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RESEARCH ARTICLE

Realizations of pairs and Oka families in tensor triangulated categories

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Abstract In this paper, we apply some methods from ring theory to the framework of prime ideals in tensor triangulated categories developed by Balmer. Given a thick tensor ideal $\mathcal{A}$ and a multiplicatively closed family $\mathcal{S}$ of objects in a tensor triangulated category $(\mathcal{C}, \otimes, 1)$, we say that a prime ideal $\mathcal{P}$ realizes $(\mathcal{A}, \mathcal{S})$ if $\mathcal{P} \supseteq \mathcal{A}$ and $\mathcal{P} \cap \mathcal{S} = \varnothing$. Analogously to the results of Bergman with ordinary rings, we show how to construct a realization of a family $\{(\mathcal{A}_i, \mathcal{S}_i)\}_{i \in I}$ of such pairs indexed by a finite chain I, i.e., a collection $\{\mathcal{P}_i\}_{i \in I}$ of prime ideals such that each $\mathcal{P}_i$ realizes $(\mathcal{A}_i, \mathcal{S}_i)$ and $\mathcal{P}_i \subseteq \mathcal{P}_j$ for each $i \leqslant j$ in I. Thereafter we obtain conditions on a family $\mathfrak{F}$ of thick tensor ideals of $(\mathcal{C}, \otimes, 1)$ so that any ideal that is maximal with respect to not being contained in $\mathfrak{F}$ must be prime. This extends the Prime Ideal Principle of Lam and Reyes from commutative algebra. We also combine these methods to consider realizations of templates $\{(\mathcal{A}_i, \mathfrak{F}_i)\}_{i \in I}$, where each $\mathcal{A}_i$ is a thick tensor ideal and each $\mathfrak{F}_i$ is a family of thick tensor ideals that is also a monoidal semifilter.

Keywords Tensor triangulated categories · Oka families

Mathematics Subject Classification 13A15 · 18E30

1 Introduction

Triangulated categories were introduced by Verdier [37] and have since assumed an increasing significance in several fields of modern mathematics; from algebraic geometry to motives and homotopy theory, modular representation theory and non-

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commutative geometry. Additionally, a triangulated category appearing in these areas is often gifted with a tensor structure, making it into a *tensor triangulated category*, i.e., a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ such that $\mathcal{C}$ is triangulated and $\otimes$ is exact in both variables. The classification of thick subcategories is a common theme that runs through the work of Devinatz et al. [17] in homotopy theory, that of Thomason [36] in algebraic geometry, the work of Benson et al. [11] in modular representation theory and that of Friedlander and Pevtsova with finite group schemes [19]. Tensor triangular geometry developed by Balmer (see [1–5,8]) unites these classifications in terms of classification of thick tensor ideal subcategories in a tensor triangulated category (also see further work by Balmer and Favi [9,10]).

Given a tensor triangulated category $(\mathcal{C}, \otimes, 1)$, Balmer [1] associates to it a spectrum $\mathrm{Spec}(\mathcal{C})$ consisting of prime ideals of $\mathcal{C}$. A thick tensor ideal $\mathcal{P}$ (see Definition 2.1) in $(\mathcal{C}, \otimes, 1)$ is called *prime* if $a \otimes b \in \mathcal{P}$ for any objects $a, b \in \mathcal{C}$ implies that at least one of a, b lies in $\mathcal{P}$. Then $\mathrm{Spec}(\mathcal{C})$ is equipped with a Zariski topology and the support theory obtained by associating to each object $a \in \mathcal{C}$ the closed subset $\mathrm{Supp}(a) = \{\mathcal{P} \in \mathrm{Spec}(\mathcal{C}) : a \notin \mathcal{P}\} \subseteq \mathrm{Spec}(\mathcal{C})$ unites various support theories in algebraic geometry, modular representation theory and homotopy theory. Further, support theory for a tensor triangulated category $(\mathcal{C}, \otimes, 1)$ acting on a triangulated category $\mathcal{M}$ has been developed by Stevenson [34]. Stevenson's work in [34] may be viewed as categorification of some of the work of Benson, Iyengar and Krause [12–14] in the case of actions of the unbounded derived category $\mathrm{D}(R)$ for a commutative noetherian ring R. Tensor triangular geometry has further emerged as an object of study in its own right: for instance, Balmer [6] introduced Chow groups of rigid tensor triangulated categories and properties of these Chow groups have been studied in detail by Klein [23,24]. For further work in tensor triangular geometry, we refer the reader, for example, to Dell'Ambrogio and Stevenson [16], Peter [28], Sanders [31], Stevenson [35] and Xu [38].

The purpose of this paper is to bring some methods in ring theory to the framework of tensor triangulated categories. Let R be an ordinary commutative ring, A be an ideal in R and let $S \subseteq R$ be a multiplicatively closed subset. Then, Bergman [15] refers to a prime ideal P such that $P \supseteq A$ and $P \cap S = \varnothing$ as a *realization* of the pair (A, S) and says that $P \in \mathrm{Spec}(A, S)$. More generally, if $(I, \leqslant)$ is a partially ordered set, Bergman [15] has studied conditions under which a template $\{(A_i, S_i)\}_{i \in I}$ of such pairs indexed by I admits a realization, i.e., a family $\{P_i\}_{i \in I}$ of prime ideals such that P_i realizes (A_i, S_i) and $P_i \subseteq P_j$ whenever $i \leqslant j$ in I (see also further work by Sharma [33]). In [1, Lemma 2.2], Balmer shows that if $\mathcal{A}$ is a thick tensor ideal in $(\mathcal{C}, \otimes, 1)$ and $\mathcal{S}$ is a multiplicatively closed family of objects of $\mathcal{C}$ such that $\mathcal{A} \cap \mathcal{S} = \varnothing$, there exists a prime ideal $\mathcal{P} \supseteq \mathcal{A}$ satisfying $\mathcal{P} \cap \mathcal{S} = \varnothing$. Our first main aim in this paper is to formulate conditions analogous to those of Bergman [15] for construction of realizations of certain templates $\{(\mathcal{A}_i, \mathcal{S}_i)\}_{i \in I}$, where each $\mathcal{A}_i$ is a thick tensor ideal in $(\mathcal{C}, \otimes, 1)$ and each $\mathcal{S}_i$ is a multiplicatively closed family of objects of $\mathcal{C}$.

