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EDITORIAL

Edge volume, part I

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William Edge was born in Stockport on 8th November 1904. Both his parents were schoolteachers. Edge was educated at his local school, Stockport Grammar School. In 1923, he went to Cambridge where he studied mathematics at Trinity College. After graduating, Edge became a Ph.D. student of Henry Baker. His dissertation generalised Luigi Cremona's results about ruled surfaces in the real projective space. In 1928, Edge became a research fellow in Trinity College. After 4 years he took a lectureship (assistant professorship) at the University of Edinburgh and remained there for the rest of his career. He was elected a Fellow of the Royal Society of Edinburgh 2 years later. Edge was promoted to reader (associate professor) in 1949. He became a full professor in 1969, 6 years prior his retirement in 1975. Edge never married, he had no children, he never drove a car, he was reluctant to travel, and he disdained radio and television. Apart from mathematics Edge loved hill walking, singing and playing the piano. He spent his final years in the retirement house in Bonnyrigg near Edinburgh, where he died on 27 September 1997.

This volume is dedicated to the memory of William Edge.

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Edge wrote nearly 100 research papers. Most of them are about classical algebraic varieties like the Veronese surface, Klein's quartic curve, Maschke's quartic surfaces, Kummer's quartic surface, Weddle's surface, Fricke's curve etc. A typical example is the paper *A pencil of four-nodal plane sextics*. In 1896, Wiman discovered the singular sextic curve in $\mathbb{P}^2$ given by

$$x^6 + y^6 + z^6 + (x^2 + y^2 + z^2)(x^4 + y^4 + z^4) = 12x^2y^2z^2.$$

This curve bears Wiman's name now. It has four nodes. Blowing up $\mathbb{P}^2$ at them, we obtain the quintic del Pezzo surface. Its automorphism group is the symmetric group S_5, and the proper transform of Wiman's curve is a smooth anticanonical curve of genus 6 that is invariant with respect to the action of the group S_5. This linear system contains another S_5-invariant member, which is the union of 10 lines. These two curves generate the pencil, which is now known as the Wiman–Edge pencil. In $\mathbb{P}^2$, its image is given by

$$\lambda(y^2 - z^2)(z^2 - x^2)(x^2 - y^2) = \mu\big(x^6 + y^6 + z^6 + \big(x^2 + y^2 + z^2\big)\big(x^4 + y^4 + z^4\big) - 12x^2y^2z^2\big),$$

where $[\lambda{:}\mu] \in \mathbb{P}^1$. In the paper *A pencil of four-nodal plane sextics*, Edge explicitly described all singular members of this pencil.

Edge's work was full of explicit geometrical constructions. For instance, the quintic del Pezzo surface in $\mathbb{P}^5$ has one apparent double point. This means that for a del Pezzo surface of degree 5 given with its anticanonical embedding into $\mathbb{P}^5$, through a general point of $\mathbb{P}^5$ there is a unique line which meets the surface in two points. In his 1932 paper *The number of apparent double points of certain loci*, Edge constructed many higher-dimensional varieties having the same property. Now they are known as Edge varieties. In 2004, Edge's work was continued by Ciro Ciliberto, Massimiliano Mella, and Francesco Russo, who classified all smooth threefolds having one apparent double point.

Like many classical geometers, Edge had a passion for algebraic varieties with interesting group of symmetries. For example, the simple group $\mathrm{PSL}_2(\mathbb{F}_7)$ is the automorphism group of the Klein quartic curve in $\mathbb{P}^2$, which is given by

$$xy^3 + yz^3 + zx^3 = 0.$$

This group is known as the Klein simple group of order 168. The Klein quartic curve admits unique $\mathrm{PSL}_2(\mathbb{F}_7)$-invariant θ-characteristic, whose tensor cube gives a $\mathrm{PSL}_2(\mathbb{F}_7)$-equivariant embedding into three-dimensional projective space. This gives a natural action of the Klein simple group on $\mathbb{P}^3$, which leaves invariant the image of the Klein quartic curve. In his 1947 paper *The Klein group in three dimensions*, Edge described many geometric facts about this action. In 2012, Cheltsov and Shramov used them to prove that $\mathbb{P}^3$ is $\mathrm{PSL}_2(\mathbb{F}_7)$-birationally rigid.

