Editors' foreword for the special issue "Finsler geometry, new methods and perspectives"

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The idea of this special issue is based on the observation that in the last decade or so, many natural and named problems in Finsler geometry were solved by mathematicians *from outside*, i.e., those who do not consider Finsler geometry as their main research field. For that reason, we announced this special issue as wide as possible, asking for submissions of survey and research papers whose subject is Finsler geometry in this non-restrictive sense. We consider the output as a successful one and in most papers of the present issue one uses methods from various branches of mathematics applied to Finsler geometry or one uses Finsler geometry in other areas of mathematics. The connection of Finsler geometry with metric and Riemannian geometry, including Lie theory methods, with analysis and complex geometry plays an important role in this special issue.

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We hope that the variety of contributions shows the richness of the subject and provides new insight and perspectives. We also do believe that the Finsler geometry of the future will be an active interplay of different groups of methods and ideas, and hope that our special issue contributes to it.

Altogether, we have 16 papers. We briefly present their content.

The paper *Foundations of singular Finsler geometry* by Pavel Andreev surveys a geometric approach leading to Finsler geometry without tensor calculus. This is the approach that was initiated by Herbert Busemann, who was the leader of 20th century synthetic Finsler geometry, and who considered Finsler spaces as certain geodesic metric spaces satisfying certain assumptions on the behavior of geodesics (e.g., local extensibility). We note incidentally that Busemann's Selected Works just appeared in two important volumes of Springer Verlag, which is a sign of the timeliness of this approach to Finsler geometry. This approach allows one in particular the study of singular Finsler metrics.

The paper *Curvatures of homogeneous sub-Riemannian manifolds* by Valerii N. Berestovskii is motivated by the earlier obtained observation of the author saying that homogeneous intrinsic metric manifolds are quotients of homogeneous Finsler or sub-Finsler manifolds. Starting with this observation, the author presents different generalizations of the notion of curvature and their applications. He discusses in particular an operator definition of lower bound for Ricci curvature by Baudoin–Garofalo, a notion of curvature for sub-Riemannian manifolds due to Agrachev, as well as notions of curvature tensors for rigged metrized distributions on manifolds due to Schouten, Wagner, and Solov'ev.

Michael Kapovich, Bernhard Leeb and Joan Porti develop in *Anosov subgroups: dynamical and geometric characterizations* new methods for the description of actions of discrete subgroups of isometries on symmetric spaces of higher rank. Several new equivalent characterizations for Anosov subgroups are presented. In this work a naturally constructed Finsler metric occurs and plays an important role.

In the paper *On systolic zeta functions*, Ivan Babenko and Daniel Massart study homology length spectra of Finsler manifolds or polyhedra and investigate some analytical properties of a generating function that arises in this setting. As an application, they obtain an inequality analogous to Gromov's classical intersystolic inequality, but taking the whole homology length spectrum into account rather than just the systole. One of the ideas expressed in this paper is that several results on this topic that are usually presented in a Riemannian context can be translated into a Finsler setting.

The paper *Nonlinear geometric analysis on Finsler manifolds* by Shin-ichi Ohta reviews the recent developments of comparison geometry and geometric analysis on Finsler manifolds of weighted Ricci curvature bounded from below. The weighted Ricci curvature, a notion introduced by the author, is useful to understand the similarities and differences between Riemannian and Finsler manifolds at the global level.

Sven Pistre and Heiko von der Mosel investigate in *The Plateau problem for the Busemann–Hausdorff area in arbitrary codimension* the Plateau problem in $\mathbb{R}^n$ with respect to a Finsler metric based on the Busemann–Hausdorff volume. The authors solve the Plateau problem using Cartan functionals for non-reversible Finsler metrics which can be bounded by Euclidean metrics and which satisfy a symmetrization assumption.

The papers *Recent progress on homogeneous Finsler spaces with positive curvature* by Shaoqiang Deng and Ming Xu and *Flag curvatures of homogeneous Finsler spaces* by Libing Huang use methods of Lie theory to study and describe homogeneous metrics of positive flag curvature.

