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Curves on Heisenberg invariant quartic surfaces in projective 3-space

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Abstract This paper is about the family of smooth quartic surfaces $X \subset \mathbb{P}^3$ that are invariant under the Heisenberg group $H_{2,2}$. For a very generic X, we show that the Picard number of X is 16 and determine its Picard lattice. It turns out that a very generic X contains 320 irreducible conics which generate the Picard group of X.

1 Introduction

Let A be an Abelian surface over $\mathbb{C}$, that is a projective group variety of dimension 2. The subgroup $A_2 = \{a \in A : 2a = 0\}$ has order 16 and therefore $A_2 \cong (\mathbb{Z}/2\mathbb{Z})^4$. The involution $i: A \to A: a \mapsto -a$ induces a $\mathbb{Z}/2\mathbb{Z}$ action on A and the quotient, which we denote by K_A, is called the Kummer surface of A. If A admits a certain kind of line bundle (see Sect. 2.1) there is an induced map $A \to \mathbb{P}^3$ which factors through an embedding $K_A \to \mathbb{P}^3$ such that the image of the Kummer surface is a quartic with 16 nodes. Moreover, the natural action of A_2 on K_A extends to a linear action on $\mathbb{P}^3$. In one set of coordinates (x, y, z, w) on $\mathbb{P}^3$ this action is given by identifying A_2 with the subgroup of $\mathrm{Aut}(\mathbb{P}^3)$ which is generated by the following four transformations:

$$(x, y, z, w) \mapsto (z, w, x, y), \qquad (x, y, z, w) \mapsto (y, x, w, z),$$
$$(x, y, z, w) \mapsto (x, y, -z, -w), \quad (x, y, z, w) \mapsto (x, -y, z, -w).$$

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The subject of this paper is the family of all quartic surfaces in $\mathbb{P}^3$ which are invariant under these transformations. We will refer to such surfaces as *Heisenberg invariant quartics* or just *invariant quartics*. This family of quartics appeared in the classical treatises [20,24] and also in several later works [4,28,36]. The family is parameterized by $\mathbb{P}^4$ and the subfamily of Kummer surfaces described above constitutes a Zariski open dense subset of a hypersurface $S_3 \subset \mathbb{P}^4$ known as the Segre cubic. However, the general Heisenberg invariant quartic is smooth. Special smooth members of the family appear throughout the literature, for example the Fermat quartic $x^4 + y^4 + z^4 + w^4 = 0$ which is studied in [34] and the pencil $x^4 + y^4 + z^4 + w^4 + \lambda xyzw = 0$, which has been studied from the point of view of mirror symmetry [11] and the periodicity of the Clebsch contravariant [13]. In [4], Barth and Nieto study the locus of Heisenberg invariant quartics that contain a line and find a quintic threefold $N_5 \subset \mathbb{P}^4$ such that the general point corresponds to a desingularized Kummer surface of an Abelian surface with a $(1, 3)$ polarization. Prior to that, these surfaces had been discovered by Traynard [35] and discussed by Godeaux [16] and Naruki [29]. The present paper was motivated by the following question: which Heisenberg invariant quartics contain a conic? It turns out that a general invariant quartic contains 320 smooth conics. The conics are found by a direct computation using the geometry of the family. Another result of this paper is that a very general member of the family of surfaces has Picard number 16. This is in accordance with the fact that certain moduli spaces of K3 surfaces whose Picard lattices contain (an isomorphic copy of) a fixed lattice M have dimension $20 - \text{rank}(M)$. The group action on the surfaces induces an action on the Picard group and we also show that, for a very general surface in the family, the sublattice of invariant divisor classes is generated by the class of the hyperplane section. In particular, any invariant curve on such a surface is a complete intersection. Next we determine the Picard lattice of a very general Heisenberg invariant quartic and show that it is generated by the 320 smooth conics. This is done by computing the configuration of the 320 conics as well as using some general facts on the existence of curves on Kummer surfaces.

The paper is organized as follows. Sections 2 and 3 set the notation and review some results that we need for the sequel. In Sect. 4 we determine the Picard number of a very generic invariant quartic. Section 5 concerns the existence of the 320 conics. In Sects. 6 and 7 we study configurations of lines on Heisenberg invariant quartic surfaces and use the results to determine the configuration of the 320 smooth conics on a very general surface. In Sect. 8 we determine the Picard lattice of a very general surface in the family up to isomorphism and show that it is generated by the 320 smooth conics.

2 The family of invariant quartics

2.1 Kummer surfaces

We begin with an overview of Kummer surfaces, for proofs and notation see [8, 18,20,28]. Let A be an Abelian surface over $\mathbb{C}$. The subgroup of 2-torsion points $A_2 = \{a \in A : 2a = 0\}$ has order 16 and hence $A_2 \cong (\mathbb{Z}/2\mathbb{Z})^4$. To an element $a \in A$ we associate a translation map $t_a \colon A \to A \colon x \mapsto x + a$. The involution

$i: A \rightarrow A: a \mapsto -a$ induces a $\mathbb{Z}/2\mathbb{Z}$ action on A and the quotient $K_A = A/\{1, i\}$ is an algebraic surface called the Kummer surface of A. Let $\pi: A \rightarrow K_A$ be the projection. The Kummer surface has 16 singular points, namely $\pi(A_2)$, and the action of A_2 on A by translations induces an action on K_A. For an ample line bundle L on A we define the Heisenberg group

$$H(L) = \{a \in A : t_a^* L \cong L\} \subset A,$$

and the set

$$\mathcal{G}(L) = \{(x, \phi) : x \in H(L), \ \phi: L \xrightarrow{\cong} t_x^* L\}.$$

If $(x, \phi) \in \mathcal{G}(L)$ and $(y, \psi) \in \mathcal{G}(L)$ then there is an induced isomorphism $t_x^* \psi: t_x^* L \rightarrow t_x^*(t_y^* L)$. Using that $t_x^*(t_y^* L) = t_{x+y}^* L$ we put a group structure on $\mathcal{G}(L)$ by letting

$$(y, \psi)(x, \phi) = (x + y, t_x^* \psi \circ \phi).$$

These two groups are connected by an exact sequence,

$$1 \rightarrow \mathbb{C}^* \rightarrow \mathcal{G}(L) \rightarrow H(L) \rightarrow 1,$$

where $\mathcal{G}(L) \rightarrow H(L): (x, \phi) \mapsto x$. The kernel of the map $\mathcal{G}(L) \rightarrow H(L)$ is the group of automorphisms of L, which is the multiplicative group $\mathbb{C}^*$ acting by multiplication by constants. We will consider the case where A has a principal polarization L', that is L' is an ample line bundle on A whose elementary divisors are both equal to 1. In addition we assume that L' is symmetric and irreducible. The former means that $i^* L' \cong L'$ where $i: A \rightarrow A: a \mapsto -a$, and the latter means that the polarized Abelian surface (A, L') does not split as a product of elliptic curves. We say that the line bundle $L = L' \otimes L'$ is of type $(2, 2)$. Then $H(L) = A_2$, $\dim(\Gamma(A, L)) = 4$ and if D is a divisor on A that corresponds to L, then $D^2 = 8$. The group $\mathcal{G}(L)$ has an action on the space of global sections of L: for $z = (x, \phi)$ and $s \in \Gamma(A, L)$, ϕ induces a section $\phi(s)$ of $t_x^* L$ and the translation t_{-x}^* induces a section $t_{-x}^*(\phi(s))$ of $L = t_{-x}^*(t_x^* L)$. Thus we put $zs = t_{-x}^*(\phi(s))$. This makes $\Gamma(A, L)$ into an irreducible $\mathcal{G}(L)$ module such that $\mathbb{C}^*$ acts by rescaling. After choosing a basis of $\Gamma(A, L)$, this gives a faithful linear action of $H(L) = A_2$ on $\mathbb{P}^3$. The rational map $A \rightarrow \mathbb{P}^3$ induced by L is defined everywhere and factors through an embedding $K_A \rightarrow \mathbb{P}^3$ and the image of the Kummer surface is a quartic with 16 nodes. This is the maximal number of nodes of a quartic surface in $\mathbb{P}^3$ and any quartic in $\mathbb{P}^3$ with 16 nodes is a Kummer surface. For each of the nodes p, there is a plane P which contains p and five other points in the orbit of p under $H(L)$. Moreover, these six points lie on a non-degenerate conic C and the plane P touches the Kummer surface along C. We say that P is a trope. In total we have 16 tropes arising this way and together with the 16 nodes they form Kummer's 16_6 configuration: each plane contains six points and there are six planes through each point. Now, the $H(L)$ action on $\mathbb{P}^3$ restricts to the given A_2 action on K_A and the embedded Kummer surface is thus invariant under the linear

action on the ambient space. The Stone–von-Neumann–Mumford theorem states that, up to isomorphism, $\mathcal{G}(L)$ has a unique irreducible representation such that $\mathbb{C}^*$ acts by rescaling. Throughout the paper we will fix coordinates on $\mathbb{P}^3$ and the particular action of $H(L)$ given in the introduction. Fixing the action of $H(L)$ merely reflects a choice of coordinates though, see for example [18, Lemma 1.52].

