A parallel metrization theorem

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Abstract

Two non-empty sets A, B of a metric space (X, d) are called parallel if $d(a, B) = d(A, B) = d(A, b)$ for any points $a \in A$ and $b \in B$. Answering a question posed on mathoverflow.net, we prove that for a cover $\mathcal{C}$ of a metrizable space X by compact subsets, the following conditions are equivalent: (i) the topology of X is generated by a metric d such that any two sets $A, B \in \mathcal{C}$ are parallel; (ii) the cover $\mathcal{C}$ is disjoint, lower semicontinuous and upper semicontinuous.

Keywords Metrization · Parallel sets · Metric space

Mathematics Subject Classification 54E35

In this paper we shall prove a “parallel” metrization theorem answering a question [1] of the Mathoverflow user116515. The question concerns parallel sets in metric spaces.

Two non-empty sets A, B in a metric space (X, d) are called *parallel* if

$$d(a, B) = d(A, B) = d(A, b) \quad \text{for any } a \in A \text{ and } b \in B.$$

Here $d(A, B) = \inf \{d(a, b) : a \in A, b \in B\}$ and $d(x, B) = d(B, x) := d(\{x\}, B)$ for $x \in X$. Observe that two closed parallel sets A, B in a metric space are either disjoint or coincide.

Let $\mathcal{C}$ be a family of non-empty closed subsets of a topological space X . A metric d on X is defined to be $\mathcal{C}$ -parallel if any two sets $A, B \in \mathcal{C}$ are parallel with respect to the metric d .

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A family $\mathcal{C}$ of subsets of X is called a *compact cover* of X if $X = \bigcup \mathcal{C}$ and each set $C \in \mathcal{C}$ is compact.

Problem ([1]) For which compact covers $\mathcal{C}$ of a topological space X the topology of X is generated by a $\mathcal{C}$ -parallel metric?

A metric generating the topology of a given topological space will be called *admissible*. A necessary condition for the existence of an admissible $\mathcal{C}$ -parallel metric is the upper and lower semicontinuity of the cover $\mathcal{C}$.

A family $\mathcal{C}$ of subsets of a topological space X is called

- *lower semicontinuous* if for any open set $U \subset X$ its $\mathcal{C}$ -star $St(U; \mathcal{C}) := \bigcup \{C \in \mathcal{C} : C \cap U \neq \emptyset\}$ is open in X ;
- *upper semicontinuous* if for any closed set $F \subset X$ its $\mathcal{C}$ -star $St(F; \mathcal{C})$ is closed in X ;
- *continuous* if $\mathcal{C}$ is both lower and upper semicontinuous;
- *disjoint* if any distinct sets $A, B \in \mathcal{C}$ are disjoint.

The following theorem is the main result of the paper, answering Problem.

Theorem *For a compact cover $\mathcal{C}$ of a metrizable topological space X the following conditions are equivalent:*

- the topology of X is generated by a $\mathcal{C}$ -parallel metric;*
- the family $\mathcal{C}$ is disjoint and continuous.*

Proof (i) $\Rightarrow$ (ii). Assume that d is an admissible $\mathcal{C}$ -parallel metric on X . The disjointness of the cover $\mathcal{C}$ follows from the obvious observation that two closed parallel sets in a metric space are either disjoint or coincide.

To see that $\mathcal{C}$ is lower semicontinuous, fix any open set $U \subset X$ and consider its $\mathcal{C}$ -star $St(U; \mathcal{C})$. To see that $St(U; \mathcal{C})$ is open, take any point $s \in St(U; \mathcal{C})$ and find a set $C \in \mathcal{C}$ such that $s \in C$ and $C \cap U \neq \emptyset$. Fix a point $u \in U \cap C$ and find $\varepsilon > 0$ such that the ε -ball $B(u; \varepsilon) = \{x \in X : d(x, u) < \varepsilon\}$ is contained in U . We claim that $B(s; \varepsilon) \subset St(U; \mathcal{C})$. Indeed, for any $x \in B(s; \varepsilon)$ we can find a set $C_x \in \mathcal{C}$ containing x and conclude that $d(C_x, u) = d(C_x, C) \leq d(x, s) < \varepsilon$ and hence $C_x \cap U \neq \emptyset$ and $x \in C_x \subset St(U; \mathcal{C})$.