After studying classical geometry, Edge moved towards finite geometry. Despite publishing a considerable number of works in this field, it was algebraic geometry that remained his abiding passion.

William Edge had only one research student, James Hirschfeld, who became a mathematician at the University of Sussex. This suggests that Edge probably had little impact on the next generations of mathematicians. This would not be a correct corollary. During Edge's time, a disproportionate number of those Edinburgh undergraduates who went on to postgraduate study actually chose something geometrical, where the chief influence inspiring this choice must have been Edge's courses. For example, the renowned symplectic geometer Dusa McDuff took Edge's lectures as an undergraduate at Edinburgh. Even the work of the applied mathematician Keith Moffatt does sometimes display leanings towards geometry. He also took Edge's geometry course, the popularity of which went beyond math graduates. When we invited Keith Moffatt to contribute to this volume, we erroneously sent an email to a biologist in Chicago, Keith Moffat (with one t in the surname). It turned out that he was a physics student at Edinburgh in 1961–1965 and took geometry course from William Edge. Edge also taught the algebra courses at Edinburgh at this time, but he taught algebra with a strong geometric flavour reflecting his deep knowledge, feel and love for geometry.

To carry Edinburgh's tradition in geometry into the future, Ahmadinezhad, Cheltsov, Guletskii, Kaloghiros, Logvinenko and Testa organized five *Edge Days* workshops at Edinburgh in 2013–2017. Among their participants were most of contributors to this volume as well as Caucher Birkar, Alexey Bondal, Alessio Corti, Ruadhai Dervan, Yoshinori Gongyo, Liana Heuberger, Ilia Itenberg, Dmitry Kaledin, Masayuki Kawakita, Alvaro Liendo, Angelo Lopez, Daniel Loughran, Frédéric Mangolte, Takuzo Okada, John Ottem, Dmitri Panov, Elisa Postinghel, Francesco Russo, Edoardo Sernesi, Nicholas Shepherd-Barron, Alexei Skorobogatov and Yuri Tschinkel.

The first part of the *Edge Volume* contains 25 research papers, whose authors are Simeon Ball, Arnaud Beauville, Anton Betten, James Hirschfeld, Fatma Karaoglu, Gavin Brown, Alexander Kasprzyk, Imran Qureshi, Aiden Bruen, James McQuillan, Elena Bunkova, Jacob Cable, Hendrik Süß, Paolo Cascini, Hiromu Tanaka, Jean-Louis Colliot-Thélène, Alastair Craw, James Green, Julie Déserti, Adrien Dubouloz, Gabino González-Diez, Gareth Jones, David Torres-Teigell, Kangjin Han, Massimil-

iano Mella, Jarosław Buczyński, Zach Teitler, Jürgen Hausen, Ivan Arzhantsev, Lukas Braun, Milena Wrobel, Ruben Hidalgo, Ilya Karzhemanov, Ilya Zhdanovskiy, Motoko Kawakita, Igor Krylov, Dusa McDuff, Emilia Mezzetti, Aleksandr Pukhlikov, Ichiro Shimada, Evgeny Shinder, Terry Wall, Andrew du Plessis, and a short postface by Sir Michael Atiyah. Most of them were participants of Edge Days, while the others are mathematicians who personally knew William Edge or used his work in their research.

Simeon Ball is the academic grandson of William Edge, having had Hirschfeld as his doctoral advisor. His paper *Extending small arcs to large arcs* deals with sets of vectors of the k-dimensional vector space over the finite field with q, in which every subset of size k is a basis of the space. Such sets are called arcs. Given an arc G in a space of odd characteristic, Ball proves that there is an upper bound on the largest arc containing G. This bound is not explicit. But the results proved in this paper may provide new tools in the computational classification and construction of large arcs.