2.2 The Heisenberg group

Let (x, y, z, w) be coordinates on $\mathbb{C}^4$ and consider the following four elements of $\mathrm{SL}_4(\mathbb{C})$:

$$\sigma_1 \colon (x, y, z, w) \mapsto (z, w, x, y), \qquad \sigma_2 \colon (x, y, z, w) \mapsto (y, x, w, z),$$
$$\tau_1 \colon (x, y, z, w) \mapsto (x, y, -z, -w), \qquad \tau_2 \colon (x, y, z, w) \mapsto (x, -y, z, -w).$$

Let $H_{2,2}$ be the subgroup of $\mathrm{SL}_4(\mathbb{C})$ generated by $\sigma_1, \sigma_2, \tau_1$, and τ_2. This group is of order 32 and the center and commutator of $H_{2,2}$ are both equal to $\{1, -1\}$, where 1 denotes the identity matrix of size 4. Let

$$H = H_{2,2}/\{1, -1\} \subset \mathrm{Aut}(\mathbb{P}^3),$$

a group of order 16. Since every element of H has order 2, it follows that

$$H \cong (\mathbb{Z}/2\mathbb{Z})^4.$$

In the sequel we will refer to H as the *Heisenberg group*.

We consider H as an $\mathbb{F}_2$-vector space $(\mathbb{F}_2)^4 \cong H$. It carries a symplectic form given by $\langle g, h \rangle = 0$ if $g, h \in H_{2,2}$ commute and $\langle g, h \rangle = 1$ if they anticommute. For each element $g \in H_{2,2}, g \neq \pm 1$, the fixed points of $g \in H$ form two skew lines in $\mathbb{P}^3$. These correspond to the eigenspaces of g in $\mathbb{C}^4$, with eigenvalues ± 1 or $\pm\sqrt{-1}$. In total we have 30 such lines in $\mathbb{P}^3$ which will be called the *fix lines*. If $h \in H_{2,2}$ commutes with g, then h leaves the fix lines belonging to g invariant and, if h anticommutes with g, then it flips the fix lines of g. Two fix lines belonging to $g, h \in H_{2,2}$, where $g \neq \pm h$, intersect if and only if g and h commute.

Let $\mathfrak{S}_n$ denote the symmetric group on n letters and let N be the normalizer of $H_{2,2}$ in $\mathrm{SL}_4(\mathbb{C})$,

$$N = \{n \in \mathrm{SL}_4(\mathbb{C}) : nH_{2,2}n^{-1} = H_{2,2}\}.$$

Let $n \in N$. The reason for introducing N is that if a subset $X \subseteq \mathbb{P}^3$ is invariant under H and $n \in N$, then nX is also invariant. In fact, for any $g \in H_{2,2}$ there is an $h \in H_{2,2}$ such that $gn = nh$, and hence $g(nX) = n(hX) = nX$. We now proceed to give a useful relation between N and $\mathfrak{S}_6$. Since N acts on $H_{2,2}$ by conjugation, and acts as the identity on the center $\{1, -1\}$, N acts on the vector space $H_{2,2}/\{1, -1\} \cong (\mathbb{F}_2)^4$. This action is linear and the transformations preserve the symplectic form. Thus we have a map

$$\phi: N \to \mathrm{Sp}_4(\mathbb{F}_2).$$

The kernel obviously contains $H_{2,2}$. In fact it contains the group $\langle \pm i, H_{2,2} \rangle$ of order 64 which is generated by $H_{2,2}$ and multiplication by $\pm i$, where $i = \sqrt{-1}$. By [10, Theorem 118], $\mathrm{Sp}_4(\mathbb{F}_2) \cong \mathfrak{S}_6$ and we get a sequence

$$1 \to \langle \pm i, H_{2,2} \rangle \to N \to \mathfrak{S}_6 \to 1.$$

In [30] it is shown that this sequence is exact, and thus $N/\langle \pm i, H_{2,2} \rangle \cong \mathfrak{S}_6$.

2.3 The family of surfaces

Consider the following elements of $\mathbb{C}[x, y, z, w]$:

$$g_0 = x^4 + y^4 + z^4 + w^4, \qquad g_1 = 2(x^2 y^2 + z^2 w^2), \qquad g_2 = 2(x^2 z^2 + y^2 w^2),$$
$$g_3 = 2(x^2 w^2 + y^2 z^2), \qquad g_4 = 4xyzw.$$

For $\lambda = (A, B, C, D, E) \in \mathbb{C}^5$ define

$$F_\lambda = A g_0 + B g_1 + C g_2 + D g_3 + E g_4. \tag{1}$$

Then $\{F_\lambda : \lambda \in \mathbb{C}^5, \lambda \neq 0\} \subset \mathbb{C}[x, y, z, w]$ is the set of all homogeneous quartic polynomials in $\{x, y, z, w\}$ invariant under $H_{2,2}$. Now let $\mathfrak{X} \to \mathbb{P}^4$ be the corresponding family of Heisenberg invariant quartic surfaces:

$$\mathbb{P}^4 \times \mathbb{P}^3 \supset \mathfrak{X} = \{(\lambda, p) : F_\lambda(p) = 0\}.$$

There is a linear embedding of $\mathbb{P}^4$ into $\mathbb{P}^5$ which better exposes the symmetry of the situation. The normalizer N of $H_{2,2}$ in $\mathrm{SL}_4(\mathbb{C})$ acts on the 5-dimensional vector space of $H_{2,2}$-invariant polynomials via the natural action on $\mathbb{C}^4$. Following [4,20,30] we use six invariant quartic polynomials $t_0, \ldots, t_5$, which satisfy the relation $\sum_i t_i = 0$, to embed the parameter space in $\mathbb{P}^5$:

$$t_0 = \frac{1}{3} g_0 - g_1 - g_2 - g_3, \qquad t_1 = \frac{1}{3} g_0 - g_1 + g_2 + g_3,$$
$$t_2 = \frac{1}{3} g_0 + g_1 - g_2 + g_3, \qquad t_3 = \frac{1}{3} g_0 + g_1 + g_2 - g_3,$$
$$t_4 = -\frac{2}{3} g_0 + 2g_4, \qquad t_5 = -\frac{2}{3} g_0 - 2g_4.$$

We use homogeneous coordinates $(u_0, u_1, u_2, u_3, u_4, u_5)$ on $\mathbb{P}^5$, expressing an invariant quartic polynomial as $\sum_i t_i u_i$. The polynomials $t_0, t_1, t_2, t_3, t_4, t_5$ generate the space of all $H_{2,2}$ invariant quartic polynomials. The action of $N/\langle \pm i, H_{2,2} \rangle \cong \mathfrak{S}_6$

permutes these six elements up to a sign change and the group acts on $\mathbb{P}^5$ as the full permutation group. Let $U \cong \mathbb{P}^4$ be the linear subspace of $\mathbb{P}^5$ given by

$$U = \{u_0 + u_1 + u_2 + u_3 + u_4 + u_5 = 0\} \subset \mathbb{P}^5.$$

The parameter space $\mathbb{P}^4$ of X with coordinates (A, B, C, D, E) is identified with U via the relations

$$A = -u_4 - u_5,$$
$$B = -u_0 - u_1 + u_2 + u_3,$$
$$C = -u_0 + u_1 - u_2 + u_3,$$
$$D = -u_0 + u_1 + u_2 - u_3,$$
$$E = 2u_4 - 2u_5.$$

The transformation from U to the parameters (A, B, C, D, E) may be written as

$$u_0 = A - B - C - D,$$
$$u_1 = A - B + C + D,$$
$$u_2 = A + B - C + D,$$
$$u_3 = A + B + C - D,$$
$$u_4 = -2A + E,$$
$$u_5 = -2A - E.$$

The point is that $\mathfrak{S}_6$ acts on U by permuting the coordinates, and on the level of the quartic surfaces the action is by means of projective transformations on $\mathbb{P}^3$.

Let $\pi : X \to U$ be the projection. For a point $u \in U$ we have a corresponding surface $X_u = \pi^{-1}(u)$. By *Heisenberg invariant quartic surface* (or simply *invariant quartic*), we shall understand a fiber X_u for some $u \in U$. In addition to these there are 15 pencils of quartic surfaces in $\mathbb{P}^3$ which are invariant under H, but whose defining polynomials are not invariant under $H_{2,2}$ (these will, however, not be considered).

2.4 The Segre cubic, the Igusa quartic and the Nieto quintic

In this section we introduce three hypersurfaces in $\mathbb{P}^4$ relevant for the study of Heisenberg invariant quartics. If a group G acts on a set A we use $\mathrm{orb}_G(a)$ to denote the orbit of a point $a \in A$.