We start in Sect. 2 by defining a relation $\preccurlyeq$ among pairs such that $(\mathcal{A}, \mathcal{S}) \preccurlyeq (\mathcal{A}', \mathcal{S}')$ if every prime ideal $\mathcal{P}'$ realizing $(\mathcal{A}', \mathcal{S}')$ contains a prime ideal $\mathcal{P}$ realizing the pair $(\mathcal{A}, \mathcal{S})$. Thereafter, given a template $T = \{(\mathcal{A}_i, \mathcal{S}_i)\}_{i \in I}$ indexed by a finite decreasing chain I, we show that its realizations can be described in terms of realizations of a template $\mathrm{D}(T) = \{(\mathcal{B}_i, \mathcal{S}_i)\}_{i \in I}$ satisfying $(\mathcal{B}_i, \mathcal{S}_i) \preccurlyeq (\mathcal{B}_j, \mathcal{S}_j)$ for each $i \leqslant j$ in I. Under certain finiteness conditions on the pairs in T, we construct a template $\mathcal{D}(T) = \{(\mathcal{B}_i, \mathcal{T}_i)\}_{i\in I}$ such that for any fixed $i_0 \in I$, we can start with a prime ideal $\mathcal{P}$ realizing the pair $(\mathcal{B}_{i_0}, \mathcal{T}_{i_0})$ and obtain a realization $\{\mathcal{P}_i\}_{i\in I}$ of T with $\mathcal{P}_{i_0} = \mathcal{P}$. Further, looking at the subsets of the form $Spec(\mathcal{A}, \mathcal{S}) \subseteq Spec(\mathcal{C})$ themselves, we show that these are exactly the *convex subsets* of $Spec(\mathcal{C})$, i.e., subsets $\mathfrak{X} \subseteq Spec(\mathcal{C})$ satisfying the property that if $\bigcup_{\mathcal{P}\in\mathfrak{X}} \mathcal{P} \subseteq \mathcal{Q} \subseteq \bigcup_{\mathcal{P}\in\mathfrak{X}} \mathcal{P}$, then $\mathcal{Q} \in \mathfrak{X}$. We also give necessary and sufficient conditions for a family of finite chains of prime ideals to be a collection of realizations of such template $T = \{(\mathcal{A}_i, \mathcal{S}_i)\}_{i\in I}$.

Since $Spec(\mathcal{C})$ is a spectral space, following Hochster [21], we know that it is equipped with an inverse topology where the open sets $W \subseteq Spec(\mathcal{C})$ are given by arbitrary unions $W = \bigcup_{i\in I} Y_i$ with each $Spec(\mathcal{C})\backslash Y_i$ open and quasi-compact in $Spec(\mathcal{C})$. Then, $Spec(\mathcal{C})$ equipped with this inverse topology is denoted by $Spec(\mathcal{C})^*$. Then, [4, Theorem 14] may be restated as follows: to every thick tensor ideal $\mathcal{I}$ in $(\mathcal{C}, \otimes, 1)$ there is associated the closed subspace

$$c(\mathcal{I}) = \{\mathcal{P} \in Spec(\mathcal{C}) : \mathcal{P} \supseteq \mathcal{I}\} \tag{1}$$

of $Spec(\mathcal{C})^*$. Then, [4, Theorem 14] shows that (1) gives a one-to-one order reversing correspondence between the radical thick tensor ideals in $(\mathcal{C}, \otimes, 1)$ and the closed subspaces of $Spec(\mathcal{C})^*$. We have restated this theorem in terms of an order reversing correspondence with closed subspaces (rather than the order preserving correspondence with open subspaces expressed in [4, Theorem 14]) in order to have a comforting similarity with the standard Nullstellensatz. In fact, if X is a topologically noetherian scheme, using the fact that there is a homeomorphism $X \simeq Spec(D^{perf}(X))$ (see [1, Corollary 5.6]), we see that the theorem gives us a correspondence between (radical) thick tensor ideals in $Spec(D^{perf}(X))$ and closed subspaces of X in the inverse topology. We can build on this idea in two ways: first, we can think about the constructible topology on X, because the constructible topology on a spectral space always coincides with the constructible topology on its inverse (see, for instance, [18, Corollary 4.8]). Secondly, we can think about the closed irreducible subspaces of the scheme X in inverse topology. More generally, for the tensor triangulated category $\mathcal{C}$, we show that subsets of the form $Spec(\mathcal{A}, \mathcal{S})$, which we have previously characterized as the convex subsets of $Spec(\mathcal{C})$, are pro-constructible subsets of $Spec(\mathcal{C})$. Then, the subspaces $Spec(\mathcal{A}, \mathcal{S}) \subseteq Spec(\mathcal{C})$ themselves become spectral spaces in the induced topology. Further, if $\mathcal{C}$ is assumed to be topologically noetherian (see [2, Definition 3.13]), i.e., $Spec(\mathcal{C})$ is a noetherian space, then the constructible topology on $Spec(\mathcal{C})$ has a basis of subsets of the form $Spec(\mathcal{A}, \mathcal{S})$. Further, we show that any closed subspace of $Spec(\mathcal{C})$ in the constructible topology may be expressed as a union of subspaces of the form $Spec(\mathcal{A}, \mathcal{S})$.

On the other hand, suppose that $\mathfrak{F}^*$ is a family of closed subspaces of $Spec(\mathcal{C})^*$ such that $\varnothing \in \mathfrak{F}^*$ and $\mathfrak{F}^*$ is closed under finite unions. Then, it is clear that any closed subspace $K_0 \subseteq Spec(\mathcal{C})^*$ that is minimal with respect to not being in $\mathfrak{F}^*$ must be irreducible. If $\mathcal{C} = D^{perf}(X)$ for a topologically noetherian scheme X, then such K_0 will correspond to a closed irreducible subspace of X in inverse topology. Now, a collection $\mathfrak{F}$ of thick tensor ideals of $(\mathcal{C}, \otimes, 1)$ will be referred to as a *monoidal family* if $\mathcal{C} \in \mathfrak{F}$ and $\mathcal{I}_1 \otimes \mathcal{I}_2 \in \mathfrak{F}$ for every $\mathcal{I}_1, \mathcal{I}_2 \in \mathfrak{F}$ (see Definition 3.1). Since

