and $\mathcal{K}_2$ described in Lemma 6.3 of length* 5 *where the five points lie in general position.*

Proof It remains to show that there are no other orbits than the ones described in Lemmas 6.1 and 6.3.

Orbits of length 6, 7 or 8 cannot exist by the orbit-stabilizer theorem as $6, 7, 8 \nmid 20 = |G_{20}|$. Assume there is an orbit of length less than 4. Then an orbit of this length would also exist in the Clebsch cubic surface but Proposition 5.6 tells us, that they do not exist there. □

7 Proof of Theorem 2.4

The link (1) in Theorem 2.4 is the only G-Sarkisov link starting from the quintic del Pezzo surface S_5. From Lemma 4.4 we know that π is the blow-up of the unique G_{20}-orbit $\{Q_1, Q_2\}$ of length 2.

Now Corollary 5.5 tells us that there are five disjoint lines on the Clebsch cubic which we can contract to obtain $\mathbb{P}^1 \times \mathbb{P}^1$ and Proposition 6.4 says that we need to consider two different cases for birational maps starting from $\mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 7.1 (orbit of length 4) *Let $\tau\colon \widetilde{S} \to \mathbb{P}^1\times\mathbb{P}^1$ be the blow-up of the four points $P_1,\ldots,P_4$ in the orbit $\mathcal{K}$ and let $E_1,\ldots,E_4$ be the corresponding exceptional curves. Then the proper transforms of the described rulings are (-2)-curves. This means that the resulting surface $\widehat{S}$ is not a del Pezzo surface.*

Proof This follows immediately from Lemma 6.1. □

Lemma 7.1 tells us that we cannot continue from $\mathbb{P}^1\times\mathbb{P}^1$ to obtain a G-Sarkisov link by blowing up the orbit $\mathcal{K}$ of length 4.

Lemma 7.2 *Let $\tau\colon \widetilde{S} \to \mathbb{P}^1\times\mathbb{P}^1$ be a blow-up of the five points $P_1,\ldots,P_5$ in the orbits $\mathcal{K}_1$ or $\mathcal{K}_2$ respectively. Then one of the following holds for τ:*

(a) *τ is the same as the blow-up σ described in (♦), so that $\pi\circ\tau^* = \psi^{-1}$.*
(b) *τ is the same as the blow-up γ described in diagram (2) of Theorem 2.4, so that $(\psi\circ\tau^*\circ\pi) = \psi\circ\phi = \chi$ is a G_{20}-birational map $S_5 \dashrightarrow S_5$.*

Proof It is clear that we can obtain case (a) if we blow up the five points in $\mathcal{K}_1$ or $\mathcal{K}_2$, i.e., $\tau=\sigma$ in the link (♦). We get back exactly the model of the Clebsch cubic we had before because the elements in $\mathcal{K}_1$ or $\mathcal{K}_2$ are the points we obtained by the contraction described in Lemma 5.9. For symmetric reasons, we may assume that these are the points in $\mathcal{K}_1$. Therefore $\pi\circ\tau^* = (\sigma\circ\pi^*)^{-1} = \psi^{-1}$ as described in (♦).

Proposition 6.4 tells us that the two orbits $\mathcal{K}_1$ and $\mathcal{K}_2$ are interchanged by an automorphism. Let us now consider the blow-up $\tau\colon \mathbb{P}^1\times\mathbb{P}^1 \to \widetilde{S}$ of the orbit $\mathcal{K}_2$, which is not the same blow-up as σ. Then we may contract the two (-1)-curves, E_1 and E_2, on the Clebsch cubic.

This gives us back S_5 because the smooth quintic del Pezzo surface is unique. This means that $\phi\circ\tau^*\circ\pi$ is a birational map $S_5 \dashrightarrow S_5$. We obtain that $\psi\circ\tau^*\circ\pi = \psi\circ\phi = \chi\colon S_5 \dashrightarrow S_5$ as shown in (♣). This is a birational map $S_5 \dashrightarrow S_5$ which is not biregular. □

$$\begin{tikzcd} & \widetilde{S} \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & & \widetilde{S} \arrow[dl, "\tau"'] \arrow[dr, "\pi"] & \\ S_5 \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^1\times\mathbb{P}^1 \arrow[rr, dashed, "\phi"'] & & S_5. \end{tikzcd} \qquad (♣)$$

Lemmas 7.1 and 7.2 tell us that there is no G_{20}-equivariant link starting from $\mathbb{P}^1\times\mathbb{P}^1$, that leads to a different minimal surface than the quintic del Pezzo surface or $\mathbb{P}^1\times\mathbb{P}^1$ itself. This together with Remark 5.2 finalises the proof of Theorem 2.4 and implies the first two parts of Theorem 1.1.