Arnaud Beauville participated in the second Edge Days in 2014. His paper *An introduction to Ulrich bundles* gives a smooth introduction to Ulrich bundles, which have been introduced in the paper *Gorenstein rings and modules with high numbers of generators* by Bernd Ulrich. Beauville recalls the definition and basic properties of Ulrich bundles, and proves that many smooth surfaces and threefolds admit an Ulrich bundle.

James Hirschfeld is the only research student of William Edge. He opened the first Edge Days back in 2013. Hirschfeld's joint paper with Anton Betten and Fatma Karaoglu *Classification of cubic surfaces with twenty-seven lines over the finite field of order thirteen* continues his earlier work on cubic surfaces over finite fields. It classifies smooth cubic surfaces with twenty-seven lines over the finite field of thirteen elements.

Both Gavin Brown and Alexander Kasprzyk participated in Edge Days at Edinburgh. Our volume contains their joint paper with Imran Qureshi *Fano* 3*-folds in* $\mathbb{P}^2 \times \mathbb{P}^2$ *format, Tom and Jerry*. It provides a lot of examples of (singular) $\mathbb{Q}$-factorial terminal Fano threefolds and very special birational transformations between them. The paper is very explicit and is a good source for everyone working with singular Fano threefolds.

Aiden Bruen and James McQuillan used Edge's results in their research on intersections of hyperconics in projective planes of even order. Their paper *On geometrical configurations* studies the structure of the self-conjugate points of the polarity obtained from a Desargues configuration. It is shown that there can be at most four self-conjugate points. Later on in the paper the famous classical theorem concerning double perspectivity of pairs of triangles inscribed in a conic is generalized to pairs of inscribed p-gons where p is an odd prime.

In 2017, Elena Bunkova participated in Edge Days. Her paper *Differentiation of genus* 3 *hyperelliptic functions* gives an explicit solution to the problem of differentiation of hyperelliptic functions in genus 3 case.

Massimiliano Mella came to Edinburgh more than 10 years ago to report on his joint work with Ciliberto and Russo on varieties with one apparent double point. His talk inspired the creation of Edge Days. Mella participated in Edge Days in 2013 and 2017. Together with Jarosław Buczyński, Kangjin Han and Zach Teitler, Mella submitted the paper *On the locus of points of high rank* to our volume. To describe their results, fix a closed subvariety X in a projective space $\mathbb{P}^n$. Then the rank of a

point $P \in \mathbb{P}^n$ with respect to X is the least integer r such that P lies in the linear span of some r points of X. Denote by W_k the closure in $\mathbb{P}^n$ of the set of points of rank k. For small values of k such loci are just secant varieties. The paper by Buczyński, Han, Mella and Teitler studies the loci W_k for values of k larger than the generic rank.

While being Seggie Brown Fellow in Edinburgh, Hendrik Süß participated in Edge Days 2014. In 2015, he was offered a lectureship in Manchester. Since then he participated in Edge Days 2016 and 2017. His joint paper with Jacob Cable *On the classification of Kähler–Ricci solitons on Gorenstein del Pezzo surfaces* expands the work by Cheltsov, Kosta, Odaka, Park, Spotti, Sun, Tian and Won on K-stability of del Pezzo surfaces with du Val singularities. Süß and Cable give a classification of all pairs consisting of a singular del Pezzo surface and a vector field on it that are K-stable in the sense of Berman–Nystrom.

Both Paolo Cascini and Hiromu Tanaka have been participants of Edge Days. Their joint paper *Smooth rational surfaces violating Kawamata–Viehweg vanishing* is about cohomological pathologies in positive characteristics. They show that over any algebraically closed field of positive characteristic, there exists a smooth rational surface that violates the Kawamata–Viehweg vanishing theorem.

Jean-Louis Colliot-Thélène had been a very active participant of Edge Days 2016. He contributed his paper *Approximation forte pour les espaces homogènes de groupes semisimples sur le corps des fonctions d'une courbe algébrique complexe* to our volume on one simple condition: the paper would not be translated into English. So, it is in French. In this paper, Colliot-Thélène proves that strong approximation holds for a homogeneous space of a semisimple linear algebraic group defined over the function field of a curve over $\mathbb{C}$ outside any finite nonempty set of places of this field.

