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RESEARCH ARTICLE

The moduli space of binary quintics

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Abstract We recall the classical construction and theory of invariants for the case of binary quintics, describe the moduli space, and identify the curves in it defined by quintics having symmetry. We describe the real case, and identify the number of real roots depending on the point in moduli space. Our main interest is in five curves of binary quintics defined as linear sections of plane curves with infinite symmetry groups: these play a role in the canonical stratification of jet space, so we describe their singularities and count their intersections. All this is done in the classical case. Thereafter we analyse the changes to be made to the whole theory when we work in characteristic 2.

Keywords Binary quintics · Invariants · Moduli space · Symmetries · Stratification

1 Invariant theory

We begin with classical invariant theory, for example as in [2], though we do not follow that version. We take $\mathbb{C}$ as ground field for the first two sections. The ring of (SL_2-)invariants of binary quintics is generated by four invariants, which we call I_4, I_8, I_{12} and I_{18}, where I_d has degree d (see [2, p. 131], but probably the first proof was given by Hermite [3]).

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We normalise the invariants I_4, I_8, I_{12} using a normal form, due to Sylvester [4] (see also [1], [2, p. 230]).

Lemma 1.1 *A general binary quintic can be expressed uniquely as a sum of three fifth powers of linear forms.*

Proof We represent the quintic $a = \sum_{r=0}^{5} \binom{5}{r} a_r x^{5-r} y^r$ by the point A with coordinates (a_r) in $\mathbb{P}^5$. Then a quintic that is a fifth power $(t_1 x + t_2 y)^5$ corresponds to a point T lying on the standard rational normal curve C.

If a is the sum of three fifth powers, A lies in the plane spanned by the corresponding points on C. No point may lie on two such planes, for otherwise the planes would lie in a hyperplane containing six points of C, whereas C has degree 5. But since there are three degrees of freedom for the three points on C, and hence for the plane, and two more for the point A on it, a generic point will lie on such a plane. □

We may extract more from this argument. It follows by specialisation that any point A in $\mathbb{P}^5$ lies on a plane meeting C in three points, but the points need not be distinct. If all three coincide, so that A lies on the osculating plane corresponding to a point T, then a is divisible by the cube of $t_1 x + t_2 y$. If just two coincide, corresponding say to x^5 (repeated) and y^5, then we may write $a = x^4(px + qy) + ry^5$. A fuller account of the exceptional cases is in [2, p. 231].

We may thus write a general binary quintic in the form $al^5 + bm^5 + cn^5$ with $l + m + n = 0$. Then the invariants are given in terms of the elementary symmetric functions σ_1, σ_2 and σ_3 of a, b and c by the formulae

$$I_4 = 4\sigma_1\sigma_3 - \sigma_2^2, \quad I_8 = \sigma_2\sigma_3^2, \quad I_{12} = -3\sigma_3^4.$$

For the above form, the discriminant vanishes if and only if, for some choice of the radicals, we have

$$a^{-1/4} + b^{-1/4} + c^{-1/4} = 0.$$

Also we have, up to a constant factor,

$$I_{18} = a^5 b^5 c^5 (b - c)(c - a)(a - b).$$

We have just seen that a quintic which may not be written in the above form either has a cubed factor or is equivalent to $x^4(px + qy) + ry^5$. For such a form, $I_{12} = 0$.

The invariants are subject to a unique syzygy (of degree 36)

$$I_{18}^2 = -144 I_{12}^3 + (I_4^3 - 72 I_4 I_8) I_{12}^2 + (24 I_8^3 - 6 I_4^2 I_8^2) I_{12} + 9 I_4 I_8^4. \tag{1}$$

Taking (I_4, I_8, I_{12}) as coordinates gives an isomorphism of the moduli space $\mathcal{M}$ onto the weighted projective space $\mathbb{P}(1, 2, 3)$. For the values of these invariants determine I_{18} up to sign, while in the weighted projective space $\mathbb{P}(4, 8, 12, 18)$, the coordinates $(I_4, I_8, I_{12}, I_{18})$ and $(t^4 I_4, t^8 I_8, t^{12} I_{12}, t^{18} I_{18})$ both give the same point; taking $t = i$ now shows that changing the sign of I_{18} still gives the same point.

The space $\mathcal{M}$ is smooth except at two points: $(0, 1, 0)$, where there is a singularity of type A_1, and $(0, 0, 1)$, where there is a singularity of type A_2. It gives a moduli space for all quintics with no factor of multiplicity 3 or more; all such are stable in the sense of geometric invariant theory (for the action of SL_2 on $\mathbb{P}^5$). Quintics with a repeated root yield a curve $\Delta \subset \mathcal{M}$: the *discriminant curve*. It has equation $I_4^2 = 128 I_8$.

2 Special quintics

For any set of five distinct points on $\mathbb{P}^1$ admitting a nontrivial symmetry group, we can choose coordinates so that the five points appear as $(0, \pm\alpha, \pm\beta)$ for some α, β; moreover, we may take $\alpha\beta = 1$. The quintic is thus $x(x^4 - 2tx^2y^2 + y^4)$ for $t = (\alpha^2 + \beta^2)/2$. For such quintics, $I_{18} = 0$. The other invariants are

$$\begin{aligned} I_4 &= 16t(3t^2 + 5), \\ I_8 &= 2(t^2 - 5)^2(9t^2 - 5), \\ I_{12} &= 12t(t^2 - 5)^4. \end{aligned} \tag{2}$$

This parametrises the curve $\Gamma_5 \subset \mathcal{M}$ whose equation is given by the right hand side of (1). This is not a good parametrisation since t and $-t$ yield the same point of the curve.