In Theorem 2.4 we additionally stated that $\mathrm{Bir}^{G_{20}}(S_5)$ is of order 40. In fact one can show that

Theorem 7.3 *Let S_5 be the smooth del Pezzo surface of degree* 5*, and let $G_{20}\cong C_5\rtimes C_4$ be a subgroup of order* 20 *in* $\mathrm{Aut}(S_5)$*. Then*

$$\mathrm{Bir}^{G_{20}}(S_5) = G_{40},$$

where $G_{40} \cong C_2 \times G_{20}$.

Proof We need to find the normalizer $G_{40} = \mathrm{Norm}_{\mathrm{Aut}(\mathbb{P}^1\times\mathbb{P}^1)}(G_{20})$. Obviously, it is enough to find $G_{40} \cap H$, where H is the subgroup of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ which preserves rulings. Certainly, $G_{40} \cap H$ lies inside the group $\mathrm{Norm}_H(D_{10})$. The normalizer of D_{10} in $\mathrm{Aut}(\mathbb{P}^1)$ is equal to D_{20} and generated by D_{10} and the involution $[x:y] \mapsto [-x:y]$. Thus $G_{40} \cap H$ lies inside the group $\langle D_{10}, a, b\rangle$, with

$$\begin{aligned} a\colon ([x_1:y_1],[x_2:y_2]) &\mapsto ([-x_1:y_1],[x_2:y_2]),\\ b\colon ([x_1:y_1],[x_2:y_2]) &\mapsto ([x_1:y_1],[-x_2:y_2]). \end{aligned}$$

One can easily check that only ab normalizes the group G_{20} and $G_{40} \cong C_2 \times G_{20}$. □

This proof was communicated to me by Artem Avilov and I thank him for thus completing the proof of Theorem 1.1.

References

1. Ahmadinezhad, H., Okada, T.: Birationally rigid Pfaffian Fano 3-folds. Algebr. Geom. **5**(2), 160–199 (2018)
2. Bannai, S., Tokunaga, H.: A note on embeddings of S_4 and A_5 into the two-dimensional Cremona group and versal Galois covers. Publ. Res. Inst. Math. Sci. **43**(4), 1111–1123 (2007)
3. Beauville, A., Blanc, J.: On Cremona transformations of prime order. C. R. Math. Acad. Sci. Paris **339**(4), 257–259 (2004)
4. Blanc, J.: Finite abelian subgroups of the Cremona group of the plane. C. R. Math. Acad. Sci. Paris **344**(1), 21–26 (2007)
5. Corti, A.: Factoring birational maps of threefolds after Sarkisov. Appendix: surfaces over nonclosed fields. J. Algebraic Geom. **4**(2), 223–254, 248–254 (1995)
6. Dolgachev, I.V.: Classical Algebraic Geometry. Cambridge University Press, Cambridge (2012)
7. Dolgachev, I.V., Iskovskikh, V.A.: Finite subgroups of the plane Cremona group. In: Tschinkel, Yu., Zarhin, Yu. (eds.) Algebra, Arithmetic, and Geometry, Vol. I. Progress in Mathematics, vol. 269, pp. 443–548. Birkhäuser, Boston (2009)
8. Hosoh, T.: Automorphism groups of cubic surfaces. J. Algebra **192**(2), 651–677 (1997)
9. Iskovskikh, V.A.: Factorization of birational maps of rational surfaces from the viewpoint of Mori theory. Russian Math. Surv. **51**(4), 585–652 (1996)
10. Iskovskikh, V.A.: On the Noether–Fano inequalities. In: Collino, A. (ed.) The Fano Conference, pp. 25–35. Università di Torino, Turin (2004)
11. Iskovskikh, V.A.: Two non-conjugate embeddings of $S_3 \times \mathbb{Z}_2$ into the Cremona group. II. In: Konno, K., Nguyen-Khac, V. (eds.) Algebraic Geometry in East Asia–Hanoi 2005. Advanced Studies in Pure Mathematics, vol. 50, pp. 251–267. Mathematical Society of Japan, Tokyo (2008)
12. Sakovics, D.: G-birational rigidity of the projective plane. Eur. J. Math. https://doi.org/10.1007/s40879-018-0261-x (2018). arXiv:1712.01587