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RESEARCH ARTICLE

On series of functions with the Baire property

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Abstract
Hugo Steinhaus in 1928 proved three theorems concerning convergence almost everywhere and divergence of series of measurable functions. We investigate similarities and differences in the behaviour of sequences of functions with the Baire property for convergence everywhere except a set of the first category.

Keywords Series of functions · Measurability · Baire property

Mathematics Subject Classification 40A30 · 28A20 · 26A21

Recall three classical theorems concerning series of real numbers.

Theorem 1 (Abel and Dini, [2, Chapter IX, Section 39, (173)]) *If* $\sum_{k=1}^{\infty} a_k = +\infty$, $a_k \geqslant 0$ *for* $k \in \mathbb{N}$, *then there exists a sequence* $\{\lambda_k\}_{k\in\mathbb{N}}$ *of positive numbers such that* $\lim_{k\to\infty} \lambda_k = 0$ *and* $\sum_{k=1}^{\infty} \lambda_k a_k = +\infty$.

Theorem 2 (Dini, [2, Chapter IX, Section 39, (175), 4 and Section 41, (178), A]) *If* $\sum_{k=1}^{\infty} a_k < \infty$, $a_k \geqslant 0$ *for* $k \in \mathbb{N}$, *then there exists a sequence* $\{\lambda_k\}_{k\in\mathbb{N}}$ *of positive numbers such that* $\lim_{k\to\infty} \lambda_k = +\infty$ *and* $\sum_{k=1}^{\infty} \lambda_k a_k < \infty$.

Theorem 3 (Stieltjes, [2, Chapter IX, Section 41, (178), I]) *For each sequence* $\{\delta_k\}_{k\in\mathbb{N}}$ *of positive numbers such that* $\lim_{k\to\infty} \delta_k = 0$ *there exists a series* $\sum_{k=1}^{\infty} a_k$, $a_k > 0$ *for* $k \in \mathbb{N}$, *such that* $\sum_{k=1}^{\infty} a_k = +\infty$ *and* $\sum_{k=1}^{\infty} a_k \delta_k < \infty$.

These theorems show that there does not exist neither a series which converges slower than any other series nor a series which diverges slower than any other series. This implies among others that no comparison test can be effective with all series.

Theorem 1 remains true also for a sequence $\{S_i\}_{i\in\mathbb{N}}$ of divergent series $\sum_{k=1}^{\infty} a_{ik}$ of positive numbers. Namely, there exists a sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ independent from i

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such that all series $\sum_{k=1}^{\infty} \lambda_k a_{ik}$, $i \in \mathbb{N}$, are divergent. It was proved in [4,5] with the use of the Banach–Steinhaus theorem. In the same paper Hugo Steinhaus dealt with the following question of Stanisław Ruziewicz: Suppose that a series of positive functions $\sum_{k=1}^{\infty} a_k(t)$, $a_k : [0, 1] \to \mathbb{R}^+$ for $k \in \mathbb{N}$, is divergent for each $t \in [0, 1]$. Is it possible to find a sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ of positive numbers such that $\lim_{k\to\infty} \lambda_k = 0$ and $\sum_{k=1}^{\infty} \lambda_k a_k(t) = +\infty$ for each t? The answer is negative, there exists a sequence $\{a_k(t)\}_{k\in\mathbb{N}}$ of measurable functions such that for each sequence $\{\lambda_k\}_{k\in\mathbb{N}} = \Lambda$ there exists a point $x(\Lambda) \in [0, 1]$ such that $\sum_{k=1}^{\infty} \lambda_k a_k(x(\Lambda)) < +\infty$.