First consider the following sets of points, planes and 3-planes in the parameter space U:

$$q_0 = (1, 1, 1, -1, -1, -1),$$
$$t_0 = (1, -1, 0, 0, 0, 0),$$
$$p_0 = \{u_0 + u_1 = u_2 + u_3 = u_4 + u_5 = 0\},$$

$$e_0 = \{u_0 + u_1 = u_0 + u_1 + u_2 + u_3 + u_4 + u_5 = 0\},$$
$$\mathcal{Q} = \mathrm{orb}_{\mathfrak{S}_6}(q_0) \quad (10 \text{ points}),$$
$$\mathcal{T} = \mathrm{orb}_{\mathfrak{S}_6}(t_0) \quad (15 \text{ points}),$$
$$\mathcal{P} = \mathrm{orb}_{\mathfrak{S}_6}(p_0) \quad (15 \text{ planes}),$$
$$\mathcal{E} = \mathrm{orb}_{\mathfrak{S}_6}(e_0) \quad (15 \text{ 3-planes}).$$

These points, planes and 3-planes relate to a $\mathfrak{S}_6$-invariant cubic hypersurface $S_3 \subset U \cong \mathbb{P}^4$ known as the *Segre cubic*. In $\mathbb{P}^5$ it is defined by the equations

$$S_3 = \{u_0^3 + u_1^3 + u_2^3 + u_3^3 + u_4^3 + u_5^3 = u_0 + u_1 + u_2 + u_3 + u_4 + u_5 = 0\}.$$

The Segre cubic has ten nodes, namely the $\mathfrak{S}_6$-orbit $\mathcal{Q}$. It also contains the 15 points of $\mathcal{T}$, and the set of 3-planes $\mathcal{E}$ is the set of tangent spaces to S_3 at these points. For $t \in \mathcal{T}$ the intersection between S_3 and the tangent space $T_t S_3$ is a union of three 2-planes. In total we have 15 such planes in S_3, called the *Segre planes*. This is the set $\mathcal{P}$. In [31], Nieto follows Jessop [24] in proving the following statement.

Proposition 2.1 *Let $u = (u_0, \ldots, u_5) \in U$. The surface $\mathcal{X}_u$ is singular if and only if*

$$\left(u_0^3 + u_1^3 + u_2^3 + u_3^3 + u_4^3 + u_5^3\right) \prod_{i<j} (u_i + u_j) = 0.$$

In other words, the subset of U parameterizing the singular invariant quartics is the threefold

$$S_3 \cup \bigcup_{t \in \mathcal{T}} T_t S_3.$$

It is shown in [20,24] that a general point on the Segre cubic corresponds to an invariant surface with exactly 16 nodes which constitute an orbit under the Heisenberg group. Such a surface acquires Kummer's 16_6 configuration of nodes and tropes and is a Kummer surface associated to some Abelian surface.

Proposition 2.2 *For generic $u \in S_3$, $\mathcal{X}_u$ is a Kummer surface.*

For $u \in U$, we say that $\mathcal{X}_u$ is a *quadric of multiplicity two*, if its defining polynomial is a square of a degree two polynomial. By *invariant tetrahedron* we understand a Heisenberg invariant quartic surface which is a union of four planes. One checks by direct computation that there are exactly 10 quadrics of multiplicity 2 and 15 invariant tetrahedra in the family. These correspond to the 10 nodes $\mathcal{Q}$ of S_3 and the 15 points $\mathcal{T} \subset S_3$. The quadrics of multiplicity two are called the *fundamental quadrics*. There is an explanation of the fundamental quadrics and invariant tetrahedra in terms of the symplectic structure on $H \cong (\mathbb{F}_2)^4$. This vector space has 35 planes, each consisting of four points. If $g, h \in H_{2,2}$ span a plane in H, then that plane will be called isotropic if g and h commute and non-isotropic if they anticommute. Of the 35 planes, 15 are isotropic and 20 are non-isotropic. For an isotropic plane $\{1, g, h, gh\}$, the three pairs

of fix lines belonging to g, h and gh are the edges of an invariant tetrahedron. Similarly the non-isotropic planes explain the fundamental quadrics. A plane is isotropic if and only if it is equal to its orthogonal complement and therefore the non-isotropic planes form 10 pairs of orthogonal planes. For each such pair P_1, P_2, the six fix lines belonging to elements of P_1 are skew and likewise for P_2. Moreover, the six fix lines belonging to elements of P_1 intersect all the six fix lines belonging to elements of P_2. All 12 lines lie on a fundamental quadric, with six lines in one ruling and six lines in the other.

The dual variety of S_3 is a quartic hypersurface in $\mathbb{P}^4$ called the *Igusa quartic*, which we denote by I_4. The singular locus of I_4 consists of 15 lines. One way to view I_4 is to look at the linear system of invariant quartics. Recall the basis $\{g_0, \ldots, g_4\}$ of this system introduced above. Because the system does not have a base point we have a morphism

$$\alpha : \mathbb{P}^3 \to \mathbb{P}^4 : q \mapsto (g_0(q), \ldots, g_4(q)).$$

The image of α is I_4 [1]. For a hyperplane $L \subset \mathbb{P}^4$ we have a corresponding Heisenberg invariant quartic surface $X_L = \alpha^{-1}(L)$. In the following proposition we we will call a surface W a quotient by H of X_L if there is a morphism $X_L \to W$ whose fibers are exactly the orbits of H.

Proposition 2.3 *For any hyperplane $L \subset \mathbb{P}^4$, $L \cap I_4$ is a quotient of $X_L = \alpha^{-1}(L)$ by the Heisenberg group with quotient map $\alpha : X_L \to L \cap I_4$.*

The major part of Proposition 2.3 follows from the discussion in [1, Chapter VII, p. 209]. More precisely it is shown there that if $p = (x, y, z, w) \in \mathbb{P}^3$ is such that $xyzw \neq 0$ then $\alpha^{-1}(\alpha(p))$ has at most 16 elements. Thus, if in addition we assume that p is not on any fix line (that is the orbit of p has 16 elements), we have that $\alpha^{-1}(\alpha(p))$ is exactly the orbit of p since clearly α is constant on the orbits. It is straightforward to verify the remaining special cases where some coordinate of p is zero or p is on a fix line.

A generic hyperplane $L \subset \mathbb{P}^4$ intersects I_4 in a quartic surface with 15 singular points and 10 tropes. The singular points come from the intersections with the singular lines of I_4. On the quotient X_L/H, there are 15 singularities that come from points with nontrivial stabilizers in H. Namely, for any of the 15 non-identity elements of H, the two fix lines intersect X_L in an eight point orbit that maps to one singular point on X_L/H by the quotient map. Now, if L is a generic tangent space to I_4, then $I_4 \cap L$ has an additional singularity at the point of tangency and $I_4 \cap L$ is a Kummer surface. In this way I_4 is a compactification of a moduli space of principally polarized Abelian surfaces with a level-2 structure [15,20].

Let us now look at one more $\mathfrak{S}_6$ invariant threefold in the parameter space $U \cong \mathbb{P}^4$, the *Nieto quintic* N_5. This is the Hessian variety to S_3, that is the zeros of the determinant of the Hessian matrix of the polynomial defining S_3. The singular locus of N_5 consists of 20 lines together with the 10 nodes of S_3. With our choice of coordinates, viewing $\mathbb{P}^4 \subset \mathbb{P}^5$, N_5 is defined by the equations

$$\sum_{i=0}^{5} u_i = \sum_{i=0}^{5} \frac{1}{u_i} = 0,$$

where $\sum_{i=0}^{5} 1/u_i = u_1 u_2 u_3 u_4 u_5 + u_0 u_2 u_3 u_4 u_5 + \cdots + u_0 u_1 u_2 u_3 u_4$. The Nieto quintic and its connection to Heisenberg invariant quartics which contain lines is studied in [4] where the following is proved ([4, Sections 7 and 8]):

Proposition 2.4 *The locus in U parameterizing Heisenberg invariant quartics which contain a line is equal to N_5 together with the* 10 *tangent cones to the isolated singular points of N_5 (the* 10 *nodes of S_3).*

3 K3 surfaces, lattices and moduli

This section consists of background material on lattices, K3 surfaces, Picard groups and moduli spaces. In particular, we state some well-known results on moduli spaces of K3 surfaces and relations to Picard groups, references on this topic include [11,12,21–23,33].