We have the following points of particular significance on the discriminant curve Δ:

- (harmonic tetrad with one point repeated) $F_0 = (48, 18, 12)$, represented by $f_0 = x^2y(x^2 - y^2)$,
- (equianharmonic tetrad with one point repeated) $F_1 = (0, 0, 1)$, represented by $f_1 = x^2(x^3 - y^3)$,
- (two repeated points, and another point) $F_2 = (16, 2, 6)$, represented by $f_2 = x^2y^2(x - y)$.

The intersection of Γ_5 with Δ consists of F_0, and F_2 counted twice.

The invariants of a pentad with distinct points, and with symmetry group of order greater than 2, give one of the following points in $\mathcal{M}$:

- $F_3 = (-3, 3, 3)$, represented by $f_3 = xy(x^3 - y^3)$,
- $F_4 = (0, 1, 0)$, represented by $f_4 = x(x^4 - y^4)$,
- $F_5 = (1, 0, 0)$, represented by $f_5 = x^5 - y^5$,

with 3-, 4- and 5-fold symmetry respectively.

The singular points of $\mathcal{M}$ occur at F_4 and F_1, with types A_1 and A_2 respectively. The curve Γ_5 has singular points of type A_2 at F_3 and of type A_5 at F_4.

3 The real case

In this section we will write $\mathcal{M}(\mathbb{R})$ for the set of real points of $\mathcal{M}$.

Lemma 3.1 *Each point of $\mathcal{M}(\mathbb{R})$ is represented by a quintic with real coefficients.*

Proof For one proof (which we believe is due to Hermite) we suppose that the point does not lie on Δ (other cases may be treated directly). Invariant theory provides linear covariants of a quintic, among which are α (of degree 5) and β (of degree 7) in the notation of [2, p. 130]. If the quintic has no repeated root, these are linearly independent and can be taken as coordinates. Then the coefficients of the quintic are themselves invariants, hence are real.

An alternative argument may be given when $I_{12} \neq 0$. Write the quintic in the above normal form $al^5 + bm^5 + cn^5$ with $l + m + n = 0$. Then the invariants are given above; since $\sigma_3 \neq 0$, the same point on $\mathcal{M}$ is given by $(4\sigma_1 - \sigma_2^2/\sigma_3, \sigma_2, -3\sigma_3)$. Thus adjusting a, b and c by a common scalar factor we may suppose that these, and hence the σ_r are all real.

Now either a, b, c are all real and we may take l, m, n to be real linear forms or one—say a—is real and the other two complex conjugate. We then take l real and m, n complex conjugate (e.g. $l = 2x$, $m = -x + iy$, $n = -x - iy$) to obtain a real quintic. □

If we think of the quintic as determining five points on $\mathbb{P}^1$, then a real form gives an anti-holomorphic involution preserving the set of points. If there are two such, their product gives a holomorphic automorphism, so the point in $\mathcal{M}$ must lie on Γ_5.

We wish to determine, for each real point in $\mathcal{M}(\mathbb{R})$, the number of real roots of the corresponding real quintic. The standard theory tells us that for one sign of the discriminant there are three real roots; for the other, either one or five real roots. It follows from the above that the number of real roots may only change as we cross Γ_5 or Δ. We next investigate Γ_5.

As before, for five distinct points on $\mathbb{P}^1_{\mathbb{R}}$ admitting a nontrivial symmetry, we choose coordinates so that the points are $(0, \pm\alpha, \pm\beta)$ for some α, β; this fixes x, y up to scalars, and the quintic is then x multiplied by a quadratic in x^2 and y^2. We adjust the scalars so that the coefficient of x^5 is 1 and that of xy^4 is ± 1 (if the coefficient of xy^4 vanishes, the quintic is unstable; if that of x^5 vanishes, we have the point F_0). This fixes x, and y up to sign. However, changing the sign of the coefficient of x^3y^2 gives the same point of $\mathcal{M}(\mathbb{R})$ (it corresponds to substituting iy for y).

First consider $x(x^4 - 2tx^2y^2 + y^4)$. From the invariants given above, we find $I_4^2 - 128I_8 = 2^8 5^3(t^2 - 1)^2 > 0$ except when $t = \pm 1$ corresponding to F_2, or, in the limit, $t = \infty$ corresponding to F_0. As t^2 increases from 0 to ∞ the point in $\mathcal{M}(\mathbb{R})$ runs from F_4 to F_0, touching Δ at F_2 when $t^2 = 1$ and passing through F_5 when $t^2 = 20$.

The quadratic for $(x/y)^2$ has no real roots for $t^2 < 1$ and two real roots for $t^2 > 1$, which have the same sign as t. Thus the quintic has five real roots for $t > 1$ and only one for $t < 1$.