Let $\mathbf{\Lambda}$ be the set of all positive sequences convergent to 0 (card $\mathbf{\Lambda} = \mathfrak{C}$). Take a set $E \subset [0, 1]$, $m(E) = 0$, card $E = \mathfrak{C}$ (here $m(\cdot)$ stands for the Lebesgue measure). There exists a function $t : \mathbf{\Lambda} \xrightarrow{1-1} E$ (denotations of Steinhaus [4,5]). For each $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}}$ there exists $\left\{a_k^{\Lambda}\right\}_{k\in\mathbb{N}}$ such that $\sum_{k=1}^{\infty} a_k^{\Lambda} = +\infty$, $a_k^{\Lambda} > 0$, and $\sum_{k=1}^{\infty} \lambda_k a_k^{\Lambda} < \infty$ (again Stieltjes). Put $a_k(t) = a_k^{\Lambda}$ for $t = t(\Lambda)$ and $a_k(t) = 1/k$ for $t \notin E$, $k \in \mathbb{N}$. All functions a_k are measurable ($= 1/k$ a.e.) and

$$\sum_{k=1}^{\infty} a_k(t) = +\infty \quad \text{for each} \quad t \in [0, 1],$$

$$\sum_{k=1}^{\infty} \lambda_k a_k(t) < \infty \qquad \text{for} \quad t = t(\Lambda), \quad \Lambda = \{\lambda_k\}_{k\in\mathbb{N}}.$$

If one requires only divergence or convergence almost everywhere, the corresponding versions of Theorems 1, 2, 3 remain true. Three theorems below are also due to Steinhaus [4,5].

Theorem 4 *If* $\{a_k(t)\}_{k\in\mathbb{N}}$ *is a sequence of measurable functions,* $a_k : [0, 1] \to \mathbb{R}^+$ *for* $k \in \mathbb{N}$ *and* $\sum_{k=1}^{\infty} a_k(t) = +\infty$ *almost everywhere on* $[0, 1]$, *then there exists a sequence* $\{\lambda_k\}_{k\in\mathbb{N}}$ *of positive numbers such that* $\lim_{k\to\infty} \lambda_k = 0$ *and* $\sum_{k=1}^{\infty} \lambda_k a_k(t) = +\infty$ *for almost every* $t \in [0, 1]$.

Theorem 5 *If* $\{a_k(t)\}_{k\in\mathbb{N}}$ *is a sequence of measurable functions,* $a_k : [0, 1] \to \mathbb{R}^+$ *for* $k \in \mathbb{N}$ *and* $\sum_{k=1}^{\infty} a_k(t) < +\infty$ *for almost every* $t \in [0, 1]$, *then there exists a sequence* $\{\lambda_k\}_{k\in\mathbb{N}}$ *of positive numbers such that* $\lim_{k\to\infty} \lambda_k = +\infty$ *and* $\sum_{k=1}^{\infty} \lambda_k a_k(t) < +\infty$ *for almost every* $t \in [0, 1]$.

Theorem 6 *If* $\{\lambda_k(t)\}_{k\in\mathbb{N}}$ *is a sequence of measurable functions,* $\lambda_k : [0, 1] \to \mathbb{R}$ *for* $k \in \mathbb{N}$ *and* $\lim_{k\to\infty} \lambda(t) = 0$ *almost everywhere on* $[0, 1]$, *then there exists a sequence of positive numbers* $\{a_k\}_{k\in\mathbb{N}}$ *such that* $\sum_{k=1}^{\infty} a_k = +\infty$ *and* $\sum_{k=1}^{\infty} a_k|\lambda_k(t)| < +\infty$ *almost everywhere.*

Below we shall prove that Theorem 4 has a satisfactory analogue for the Baire property while Theorems 5 and 6 do not.