A lattice M is a finitely generated free Abelian group equipped with an integral symmetric bilinear form. An important invariant of M is the discriminant, denoted $\mathrm{discr}(M)$, which is equal to the determinant of any matrix representing the bilinear form. We shall deal only with non-degenerate lattices, that is we assume that the discriminant is non-zero. If the discriminant is equal to ± 1, the lattice is called unimodular. If $N \subseteq M$ is a sublattice of full rank, then $\mathrm{discr}(N)/\mathrm{discr}(M) = [M:N]^2$, where $[M:N]$ is the index of N in M, see [3, I.2.1]. If $N \subseteq M$ is a sublattice, we use the notation $N^{\perp}$ for the orthogonal complement of N in M. A lattice is called even if the $\mathbb{Z}$-valued quadratic form associated to the bilinear form takes on even values only. There is an induced bilinear form on the real vector space $M \otimes_{\mathbb{Z}} \mathbb{R}$ which may be diagonalized in such a way that only ± 1 appear on the diagonal. The signature of a lattice M is a pair of numbers (a, b), where a is the number of positive entries in such a diagonal matrix and b the number of negative entries. The sum of a and b is the rank of M. A sublattice N of a lattice M is called primitive if M/N is torsion free. An isomorphism of lattices is a group isomorphism that also respects the bilinear form.

For a scheme X, the Picard group is the group of isomorphism classes of line bundles on X, and is denoted $\mathrm{Pic}(X)$. If X is a smooth variety over an algebraically closed field, then $\mathrm{Pic}(X)$ is isomorphic to the group of Weil divisors on X modulo linear equivalence. For a smooth complex projective variety X, let $\mathrm{Pic}^0(X)$ denote the subgroup of $\mathrm{Pic}(X)$ corresponding to divisors algebraically equivalent to 0 and define the Néron–Severi group by $\mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$. The Néron–Severi group is a finitely generated Abelian group and its rank is called the Picard number of X. Now suppose that X is a complex projective K3 surface, which means that X is a smooth simply connected projective complex variety of dimension 2, whose canonical bundle is trivial. In this case $\mathrm{Pic}^0(X) = \{0\}$ and thus $\mathrm{NS}(X) = \mathrm{Pic}(X)$ and $\mathrm{Pic}(X)$ is free and finitely generated. It is well known that $1 \leqslant \rho(X) \leqslant 20$, where $\rho(X) = \mathrm{rank}(\mathrm{Pic}(X))$ is the Picard number of X. The second cohomology group $H^2(X, \mathbb{Z})$ is also free and of rank 22, and the cup product defines a bilinear form giving it the structure of a lattice. Abstractly, this lattice is isomorphic to the so-called K3 lattice

$$L = U \oplus U \oplus U \oplus (-E_8) \oplus (-E_8).$$

Here, U is the hyperbolic plane, a rank 2 lattice with bilinear form given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and E_8 is the lattice of rank 8 whose bilinear form is given by the Cartan matrix of the root system E_8. The K3 lattice is even, unimodular and of signature $(3, 19)$. The Picard group of X may be viewed as a non-degenerate sublattice of $H^2(X, \mathbb{Z})$ with signature $(1, \rho(X) - 1)$, where the bilinear form on $\mathrm{Pic}(X)$ is the intersection product. Since the lattice L is even, so is $\mathrm{Pic}(X)$. The orthogonal complement of $\mathrm{Pic}(X)$ in $H^2(X, \mathbb{Z})$ is called the transcendental lattice and is denoted by T_X.

We now turn to a particular class of K3 surfaces, namely smooth quartic surfaces in $\mathbb{P}^3$. A homogeneous quartic polynomial in four variables has 35 coefficients and therefore $\mathbb{P}^{34}$ parameterizes quartic surfaces in $\mathbb{P}^3$. Consider the open set $O \subseteq \mathbb{P}^{34}$ that parameterizes smooth surfaces. The group $\mathrm{PGL}_4(\mathbb{C})$ acts on O by means of isomorphisms of $\mathbb{P}^3$ and we will use $\mathcal{M}_2$ to denote the GIT quotient with respect to this action, see [22, Chapter 5, Example 2.5]. Thus $\mathcal{M}_2$ is the (coarse) moduli space of smooth quartic surfaces in $\mathbb{P}^3$ parameterizing such surfaces up to projective transformations. It is an irreducible quasi projective variety of dimension 19.

A marking of a K3 surface X is an isomorphism of lattices $L \cong H^2(X, \mathbb{Z})$. Below we will make use of the (fine) moduli space of marked polarized K3 surfaces of degree 4. This is a complex manifold which we will denote by $\mathcal{N}_2$, see [22, Chapter 6, 3.4]. For $m \in \mathcal{N}_2$, let $\rho(m)$ denote the Picard number of the surface corresponding to m and consider the *higher Noether–Lefschetz loci*,

$$\mathrm{NL}_k = \{m \in \mathcal{N}_2 : \rho(m) \geqslant k\},$$

where k is an integer $1 \leqslant k \leqslant 20$. The Noether–Lefschetz locus NL_k may be written as a countable union of analytic subvarieties of $\mathcal{N}_2$ of dimension $20 - k$, see [22, Chapter 17, 1.3].

Definition 3.1 We say that a condition holds for a *very generic* point, or *very general* point, of a variety Y if there is a countable union $K = \bigcup_{i \in \mathbb{Z}} K_i$ of proper Zariski-closed subsets $K_i \subset Y$, such that the condition holds for all points $y \in Y \setminus K$.

Let $O \subset \mathbb{P}^{34}$ be the subset parameterizing smooth quartic surfaces, let $Z \subseteq O$ be a subvariety and $\mathcal{Y} \to Z$ the corresponding family of surfaces. We use $\rho_0(Z)$ to denote the minimal Picard number of any fiber in the family.

Lemma 3.2 *If $Z \subseteq O \subseteq \mathbb{P}^{34}$ is a subvariety of O such that $Z \to \mathcal{M}_2$ is generically finite to one, then $\dim(Z) \leqslant 20 - \rho_0(Z)$.*

Proof Let $k = \rho_0(Z)$. For a general smooth point $p \in Z$ there is an open analytic neighborhood $V \subseteq O$ of p such that $V \cap Z \to \mathcal{M}_2$ is injective. By shrinking V we may assume that $B = V \cap Z$ is smooth, connected and simply connected. The family of surfaces with base B may be marked by choosing a marking of some fiber, which yields a canonical marking of all fibers, see [22, Chapter 6, 2.2]. This gives a holomorphic map $\phi \colon B \to \mathcal{N}_2$ which is also injective since if $\phi(b) = \phi(b')$ then the fibers over b

and b' are projectively equivalent. Since $\phi(B) \subseteq \mathrm{NL}_k$ and NL_k is a countable union of $(20-k)$-dimensional analytic subvarieties of $\mathcal{N}_2$, $\dim(Z) = \dim(B) \leqslant 20 - k$. $\square$

4 The Picard number

We will show that the Picard number of a very general Heisenberg invariant quartic is 16. A corollary to this is that any invariant divisor class is a multiple of the hyperplane section. We start with some generalities on symplectic group actions on K3 surfaces.

If a group G acts on a K3 surface X, then G acts on $H^2(X, \mathbb{Z})$ and the action restricts to the sublattices $\mathrm{Pic}(X)$ and T_X. We will write this action g^*l for $g \in G$ and $l \in H^2(X, \mathbb{Z})$. Let

$$H^2(X, \mathbb{Z})^G = \{l \in H^2(X, \mathbb{Z}) : g^*l = l, \text{ for all } g \in G\},$$

and $\mathrm{Pic}(X)^G = H^2(X, \mathbb{Z})^G \cap \mathrm{Pic}(X)$. Since the canonical bundle of a K3 surface X is trivial, X has a unique non-zero holomorphic 2-form ω, up to rescaling. Hence for $f \in \mathrm{Aut}(X)$, $f^*\omega = \lambda\omega$, where $\lambda \in \mathbb{C}$. If $\lambda = 1$, then f is called symplectic. In the following proposition, the first claim is due to Nikulin [32] and the second claim is due to Mukai [27].

Proposition 4.1 *Let G be a finite group of symplectic automorphisms of a K3 surface X and let $\Omega_G = (H^2(X, \mathbb{Z})^G)^\perp$ be the orthogonal complement of $H^2(X, \mathbb{Z})^G$ in $H^2(X, \mathbb{Z})$. For $g \in G$, $g \neq 1$, let $f(g)$ be the number of fixed points of $g \in G$ and put $f(1) = 24$. Then*

(a) $T_X \subseteq H^2(X, \mathbb{Z})^G$ *and* $\Omega_G \subseteq \mathrm{Pic}(X)$,
(b) $\mathrm{rank}(H^2(X, \mathbb{Z})^G) + 2 = \frac{1}{|G|} \sum_{g \in G} f(g)$.

The generic member of our family is a smooth quartic in $\mathbb{P}^3$ and hence a K3 surface. Since the action of the Heisenberg group is induced by a representation $H_{2,2} \to \mathrm{SL}_4(\mathbb{C})$ and the action of $H_{2,2}$ preserves the defining polynomials of our surfaces, the action of H is symplectic by [27, Lemma 2.1].

Corollary 4.2 *If $X \subset \mathbb{P}^3$ is a generic Heisenberg invariant quartic, then*

(a) $\mathrm{rank}(H^2(X, \mathbb{Z})^H) = 7$,
(b) $16 \leqslant \rho(X)$.