For $x(x^4 - 2ux^2y^2 - y^4)$, we have $I_4^2 - 128I_8 = -2^8 5^3(u^2 + 1)^2 < 0$. As u^2 increases from 0 to ∞, the point in $\mathcal{M}(\mathbb{R})$ runs from F_4 to F_0, passing through F_3 when $u^2 = 3$. The quadratic for $(x/y)^2$ has two real roots of opposite signs. Thus the quintic has three real roots.

To interpret these facts it is convenient to consider the affine chart of $\mathcal{M}(\mathbb{R})$ where $I_4 = 1$, so that we may take (I_8, I_{12}) as coordinates. Taking the weights into account, we see that the compactification of this plane in $\mathcal{M}(\mathbb{R})$ may be topologically described by regarding the plane as the interior of a disc, and identifying points on the boundary via reflection in the horizontal axis. In addition, the two points at the ends of this axis are identified in $\mathcal{M}(\mathbb{R})$ (the space has a topological singularity—a quadratic cone—at F_4), but it is convenient to describe the space where these are regarded as separate: this is homeomorphic to a sphere.

In this plane picture, the curve Δ is represented by a vertical line. The curve Γ_5 runs from the point F_4 at infinity on the left to the corresponding point to the right, touching Δ at F_2, then crossing it at F_0, then having a simple cusp at F_5. It follows that the region to the right of Δ, which corresponds to having three real roots, is not separated by Γ_5; while the region to the left is separated into two components. Both sides of that part of Γ_5 going off to the left (where $t^2 < 1$) give quintics with one real root, thus those with five real roots correspond to the small region bounded by Γ_5 and Δ, near the centre of the figure.

4 Exceptional curves

We are interested in the five curves in $\mathcal{M}$ arising in [5]: as explained in that paper, this is part of the analysis of the partition of the N_{16} stratum induced by the canonical stratification of jet space. These curves are constructed as follows (here we return to working over $\mathbb{C}$). We start with the quintic curves H_i $(1 \leqslant i \leqslant 5)$ defined in $\mathbb{P}^2$ by the equations $h_i = 0$, with h_i given by the following table, which also gives the types of singularities of the curves H_i at the points indicated:

Equation	(0, 0, 1)	(0, 1, 0)	(1, 0, 0)
$h_1 = xz(y^2z - x^3)$	D_8	D_5	
$h_2 = z(y^3z - x^4)$	E_6	A_7	
$h_3 = y^3z^2 - x^5$	E_8	A_4	
$h_4 = yz(y^2z - x^3)$	E_7	A_5	A_1
$h_5 = xyz(yz - x^2)$	D_6	D_6	A_1

Each curve H_i admits a (semisimple) 1-parameter symmetry group G_i. Thus the lines L in the plane $\mathbb{P}^2$ fall into a 1-parameter family of orbits under G_i, so their intersections with H_i give essentially a 1-parameter family of binary quintics, whose moduli trace out a curve in $\mathcal{M}$, whose closure we denote Γ_i.

The other exceptional cases ($2D_6$ and $T_{2,3,10}$) in the main theorem of [5], like h_5, are represented by quintic curves C composed of two conics in a pencil containing a repeated line, together with that line. The intersection of C with any line thus consists of two pairs in the involution cut on the line by the pencil, together with one of its fixed points; hence is a pentad possessing a symmetry. So these all yield the curve Γ_5 in $\mathcal{M}$; we need not consider them further.

If the line L is an edge of the triangle of reference, or even if it passes through the point (0, 0, 1), the corresponding binary quintic is unstable, so does not determine a point of $\mathcal{M}$. For lines through $(1, 0, 0)$ or $(0, 1, 0)$ respectively, the corresponding point in $\mathcal{M}$ is given by the following table:

Line	H_1	H_2	H_3	H_4	H_5
$y = z$	F_3	F_4	F_5	F_1	F_0
$x = z$	(F_2)	F_1	F_1	F_0	(F_2)

Here the symbol (F_2) means that although the intersection gives an unstable binary quintic, the completed curve Γ_i in $\mathcal{M}$ passes through F_2 at the corresponding limiting point.

We will see later that $F_i \in \Gamma_j$ only as indicated in this table, except that (as we already know), F_3, F_4 and F_5 belong to Γ_5.

To obtain a parametrisation of Γ_i we substitute $(x, 1, z+u)$ for (x, y, z) in h_i, and calculate the invariants $\{I_4, I_8, I_{12}\}$ of the resulting polynomial in x. This yields

$$\begin{aligned}
I(h_1) &= \big\{-8(-2+10u^2+9u^4), 2(1-10u^2+16u^4-80u^6+864u^8),\\
&\qquad 6(1-40u^2+624u^4-4640u^6+16192u^8-21760u^{10}+6912u^{12})\big\},\\
I(h_2) &= \big\{-48u(8+67u^3), 32u^2(36+603u^3+2736u^6+2000u^9),\\
&\qquad -6(-675-15688u^3-109536u^6-171264u^9+377088u^{12}+768000u^{15})\big\},\\
I(h_3) &= \big\{-80u(6+125u^3), 600u^2(3+80u^3), 300(27+1760u^3+28800u^6)\big\},\\
I(h_4) &= \big\{-48(-2+12u^2+u^4), 8(9-108u^2+315u^4-71u^6+846u^8),\\
&\qquad 12(8+531u^2-11748u^4+78616u^6\\
&\qquad -199026u^8+161064u^{10}+2924u^{12}+675u^{14})\big\},\\
I(h_5) &= \big\{8(1+2u)(2+8u+3u^2), 2(-1-4u+u^2)^2(1+4u+9u^2),\\
&\qquad 6(1+2u)(-1-4u+u^2)^4\big\}.
\end{aligned}$$

These do not give good parametrisations of the curves in $\mathbb{P}(1, 2, 3)$: to obtain these, we need to substitute $u^2 = T$ for Γ_1 and Γ_4 and $u^3 = T$ for Γ_2 and Γ_3. For Γ_5 it is better to let $t^2 = T$ in (2).