Alastair Craw was an undergraduate in mathematics at the University of Edinburgh, and while he worked in Glasgow he participated in many research meetings at Edinburgh, including Edge Days. His joint paper with James Green, entitled *Reconstructing toric quiver flag varieties from a tilting bundle*, provides a multigraded analogue of the fact that projective space can be reconstructed from the endomorphism algebra of Beilinson's tilting bundle.

Both Julie Déserti and Adrien Dubouloz participated in Edge Days 2014. Déserti's paper *Degree growth of polynomial automorphisms and birational maps: some example* provides many new examples of polynomial automorphisms of the affine space $\mathbb{C}^n$. The paper *Families of exotic affine 3-spheres* by Dubouloz provides examples of smooth affine threefolds diffeomorphic to a non-degenerate smooth complex affine quadric of dimension 3 but non-algebraically isomorphic to it.

Gabino González-Diez used Edge's results in his research on automorphisms group of algebraic curves. In his joint paper with Gareth Jones and David Torres-Teigell *Arbitrarily large Galois orbits of non-homeomorphic surfaces*, they construct orbits of the absolute Galois group, of explicit unbounded size, consisting of surfaces with mutually non-isomorphic fundamental groups.

Jürgen Hausen participated in the first Edge Days. The paper *Log terminal singularities, platonic tuples and iteration of Cox rings* by Ivan Arzhantsev, Lukas Braun, Jürgen Hausen and Milena Wrobel deals with log terminal singularities equipped with a torus action of complexity one. In dimension two, these are precisely the quotient singularities $\mathbb{C}^2/G$, where G is a finite subgroup of $\mathrm{GL}_2(\mathbb{C})$. Here one observes that

the derived series of G reflects an iteration of Cox rings. The paper generalises this to any dimension and provides an explicit study in the case of compound du Val threefold singularities.

Ruben Hidalgo used Edge's results in his research on automorphisms groups of Riemann surfaces. In his paper *About the Fricke–Macbeath curve*, Hidalgo studies the unique smooth genus 7 curve whose automorphisms group is the simple group $\mathrm{PSL}_2(\mathbb{F}_8)$ of order 504. This curves is called the Fricke–Macbeath curve. It was also studied by Edge in his 1967 paper *A canonical curve of genus* 7. Recently Bradley Brock found a very simple affine equation of this curve:

$$1 + 7xy + 21x^2y^2 + 35x^3y^3 + 28x^4y^4 + 2x^7 + 2y^7 = 0.$$

Ilya Karzhemanov participated in Edge Days in 2013 and 2014. In his joint paper with Ilya Zhdanovskiy *Some properties of surjective rational maps*, they study rational maps $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ having the following property: there exists a codimension 2 closed subset $Z \subset \mathbb{P}^n$ such that the restriction map is a surjective morphism.

Motoko Kawakita's husband, Masayuki Kawakita participated in Edge Days 2017. Her paper *Wiman's and Edge's sextics attaining Serre's bound* provides examples of algebraic curves of genus 6 over certain finite fields that attain Serre's bound on the number of points. All these examples are sextic curves studied by Wiman and Edge.

Igor Krylov participated in Edge Days in 2014, 2015, 2016 and 2017. In his paper *Rationally connected non-Fano type varieties*, Krylov constructs many examples that answer negatively

Question *Is every rationally connected variety birational to a variety of Fano type?*

This question was raised by Paolo Cascini and Yoshinori Gongyo in their joint paper *On the anti-canonical ring and varieties of Fano type*. In dimension two, the answer to this question is obviously positive. Krylov uses birationally rigid del Pezzo fibrations to produce his examples in dimension 3. Similarly, he uses birationally rigid fibrations into Fano varieties to produce examples in higher dimensions.

Dusa McDuff did her undergraduate studies at the University of Edinburgh. At that time, she took Edge's course on geometry. Her paper *A remark on the stabilized symplectic embedding problem for ellipsoids* studies obstructions for stabilized symplectic embeddings of an ellipsoid into a ball.