Proof Let $g \in H$, $g \neq 1$. The fixed points of g form two skew lines in $\mathbb{P}^3$, and hence g has eight fixed points on X. By Proposition 4.1, $T_X \subseteq H^2(X, \mathbb{Z})^H$ and $\mathrm{rank}(H^2(X, \mathbb{Z})^H) = 7$. Assume that $\mathrm{rank}(T_X) = 7$. The hyperplane class $h \in \mathrm{Pic}(X)$ is invariant and thus there is an integer $n > 0$ such that $nh \in T_X$. Since T_X is the orthogonal complement of $\mathrm{Pic}(X)$, $0 = (nh)h = 4n$, a contradiction. Hence $\mathrm{rank}(T_X) \leqslant 6$. It follows that $16 \leqslant \mathrm{rank}(\mathrm{Pic}(X))$, since $\mathrm{rank}(\mathrm{Pic}(X)) + \mathrm{rank}(T_X) = \mathrm{rank}(H^2(X, \mathbb{Z})) = 22$. $\square$

For homogeneous $F \in \mathbb{C}[x, y, z, w]$, let $X = \{F = 0\} \subset \mathbb{P}^3$ and consider the Hessian surface of X,

$$\mathrm{Hess}(X) = \{x \in \mathbb{P}^3 : \det(\mathrm{Hess}(F)(x)) = 0\},$$

where $\text{Hess}(F)$ is the Hessian matrix of second derivatives of F. Let $T \in \text{GL}_4(\mathbb{C})$ and denote the induced automorphism of $\mathbb{P}^3$ also by T. Let $Y = T(X)$ and let $(T^{-1})^t$ denote the transpose of T^{-1}. Note that for $p \in \mathbb{C}^4$,

$$\text{Hess}(F \circ T^{-1})(p) = (T^{-1})^t \cdot \text{Hess}(F)(T^{-1}(p)) \cdot T^{-1}.$$

It follows that T takes the Hessian surface of X to the Hessian surface of Y, with preserved multiplicities of irreducible components. In particular, the Hessian surface of a Heisenberg invariant quartic is a Heisenberg invariant octic.

Example 4.3 Let $u = (1, 1, 1, 1, -2, -2) \in U$. We will look at linear transformations of the Fermat quartic $X = \mathcal{X}_u = \{(x, y, z, w) \in \mathbb{P}^3 : x^4 + y^4 + z^4 + w^4 = 0\}$ for the purpose of applying Lemma 3.2.

There are 15 surfaces in the family that are projectively equivalent to X corresponding to the $\mathfrak{S}_6$-orbit of $(1, 1, 1, 1, -2, -2)$, and we shall see that there are no others. The Hessian surface of the Fermat quartic is the zeros of $x^2 y^2 z^2 w^2$, an invariant tetrahedron of double multiplicity. Note that the invariant tetrahedron defined by $xyzw$ corresponds to the point $(0, 0, 0, 0, 1, -1) \in U$. Suppose that $T \in \text{GL}_4(\mathbb{C})$ takes X to some Heisenberg invariant quartic Y. By the discussion above, the Hessian surface of Y must be one of the 15 invariant tetrahedra, counted with multiplicity 2. These correspond to the $\mathfrak{S}_6$-orbit of $(0, 0, 0, 0, 1, -1)$. Let $S \in \text{SL}_4(\mathbb{C})$ be a transformation in the normalizer of $H_{2,2}$ in $\text{SL}_4(\mathbb{C})$ which takes $\text{Hess}(Y)$ to $\text{Hess}(X)$. Then $S \circ T : \mathbb{P}^3 \to \mathbb{P}^3$ induces an automorphism of $\text{Hess}(X)$ and therefore corresponds to a permutation of the coordinates (x, y, z, w) possibly combined with a rescaling of the axes. Since $S(T(X))$ is an invariant quartic, $S(T(X)) = X$ and $T(X) = S^{-1}(X)$. Therefore $T(X)$ is in the $\mathfrak{S}_6$-orbit of $(1, 1, 1, 1, -2, -2)$.

Theorem 4.4 *A very generic Heisenberg invariant quartic $X \subset \mathbb{P}^3$ has Picard number 16.*

Proof By the second claim of Corollary 4.2, $\rho(X) \geqslant 16$. Let $Z \subset \mathbb{P}^{34}$ be the subset parameterizing smooth Heisenberg invariant quartics. The quotient map $Z \to \mathcal{M}_2$ is generically finite to one by upper-semi-continuity of the dimension of fibers and Example 4.3. By Lemma 7.1, $\rho(X) = \rho_0(Z)$ and using Lemma 3.2 we get $\rho(X) \leqslant 20 - 4 = 16$. $\qquad\square$

Corollary 4.5 *For very generic $u \in U$ and $X = \mathcal{X}_u$, $\text{Pic}(X)^H = \mathbb{Z}h$, where $h \in \text{Pic}(X)$ is the class of the hyperplane section. In particular, every invariant curve on X is a complete intersection between X and another surface in $\mathbb{P}^3$.*

Proof Since $\text{rank}(\text{Pic}(X)) = 16$,

$$\text{rank}(T_X) = \text{rank}(H^2(X, \mathbb{Z})) - \text{rank}(\text{Pic}(X)) = 6.$$

Because $T_X \cap \text{Pic}(X) = \{0\}$, and T_X is contained in the rank 7 lattice $H^2(X, \mathbb{Z})^H$, it follows that $\text{Pic}(X)^H$ has rank at most 1. Clearly, the class h is invariant under H and thus $\text{rank}(\text{Pic}(X)^H) = 1$. Let D be a generator of $\text{Pic}(X)^H$ and let $n \in \mathbb{Z}$ be such that $h = nD$. Since $4 = h^2 = n^2 D^2$, and D^2 is even since $\text{Pic}(X)$ is an even lattice, it follows that $n = \pm 1$. $\qquad\square$

5 Conics on the invariant surfaces

Definition 5.1 Let X be a quartic surface in $\mathbb{P}^3$. We say that a plane L in $\mathbb{P}^3$ is a *trope* of X if $X \cap L$ is an irreducible conic counted with multiplicity 2.

Lemma 5.2 *A quartic surface $X \subset \mathbb{P}^3$ which has a trope L is necessarily singular.*

Proof Let (x, y, z, w) be coordinates on $\mathbb{P}^3$. Change coordinates so that $L = \{x = 0\}$. Then X is defined by $x F(x, y, z, w) + (G(y, z, w))^2 = 0$, for some cubic polynomial F and quadratic polynomial G. Then the set $M = \{x = 0\} \cap \{F = 0\} \cap \{G = 0\}$ is nonempty and inside the singular locus of X. $\qquad\square$

Below we shall make use of the following fact: for generic $u \in S_3$, the Kummer surface $\mathcal{X}_u$ is uniquely determined by any of its nodes as well as any of its tropes. To establish this, let $F_\lambda(x, y, z, w)$ be a Heisenberg invariant polynomial depending on the parameters $\lambda = (A, B, C, D, E)$ as in (1). For a general $p \in \mathbb{P}^3$ there is a unique $u \in U$ such that $\mathcal{X}_u$ is singular at p. To see this, it is enough to check that there is a point $p = (x, y, z, w) \in \mathbb{P}^3$ such that the system of linear equations

$$\frac{\partial F_\lambda}{\partial x}(p) = \frac{\partial F_\lambda}{\partial y}(p) = \frac{\partial F_\lambda}{\partial z}(p) = \frac{\partial F_\lambda}{\partial w}(p) = 0$$

has a unique solution $(A, B, C, D, E) \in \mathbb{P}^4$ (this is true for example if $p = (1, 2, 3, 4)$). In fact, there exists a unique invariant quartic singular at $p \in \mathbb{P}^3$ exactly when p is not on a fix line [4]. Using $\mathfrak{S}_6$-invariance one checks that the set of $u \in U$ such that $\mathcal{X}_u$ has a singular point that lies on some fix line is equal to the union of the 15 tangent spaces $T_t S_3$ where $\mathcal{X}_t$ is an invariant tetrahedron. We conclude that a generic Heisenberg invariant Kummer surface is determined by any of its nodes. The corresponding statement for tropes is clear once we see that $p \in \mathbb{P}^3$ is a node precisely when the plane $p^* \in \mathbb{P}^{3*}$ with the same coordinates as p is a trope. Let p be a node, and note that we may assume p to be a generic point in $\mathbb{P}^3$. As in [20, Chapter I, Section 3], there are six points in the orbit of p that lie in p^*. It follows that the six points are multiple points of the curve of intersection between p^* and $\mathcal{X}_u$. This curve must then either contain a line as an irreducible component or be a conic of multiplicity two. The first case is excluded by Proposition 2.4. The set of 16 tropes is hence the orbit of p^*.