We determine equations for these curves by eliminating the parameter u. If $R_i = 0$ is the equation of Γ_i, we have deg $\Delta = 2$ and

i	1	2	3	4	5
deg Γ_i	12	11	8	14	9

In the following equations, x, y, z stand for I_4, I_8 and I_{12} respectively.

$$
\begin{aligned}
R_1 = {} & +257460937500z^4 + (-849570312500xy + 5283203125x^3)z^3 \\
& + (-3920231212500y^3 + 638876915625x^2y^2 \\
& \qquad - 15092175000x^4y + 83418750x^6)z^2 \\
& + (6869004789375xy^4 - 293089815000x^3y^3 \\
& \qquad + 4993132500x^5y^2 - 33322500x^7y + 67500x^9)z \\
& + (14838034276107y^6 - 3318382102464x^2y^5 \\
& \qquad + 182225527470x^4y^4 - 3154670820x^6y^3 \\
& \qquad + 23909580x^8y^2 - 82944x^{10}y + 108x^{12}); \\
R_2 = {} & (-1043568939604377600000y + 8152882340659200000x^2)z^3 \\
& + (-30619773585924249600000xy^2 \\
& \qquad + 511847900185424400000x^3y - 2129929055042118 75x^5)z^2 \\
& + (-99237906807021001764442 8y^4 \\
& \qquad + 2968368321408247540040 4x^2y^3 - 327745815296433611202x^4y^2 \\
& \qquad + 1575492903769317156x^6y - 2760926734011888x^8)z \\
& + (4443488364493477690945 2xy^5 - 1596859992578905245387x^3y^4 \\
& \qquad + 22126314998528665236x^5y^3 - 146702369241038988x^7y^2 \\
& \qquad + 461918940996344x^9y - 551433967396x^{11}); \\
R_3 = {} & (3556224y - 27783x^2)z^2 + (3951234xy^2 - 71640x^3y + 288x^5)z \\
& \qquad + (85470025y^4 - 1919000x^2y^3 + 10800x^4y^2); \\
R_4 = {} & (+8769775307736062361600000000000y \\
& \qquad - 257612149664746831872000000000 0x^2)z^4 \\
& + (-2241161829517312917504000000000xy^2 \\
& \qquad + 504838095366006610329600000000 0x^3y \\
& \qquad - 350172129850042613760000000000x^5)z^3 \\
& + (-1345689133882266793401542246400000y^4 \\
& \qquad + 398407709755188248466444779520000 0x^2y^3 \\
& \qquad - 88678603566351816985254297600000x^4y^2 \\
& \qquad + 6734293547335780115939328000000x^6y \\
& \qquad - 17446111644071046610944000000x^8)z^2 \\
& + (-72754287547694880513859780608000 0xy^5 \\
& \qquad + 453931671321827324416764149760000 0x^3y^4 \\
& \qquad - 1371235960355747933067411456000000x^5y^3 \\
& \qquad + 151028418367373674359029760000 0x^7y^2 \\
& \qquad - 717393089225769043968000000 00x^9y \\
& \qquad + 12362556735676003167360000x^{11})z \\
& + (+240997492810845047331283179405312y^7 \\
& \qquad - 154334790552800926632069172297728x^2y^6 \\
& \qquad + 117528893374342480447344437152972 8x^4y^5
\end{aligned}
$$

$$
\begin{aligned}
&- 6033503066980781574006396420096 0x^6y^4\\
&+ 119634271168717584053677228032 0x^8y^3\\
&- 114763380700341334611889520 64x^{10}y^2\\
&+ 535024460126455480420808 96x^{12}y\\
&- 97393677359695041798001x^{14});
\end{aligned}
$$

$$R_5 = -144z^3 + (-72xy + x^3)z^2 + (24y^3 - 6x^2y^2)z + 9xy^4.$$

5 Intersections and singularities

In general in $\mathcal{M} \cong \mathbb{P}(1, 2, 3)$ curves of degrees d_1 and d_2 will have d_1d_2 points of intersection. At smooth points, intersection multiplicities are as usual. At a singular point we lift the curve-germs to the appropriate branched cover, determine the intersection number there, and divide by 2 (at the singular point of type A_1) or 3 (for that of type A_2).