Theorem 7 *If* $\{a_k(t)\}_{k\in\mathbb{N}}$ *is a sequence of functions with the Baire property,* $a_k : [0, 1] \to \mathbb{R}^+$ *for* $k \in \mathbb{N}$ *and* $\sum_{k=1}^{\infty} a_k(t) = +\infty$ *everywhere except a set of the first category on* $[0, 1]$, *then there exists a sequence* $\{\lambda_k\}_{k\in\mathbb{N}}$ *of positive numbers such that* $\lim_{k\to\infty} \lambda_k = 0$ *and* $\sum_{k=1}^{\infty} \lambda_k a_k(t) = +\infty$ *everywhere except a set of the first category.*

Proof We shall need a simple lemma:

Lemma *If* $\{f_n\}_{n\in\mathbb{N}}$ *is an increasing sequence of functions with the Baire property,* $f_n\colon [0,1] \to \mathbb{R}^+$ *for* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} f_n(t) = +\infty$ *everywhere except a set of the first category, then for each* $p > 0$ *and for each interval* $[a,b] \subset [0,1]$ *there exists* $n_0 \in \mathbb{N}$ *such that* $E_n = \{t \in [a,b] : f_n(t) \geqslant p\}$ *for* $n \geqslant n_0$ *is of the second category* (*so* E_n *is residual in some subinterval* $[c,d] \subset [a,b]$ *as a set having the Baire property*).

Proof of Lemma Let $E_n = \{t \in [a,b] : f_n(t) \geqslant p\}$ for $n \in \mathbb{N}$. Since $\bigcup_{n=1}^{\infty} E_n$ is residual in $[a,b]$, there exists $n_0 \in \mathbb{N}$ such that E_{n_0} is of the second category, each E_n for $n \geqslant n_0$ is also of the second category because the sequence $\{E_n\}_{n\in\mathbb{N}}$ is ascending. □

Now, the existence of an increasing sequence of natural numbers $\{n_i\}_{i\in\mathbb{N}}$ will follow from the lemma. Let n_1 be a natural number for which the set $A_1 = \left\{t \in [0,1] : \sum_{k=1}^{n_1} a_k(t) \geqslant 1\right\}$ is of the second category. Let $n_2 > n_1$ be a natural number for which both sets $A_2^1 = \left\{t \in [0,1/2] : \sum_{k=n_1+1}^{n_2} a_k(t) \geqslant 2\right\}$, $A_2^2 = \left\{t \in [1/2,1] : \sum_{k=n_1+1}^{n_2} a_k(t) \geqslant 2\right\}$ are of the second category. Suppose that we have chosen $n_1 < n_2 < \cdots < n_i$. Let $n_{i+1} > n_i$ be a natural number for which all sets

$$A_{i+1}^j = \left\{t \in \left[\frac{j-1}{2^i}, \frac{j}{2^i}\right] : \sum_{k=n_i+1}^{n_{i+1}} a_k(t) \geqslant i+1\right\}, \quad j \in \{1,2,\ldots,2^i\},$$

are of the second category.

Now put (similarly as in [4,5]): $\lambda_k = 1$ for $1 \leqslant k \leqslant n_1$ and $\lambda_k = 1/(k+1)$ for $n_i + 1 \leqslant k \leqslant n_{i+1}$. We obviously have $\lim_{k\to\infty} \lambda_k = 0$. At the same time $\sum_{k=n_i+1}^{n_{i+1}} \lambda_k a_k(t) \geqslant 1$ for $t \in \bigcup_{j=1}^{2^i} A_{i+1}^j$. Hence $\sum_{k=1}^{\infty} \lambda_k a_k(t) = +\infty$ for $t \in \limsup_{i\to\infty}\left(\bigcup_{j=1}^{2^i} A_{i+1}^j\right) = A$. From the construction it follows that A is residual in $[0,1]$. □

To show that analogues of Theorems 5 and 6 do not hold for the Baire property we shall construct a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions with the Baire property which converges pointwise to zero and does not converge uniformly on any set of the second category with the Baire property. One can find an example of such sequence of functions in [3, Chapter 8], consisting of continuous functions, but in our construction the sequence is non-increasing, which enables us to build a series of functions with non-negative terms.