Theorem 5.3 *A generic invariant quartic $\mathcal{X}_u$ contains at least* 320 *smooth conics.*

Proof Pick $u \in U$ generic and let q be a node of the Segre cubic S_3. The line through u and q intersects S_3 in one additional point p, let $K(q) = \mathcal{X}_p$. Since u is generic, $K(q)$ is a Kummer surface by Proposition 2.2, and thus has 16 tropes which form an orbit under H. Let T be a trope of $K(q)$ and note that $\mathcal{X}_q$ is a quadric of multiplicity two. The polynomial defining $\mathcal{X}_u \cap T$, in homogeneous coordinates on T, is a linear combination of squares, and thus reducible. The generic member of the family does not contain any line (Proposition 2.4), and since $\mathcal{X}_u$ is smooth nor does it have a trope (Lemma 5.2). We conclude that $\mathcal{X}_u \cap T$ is a union of two smooth conics.

For two different nodes q_1 and q_2 of S_3 the corresponding Kummer surfaces $K(q_1)$ and $K(q_2)$ are different, since we may assume that u is not on the line through q_1

and q_2. Since a generic Heisenberg invariant Kummer surface is determined by any of its tropes, all tropes of the Kummer surfaces $K(q)$, for all of the 10 nodes q of S_3, are different. Since two different planes cannot have a smooth conic in common, we conclude that there are at least $2 \cdot 16 \cdot 10 = 320$ smooth conics on $\mathcal{X}_u$. □

For a general invariant quartic X, we will refer to the conics in Theorem 5.3 as *the 320 conics* on X. As we show in Proposition 7.2, these conics are the only conics on a very general surface in the family.

6 Invariant surfaces containing lines

The generic Heisenberg invariant quartic does not contain any line. The invariant surfaces that contain a line are parameterized by the Nieto quintic N_5 and the 10 tangent cones to the isolated singularities of N_5 [4]. In this section we will look at the configuration of lines on a Heisenberg invariant quartic that corresponds to a general point on N_5. This will be used to compute the intersection matrix of the 320 smooth conics on a very generic invariant quartic. General references for this section are [2,4,6] and in particular the proof of Proposition 6.1 follows arguments made in [2] and [6].

Let $(p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23})$ be the Plücker coordinates on $\mathbb{P}^5$. The Plücker embedding identifies the Grassmannian of lines in $\mathbb{P}^3$ with the quadric in $\mathbb{P}^5$ defined by the Plücker relation

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0.$$

We will use so-called Klein coordinates $(x_0, \ldots, x_5)$, which are defined in terms of Plücker coordinates by

$$\begin{aligned} x_0 &= p_{01} - p_{23}, & x_2 &= p_{02} + p_{13}, & x_4 &= p_{03} - p_{12}, \\ x_1 &= i(p_{01} + p_{23}), & x_3 &= i(p_{02} - p_{13}), & x_5 &= i(p_{03} + p_{12}), \end{aligned}$$

where $i^2 = -1$. In Klein coordinates the Plücker relation reads

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0,$$

and the condition that a line with coordinates $(x_0, \ldots, x_5)$ is coplanar with a line with coordinates $(y_0, \ldots, y_5)$ is

$$x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4 + x_5 y_5 = 0. \tag{2}$$

The Heisenberg group acts on the space of lines in $\mathbb{P}^3$ and in Klein coordinates the action is given neatly by sign changes in accordance with the table

	x_0	x_1	x_2	x_3	x_4	x_5
σ_1	−	+	−	−	+	−
σ_2	−	−	+	−	−	+
τ_1	+	+	−	−	−	−
τ_2	−	−	+	+	−	−

where σ_1, σ_2, τ_1, τ_2 are the generators from Sect. 2.2.

In [4] it is shown that for a generic $u \in N_5$, $\mathfrak{X}_u$ contains exactly 32 lines. In the same paper it is proved that such a surface is a desingularized Kummer surface coming from an Abelian surface A with a $(1, 3)$-polarization. The Klein coordinates for any of the 32 lines satisfy the condition

$$\frac{1}{x_0^2} + \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} + \frac{1}{x_4^2} + \frac{1}{x_5^2} = 0,$$

by which we mean that

$$x_1^2 x_2^2 x_3^2 x_4^2 x_5^2 + x_0^2 x_2^2 x_3^2 x_4^2 x_5^2 + \cdots + x_0^2 x_1^2 x_2^2 x_3^2 x_4^2 = 0.$$

The 32 lines constitute two orbits under the Heisenberg group, each containing 16 lines. There is an involution relating the two orbits: if $(x_0, \ldots, x_5)$ are Klein coordinates for a line in one of the orbits, then

$$\left(-\frac{1}{x_0}, \frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}, \frac{1}{x_4}, \frac{1}{x_5} \right)$$

are Klein coordinates for a line in the other orbit. The 16 lines in any of the two orbits are mutually disjoint, in fact one of the orbits is the union of the exceptional divisors coming from blowing up the singular Kummer surface in its 16 nodes. The configuration of lines is called a 32_{10} since any line in one orbit intersects exactly 10 lines in the other orbit. There are thus exactly 160 reducible conics on the surface. If two lines, one from each orbit, are coplanar then their *complement* on the surface is an irreducible conic. In other words, the plane that contains the reducible conic intersects the surface in the union of two lines and an irreducible conic. This gives rise to 160 irreducible conics on $\mathfrak{X}_u$.

Proposition 6.1 *For a very general $u \in N_5$ there are no other irreducible conics on $\mathfrak{X}_u$ except for the* 160 *complements to reducible conics.*

Proof Let $X = \mathfrak{X}_u$. The surface X is a desingularized Kummer surface of an Abelian surface A admitting a $(1, 3)$-polarization. In fact, there is an open subset of the moduli space of Abelian surfaces which admit a $(1, 3)$-polarization together with the choice of a level-2 structure which is naturally an unbranched double cover of the open subset of N_5 parameterizing smooth surfaces [4]. Since $u \in N_5$ is very general we may assume that A has Picard number 1 (see [8, Section 15.1]). Let $e_1, \ldots, e_{16}$ denote the halfperiods of A, that is e_i is a 2-torsion point or the identity element. Let $\widetilde{A}$ denote the blow-up of A in $\{e_1, \ldots, e_{16}\}$. We have morphisms

$$A \xleftarrow{\alpha} \widetilde{A} \xrightarrow{\gamma} X,$$

where γ is a double cover branched over the 16 lines $L_1, \ldots, L_{16}$ in one of the orbits and α is the blow-up map [4]. Let $E_i = \alpha^{-1}(e_i)$, $i = 1, \ldots, 16$. Then $E_i \cdot E_j = 0$ if $i \neq j$, $E_i^2 = -1$ and up to reordering $\gamma(E_i) = L_i$. Now, as is explained in

[4], if M is the line bundle corresponding to the sheaf $\mathcal{O}_X(1)$, then the line bundle $\gamma^*(M) \otimes \mathcal{O}_{\tilde{A}}(\sum_{i=1}^{16} E_i)$ on $\tilde{A}$ descends to a line bundle on A which defines a polarization of type $(2, 6)$. Further, there is a symmetric line bundle θ on A of type $(1, 3)$ such that $\alpha^*(\theta \otimes \theta) = \gamma^*(M) \otimes \mathcal{O}_{\tilde{A}}(\sum_{i=1}^{16} E_i)$, see [4]. Let T denote the divisor class in the Néron–Severi group of A that corresponds to θ and let H_X be the hyperplane class of X. Then we have that $T^2 = 2d_1d_2$ where $(d_1, d_2) = (1, 3)$ [28]. Thus $T^2 = 6$.

Now let $C \subseteq X$ be an irreducible conic different from the complements of reducible conics. By the adjunction formula, we have that $C^2 = -2$. Let $F = \alpha_*(\gamma^*(C))$. Since $T^2 = 6$ is square free, T generates the Néron–Severi group of A and we may write $F = dT$ for some integer d. Now let $m_i = C \cdot L_i = \gamma^*(C) \cdot E_i$ for $i = 1, \ldots, 16$. Then $\alpha^*(F) = \gamma^*(C) + \sum_{i=1}^{16} m_i E_i$. Thus $6d^2 = (dT)^2 = F^2 = \alpha^*(F)^2 = (\gamma^*(C) + \sum_{i=1}^{16} m_i E_i)^2$, and therefore $6d^2 = \gamma^*(C)^2 + 2\sum_{i=1}^{16} m_i \gamma^*(C) \cdot E_i + \sum_{i=1}^{16} m_i^2 E_i^2 = 2C^2 + 2\sum_{i=1}^{16} m_i^2 - \sum_{i=1}^{16} m_i^2$. In conclusion,

$$6d^2 = \sum_{i=1}^{16} m_i^2 - 4. \tag{3}$$

Furthermore, $12d = 2T \cdot dT = 2T \cdot F = \alpha^*(2T) \cdot \alpha^*(F) = (\gamma^*(H_X) + \sum_{i=1}^{16} E_i) \cdot (\gamma^*(C) + \sum_{i=1}^{16} m_i E_i)$, and hence $12d = \gamma^*(H_X) \cdot \gamma^*(C) + \sum_{i=1}^{16} m_i E_i \cdot \gamma^*(H_X) + \sum_{i=1}^{16} E_i \cdot \gamma^*(C) + \sum_{i=1}^{16} m_i E_i^2$, which implies that

$$12d = 4 + \sum_{i=1}^{16} m_i + \sum_{i=1}^{16} m_i - \sum_{i=1}^{16} m_i = 4 + \sum_{i=1}^{16} m_i. \tag{4}$$