First we check intersections with Δ. We find

$$
\begin{aligned}
\Gamma_1 \cap \Delta &= 3F_2 + G_1,\\
\Gamma_2 \cap \Delta &= \frac{8}{3}\,F_1 + G_2,\\
\Gamma_3 \cap \Delta &= \frac{5}{3}\,F_1 + G_3,\\
\Gamma_4 \cap \Delta &= \frac{2}{3}\,F_1 + 3F_0 + G_4,\\
\Gamma_5 \cap \Delta &= F_0 + 2F_2,
\end{aligned}
$$

where the points G_i are distinct from each other and from the F_i. In particular, the only points $\Gamma_i \cap \Gamma_j \cap \Delta$ occur among F_0, F_1, F_2.

The mutual intersections of the Γ_i can be found by substituting a parametrisation of Γ_i in the equation R_j and factorising. A first result is that (except at F_0, F_4), all intersection numbers with Γ_5 are even, so for this purpose, Γ_5 does behave as though its degree were $4\frac{1}{2}$. The factors are as in the following table, where α_i is of degree 1, and corresponds to F_i; ϕ_j is irreducible of degree j over $\mathbb{Q}$ (and not necessarily the same on different occasions).

$$
\begin{aligned}
\Gamma_1 \cap \Gamma_2 &= \phi_4\phi_6\phi_{12},\\
\Gamma_1 \cap \Gamma_3 &= \phi_{16},\\
\Gamma_1 \cap \Gamma_4 &= \phi_1\phi_1\phi_2\phi_2\phi_4\phi_4\phi_6\phi_8,\\
\Gamma_1 \cap \Gamma_5 &= \alpha_2\alpha_3\phi_1\phi_2\phi_4,\\
\Gamma_2 \cap \Gamma_3 &= \alpha_1^{5/3}\phi_{13},\\
\Gamma_2 \cap \Gamma_4 &= \alpha_1^{2/3}\phi_2\phi_3\phi_4\phi_{16},\\
\Gamma_2 \cap \Gamma_5 &= \alpha_4^{1/4}\phi_2\phi_6,\\
\Gamma_3 \cap \Gamma_4 &= \alpha_1^{2/3}\phi_{18},
\end{aligned}
$$

$$\Gamma_3 \cap \Gamma_5 = \alpha_5^2 \phi_4,$$
$$\Gamma_4 \cap \Gamma_5 = \alpha_0^{1/2} \phi_1 \phi_2 \phi_3 \phi_4.$$

Apart from the α_j, different ϕ_j all represent points distinct from the F_j and from each other. Collectively, the ϕ's respresent 151 points, at each of which just two of the Γ_i meet transversely (save when one curve is Γ_5, when we have simple contact.

We now consider singularities of the curves Γ_j. In general, a smooth curve of degree d in weighted projective space $\mathbb{P}(a_0, a_1, a_2)$ has genus g equal to the number of monomials of degree $d - \sum_0^2 a_i$. This is to be decreased by terms corresponding to any singularities (the usual rule $\mu + r - 1$ will apply at points smooth on $\mathcal{M}$). Now our curves $\Gamma_1, \ldots, \Gamma_5$ have degrees 12, 11, 8, 14, 9; decreasing by 6 gives 6, 5, 2, 8, 3 and the corresponding numbers of monomials are 7, 5, 2, 10, 3: e.g. in the first case, we have $x^7, x^5y, x^3y^2, xy^3, x^3z, xyz, xz^2$. Since our curves are parametrised, they are rational, so these numbers must be accounted for by singularities.

Computationally, we seek repeated factors of the discriminant of R_i with respect to $z = I_{12}$. These are, with notation as above and where δ denotes $I_4^2 - 128I_8$,

Curve	Discriminant	Singularities
Γ_1	$\delta^2\phi_1^2\phi_2^2\phi_4^2$	$\phi_1\phi_2\phi_4$
Γ_2	ϕ_5^2	ϕ_5
Γ_3	ϕ_2^2	ϕ_2
Γ_4	$\delta^2\phi_1^2\phi_3^2\phi_6^2$	$\phi_1\phi_3\phi_6$
Γ_5	$j_2^5\phi_1^3$	$j_2\beta_3$

The factors δ for Γ_1, Γ_4 do not yield singular points: Γ_1 touches Δ at F_2 and Γ_4 touches it at F_0 (otherwise we would have transverse intersections at smooth points). For $\Gamma_1, \ldots, \Gamma_4$ the remaining factors yield (distinct) singular points of type A_1.

For Γ_5 the factor j_2 vanishes at F_5; β_3 vanishes at F_3. Calculating in local coordinates shows that F_5 is a singular point of Γ_5 of type A_4, and that F_3 is one of type A_2: together, these lower the genus by 3, confirming our calculations.

The ϕ's corresponding to singular points represent a further 24 points of $\mathcal{M}$. For our stratification, in the $\mathbb{C}$ case we need

$$151(\phi\text{'s before}) + 24(\text{these } \phi\text{'s}) + 3(F_3, F_4, F_5) = 178$$

special points (as well as the seven points F_i $(i = 0, 1, 2)$, G_i $(i = 1, 2, 3, 4)$ on Δ).