Let $I_i^n = ((i-1)/2^n, i/2^n)$ for $n \in \mathbb{N}$ and $i \in \{1,2,\ldots,2^n\}$, $A_n = \bigcup_{i=1}^{2^{n-1}} I_{2i-1}^n$, $B_n = \bigcup_{i=1}^{2^{n-1}} I_{2i}^n$ for $n \in \mathbb{N}$. Put

$$f_1(x) = \begin{cases} 1 & \text{for } x \in A_1, \\ \dfrac{1}{2} & \text{for } x \in B_1, \\ 0 & \text{for the remaining } x \in [0,1]. \end{cases} \tag{1}$$

If $f_1, f_2, \ldots, f_{n-1}$ are already defined, put

$$f_n(x) = \begin{cases} f_{n-1}(x) & \text{for } x \in A_n, \\ \dfrac{1}{n+1} & \text{for } x \in B_n, \\ 0 & \text{for the remaining } x \in [0,1]. \end{cases} \tag{2}$$

Obviously $f_{n+1}(x) \leqslant f_n(x)$ for each $n \in \mathbb{N}$ and $x \in [0,1]$. We shall show that $\lim_{n\to\infty} f_n(x) = 0$ for each $x \in [0,1]$. Clearly, if the binary expansion of x contains a finite number of 1's (or equivalently, a finite number of 0's), then $f_n(x) = 0$ for sufficiently big $n \in \mathbb{N}$. Suppose now that the binary expansion of x contains infinitely many 0's as well as infinitely many 1's. If $x = (0, a_1, a_2, \ldots, a_k, \ldots)_2$ and $a_k = 1$, then from the definition of f_k it follows that $f_k(x) = 1/(k+1)$; if $a_k = 0$, then $f_k(x) = 1/(j+1)$, where $j = \max\{i : a_i = 1 \text{ and } i < k\}$ and $j \to \infty$ as $k \to \infty$. Hence $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$.

Suppose now that $E \subset [0,1]$ is a set of the second category with the Baire property. Then there exists an interval $(a,b) \subset [0,1]$ and a set P of the first category such that $(a,b)\setminus P \subset E$. We shall show that $\{f_n\}_{n\in\mathbb{N}}$ is not uniformly convergent on $(a,b)\setminus P$. Indeed, there exists $n_0 \in \mathbb{N}$ such that at least one component of B_{n_0} is included in $[a,b]$, so there exists a point $x_{n_0} \in B_{n_0} \cap ((a,b)\setminus P))$ and $f_{n_0}(x_{n_0}) = 1/(n_0+1)$. From the construction of $\{f_n\}_{n\in\mathbb{N}}$ it follows that for each $n > n_0$ at least one component of A_n is included in (a,b), so for each $n > n_0$ there exists a point $x_n \in A_n \cap ((a,b)\setminus P)$ such that $f_n(x_n) = 1/(n_0+1)$. Hence $\{f_n\}_{n\in\mathbb{N}}$ is not uniformly convergent on $(a,b)\setminus P$ and obviously on E.

Theorem 8 *There exists a sequence $\{a_k(t)\}_{k\in\mathbb{N}}$ of functions with the Baire property, $a_k\colon [0,1] \to \mathbb{R}^+$ for $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} a_k(t) < \infty$ for each $t \in [0,1]$, such that for every sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ of positive numbers tending to infinity the series $\sum_{k=1}^{\infty} \lambda_k a_k(t)$ is divergent to infinity on a set residual in* $[0,1]$.

Proof Put $a_1(t) = 1 - f_1(t)$ and $a_k(t) = f_{k-1}(t) - f_k(t)$ for $k \geqslant 2$, where $\{f_n\}_{n\in\mathbb{N}}$ is the sequences (1)–(2). Then $\sum_{i=1}^{k} a_i(t) = 1 - f_k(t)$ for $k \in \mathbb{N}$, so $\sum_{k=1}^{\infty} a_k(t) = 1$ everywhere except a denumerable set. Observe that $a_k(t) \geqslant 1/k - 1/(k+1) = 1/(k(k+1))$ for $t \in B_k$, since $f_{k-1}(t) \geqslant 1/k$ and $f_k(t) = 1/(k+1)$ on B_k. From the construction of $\{f_n\}_{n\in\mathbb{N}}$ it follows immediately that $a_{k+p}(t) \geqslant 1/(k(k+1))$ on $A_{k+p} \cap B_k$ for each $p \in \mathbb{N}$.