$c(\mathcal{I}_1)\cup c(\mathcal{I}_2) = c(\mathcal{I}_1\otimes\mathcal{I}_2)$ for thick tensor ideals $\mathcal{I}_1, \mathcal{I}_2$ in $\mathcal{C}$, using the correspondence above, we can translate this fact as follows: let $\mathfrak{F}$ be a monoidal family of radical ideals in $\mathcal{C}$ such that $\mathcal{C} \in \mathfrak{F}$ and let $\mathcal{I}_0$ be a radical ideal that is maximal with respect to being a radical ideal not contained in $\mathfrak{F}$. Then, if $\mathcal{J}$, $\mathcal{K}$ are radical ideals such that $\mathcal{I}_0 \supseteq \mathcal{J}\otimes\mathcal{K}$, we must have either $\mathcal{I}_0 \supseteq \mathcal{J}$ or $\mathcal{I}_0 \supseteq \mathcal{K}$. However, this statement does not seem very satisfactory and we would like to obtain better results on families of thick tensor ideals in $(\mathcal{C}, \otimes, 1)$. We therefore use some methods from commutative algebra, where there are several results of the type "maximal implies prime". For example, given an R-module M, an ideal $I \subseteq R$ that is maximal among annihilators of non-zero elements of M must be prime. In [25], Lam and Reyes unified such results in the form of a "Prime Ideal Principle", i.e., criteria on a family $\mathcal{F}$ of ideals in R such that an ideal I that is maximal with respect to not being in $\mathcal{F}$ must be prime (see also further work by Lam and Reyes [26] and Reyes [29,30]). Then, the main purpose of Sect. 3 is to construct an analogous Prime Ideal Principle for thick tensor ideals in $(\mathcal{C}, \otimes, 1)$ (see Theorem 3.5). For any thick tensor ideal $\mathcal{I}$ and any collection X of objects of $\mathcal{C}$, we set

$$(\mathcal{I}{:}X) = \{a \in \mathcal{C} : a\otimes x \in \mathcal{I} \text{ for each } x \in X\}.$$

Similarly to the terminology of Lam and Reyes [25], we say that a family $\mathfrak{F}$ of thick tensor ideals is an *Oka family* (resp. an *Ako family*) if given objects $a, b \in \mathcal{C}$ and a thick tensor ideal $\mathcal{I}$ in $(\mathcal{C}, \otimes, 1)$, then $(\mathcal{I}, a) \in \mathfrak{F}$ and $(\mathcal{I}{:}a) \in \mathfrak{F}$ implies $\mathcal{I} \in \mathfrak{F}$ (resp. $(\mathcal{I}, a) \in \mathfrak{F}$ and $(\mathcal{I}, b) \in \mathfrak{F}$ implies $(\mathcal{I}, a\otimes b) \in \mathfrak{F}$). Thereafter, we show that any family $\mathfrak{F}$ of thick tensor ideals that is either Oka or Ako must satisfy the Prime Ideal Principle. In particular, if $\mathcal{S}$ is a multiplicatively closed family of objects of $\mathcal{C}$, we show that the family

$$\mathfrak{F}_{\mathcal{S}} = \{\mathcal{I} : \mathcal{I} \text{ is a thick tensor ideal and } \mathcal{I} \cap \mathcal{S} \neq \varnothing\}$$

is an Oka family (as well as an Ako family) and hence any thick tensor ideal maximal with respect to being disjoint from $\mathcal{S}$ must be prime. In fact, $\mathfrak{F}_{\mathcal{S}}$ is also a *monoidal semifilter* (see Definition 3.1), i.e., the product of any two thick tensor ideals in $\mathfrak{F}_{\mathcal{S}}$ lies in $\mathfrak{F}_{\mathcal{S}}$ and, given any $\mathcal{I} \in \mathfrak{F}_{\mathcal{S}}$, any thick tensor ideal $\mathcal{J} \supseteq \mathcal{I}$ also lies in $\mathfrak{F}_{\mathcal{S}}$. From Theorem 3.5, a monoidal semifilter $\mathfrak{F}$ is both an Oka family and an Ako family. We also prove other results from [25] in the framework of thick tensor ideals in $(\mathcal{C}, \otimes, 1)$, such as any ideal that is maximal among thick tensor ideals satisfying $\mathcal{I}^{\otimes n} \supsetneq \mathcal{I}^{\otimes n+1}$ for each $n \geqslant 0$ must be prime. Using the notion from Stevenson [34] of a tensor triangulated category $(\mathcal{C}, \otimes, 1)$ having a module action on a triangulated category $\mathcal{M}$, we show that a thick tensor ideal that is maximal among annihilators of non-zero objects of $\mathcal{M}$ is also prime.

In Sect. 4, we combine the methods of Sects. 2 and 3. We consider pairs $(\mathcal{A}, \mathfrak{F})$ such that $\mathcal{A}$ is a thick tensor ideal and $\mathfrak{F}$ is a monoidal semifilter. Further, we assume that any non-empty increasing chain of ideals in the complement $\mathfrak{F}^c$ of $\mathfrak{F}$ has an upper bound in $\mathfrak{F}^c$ (see Definition 4.1), whence it follows that if $\mathcal{A} \notin \mathfrak{F}$, there always exists a prime ideal $\mathcal{P} \supseteq \mathcal{A}$ such that $\mathcal{P} \notin \mathfrak{F}$. We refer to such prime ideal $\mathcal{P}$ as a realization of $(\mathcal{A}, \mathfrak{F})$. Accordingly, we can define realizations of templates $\{(\mathcal{A}_i, \mathfrak{F}_i)\}_{i\in I}$ indexed by a partially ordered set I. As in Sect. 2, we define a relation $\preccurlyeq$ among pairs such that $(\mathcal{A}, \mathfrak{F})\preccurlyeq(\mathcal{A}', \mathfrak{F}')$ if every prime ideal $\mathcal{P}'$ realizing $(\mathcal{A}', \mathfrak{F}')$ contains a prime ideal $\mathcal{P}$ realizing $(\mathcal{A}, \mathfrak{F})$. We show how to construct realizations of a template $T = \{(\mathcal{A}_i, \mathfrak{F}_i)\}_{i\in I}$ indexed by a finite decreasing chain I by replacing it with a template $\mathrm{D}(T) = \{(\mathcal{B}_i, \mathfrak{F}_i)\}_{i\in I}$ that satisfies $(\mathcal{B}_i, \mathfrak{F}_i) \preccurlyeq (\mathcal{B}_j, \mathfrak{F}_j)$ for each $i \leqslant j$ in I. Again, under certain finiteness conditions, we construct a template $\mathcal{D}(T) = \{(\mathcal{B}_i, \mathfrak{G}_i)\}_{i\in I}$ such that for any chosen $i_0 \in I$, we can start with a prime ideal $\mathcal{P}$ realizing $(\mathcal{B}_{i_0}, \mathfrak{G}_{i_0})$ and obtain a realization $\{\mathcal{P}_i\}_{i\in I}$ of T such that $\mathcal{P}_{i_0} = \mathcal{P}$. Towards the end of Sect. 4, we also construct realizations for templates indexed by finite descending trees.