6 Characteristic 2

6.1 Introduction

Over a field K of characteristic 2 (which, when convenient, we assume algebraically closed) there are several differences to the above theory. We now write a general quintic in the form

$$a = \sum_{r=0}^{5} a_r x^{5-r} y^r, \tag{3}$$

omitting the binomial coefficients. We compute the above invariants for a where, for the moment, we still work in characteristic zero. We find that the expressions for I_4, $2^5 I_8$ and $2^{10} I_{12}$ have no denominators in their expansions; reducing them modulo 2 we obtain i_4, i_8 and i_6^2, where the invariants i_4, i_6 and i_8 are given by the following expressions. First set

$$A_1 := a_0a_3 + a_1a_2, \quad A_2 := a_0a_5 + a_1a_4, \quad A_3 := a_2a_3, \quad A_4 := a_2a_5 + a_3a_4,$$
$$B_1 := a_1a_2^2a_5 + a_0a_3^2a_4, \quad B_2 := a_1a_3^3 + a_2^3a_4, \quad C := a_0a_3^5 + a_2^5a_5.$$

Then we have

$$\begin{aligned} i_4 &= A_2A_3 + A_2^2 + B_1, \\ i_6 &= a_2^4A_4 + a_3^4A_1, \\ i_8 &= A_2C + B_2^2 + B_2A_2A_3 + B_1A_3^2 + A_3^3A_2 + A_3^4. \end{aligned}$$

We can verify directly that i_4, i_6 and i_8 are indeed invariants of a under the action of $\mathrm{SL}_2(K)$. Using the same technique of clearing denominators and reducing modulo 2, we find that the invariant I_{18} yields i_6^3, and the discriminant gives i_4^2, so we obtain no further basic invariants. We will see in Theorem 6.2 that the ring of polynomial invariants is the polynomial ring $K[i_4, i_6, i_8]$. The moduli space $\mathcal{M}_2$ is thus isomorphic to the weighted projective space $\mathbb{P}(2, 3, 4)_K$. Its singular points are of type A_2 at (0, 1, 0) and of type A_1 at (0, 0, 1).

6.2 Binary quartics in characteristic 2

We begin by revising this rather simpler case. For quartics $f(x, y) := ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$, the ring of invariants is polynomial, generated by $i_2 := bd + c^2$ and $i_3 := ad^2 + bcd + eb^2$. The vanishing of i_3 characterises the case of repeated roots.

The automorphism group of $\mathbb{P}^1_K$ is $\mathrm{SL}_2(K) \cong \mathrm{PGL}_2(K)$. Each involution (element of order 2) is conjugate to the map given affinely as $t \to t + 1$, which has the unique fixed point (pole) ∞. Given four distinct points in $\mathbb{P}^1_K$, we can arrange them in three ways into two pairs. For each such arrangement, there is a unique involution interchanging the two elements of each pair. The poles of all three involutions coincide, and the involutions commute, forming a copy of the four group.

The derivatives $\partial f/\partial x = bx^2y + dy^3$, $\partial f/\partial y = bx^3 + dxy^2$ depend only on the class of f modulo the ideal $I_1 = \langle x^2, y^2\rangle \lhd K[x, y]$ generated by perfect squares, and (provided $f \notin I_1$) vanish together only at the point where $bx^2 + dy^2 = 0$. This point is the pole of the involutions permuting the roots of f: we can call it the pole of f. If, however, $f \in I_1$, so that $b = d = 0$, f itself is a perfect square, so can be reduced under $\mathrm{SL}_2(K)$ to cx^2y^2 (with $c \neq 0$) or x^4.

For f not a square, we take the pole of f as $(1, 0)$, so that $b = 0$, $d \neq 0$, the involutions take the form $t(=x/y) \to t + \alpha$, $t \to t + \beta$, so

$$f = a(x + \gamma y)(x + (\gamma + \alpha)y)(x + (\gamma + \beta)y)(x + (\gamma + \alpha + \beta)y); \qquad (4)$$

here we can also reduce γ to 0, so $f = ax^4 + cx^2y^2 + dxy^3$, and the invariants reduce to $i_2 = c^2$, $i_3 = ad^2$. Under $\mathrm{SL}_2(K)$ we can further reduce d to 1, so f has the unique normal form $i_3x^4 + \sqrt{i_2}x^2y^2 + xy^3$. If there is a further symmetry of the roots, we can take it to fix 0, and it must cyclically permute the others. In this case, $i_2 = 0$ and the set of four points is equivalent (under $\mathrm{PGL}_2(K)$) to the affine line over $\mathbb{F}_4$.

6.3 Quintics with symmetry and special quintics

A set of five points admitting an involution must consist of a set of four points together with its pole, and hence we can take the equation as yf with f as in (4), with invariants $((\alpha\beta(\alpha+\beta))^2, 0, (\alpha^2+\alpha\beta+\beta^2)^6)$. In particular, the condition for a quintic to possess such a symmetry is $i_6 = 0$.

For a symmetry of order 3 or 5, as in the characteristic zero case, we may suppose the group a diagonal subgroup of SL_2, and obtain the above forms f_3 and f_5. Over K, these are equivalent to each other, and to f_4, which is the equation for the set of five points forming the projective line $\mathbb{P}^1_{\mathbb{F}_4}$ over the Galois field $\mathbb{F}_4$. This admits the group $\mathrm{SL}_2(\mathbb{F}_4)$ of automorphisms, which has order 60 and acts as the alternating group. The corresponding invariants are $(1, 0, 0)$.