To see that $\mathcal{C}$ is upper semicontinuous, fix any closed set $F \subset X$ and consider its $\mathcal{C}$ -star $St(F; \mathcal{C})$. To see that $St(F; \mathcal{C})$ is closed, take any point $s \in X \setminus St(F; \mathcal{C})$ and find a set $C \in \mathcal{C}$ such that $s \in C$. It follows from $s \notin St(F; \mathcal{C})$ that $C \cap F = \emptyset$ and hence $\varepsilon := d(C, F) > 0$ by the compactness of C . We claim that $B(s; \varepsilon) \cap St(F; \mathcal{C}) = \emptyset$. Assuming the opposite, we can find a point $x \in B(s; \varepsilon) \cap St(F; \mathcal{C})$ and a set $C_x \in \mathcal{C}$ such that $x \in C_x$ and $C_x \cap F \neq \emptyset$. Fix a point $z \in C_x \cap F$ and observe that $d(C, F) \leq d(C, z) = d(C, C_x) \leq d(s, x) < \varepsilon = d(C, F)$, which is a desired contradiction.

(ii) $\Rightarrow$ (i). The proof of this implication is more difficult. Assume that $\mathcal{C}$ is disjoint and continuous. Fix any admissible metric $\rho \leq 1$ on X .

Let $\mathcal{U}_0(C) = \{X\}$ for every $C \in \mathcal{C}$.

Claim For every $n \in \mathbb{N}$ and every $C \in \mathcal{C}$ there exists a finite cover $\mathcal{U}_n(C)$ of C by open subsets of X such that

- (a) each set $U \in \mathcal{U}_n(C)$ has ρ -diameter $\leq 1/2^n$;
- (b) if a set $A \in \mathcal{C}$ meets some set $U \in \mathcal{U}_n(C)$, then $A \subset \bigcup \mathcal{U}_n(C)$ and A meets each set $U' \in \mathcal{U}_n(C)$.

Proof Using the paracompactness [2, 5.1.3] of the metrizable space X , choose a locally finite open cover $\mathcal{V}$ of X consisting of sets of ρ -diameter $< 1/2^n$.

For every compact set $C \in \mathcal{C}$ consider the finite subfamily $\mathcal{V}(C) := \{V \in \mathcal{V} : V \cap C \neq \emptyset\}$ of the locally finite cover $\mathcal{V}$. Since the cover $\mathcal{C}$ is upper semicontinuous, the set $F_C = \text{St}(X \setminus \bigcup \mathcal{V}(C); \mathcal{C})$ is closed and disjoint with the set C . Since $\mathcal{C}$ is lower semicontinuous, for any open set $V \in \mathcal{V}(C)$ the set $\text{St}(V; \mathcal{C})$ is open and hence $W(C) := \bigcap_{V \in \mathcal{V}(C)} \text{St}(V; \mathcal{C}) \setminus F_C$ is an open neighborhood of C .

Put $\mathcal{U}_n(C) := \{W(C) \cap V : V \in \mathcal{V}(C)\}$ and observe that $\mathcal{U}_n$ satisfies condition (a).

Let us show that the cover $\mathcal{U}_n(C)$ satisfies condition (b). Assume that a set $A \in \mathcal{C}$ meets some set $U \in \mathcal{U}_n(C)$. First we show that $A \subset \bigcup \mathcal{U}_n(C)$. Find a set $V \in \mathcal{V}(C)$ such that $U = W(C) \cap V$. It follows from $\emptyset \neq A \cap U \subset A \cap W(C)$ that the set A meets $W(C)$ and hence is contained in $W(C)$ and is disjoint with F_C . Hence

$$A \subset W(C) \cap \left(\bigcup \mathcal{V}(C)\right) = \bigcup_{V \in \mathcal{V}(C)} W(C) \cap V = \bigcup \mathcal{U}_n(C).$$