Finally, in Sect. 5, we assume that all thick tensor ideals in $(\mathcal{C}, \otimes, 1)$ are *radical*, i.e., $r(\mathcal{I}) = \mathcal{I}$ for all thick tensor ideals in $(\mathcal{C}, \otimes, 1)$. In fact, this property is quite common in examples of tensor triangulated categories (see [1, Remark 4.3]). For us, the key consequence of this assumption is that it implies $\mathcal{I}\otimes\mathcal{J} = \mathcal{I}\cap\mathcal{J}$ for all thick tensor ideals $\mathcal{I}$ and $\mathcal{J}$. We then show that all thick tensor ideals being radical, every monoidal family $\mathfrak{F}$ of thick tensor ideals satisfies the Prime Ideal Principle, i.e., any ideal that is maximal with respect to not being in $\mathfrak{F}$ must be prime. Accordingly, we show that any ideal in $(\mathcal{C}, \otimes, 1)$ that is maximal with respect to not being principal is prime. An analogous result holds for ideals that are maximal with respect to not being generated by a set of cardinality $\leqslant \alpha$ for some infinite cardinal α. Thereafter, we formulate conditions for the construction of realizations of certain templates $\{(\mathcal{A}_i, \mathfrak{F}_i)\}_{i\in I}$ indexed by an infinite decreasing chain I. We conclude by showing how we can construct families $\mathfrak{F}^*$ of closed subspaces of $\mathrm{Spec}(\mathcal{C})^*$ that are not closed under finite unions such that any closed subspace that is minimal with respect to not being in $\mathfrak{F}^*$ is irreducible. This is done with the help of Ako families of thick tensor ideals in $(\mathcal{C}, \otimes, 1)$.

We mention here that in [8], Balmer has proved a Going-Up Theorem in tensor triangular geometry with profound connections to Quillen stratification in modular representation theory. When $\mathcal{C}$ is idempotent-complete, Balmer's result (see [8, Section 1.5]) gives going-up and incomparability results for prime ideals in the spectra of categories of modules over *tt-rings* in $(\mathcal{C}, \otimes, 1)$. The tt-rings are commutative ring objects in $(\mathcal{C}, \otimes, 1)$ that are also *separable* in a suitable sense (see [8, Section 2] for details). As such, it is hoped that the methods in this paper can be developed in the future to study prime ideals in the spectra of categories of modules over tt-rings, thus using tensor triangular geometry to obtain further connections between classical commutative algebra and modular representation theory. For more on tt-rings in $(\mathcal{C}, \otimes, 1)$, we also refer the reader to Balmer [7].

In this paper, we will always assume that our categories are essentially small. Further, by abuse of notation, for any category $\mathcal{D}$, we will always write $x \in \mathcal{D}$ to mean that x is an object of $\mathcal{D}$.

2 Prime ideals in $(\mathcal{C}, \otimes, 1)$ and realizations of pairs

Throughout this section and the rest of this paper, $(\mathcal{C}, \otimes, 1)$ will be a symmetric monoidal category with $\mathcal{C}$ also having the structure of a triangulated category (see [37]). Further, we will always assume that the symmetric monoidal *tensor* product $\otimes\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ is exact in each variable and the category $\mathcal{C}$ contains all finite direct sums. We will say that $(\mathcal{C}, \otimes, 1)$ is a *tensor triangulated category*. Further, a tensor triangulated functor $F\colon (\mathcal{C}, \otimes, 1) \to (\mathcal{D}, \otimes, 1)$ between tensor triangulated categories

$\mathcal{C}$ and $\mathcal{D}$ will be an *exact functor* $F: \mathcal{C} \to \mathcal{D}$ that preserves the symmetric monoidal product and carries the unit object in $\mathcal{C}$ to the unit object in $\mathcal{D}$. Unless otherwise mentioned, we do not require our tensor triangulated categories to satisfy the additional axioms due to May [27].

Given an object a in the triangulated category $\mathcal{C}$, we will denote the translation of a in $\mathcal{C}$ by $\mathrm{T}a$. We now recall from Balmer [1] the notion of a prime ideal in the tensor triangulated category $(\mathcal{C}, \otimes, 1)$.

Definition 2.1 Let $(\mathcal{C}, \otimes, 1)$ be a tensor triangulated category as given above and let $\mathcal{A}$ be a full subcategory of $\mathcal{C}$ containing 0. Then, $\mathcal{A}$ will be called a *thick tensor ideal* if it satisfies the following conditions:

- The subcategory $\mathcal{A}$ is triangulated, i.e. if $a \to b \to c \to \mathrm{T}a$ is a distinguished triangle in $\mathcal{C}$ and any two of a, b and c lie in $\mathcal{A}$, so does the third.
- The subcategory $\mathcal{A}$ is thick, i.e., if $a \in \mathcal{A}$ splits in $\mathcal{C}$ as a direct sum $a \cong b \oplus c$, both direct summands b and c lie in $\mathcal{A}$.
- The subcategory $\mathcal{A}$ is a tensor ideal, i.e., if $a \in \mathcal{A}$ and $b \in \mathcal{C}$, then we must have $a \otimes b \in \mathcal{A}$.

We will use the expression $\mathcal{A} \lhd \mathcal{C}$ to mean that $\mathcal{A}$ is a thick tensor ideal of $(\mathcal{C}, \otimes, 1)$. Given thick tensor ideals $\mathcal{A}, \mathcal{B} \lhd \mathcal{C}$, we will denote by $\mathcal{A} + \mathcal{B}$ the smallest thick tensor ideal containing both $\mathcal{A}$ and $\mathcal{B}$.

Finally, if $\mathcal{P}$ is a proper thick tensor ideal in $(\mathcal{C}, \otimes, 1)$, $\mathcal{P}$ is said to be *prime* if

$$a \otimes b \in \mathcal{P} \quad \Longrightarrow \quad a \in \mathcal{P} \quad \text{or} \quad b \in \mathcal{P}.$$

A family $\mathcal{S}$ of objects of $\mathcal{C}$ will be said to be *multiplicatively closed* if $1 \in \mathcal{S}$ and for any $a, b \in \mathcal{S}$, we have $a \otimes b \in \mathcal{S}$. We will work with pairs $(\mathcal{A}, \mathcal{S})$, where $\mathcal{A}$ is a thick tensor ideal and $\mathcal{S}$ is a multiplicatively closed family of objects of $\mathcal{C}$. From [1, Lemma 2.2], we know that if $(\mathcal{A}, \mathcal{S})$ is such pair with $\mathcal{A} \cap \mathcal{S} = \varnothing$, there always exists a prime ideal $\mathcal{P}$ such that $\mathcal{A} \subseteq \mathcal{P}$ and $\mathcal{P} \cap \mathcal{S} = \varnothing$. We start with a prime avoidance result for the category $(\mathcal{C}, \otimes, 1)$.