As in characteristic zero, we also have cases with repeated roots: $f_1 = x^2(x^3 - y^3)$ (equianharmonic tetrad with one point repeated), with invariants $(0, 1, 0)$, and $f_2 = x^2y^2(x - y)$ (two repeated points, and another point), with invariants $(0, 0, 1)$ (over K, f_0 is equivalent to this).

6.4 Normal forms

For a fifth power of a linear form we have $a_2 = a_3 = 0$; thus a general quintic is no longer a linear combination of fifth powers: indeed, a quintic is so if and only if it belongs to the ideal $I_2 \lhd K[x, y]$ generated by 4th powers of linear forms: equivalently, if $a_2 = a_3 = 0$. Noting that $\partial f/\partial x = a_0x^4 + a_2x^2y^2 + a_4y^4$, we see that, if a_2 and a_3 do not both vanish, there is a unique differential operator $D = \alpha\partial/\partial x + \beta\partial/\partial y$ such that Df has zero coefficient of x^2y^2 and hence is a fourth power.

Theorem 6.1 *A binary quintic f can be reduced under $\mathrm{SL}_2(K)$ to a unique normal form as follows:*

(a) *if $i_4 \neq 0$, $i_6 \neq 0$, $a_0x^5 + x^3y^2 + a_4xy^4 + a_5y^5$ with $a_0 \neq 0$, $a_5 \neq 0$, with invariants $(a_0^2a_5^2, a_5, a_0a_5^2 + a_4^2)$;*
(b) *if $i_4 \neq 0$, $i_6 = 0$, $a_1x^4y + a_3x^2y^3 + a_4xy^4$ with $a_1 \neq 0$, $a_4 \neq 0$; with invariants $(a_1^2a_4^2, 0, a_1^2a_3^6)$, and either $a_3 = 1$ or $a_3 = 0$, $a_1 = 1$;*
(c) *if $i_4 = 0$, $(i_6, i_8) \neq (0, 0)$, $x^3y^2 + a_4xy^4 + a_5y^5$ with $(a_4, a_5) \neq (0, 0)$, with invariants $(0, a_5, a_4^2)$;*

(d) *if* $i_4 = i_6 = i_8 = 0$, *one of* $ax^3y(x+y)$ *with* $a \neq 0$, x^3y^2, x^4y *or* x^5.

Proof (a) Since $i_6 \neq 0$, we cannot have $a_2 = a_3 = 0$, so have a derivation D as above. We change coordinates to take Df as (a non-zero multiple of) y^4. Then $A_1 = 0$ and $A_4 \neq 0$, so $i_6 = a_2^4 A_4 \neq 0$. We can thus also take $a_2x + a_3y$ as a non-zero multiple of x, or equivalently D as $\partial/\partial y$. Then the coordinates x and y are both fixed up to scalar multiples; and $a_1 = a_3 = 0$, $a_5 \neq 0$. Now $i_4 = a_0^2 a_5^2$ is non-zero, so $a_0 \neq 0$. Replacing x by $a_2^{-1}x$ and y by a_2y, we reduce a_2 to 1.

(b) If a_2 and a_3 do not both vanish, we proceed as before up to the point where $A_1 = 0$ and $A_4 \neq 0$, but now since $i_6 = 0$, we deduce $a_2 = 0$, so $D = \partial/\partial x$. We now have $a_0 = a_2 = 0$, $a_4 \neq 0$, which we recognise as the same form as for quintics with symmetry. As in the preceding paragraph, we can choose x as one of the factors of f, thus reducing a_5 to 0, and can reduce a_3 to 1.

In the case $a_2 = a_3 = 0$, the invariants are $((a_1a_4 + a_0a_5)^2, 0, 0)$. As $i_4 \neq 0$, there are at least two distinct linear factors. Choosing x and y to be two of them we have $a_0 = a_5 = 0$, hence $a_1 \neq 0$, $a_4 \neq 0$. The quintic is equivalent to $\mathbb{P}^1_{\mathbb{F}_4}$ (in particular, f is indeed a sum of two fifth powers), and this case is subsumed as $a_3 = 0$ in the preceding normal form.

(c) If $i_4 = 0$, the quintic has a repeated factor, and we can take the factor as y^2, and thus suppose $a_0 = a_1 = 0$. The invariants then reduce to $(0, a_2^4(a_2a_5 + a_3a_4), a_2^6a_4^2 + a_2^4a_3^4)$. If $a_2 = 0$, all invariants vanish; otherwise we normalise coordinates first so that $a_2 = 1$ and then so that $a_3 = 0$. The invariants now reduce to $(0, a_5, a_4^2)$, so our hypothesis gives $(a_4, a_5) \neq (0, 0)$.

(d) We saw in the preceding paragraph that if all invariants vanish, we may reduce till either $a_0 = a_1 = a_2 = 0$ or $a_3 = a_4 = a_5 = 0$: in either case, there is a linear factor of multiplicity at least 3. Thus there are at most three distinct roots, so by inspection the quintic can be reduced to $ax^3y(x+y)$ (with $a \neq 0$), x^3y^2, x^4y or x^5. □

We observe that the results of Geometric Invariant Theory apply to this case, and imply that all invariants vanish if and only if there is a 3-fold factor; and in all other cases the quintic is stable, and is determined up to $SL_2(K)$ by its invariants, as we see directly.