Let $\{p_k\}_{k\in\mathbb{N}}$ be an increasing sequence of natural numbers with $\lambda_{k+p_k} > k(k+1)$ for each $k \in \mathbb{N}$. Then we have $\lambda_{k+p_k} \cdot a_{k+p_k}(t) > 1$ for $t \in A_{k+p_k} \cap B_k$. Hence $\sum_{k=1}^{\infty} \lambda_k a_k(t) \geqslant \sum_{k=1}^{\infty} \lambda_{k+p_k} a_{k+p_k}(t) = +\infty$ for $t \in \limsup_{k\to\infty} A_{k+p_k} \cap B_k$ and the last set is residual in $[0,1]$. □

Theorem 9 *There exists a sequence $\{f_k(t)\}_{k\in\mathbb{N}}$ of functions with the Baire property, $f_k\colon [0,1] \to \mathbb{R}^+$ for $k \in \mathbb{N}$ and $\lim_{k\to\infty} f_k(t) = 0$ for $t \in [0,1]$, such that for each sequence $\{a_k\}_{k\in\mathbb{N}}$ of positive numbers with $\sum_{k=1}^{\infty} a_k = +\infty$ the series $\sum_{k=1}^{\infty} a_k f_k(t)$ is divergent to infinity on a set residual in* $[0,1]$.

Proof Let $\{f_k(t)\}_{k\in\mathbb{N}}$ be the sequence of functions (1)–(2). Suppose that $\sum_{k=1}^{\infty} a_k = +\infty$. There exists $n_1 \in \mathbb{N}$ such that $\sum_{k=1}^{n_1} a_k > 1$. Observe that $f_1(t) = f_2(t) = \cdots =$

$f_{n_1}(t) = 1$ for $t \in (0, 1/2^{n_1})$, so $\sum_{k=1}^{n_1} a_k f_k > 1$ on this interval. There exists $n_2 > n_1$ such that $\sum_{k=n_1+1}^{n_2} a_k > 2$. Again observe that $f_{n_1+1}(t), f_{n_1+2}(t), \ldots, f_{n_2}(t) \geqslant 1/2$ for $t \in (0, 1/2^{n_2}) \cup (1/2, 1/2+1/2^{n_2})$ (actually the value of all functions above is equal to 1 in the first interval and to 1/2 in the second one), so $\sum_{k=n_1+1}^{n_2} a_k f_k(t) > 1$ on this set. Suppose that we have found $n_1 < n_2 < \cdots < n_i$. Let n_{i+1} be a number such that $\sum_{k=n_i+1}^{n_{i+1}} a_k > i + 1$. Similarly, we observe that $f_{n_i+1}(t), f_{n_i+2}(t), \ldots, f_{n_{i+1}}(t) \geqslant 1/(i+1)$ on an open set E_i such that the intersection of E_i with each component of A_{i+1} is nonempty (namely it is equal to the interval $(2(j-1)/2^{i+1}, 2j - 1/2^{i+1}) \cap (2(j-1)/2^{i+1}, 2(j-1)/2^{i+1} + 1/2^{n_{i+1}}))$. Hence $\sum_{k=n_i+1}^{n_{i+1}} a_k f_k(t) > 1$ for $t \in E_i$. Finally $\sum_{k=1}^{\infty} a_k f_k(t) = +\infty$ for $t \in \limsup_{i\to\infty} E_i$ and the last set is residual in $[0, 1]$. □

We show another similarity and difference between measure and category. Bartle in [1] proved the following theorem:

Theorem 10 *If a sequence $\{f_n\}_{n\in\mathbb{N}}$ of measurable functions on $[0, 1]$ converges almost uniformly to f, then it satisfies the vanishing restriction with respect to f. If $\{f_n\}_{n\in\mathbb{N}}$ converges in measure to f and $\{f_n\}_{n\in\mathbb{N}}$ satisfies the vanishing restriction with respect to f, then $\{f_n\}_{n\in\mathbb{N}}$ converges almost uniformly to f.*