Proposition 2.2 *Let $\mathcal{A}$ be a thick tensor ideal of $(\mathcal{C}, \otimes, 1)$ that is contained in the union $\bigcup_{i=1}^{n} \mathcal{P}_i$ of finitely many prime ideals $\mathcal{P}_i$. Then, there exists some $1 \leqslant i \leqslant n$ such that $\mathcal{A} \subseteq \mathcal{P}_i$.*

Proof We proceed by induction on n. The result is obvious for $n = 1$. We suppose that the result holds for all integers up to $n - 1$ and consider some $\mathcal{A} \subseteq \bigcup_{i=1}^{n} \mathcal{P}_i$. Suppose that we can choose some object $a_j \in \mathcal{A}$ for each $1 \leqslant j \leqslant n$ such that $a_j \in \mathcal{P}_j$ and $a_j \notin \bigcup_{i=1, i \neq j}^{n} \mathcal{P}_i$. We consider the object $a = (a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1}) \oplus a_n \in \mathcal{A}$ (since $\mathcal{A}$ is triangulated, it is easy to check that it contains direct sums).

Now suppose that $a \in \mathcal{P}_n$. Then, since $\mathcal{P}_n$ is thick, we must have $a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \in \mathcal{P}_n$. However, this is impossible, since $\mathcal{P}_n$ is prime and we have chosen $a_i \notin \mathcal{P}_n$ for all $1 \leqslant i \leqslant n - 1$. On the other hand, if $a \in \mathcal{P}_i$ for some $1 \leqslant i \leqslant n - 1$, it follows that $a_n \in \mathcal{P}_i$ which is also a contradiction. This contradicts the fact that $\mathcal{A} \subseteq \bigcup_{i=1}^{n} \mathcal{P}_i$. Hence, it follows that the thick tensor ideal $\mathcal{A}$ is already contained in the union of some proper subcollection of $\{\mathcal{P}_i\}_{1 \leqslant i \leqslant n}$. Using the induction assumption, the result follows. ◻

Analogously to the terminology of Bergman [15, Definition 7], we now introduce the following definition.

Definition 2.3 Let $(\mathcal{C}, \otimes, 1)$ be a tensor triangulated category as given above. Let $(\mathcal{A}, \mathcal{S})$ be a pair such that $\mathcal{A}$ is a thick tensor ideal and $\mathcal{S}$ is a multiplicatively closed family of objects of $\mathcal{C}$. Then, a prime ideal $\mathcal{P}$ in $\mathcal{C}$ is said to be a *realization of the pair* $(\mathcal{A}, \mathcal{S})$ if $\mathcal{A} \subseteq \mathcal{P}$ and $\mathcal{P} \cap \mathcal{S} = \varnothing$. The collection of all realizations of such a pair $(\mathcal{A}, \mathcal{S})$ will be denoted by $\mathrm{Spec}(\mathcal{A}, \mathcal{S})$. Further, we let $\mathcal{M}(\mathcal{A}, \mathcal{S})$ be the multiplicatively closed family given by the complement of $\bigcup_{\mathcal{Q} \in \mathrm{Spec}(\mathcal{A}, \mathcal{S})} \mathcal{Q}$.

More generally, let $(I, \leqslant)$ be a partially ordered set. By a *template* T indexed by I, we will mean a family $T = \{(\mathcal{A}_i, \mathcal{S}_i)\}_{i \in I}$ of pairs indexed by I. Then, we will say that a collection $\{\mathcal{P}_i\}_{i \in I}$ of prime ideals in $\mathcal{C}$ is a *realization of the template* T if each $\mathcal{P}_i$ realizes the pair $(\mathcal{A}_i, \mathcal{S}_i)$ and $\mathcal{P}_i \subseteq \mathcal{P}_j$ for every $i \leqslant j$ in I.

Given a thick tensor ideal $\mathcal{A}$ and a multiplicatively closed family $\mathcal{S}$, we define $\mathcal{A} \div \mathcal{S}$ to be the full subcategory of $\mathcal{C}$ consisting of the following objects:

$$\mathrm{Ob}(\mathcal{A} \div \mathcal{S}) = \{x \in \mathcal{C} : \text{ there exists } s \in \mathcal{S} \text{ such that } x \otimes s \in \mathcal{A}\}. \tag{2}$$

Lemma 2.4 *Let $\mathcal{A}$ be a thick tensor ideal and $\mathcal{S}$ be a multiplicatively closed family. Then, the full subcategory $\mathcal{A} \div \mathcal{S}$ as defined in* (2) *determines a thick tensor ideal in* $(\mathcal{C}, \otimes, 1)$ *containing $\mathcal{A}$.*

Proof From the definition in (2), it is clear that the full subcategory $\mathcal{A} \div \mathcal{S}$ contains $\mathcal{A}$ and that given any $a \in \mathcal{A} \div \mathcal{S}$ and $a' \in \mathcal{C}$, we must have $a' \otimes a \in \mathcal{A} \div \mathcal{S}$. In order to check that $\mathcal{A} \div \mathcal{S}$ is thick, consider some $a \in \mathcal{A} \div \mathcal{S}$ such that a decomposes as a direct sum $a \cong b \oplus c$. We know that there exists $s \in \mathcal{S}$ such that $a \otimes s \in \mathcal{A}$. It follows that

$$a \otimes s \cong (b \otimes s) \oplus (c \otimes s) \in \mathcal{A}.$$

Since $\mathcal{A}$ is thick, it now follows that $b \otimes s, c \otimes s \in \mathcal{A}$, whence we have $b, c \in \mathcal{A} \div \mathcal{S}$. Finally, we consider a distinguished triangle

$$a \to b \to c \to \mathrm{T}a,$$

where two out of a, b, c lie in $\mathcal{A} \div \mathcal{S}$. For the sake of definiteness, suppose that $a, b \in \mathcal{A} \div \mathcal{S}$ and choose $s, t \in S$ such that $a \otimes s, b \otimes t \in \mathcal{A}$. Since S is multiplicatively closed and $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is exact in both variables, we have a new distinguished triangle

$$a \otimes s \otimes t \to b \otimes s \otimes t \to c \otimes s \otimes t \to \mathrm{T}a \otimes s \otimes t$$

in $\mathcal{C}$. Since $a \otimes s \otimes t, b \otimes s \otimes t \in \mathcal{A}$, we see that $c \otimes s \otimes t \in \mathcal{A}$ and we have $c \in \mathcal{A} \div \mathcal{S}$. This proves the result. □