Since L_i and C are not coplanar, $m_i = 0$ or $m_i = 1$ for all i. It follows by (4) that $d = 1$. Note also that $m_i = m_i^2$ for all i. But then (3) gives $\sum_{i=1}^{16} m_i = 10$ and (4) gives $\sum_{i=1}^{16} m_i = 8$, a contradiction.   □

Let $u \in N_5$ be very generic and put $X = \mathcal{X}_u$. As long as all the Klein coordinates $(x_0, \ldots, x_5)$ of a line l are non-zero, the matter of whether some Heisenberg translate of l intersects some Heisenberg translate of the image of l under the involution does not depend on l. From the description above of the action of the Heisenberg group on the Grassmannian of lines in $\mathbb{P}^3$, the condition (2), and the involution relating the two sets of 16 lines on X, it is straightforward to compute the intersection matrix of the 160 reducible conics on X. Since the 160 irreducible conics on X are coplanar with reducible conics, the intersection matrix of all 320 conics on X is readily deducible from the intersection matrix of the 160 reducible conics. We will not do this in detail but we note in passing that if C and D are conics on X such that C and D belong to different orbits under the Heisenberg group H, then

$C \cdot gC = 0$ for six different $g \in H$, $g \neq 1$,
$C \cdot gC = 2$ for nine different $g \in H$, $g \neq 1$,
$C \cdot gD = 0$ for four different $g \in H$,
$C \cdot gD = 1$ for eight different $g \in H$,

$$C \cdot gD = 2 \quad \text{for four different } g \in H.$$

We will see in Sect. 7 that the intersection matrix of the conics on X is the same as the intersection matrix associated to the 320 smooth conics on a very generic Heisenberg invariant quartic.

Remark 6.2 For later purposes we will consider the intersections of a special set of 16 reducible conics on X. We need to put an order on the set of all reducible conics on X. We will order the elements of the Heisenberg group as follows. Let $g \in H$, $g = \sigma_1^i \sigma_2^j \tau_1^k \tau_2^l$ where $0 \leqslant i, j, k, l \leqslant 1$ and $\sigma_1, \sigma_2, \tau_1$ and τ_2 are the generators from Sect. 2.2. Then we order H by interpreting (i, j, k, l) as a number in binary form that marks the place of g. Let L be a line on X. The order on H puts an order on the orbit of L as well as the orbit of the image of L under the involution. This induces an order on the set of reducible conics by saying that if line number a in the orbit of L intersects line number b in the other orbit, and likewise for a' and b' with $(a, b) \neq (a', b')$, then the reducible conic indexed by (a, b) is prior to the conic indexed by (a', b') if and only if $a < a'$, or $a = a'$ and $b < b'$. With that order of rows and columns, let N denote the intersection matrix of the reducible conics on X. Let M be the submatrix of N given by picking out the following rows and columns:

$$(4, 7, 21, 27, 36, 50, 75, 81, 88, 110, 114, 128, 131, 138, 141, 154).$$

This particular choice of conics has been made because, as we shall see in Sect. 8, it defines a sublattice with minimal discriminant (the conics were found using a computer aided search). Then

$$M = \begin{pmatrix}
-2 & 0 & 2 & 1 & 2 & 2 & 1 & 0 & 0 & 2 & 0 & 1 & 1 & 2 & 1 & 1 \\
0 & -2 & 1 & 0 & 2 & 1 & 2 & 0 & 0 & 1 & 2 & 2 & 1 & 2 & 1 & 2 \\
2 & 1 & -2 & 0 & 1 & 1 & 2 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 2 \\
1 & 0 & 0 & -2 & 1 & 0 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
2 & 2 & 1 & 1 & -2 & 1 & 1 & 2 & 2 & 0 & 1 & 2 & 1 & 1 & 1 & 0 \\
2 & 1 & 1 & 0 & 1 & -2 & 1 & 2 & 1 & 0 & 2 & 2 & 1 & 1 & 2 & 1 \\
1 & 2 & 2 & 2 & 1 & 1 & -2 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 2 & 2 & 1 & -2 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 2 \\
0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 & -2 & 1 & 1 & 2 & 0 & 1 & 2 & 1 \\
2 & 1 & 1 & 1 & 0 & 0 & 2 & 1 & 2 & -2 & 2 & 2 & 1 & 1 & 1 & 0 \\
0 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & -2 & 0 & 1 & 1 & 2 & 0 \\
1 & 2 & 0 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 0 & -2 & 2 & 0 & 1 & 2 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & -2 & 0 & 0 & 0 \\
2 & 2 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -2 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 2 & 1 & 0 & 2 & 1 & 2 & 1 & 0 & 1 & -2 & 1 \\
1 & 2 & 2 & 2 & 0 & 1 & 1 & 2 & 1 & 0 & 0 & 2 & 0 & 1 & 1 & -2
\end{pmatrix},$$

and $\det(M) = -512$.

7 The intersection matrix of the 320 conics

In Sect. 6 we determined the configuration of conics (reducible or irreducible) on an invariant quartic $\mathcal{X}_u$ for very generic $u \in N_5$. In this section we shall show that this configuration is the same as the configuration of the 320 irreducible conics on a very general invariant quartic. That is, the intersection matrices of the two collections of curves are the same up to reordering of the curves. The 320 smooth conics vary in a family of curves whose configuration we can compute in a special case, namely the case of a surface that contains lines.

In the process we shall also show that there are no more conics on a very general Heisenberg invariant quartic than the 320 conics of Theorem 5.3.

Lemma 7.1 *Let $O \subset \mathbb{P}^{34}$ be the open subset parameterizing smooth quartic surfaces, let $Z \subseteq O$ be a smooth subvariety and $\mathcal{Y} \to Z$ the corresponding family of surfaces. For a very general point $z \in Z$ and any $z' \in Z$ there is an injection of lattices*

$$i : \mathrm{Pic}(\mathcal{Y}_z) \hookrightarrow \mathrm{Pic}(\mathcal{Y}_{z'})$$

which respects hyperplane classes.

Proof The Néron–Severi group of the geometric generic fiber of $\mathcal{Y}$ injects into $\mathrm{Pic}(\mathcal{Y}_z)$ for any $z \in Z$ (see [26, Proposition 3.6(a)]). Moreover, for very general $z \in Z$, that injection is an isomorphism (see [26, Proposition 3.6(a) and Corollary 3.9]). This induces an injection $i : \mathrm{Pic}(\mathcal{Y}_z) \to \mathrm{Pic}(\mathcal{Y}_{z'})$, for a very general $z \in Z$ and any $z' \in Z$. This injection respects intersection numbers (see [22, Chapter 17, Proposition 2.10]). Also, the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{Pic}(\mathcal{Y}) & \xrightarrow{\quad r_1 \quad} & \mathrm{Pic}(\mathcal{Y}_{z'}), \\
& {}_{r_2}\searrow \quad \nearrow_{i} & \\
& \mathrm{Pic}(\mathcal{Y}_z) &
\end{array}
$$

where r_1 is the restriction map j^* with $j : \mathcal{Y}_{z'} \hookrightarrow \mathcal{Y}$ the inclusion and similarly for r_2 (this follows from the construction in [7, X, 7.8]). Consider the polarizations on $\mathcal{Y}_z$ and $\mathcal{Y}_{z'}$ that define the embeddings as quartic surfaces in $\mathbb{P}^3$. For a hyperplane $L \subset \mathbb{P}^3$, these polarizations both come from the divisor class $(\mathbb{P}^{34} \times L) \cap \mathcal{Y}$ on $\mathcal{Y}$ via the restriction maps. Hence, i maps the hyperplane class of $\mathcal{Y}_z$ to the hyperplane class of $\mathcal{Y}_{z'}$. □

Proposition 7.2 *For very general points $u \in U$ and $u' \in N_5$ we have that*

(a) *the quartic $\mathcal{X}_u$ has exactly* 320 *conics and no lines,*
(b) *the configuration of irreducible conics on $\mathcal{X}_u$ is the same as the configuration of conics (reducible or irreducible) on $\mathcal{X}_{u'}$.*

Proof Let $u \in U$ and $u' \in N_5$ be such that there are only 320 conics on $\mathcal{X}_{u'}$, the surfaces $\mathcal{X}_u$ and $\mathcal{X}_{u'}$ are smooth, $\mathcal{X}_u$ contains the configuration of 320 smooth conics and there is an injection of lattices

$$i : \mathrm{Pic}(\mathcal{X}_u) \hookrightarrow \mathrm{Pic}(\mathcal{X}_{u'})$$

which respects hyperplane classes. As we have seen, these conditions are satisfied for very general points $u \in U$ and $u' \in N_5$, see Lemma 7.1 for the part about i. Let $C \in \mathrm{Pic}(\mathcal{X}_u)$ be the class of a conic. Since $i(C)^2 = -2$, either $i(C)$ or $-i(C)$ is effective, see [3, Chapter VIII, Proposition 3.7(i)]. But the degree of $i(C)$ is equal to 2, and hence $i(C)$ is effective. Since $i(C)^2 = -2$, $i(C)$ is represented either by an

irreducible conic or a reducible conic. But there are only 320 conics on $\mathcal{X}_{u'}$, and it follows that there are no more conics on $\mathcal{X}_u$ either. Moreover, since i maps the classes of the 320 smooth conics on $\mathcal{X}_u$ onto the set of classes of conics on $\mathcal{X}_{u'}$, we have that the configurations of conics on the two surfaces coincide. $\square$

8 The Picard group

In this section we determine the Picard group of a very general Heisenberg invariant quartic and show that it is generated by the 320 conics on the surface.