Emilia Mezzetti used Edge's results in her research on Laplace equations and the weak Lefschetz property. Her paper *Osculating behavior of Kummer surface in* $\mathbb{P}^5$ extends Edge's results on complete intersection of three quadrics in $\mathbb{P}^5$.

Aleksandr Pukhlikov participated in Edge Days in 2014 and 2016. In his paper *Canonical and log canonical thresholds of Fano complete intersections*, he proves that the α-invariant of Tian of a general Fano complete intersection of index 1 and codimension k in $\mathbb{P}^{M+k}$ is equal to 1 provided that $M \geqslant 2k + 3$ and the maximum of the degrees of defining equations is at least 8. By Tian's theorem, this implies the existence of Kähler–Einstein metrics on these Fano manifolds.

Ichiro Shimada uses Edge's results in his research on lattices of algebraic cycles on Fermat varieties in positive characteristics. To be precise, Shimada used a 1970 paper *Permutation representations of a group of order* 9, 196, 830, 720 by Edge. In this

paper, Edge constructed a permutation representation of the group $\cdot 222 \cong \mathrm{PSU}_6(\mathbb{F}_4)$ on a certain set of lattice points of the Leech lattice. He suggested that this representation corresponds to the natural permutation representation of the group $\cdot 222$ on the set of planes contained in the Fermat cubic 4-fold in $\mathbb{P}^5$ defined over an algebraically closed field of characteristic 2. In the paper *On Edge's correspondence associated with* $\cdot 222$, Shimada clarifies this correspondence and describes it explicitly.

Before moving to Sheffield, Evgeny Shinder was a research associate at Edinburgh. At this time he participated in the second Edge Days. In 2017, he participated in fifth Edge Days. In his paper *Group actions on categories and Elagin's theorem revisited*, Shinder gives a new proof of the following result of Elagin:

Theorem *Let $\mathcal{C}$ be a triangulated category. Suppose that there exists its semiorthogonal decomposition $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$, which is preserved by the action of G, and $\mathcal{C}^G$ is triangulated. Then there is a semiorthogonal decomposition $\mathcal{C}^G = \langle \mathcal{A}^G, \mathcal{B}^G \rangle$.*

Terry Wall personally knew William Edge. His joint paper with Andrew du Plessis *The moduli space of binary quintics* describes the moduli space of binary quintics, and identifies the curves in it defined by quintics having prescribed symmetries.

The first part of *Edge Volume* is concluded by a short note *William Leonard Edge. Some personal memories* by Sir Michael Atiyah, whose wife was an undergraduate student at the University of Edinburgh and took geometry course of William Edge.

William Edge's scientific results influenced the research of many mathematicians worldwide. His papers, even from 50 years ago, continue to attract attention. However, at the time of their publications, the journals had trouble finding suitable referees and also wondered whether there were readers. It is sad that, perhaps being perceived as old-fashioned, Edge's labours did not receive greater and earlier recognition. Probably because of this he was never offered a post at Oxbridge College and always felt that his life had been a failure. In addition to this, Edge was also distressed by the decline in geometry in the syllabus of schools and universities. Nevertheless, he managed to keep his enthusiasm high until his final years. Shortly before his retirement, Edge gave a morning talk at the British Mathematical Colloquium. It did not take long before the majority of the audience was lost, as most did not have a clue what the variously named algebraic varieties were. Interest was only revived when he took his coat off revealing the braces that were holding up his trousers. At the end of the talk the chairman pointed out that anyone present would be doing well, if after 50 years of mathematics he still derived the same fascination for the subject. In fact, Edge stayed research active long after retirement. Geometry gave him energy, kept his spirit and probably prolonged his life. Edge's very last paper 28 *real bitangents* has been published in 1994 when he was 90 years old. In it Edge gave an explicit example of a non-singular plane quartic with 28 real bitangents and computed the equations of all 28 bitangents.

We hope that *Edge Volume* will help to keep Edge's legacy alive. The diversity of its contributions reflects the vitality of algebraic geometry in the directions impulsed by William Edge.

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