We note here that, by Definition 2.1, every thick tensor ideal is a full subcategory of $\mathcal{C}$. Hence, $\mathcal{A} \div \{1\} = \mathcal{A}$. Further, given any thick tensor ideal $\mathcal{A}$ and a multiplicatively closed family of objects $\mathcal{S}$, it is clear that $\mathcal{A}\oplus\mathcal{S} = \{a\oplus s : a \in \mathcal{A},\ s \in \mathcal{S}\}$ is also a multiplicatively closed family of objects. Given multiplicatively closed families $\mathcal{S}$, $\mathcal{S}'$, we can also form the multiplicatively closed family $\mathcal{S}\mathcal{S}' = \{s\otimes s' : s \in \mathcal{S},\ s' \in \mathcal{S}'\}$. In order to understand realizations of chains, we will need the following result.

Proposition 2.5 *Let $(\mathcal{A}, \mathcal{S})$ be a pair such that $\mathcal{A}$ is a thick tensor ideal and $\mathcal{S}$ is a multiplicatively closed family of objects of $\mathcal{C}$. Then, we have:*

(a) *A prime ideal $\mathcal{P}$ contains $\mathcal{A}\div\mathcal{S}$ if and only if it contains a prime ideal realizing $(\mathcal{A}, \mathcal{S})$.*
(b) *If a prime ideal $\mathcal{P}$ is contained in a prime ideal that realizes $(\mathcal{A}, \mathcal{S})$, then $\mathcal{P}$ must be disjoint from $\mathcal{A}\oplus\mathcal{S}$.*
(c) *Suppose that $\mathrm{Spec}(\mathcal{A}, \mathcal{S})$ is finite. Then, a prime ideal $\mathcal{P}$ is contained in a prime ideal realizing $(\mathcal{A}, \mathcal{S})$ if and only if $\mathcal{P}$ is disjoint from $\mathcal{M}(\mathcal{A}, \mathcal{S})$.*

Proof (a) From the definition of $\mathcal{A}\div\mathcal{S}$ it is clear that any prime ideal $\mathcal{Q}$ realizing $(\mathcal{A}, \mathcal{S})$ also contains $\mathcal{A}\div\mathcal{S}$. Hence, so does any prime ideal containing $\mathcal{Q}$. Conversely, if $\mathcal{P}$ is a prime ideal containing $\mathcal{A}\div\mathcal{S}$, it follows that $(\mathcal{C} - \mathcal{P}) \cap (\mathcal{A}\div\mathcal{S}) = \varnothing$ whence it follows that $\mathcal{S}(\mathcal{C} - \mathcal{P}) \cap \mathcal{A} = \varnothing$. Accordingly, we can choose a prime ideal $\mathcal{Q}$ such that $\mathcal{Q} \cap \mathcal{S}(\mathcal{C} - \mathcal{P}) = \varnothing$ and $\mathcal{A} \subseteq \mathcal{Q}$. Since $\mathcal{Q} \cap \mathcal{S} \subseteq \mathcal{Q} \cap \mathcal{S}(\mathcal{C} - \mathcal{P}) = \varnothing$, we know that $\mathcal{Q}$ realizes $(\mathcal{A}, \mathcal{S})$. Finally, since $\mathcal{Q} \cap (\mathcal{C} - \mathcal{P}) \subseteq \mathcal{Q} \cap \mathcal{S}(\mathcal{C} - \mathcal{P}) = \varnothing$, it follows that $\mathcal{Q} \subseteq \mathcal{P}$.

(b) Suppose that $\mathcal{Q}$ realizes $(\mathcal{A}, \mathcal{S})$ and let $a \in \mathcal{A}$, $s \in \mathcal{S}$ be such that $a\oplus s \in \mathcal{Q}$. Since $\mathcal{Q}$ is thick, this implies that $s \in \mathcal{Q}$ which is a contradiction. Hence, $\mathcal{Q} \cap (\mathcal{A}\oplus\mathcal{S}) = \varnothing$ and hence any prime ideal contained in $\mathcal{Q}$ is also disjoint from $\mathcal{A}\oplus\mathcal{S}$.

(c) Suppose that $\mathcal{P} \subseteq \mathcal{Q}$ for some $\mathcal{Q} \in \mathrm{Spec}(\mathcal{A}, \mathcal{S})$. Then, $\mathcal{P} \subseteq \bigcup_{\mathcal{Q}\in\mathrm{Spec}(\mathcal{A},\mathcal{S})} \mathcal{Q}$ and hence $\mathcal{P}\cap\mathcal{M}(\mathcal{A}, \mathcal{S}) = \varnothing$. Conversely, if $\mathcal{P}\cap\mathcal{M}(\mathcal{A}, \mathcal{S}) = \varnothing$, then $\mathcal{P} \subseteq \bigcup_{\mathcal{Q}\in\mathrm{Spec}(\mathcal{A},\mathcal{S})} \mathcal{Q}$. However, since $\mathrm{Spec}(\mathcal{A}, \mathcal{S})$ is finite, it follows from the prime avoidance result in Proposition 2.2 that $\mathcal{P} \subseteq \mathcal{Q}$ for some $\mathcal{Q} \in \mathrm{Spec}(\mathcal{A}, \mathcal{S})$. □

Definition 2.6 Let $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{A}', \mathcal{S}')$ be two pairs as in Definition 2.3. Then, we will say that $(\mathcal{A}, \mathcal{S}) \preccurlyeq (\mathcal{A}', \mathcal{S}')$ if every prime ideal $\mathcal{P}'$ realizing the pair $(\mathcal{A}', \mathcal{S}')$ contains a prime ideal $\mathcal{P}$ realizing $(\mathcal{A}, \mathcal{S})$.