Here the *convergence almost uniformly* means that there exists a sequence $\{B_i\}_{i\in\mathbb{N}}$ of measurable sets such that $m\left(\bigcup_{i=1}^{\infty} B_i\right) = 1$ ($[0, 1] \setminus \bigcup_{i=1}^{\infty} B_i$ is a nullset) and $f_{n|B_i} \rightrightarrows f_{|B_i}$ for each $i \in \mathbb{N}$ as $n \to \infty$ (cf. Egorov's theorem).

The sequence $\{f_n\}_{n\in\mathbb{N}}$ of measurable functions satisfies the *vanishing restriction with respect to* f if for all $\alpha > 0$ we have

$$\lim_{n\to\infty} m\left(E_n^f(\alpha)\right) = 0, \quad \text{where} \quad E_n^f(\alpha) = \bigcup_{j=n}^{\infty} \{x \in [0, 1] : |f_j(x) - f(x)| > \alpha\}.$$

Since the sequence $\left\{E_n^f(\alpha)\right\}_{n\in\mathbb{N}}$ is descending, the last condition means $\bigcap_{n=1}^{\infty} E_n^f(\alpha)$ is a nullset.

We shall say that a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions with the Baire property *converges to* f *B-almost uniformly* if there exists a sequence $\{B_i\}_{i\in\mathbb{N}}$ of sets with the Baire property such that $[0, 1] \setminus \bigcup_{i=1}^{\infty} B_i$ is of the first category and $f_{n|B_i} \rightrightarrows f_{|B_i}$ for each $i \in \mathbb{N}$ as $n \to \infty$. We can (and shall) suppose that the sequence $\{B_i\}_{i\in\mathbb{N}}$ is ascending.

We shall say that a sequence $\{f_n\}_{n\in\mathbb{N}}$ satisfies the *B-vanishing restriction with respect to* f if for all $\alpha > 0$ the set $\bigcap_{n=1}^{\infty} E_n^f(\alpha)$ is of the first category.

Theorem 11 *If a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions with the Baire property on $[0, 1]$ converges B-almost uniformly to f, then it satisfies the B-vanishing condition with respect to f. There exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions with the Baire property on $[0, 1]$ which converges everywhere to $f \equiv 0$ and satisfies the B-vanishing restriction with respect to f but which does not converge B-almost uniformly to f.*

Proof Suppose that $\{B_i\}_{i\in\mathbb{N}}$ is an increasing sequence of sets with the Baire property such that $[0, 1] \setminus \bigcup_{i=1}^{\infty} B_i$ is of the first category and $f_{n|B_i} \rightrightarrows f_{|B_i}$ for each $i \in \mathbb{N}$ as $n \to \infty$.

Fix $\alpha > 0$. For each $i \in \mathbb{N}$ there exists $n_{\alpha,i} \in \mathbb{N}$ such that for each $x \in B_i$ and $n \geqslant n_{\alpha,i}$ we have $|f_n(x) - f(x)| \leqslant \alpha$. So $E_{n_{\alpha,i}}(\alpha) \subset [0,1] \setminus B_i$ and $\bigcap_{n=1}^{\infty} E_n(\alpha) \subset \bigcap_{i=1}^{\infty}([0,1] \setminus B_i) = [0,1] \setminus \bigcup_{i=1}^{\infty} B_i$ which completes the proof of the first part of the theorem.

To prove the second part it is sufficient to observe that the sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions (1)–(2) satisfies the B-vanishing restriction. If this sequence were convergent uniformly to the zero-function on a set $E \subset [0,1]$ with the Baire property, then E would be of the first category. Hence it is clear that $\{f_n\}_{n\in\mathbb{N}}$ does not converge B-almost uniformly to the zero-function. □

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