It has been known since over a hundred years that every curve C on a general enough Kummer surface Y in $\mathbb{P}^3$ is such that C counted with multiplicity 2 is a complete intersection between Y and some other surface in $\mathbb{P}^3$, see [20, Chapter XIII]. A modern treatment is given in [17,18]. In particular, every curve on Y has even degree. To be more precise, if $Y \subset \mathbb{P}^3$ is the Kummer surface of a principally polarized Abelian surface A with Picard number 1, which is true for a very general point of the moduli space of principally polarized Abelian surfaces, then every curve on Y has even degree (see [18, Theorem 4.32]). The Kummer surfaces that appear in the family $\mathcal{X}$ are Heisenberg invariant but this merely reflects a choice of coordinates on $\mathbb{P}^3$ (see [18, Lemma 1.52]).

Lemma 8.1 *For a very generic Heisenberg invariant quartic surface X, no curve on X has odd degree.*

Proof Let $\mathrm{Hilb}_{(d,g)}(\mathbb{P}^3)$ denote the Hilbert scheme of curves in $\mathbb{P}^3$ with Hilbert polynomial $P(x) = dx + (1-g)$. Recall that $\mathrm{Hilb}_{(d,g)}(\mathbb{P}^3)$ is projective, see [25, Chapter I, 1.4]. Let $\mathcal{Y} \to \mathrm{Hilb}_{(d,g)}(\mathbb{P}^3)$ be the universal family. Define an incidence $I_{(d,g)}$ by

$$\mathrm{Hilb}_{(d,g)}(\mathbb{P}^3) \times U \supseteq I_{(d,g)} = \{(s,u) : \mathcal{Y}_s \subset \mathcal{X}_u\},$$

and let $\pi : \mathrm{Hilb}_{(d,g)}(\mathbb{P}^3) \times U \to U$ denote the projection. Now, since for odd d and any g, the parameter point of a general enough Heisenberg invariant Kummer surface is not in $\pi(I_{(d,g)})$, we have that $\pi(I_{(d,g)}) \neq U$ if d is odd. A surface whose parameter point is outside the countable union $\bigcup_{k,g} \pi(I_{(2k+1,g)})$ has no curve of odd degree. $\square$

Finite Abelian groups with a symplectic action on some K3 surface have been classified by Nikulin [32]. In that paper, Theorem 4.7, it is shown that if G is finite Abelian and acts symplectically on a K3 surface X, then the induced action of G on $H^2(X, \mathbb{Z})$ is independent of the surface, up to isomorphism of lattices. It follows that the lattices $H^2(X, \mathbb{Z})^G$ and $\Omega_G = (H^2(X, \mathbb{Z})^G)^\perp$ are determined by G up to isomorphism. In [14] these lattices are worked out in each case of the classification and in the case of the Heisenberg group, Ω_H turns out to be well known. It is isomorphic to $-\Lambda_{15}$, where Λ_{15} is the so-called laminated lattice of rank 15, see [9].

Remark 8.2 The discriminant of Λ_{15} is 2^9 and Λ_{15} is positive definite, see [9] for a detailed treatment. In one basis, the bilinear form on Λ_{15} is given by

$$
\begin{pmatrix}
4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
-2 & 4 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -2 & 4 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 \\
0 & 2 & 0 & 4 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 2 & 4 & 2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 2 & 2 & 4 & 2 & 2 & 1 & 2 & 0 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 & 2 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 2 & 2 & 2 & 4 & 0 & 2 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 4 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 4 & 2 & 2 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 2 & 2 & 1 & 1 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 4 & 1 & 2 & 2 & \\
0 & -1 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 4 & 0 & 2 \\
1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 2 & 0 & 4 & 2 \\
1 & 0 & 1 & 1 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 2 & 2 & 4
\end{pmatrix}.
$$

Theorem 8.3 *Let X be a very generic Heisenberg invariant quartic and let $h \in \mathrm{Pic}(X)$ be the hyperplane class. Then*

(a) *the sublattice $\mathbb{Z}h \oplus \Omega_H \subset \mathrm{Pic}(X)$ has index* 2,
(b) $\mathrm{discr}(\mathrm{Pic}(X)) = -2^9$.

Proof We first show that the group $\mathrm{Pic}(X)/(\mathbb{Z}h \oplus \Omega_H)$ is cyclic. Since $\mathrm{Pic}(X)^H = \mathbb{Z}h$ and $H^2(X, \mathbb{Z})^H$ has rank 7, $\mathbb{Z}h \oplus T_X \subseteq H^2(X, \mathbb{Z})^H$ is a full rank sublattice. If $l \in \mathrm{Pic}(X)$ and $lh = 0$, then l kills $\mathbb{Z}h \oplus T_X$ and therefore $l \in \Omega_H$. Since $\Omega_H \subset \mathrm{Pic}(X)$, it follows that Ω_H is the orthogonal complement of $\mathbb{Z}h$ in $\mathrm{Pic}(X)$. Observe that Ω_H is a primitive sublattice of $\mathrm{Pic}(X)$, that is if $nl \in \Omega_H$ for some non-zero $n \in \mathbb{Z}$ and some $l \in \mathrm{Pic}(X)$, then $l \in \Omega_H$. In other words $\mathrm{Pic}(X)/\Omega_H$ is torsion free. Because $\mathrm{Pic}(X)/\Omega_H$ has rank 1, there is an element $l \in \mathrm{Pic}(X)$ such that $\mathrm{Pic}(X) = \mathbb{Z}l \oplus \Omega_H$, as Abelian groups. The class of l in $\mathrm{Pic}(X)/(\mathbb{Z}h \oplus \Omega_H)$ generates $\mathrm{Pic}(X)/(\mathbb{Z}h \oplus \Omega_H)$. Further, note that for $v \in (\mathbb{Z}h \oplus \Omega_H)$, $v = nh + \omega$ with $\omega \in \Omega_H$ and $n \in \mathbb{Z}$, we have that $vh = nh^2 = 4n$. If it were true that $\mathrm{Pic}(X) = \mathbb{Z}h \oplus \Omega_H$ then the degree of any curve on X would be a multiple of 4, but this is not the case since X contains a conic. Now let $D \in \mathrm{Pic}(X)$. It follows by Lemma 8.1 that Dh is even, say $Dh = 2m$. Then $(2D - mh)h = 0$ and hence $(2D - mh) \in \Omega_H$. Consequently, $2D \in \mathbb{Z}h \oplus \Omega_H$ and thus $\mathbb{Z}h \oplus \Omega_H$ has index 2 in $\mathrm{Pic}(X)$.

Since

$$
\mathrm{discr}(\mathbb{Z}h \oplus \Omega_H) = 4 \cdot \mathrm{discr}(\Omega_H) = -2^{11}
$$

and

$$
\mathrm{discr}(\mathbb{Z}h \oplus \Omega_H)/\mathrm{discr}(\mathrm{Pic}(X)) = 2^2,
$$

we have that $\mathrm{discr}(\mathrm{Pic}(X)) = -2^9$. $\square$

Corollary 8.4 *The Picard group of a very generic Heisenberg invariant quartic X is generated by the* 320 *conics.*

Proof By Proposition 7.2, some set of 16 conics on X correspond to the 16 conics of Remark 6.2. Let $P \subseteq \mathrm{Pic}(X)$ be the sublattice generated by these 16 conics on X. Because $\mathrm{discr}(P) = \mathrm{discr}(\mathrm{Pic}(X))$, the index of P in $\mathrm{Pic}(X)$ is equal to 1, that is $P = \mathrm{Pic}(X)$. □

The proof of the corollary shows that (in some basis) the lattice structure of the Picard group of a very generic Heisenberg invariant quartic is given by the matrix M in Remark 6.2.

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