Theorem 6.2 *The* $SL_2(K)$*-invariants in* $K[a_0, \ldots, a_5]$ *form the polynomial ring* $K[i_4, i_6, i_8]$.

Proof Any invariant is determined by its value on a generic quintic, which we can take in the normal form (a). As we have just seen, the invariants (i_4, i_6, i_8) here take the values $(a_0^2a_5^2, a_5, a_0a_5^2 + a_4^2)$. We can thus write $a_5 = i_6$, $a_0 = i_6^{-1}\sqrt{(i_4)}$ and $a_4 = \sqrt{(i_8 + i_6\sqrt{(i_4)})}$. The field of invariants is thus contained in $K(i_6, i_6^{-1}\sqrt{(i_4)}, \sqrt{(i_8 + i_6\sqrt{(i_4)})})$, which has degree 4 over $K(i_4, i_6, i_8)$. We see by inspection that each strictly intermediate field contains $\sqrt{(i_4)}$. However, for a generic quintic (3), i_4 is not a perfect square. Thus the field of invariants coincides with $K(i_4, i_6, i_8)$.

It remains to show that an element of this field which restricts to a polynomial in $a_0, \ldots, a_5$ is a polynomial in i_4, i_6 and i_8. We see from the preceding paragraph that we have a polynomial in i_6, $i_6^{-1}\sqrt{(i_4)}$ and $\sqrt{(i_8 + i_6\sqrt{(i_4)})}$, so the only denominator that may occur is a power of i_6. If such a case occurs, there must be an example with denominator i_6, and we may suppose without loss of generality that the numerator depends on i_4 and i_8 only. But for the normal form for case (b), i_4 and i_8 are independent, while $i_6 = 0$. Hence no such example exists. □

6.5 Exceptional curves

As before, given a plane quintic curve H_i with 1-parameter symmetry group G_i, the lines L in the plane $\mathbb{P}^2_K$ fall into a 1-parameter family of orbits under G_i, so we expect their intersections with H_i to give a family of binary quintics whose moduli trace out a curve Γ_i in $\mathcal{M}_2$.

If we take the above calculations, remove the appropriate power of 2, and reduce mod 2, R_1 reduces to i_6^8, R_2 to $i_6^2 i_8^4$, R_3 to $(i_4 i_6^2 + i_8^2)^2$ and R_5 to i_6^6, but R_4 remains irreducible. However, it is easy to calculate directly.

For each of $h_1, \ldots, h_5$ in turn, we first substitute $z = x + uy$, then list the coefficients of $x^{5-r} y^r$ for $r = 0, 1, 2, 3, 4, 5$. We then calculate in turn $A_1, A_2, A_3, A_4, B_1, B_2, C$ and finally the invariants (i_4, i_6, i_8). The results are as follows (recall that we are calculating mod 2):

For h_1, the quintic $x^4(x + uy) + xy^2(x + uy)^2$ has coefficients $[1, u, 1, 0, u^2, 0]$. We obtain $A_1 = u$, $A_2 = u^3$, $A_3 = 0$, $A_4 = 0$, $B_1 = 0$, $B_2 = u^6$, $C = 0$, and $(i_4, i_6, i_8) = (u^6, 0, u^{12})$.

For h_2 we find in turn: $x^4(x + uy) + y^3(x + uy)^2$, coefficients $[1, u, 0, 1, 0, u^2]$ $A_1 = 1$, $A_2 = u^2$, $A_3 = 0$, $A_4 = 0$, $B_1 = 0$, $B_2 = u$, $C = 1$ and $(i_4, i_6, i_8) = (u^4, 1, 0)$.

Next, h_3 gives $x^5 + y^3(x + uy)^2$, with coefficients $[1, 0, 0, 1, 0, u^2]$, so $A_1 = 1$, $A_2 = u^2$, $A_3 = 0$, $A_4 = 0$, $B_1 = 0$, $B_2 = 0$, $C = 1$, and $(i_4, i_6, i_8) = (u^4, 1, u^2)$.

Next, h_4 yields the quintic $x^3 y(x+uy)+y^3(x+uy)^2$, coefficients $[0, 1, u, 1, 0, u^2]$, giving $A_1 = u$, $A_2 = 0$, $A_3 = u$, $A_4 = u^3$, $B_1 = u^4$, $B_2 = 1$, $C = u^7$, and $(i_4, i_6, i_8) = (u^4, u + u^5, 1 + u^4 + u^6)$.

Finally, h_5 gives $xy^3(x + uy) + x^2 y(x + uy)^2$, $[0, 1, 0, 1 + u^2, u, 0]$, $A_1 = 0$, $A_2 = u$, $A_3 = 0$, $A_4 = u + u^3$, $B_1 = 0$, $B_2 = 1 + u^2 + u^4 + u^6$, $C = 0$, and $[i_4, i_6, i_8] = [u^2, 0, 1 + u^4 + u^8 + u^{12}]$.

Thus the loci in $\mathcal{M}_2$ are indeed defined by $i_6 = 0$, $i_8 = 0$, $i_4 i_6^2 = i_8^2$, a complicated expression, and $i_6 = 0$.

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