Finsler geometry appears naturally in complex geometry, and one of its main incarnations are the Kobayashi and the Carathéodory metrics defined on complex manifolds. The paper *Metric geometry of the Kobayashi metric* by Hervé Gaussier and Harish Seshadri as well as *The Finsler geometry of the Teichmüller metric* by Weixu Su and Youliang Zhong are surveys covering different aspects of this topic. The paper by Gaussier and Seshadri concerns the Kobayashi metric on domains in $\mathbb{C}^n$. The authors survey questions related to pseudo-convexity, complex geodesics in convex domains, holomorphicity of isometries and criteria for Gromov hyperbolicity of the Kobayashi metric. The paper by Su and Zhong concerns the Finsler structure of the Teichmüller metric, which is known to coincide with the Kobayashi metric of Teichmüller space. It is probably worth noting the recent results obtained by Vladimir Marković on the Carathéodory metric of this space (see his recent paper *Carathéodory's metrics on Teichmüller spaces and L-shaped pillowcases*). Teichmüller theory is certainly the best example of a setting where Finsler geometry enters into the realm of low dimensional geometry and topology.

Max Dörner, Hansjörg Geiges and Kai Zehmisch present a survey *Finsler geodesics, periodic Reeb orbits, and open books* on the question of existence of periodic orbits of Reeb flows. There are considered non-existence results as well as existence results on connected sums and open books. The authors also include a proof of the folklore result that the geodesic flow defined by a Finsler metric is a Reeb flow.

One of the main open problems in Finsler geometry is the so-called Landsberg Unicorn problem, i.e., to find a Finsler metric which is Landsberg but not Berwald. The papers *Pseudo-Finsleroid metrics with two axes* by Gennadii S. Asanov and *On a special type of generalized Berwald manifolds: semi-symmetric linear connections preserving the Finslerian length of tangent vectors* by Csaba Vincze are to some extent motivated by this problem via the famous example of Asanov's nonsmooth Landsberg Unicorn. In both papers different generalizations of this construction are studied. In his paper Vincze makes a review of the notion of generalized Berwald spaces pointing out the importance and motivation for the study of such spaces, as well as the relation to Randers and Landsberg metrics. The paper proves the utility of the combined use of the theory of Finsler connections, developed in particular by the Japanese school of Makoto Matsumoto, with the modern trend of average Riemannian metrics on Finsler manifolds. Some new insight into Asanov's Finsleroid-Finsler metrics is also obtained by characterizing these rather special types of Landsberg spaces in terms of generalized Berwald manifolds with semi-symmetric compatible linear connection.

The papers *Geodesics on strong Kropina manifolds* by Sorin V. Sabau, Kazuhiro Shibuya and Ryoza Yoshikawa and *Wind Riemannian spaceforms and Randers-Kropina metrics of constant flag curvature* by Miguel A. Javaloyes and Miguel Sánchez concern different properties of the metrics obtained by Zermelo deformation from Riemannian metrics. Sabau et al. systematically investigate the global geometry of strong Kropina metrics. Javaloyes and Sánchez use the notion of wind Riemannian structures and obtain a better understanding of Randers and Kropina metrics of constant

flag curvature, putting them in a unified theory that is able to recover and generalize the known classification of Randers and Kropina spaces of constant flag curvature.

Carlos Durán and Henrique Vitória give in *Moving planes, Jacobi curves and the dynamical approach to Finsler geometry* a dynamical approach to Finsler geometry using moving planes and their Jacobi curves. They give applications of this method to an O'Neill formula for Finsler submersions as well as to the description of flag curvature of projectively related Finsler metrics and Katok perturbations.

Finally, let us mention that this special issue was motivated by two international conferences in Finsler geometry: the *50th Symposium on Finsler Geometry* (Hiroshima, November 2015), organized by Tadashi Aikou and Sorin V. Sabau, an event marking a half century of Finsler geometry in Japan, and the conference *New Methods in Finsler Geometry* (Leipzig, July 2016), organized by Vladimir S. Matveev and Hans-Bert Rademacher, a satellite event of the 7th European Congress of Mathematics in Berlin.

The following authors of the present volume were also speakers at one of the conferences above: B. Leeb, S.-I. Ohta, S. Pistre, S.V. Sabau, M. Sánchez, C. Vincze, H. Vitória, R. Yoshikawa, and K. Zehmisch. However, the submission to this issue was not restricted to participants of the conference, and this issue of the EJM does not constitute a volume of conference proceedings.

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