Next, take any set $U' \in \mathcal{U}_n(C)$ and find a set $V' \in \mathcal{V}(C)$ with $U' = W(C) \cap V'$. The relation $A \cap W(C) \cap V = A \cap U \neq \emptyset$ and the definition of the set $W(C) \supset A$ imply that A intersects the set $V' \in \mathcal{V}(C)$ and hence intersects the set $U' = W(C) \cap V'$. ■

Given two points $x, y \in X$ let

$$\delta(x, y) := \inf \left\{ \frac{1}{2^n} : \exists C \in \mathcal{C} \text{ and } U \in \mathcal{U}_n(C) \text{ such that } x, y \in U \right\}.$$

Adjust the function δ to a pseudometric d letting

$$d(x, y) = \inf \sum_{i=1}^m \delta(x_{i-1}, x_i)$$

where the infimum is taken over all sequences $x = x_0, \dots, x_m = y$. Condition (a) of Claim implies that $\rho(x, y) \leq \delta(x, y)$ and hence $\rho(x, y) \leq d(x, y)$ for any $x, y \in X$. So, the pseudometric d is a metric on X such that the identity map $(X, d) \rightarrow (X, \rho)$ is continuous. To see that this map is a homeomorphism, take any point $x \in X$ and $\varepsilon > 0$. Find $n \in \mathbb{N}$ such that $1/2^n < \varepsilon$ and choose a set $C \in \mathcal{C}$ with $x \in C$ and a set $U \in \mathcal{U}_n(C)$ with $x \in U$. Then for any $y \in U$ we get $d(y, x) \leq \delta(x, y) \leq 1/2^n < \varepsilon$, which means that the map $X \rightarrow (X, d)$ is continuous.

Finally, let us prove that the metric d is $\mathcal{C}$ -parallel. Pick any two distinct compact sets $A, B \in \mathcal{C}$. We need to show that $d(a, B) = d(A, B) = d(A, b)$ for any $a \in A$,

$b \in B$. Assuming that this inequality is not true, we conclude that either $d(a, B) > d(A, B) > 0$ or $d(A, b) > d(A, B) > 0$ for some $a \in A$ and $b \in B$.

First assume that $d(a, B) > d(A, B)$ for some $a \in A$. Choose points $a' \in A, b' \in B$ such that $d(a', b') = d(A, B) < d(a, B)$. By the definition of the distance $d(a', b') < d(a, B)$, there exists a chain $a' = x'_0, x'_1, \dots, x'_m = b'$ such that $\sum_{i=1}^m \delta(x'_{i-1}, x'_i) < d(a, B)$. We can assume that the points $x'_0, \dots, x'_m$ are pairwise distinct, so for every $i \leq m$ there exist $n_i \geq 0$ such that $\delta(x'_{i-1}, x'_i) = 1/2^{n_i}$ and hence $x'_{i-1}, x'_i \in U'_i$ for some $C_i \in \mathcal{C}$ and $U'_i \in \mathcal{U}_{n_i}(C_i)$. For every $i \leq m$ let $A_i \in \mathcal{C}$ be the unique set with $x'_i \in A_i$. Then $A_0 = A$ and $A_m = B$.

Using condition (b), we can inductively construct a sequence of points $a = x_0, x_1, \dots, x_m \in B$ such that for every positive $i \leq m$ the point x_i belongs to A_i and the points x_{i-1}, x_i belong to some set $U_i \in \mathcal{U}_{n_i}(C_i)$. The chain $a = x_0, x_1, \dots, x_m \in A_m = B$ witnesses that

$$d(a, B) \leq d(a, x_m) \leq \sum_{i=1}^m \delta(x_{i-1}, x_i) \leq \sum_{i=1}^m \frac{1}{2^{n_i}} = \sum_{i=1}^m \delta(x'_{i-1}, x'_i) < d(a, B),$$

which is a desired contradiction. By analogy we can prove that the case $d(A, B) < d(A, b)$ leads to a contradiction. $\square$

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