We now recall from [1, Section 4] that, given any thick tensor ideal $\mathcal{I}$, we can define its *radical* $r(\mathcal{I})$ as follows:

$$r(\mathcal{I}) = \{a \in \mathcal{C} : \text{there exists } n \geqslant 1 \text{ such that } a^{\otimes n} \in \mathcal{I}\}.$$

Then, from [1, Lemma 4.2], we know that $r(\mathcal{I})$ is also a thick tensor ideal and indeed $r(\mathcal{I})$ is given by the intersection $\bigcap_{\mathcal{I}\subseteq\mathcal{P}\in\mathrm{Spec}(\mathcal{C})} \mathcal{P}$ of all prime ideals containing $\mathcal{I}$.

Proposition 2.7 *Let $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{A}', \mathcal{S}')$ be two pairs. Then, the following are equivalent:*

(a) *The pairs are related as $(\mathcal{A}, \mathcal{S}) \preccurlyeq (\mathcal{A}', \mathcal{S}')$, i.e., any prime ideal realizing $(\mathcal{A}', \mathcal{S}')$ contains a prime ideal realizing $(\mathcal{A}, \mathcal{S})$.*
(b) *The radical of $\mathcal{A}\div\mathcal{S}$ is contained in the radical of $\mathcal{A}'\div\mathcal{S}'$, i.e., $r(\mathcal{A}\div\mathcal{S}) \subseteq r(\mathcal{A}'\div\mathcal{S}')$.*

Proof (b)$\Rightarrow$(a). Suppose that $r(\mathcal{A}\div\mathcal{S}) \subseteq r(\mathcal{A}'\div\mathcal{S}')$ and let $\mathcal{P}'$ be a prime ideal realizing $(\mathcal{A}', \mathcal{S}')$. Then, $\mathcal{A}'\div\mathcal{S}' \subseteq \mathcal{P}'$ and $\mathcal{P}'$ being prime, we get $r(\mathcal{A}'\div\mathcal{S}') \subseteq \mathcal{P}'$. Then, we have

$$\mathcal{A}\div\mathcal{S} \subseteq r(\mathcal{A}\div\mathcal{S}) \subseteq r(\mathcal{A}'\div\mathcal{S}') \subseteq \mathcal{P}'$$

and it follows from Proposition 2.5 that $\mathcal{P}'$ contains a prime ideal realizing the pair $(\mathcal{A}, \mathcal{S})$.

(a)$\Rightarrow$(b). Consider any prime ideal $\mathcal{P}'$ such that $\mathcal{A}'\div\mathcal{S}' \subseteq \mathcal{P}'$. Then, from Proposition 2.5, we know that $\mathcal{P}'$ contains a prime ideal $\mathcal{P}''$ realizing $(\mathcal{A}', \mathcal{S}')$. By assumption, there exists a prime ideal $\mathcal{P} \subseteq \mathcal{P}''$ such that $\mathcal{P}$ realizes $(\mathcal{A}, \mathcal{S})$. Hence, $\mathcal{A}\div\mathcal{S} \subseteq \mathcal{P} \subseteq \mathcal{P}'' \subseteq \mathcal{P}'$ and therefore $r(\mathcal{A}\div\mathcal{S}) \subseteq \mathcal{P}'$ for any prime ideal $\mathcal{P}'$ containing $\mathcal{A}'\div\mathcal{S}'$. It now follows that $r(\mathcal{A}\div\mathcal{S}) \subseteq r(\mathcal{A}'\div\mathcal{S}') = \bigcap_{\mathcal{A}'\div\mathcal{S}'\subseteq\mathcal{P}'\in\mathrm{Spec}(\mathcal{C})} \mathcal{P}'$. □

Remark 2.8 We note that $\preccurlyeq$ is not a partial order relation. In particular, if $(\mathcal{A}, \mathcal{S}) \preccurlyeq (\mathcal{A}', S')$ and $(\mathcal{A}', \mathcal{S}') \preccurlyeq (\mathcal{A}, \mathcal{S})$, we do get $r(\mathcal{A}\div\mathcal{S}) = r(\mathcal{A}'\div\mathcal{S}')$ but not necessarily that $(\mathcal{A}, \mathcal{S}) = (\mathcal{A}', \mathcal{S}')$. However, it is clear that $\preccurlyeq$ is reflexive and transitive.

We now start by considering realizations of templates $T = \{(\mathcal{A}_i, \mathcal{S}_i)\}_{1\leqslant i\leqslant n}$ indexed by a finite chain of length $n \geqslant 1$. In particular, if the pairs in the finite chain template also satisfy

$$(\mathcal{A}_n, \mathcal{S}_n) \preccurlyeq \cdots \preccurlyeq (\mathcal{A}_2, \mathcal{S}_2) \preccurlyeq (\mathcal{A}_1, \mathcal{S}_1) \tag{3}$$

it is clear how to find a realization of such template: we choose any prime ideal $\mathcal{P}_1$ realizing $(\mathcal{A}_1, \mathcal{S}_1)$. Then, since $(\mathcal{A}_2, \mathcal{S}_2) \preccurlyeq (\mathcal{A}_1, \mathcal{S}_1)$, we can choose a prime ideal $\mathcal{P}_2 \subseteq \mathcal{P}_1$ such that $\mathcal{P}_2$ realizes $(\mathcal{A}_2, \mathcal{S}_2)$ and so on. We will now show that given any finite chain template, its realizations can be described in terms of realizations of a chain template satisfying the condition in (3). Given a partially ordered set $(I, \leqslant)$, we will denote by I^{op} the partially ordered set obtained by reversing all order relations in I.

Proposition 2.9 *Let $n \geqslant 1$ and let $T = \{(\mathcal{A}_i, \mathcal{S}_i)\}_{i\in I^{\mathrm{op}}}$ be a finite chain template indexed by the opposite I^{op} of the ordered set $I = \{1 < 2 < \cdots < n\}$. We define $\{\mathcal{B}_i\}_{1\leqslant i\leqslant n}$ inductively by letting $\mathcal{B}_n = \mathcal{A}_n$ and setting*

$$\mathcal{B}_i = \mathcal{A}_i + (\mathcal{B}_{i+1}\div\mathcal{S}_{i+1}) \quad \text{for all} \quad n > i \geqslant 1. \tag{4}$$

Then, we have:

- (a) *A chain $\mathcal{P}_n \subseteq \cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1$ is a realization of the template $T = \{(\mathcal{A}_i, \mathcal{S}_i)\}_{i\in I^{\mathrm{op}}}$ if and only if it is also a realization of the template $\mathrm{D}(T) = \{(\mathcal{B}_i, \mathcal{S}_i)\}_{i\in I^{\mathrm{op}}}$.*
- (b) *The template $T = \{(\mathcal{A}_i, \mathcal{S}_i)\}_{i\in I^{\mathrm{op}}}$ has a realization if and only if $\mathcal{B}_1 \cap \mathcal{S}_1 = \varnothing$, i.e.*

$$(\mathcal{A}_1 + ((\mathcal{A}_2 + ((\cdots(\mathcal{A}_{n-1} + (\mathcal{A}_n\div\mathcal{S}_n))\div\mathcal{S}_{n-1}))\cdots\div\mathcal{S}_3)\div\mathcal{S}_2)) \cap \mathcal{S}_1 = \varnothing.$$

Proof (a) Let $\mathcal{P}_n \subseteq \cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1$ be a realization of the template T. We know that $\mathcal{B}_n = \mathcal{A}_n$ and hence $\mathcal{P}_n$ realizes $(\mathcal{B}_n, \mathcal{S}_n)$. Now suppose that $\mathcal{P}_i$ realizes $(\mathcal{B}_i, \mathcal{S}_i)$ for each $n \geqslant i > j$ for some given j. Then, since $\mathcal{P}_j \supseteq \mathcal{P}_{j+1}$ and $\mathcal{P}_{j+1}$ realizes

$(\mathcal{B}_{j+1}, \mathcal{S}_{j+1})$, it follows from Proposition 2.5 that $\mathcal{P}_j \supseteq \mathcal{B}_{j+1} \div \mathcal{S}_{j+1}$. Since $\mathcal{P}_j$ realizes $(\mathcal{A}_j, \mathcal{S}_j)$, we already know that $\mathcal{P}_j \supseteq \mathcal{A}_j$ and $\mathcal{P}_j \cap \mathcal{S}_j = \varnothing$. From (4), it follows that $\mathcal{P}_j \supseteq \mathcal{A}_j + (\mathcal{B}_{j+1} \div \mathcal{S}_{j+1}) = \mathcal{B}_j$ and $\mathcal{P}_j \cap \mathcal{S}_j = \varnothing$, i.e., $\mathcal{P}_j$ realizes the pair $(\mathcal{B}_j, \mathcal{S}_j)$. This proves the result by induction.

Conversely, let $\mathcal{P}_n \subseteq \cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1$ be a realization of the template $\mathrm{D}(T)$. Then, for each $1 \leqslant i \leqslant n$, we know that $\mathcal{P}_i$ realizes the pair $(\mathcal{B}_i, \mathcal{S}_i)$. From (4), it is clear that $\mathcal{A}_i \subseteq \mathcal{B}_i$ and hence $\mathcal{P}_i$ realizes the pair $(\mathcal{A}_i, \mathcal{S}_i)$.

(b) From part (a), it is clear that the template T can be realized if and only if the template $\mathrm{D}(T)$ can be realized. In particular, this means that if T has a realization, the pair $(\mathcal{B}_1, \mathcal{S}_1)$ can be realized and we must have $\mathcal{B}_1 \cap \mathcal{S}_1 = \varnothing$. Conversely, if $\mathcal{B}_1 \cap \mathcal{S}_1 = \varnothing$, we can choose a prime ideal $\mathcal{P}_1$ realizing $(\mathcal{B}_1, \mathcal{S}_1)$. From (4), it follows that

$$\mathcal{B}_i \div \mathcal{S}_i = (\mathcal{A}_i + (\mathcal{B}_{i+1} \div \mathcal{S}_{i+1})) \div \mathcal{S}_i \supseteq \mathcal{A}_i + (\mathcal{B}_{i+1} \div \mathcal{S}_{i+1}) \supseteq \mathcal{B}_{i+1} \div \mathcal{S}_{i+1}. \quad (5)$$

From (5) it is clear that $r(\mathcal{B}_{i+1} \div \mathcal{S}_{i+1}) \subseteq r(\mathcal{B}_i \div \mathcal{S}_i)$ and hence it follows from Proposition 2.7 that

$$(\mathcal{B}_n, \mathcal{S}_n) \preccurlyeq \cdots \preccurlyeq (\mathcal{B}_2, \mathcal{S}_2) \preccurlyeq (\mathcal{B}_1, \mathcal{S}_1).$$

Thus, we can form a realization $\mathcal{P}_n \subseteq \cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1$ of the template $\mathrm{D}(T)$ (and hence of T) starting from $\mathcal{P}_1$. □

From now onwards, we will say that two templates indexed by the same partially ordered set are *equivalent* if they have the same realizations. In Proposition 2.9, we have shown that the finite chain template $T = \{(\mathcal{A}_i, \mathcal{S}_i)\}_{i \in I^{\mathrm{op}}}$ indexed by the opposite I^{op} of $I = \{1 < 2 < \cdots < n\}$ is equivalent to the modified chain template $\mathrm{D}(T) = \{(\mathcal{B}_i, S_i)\}_{i \in I^{\mathrm{op}}}$. Further, since the modified chain template $\mathrm{D}(T)$ satisfies

$$(\mathcal{B}_n, \mathcal{S}_n) \preccurlyeq \cdots \preccurlyeq (\mathcal{B}_2, \mathcal{S}_2) \preccurlyeq (\mathcal{B}_1, \mathcal{S}_1),$$

we can start with an arbitrary realization $\mathcal{P}_1$ of $(\mathcal{B}_1, \mathcal{S}_1)$ and pick a prime ideal $\mathcal{P}_2 \subseteq \mathcal{P}_1$ realizing $(\mathcal{B}_2, \mathcal{S}_2)$ and so on to obtain a realization $\mathcal{P}_n \subseteq \cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1$ of $\mathrm{D}(T)$. However, if we started with an arbitrary realization say $\mathcal{P}_2'$ of $(\mathcal{B}_2, \mathcal{S}_2)$, it is not necessary that we can find some prime $\mathcal{P}_1' \supseteq \mathcal{P}_2'$ realizing $(\mathcal{B}_1, \mathcal{S}_1)$, i.e., the process of realizing a finite chain in Proposition 2.9 can proceed in one direction only. We will now show that under certain finiteness conditions, we can construct a template $\mathcal{D}(T) = \{(\mathcal{B}_i, \mathcal{T}_i)\}_{i \in I^{\mathrm{op}}}$ equivalent to T such that starting from any arbitrary realization $\mathcal{Q}_j$ of some $(\mathcal{B}_j, \mathcal{T}_j)$, we can proceed in both directions to form a realization $\mathcal{Q}_n \subseteq \cdots \subseteq \mathcal{Q}_{j+1} \subseteq \mathcal{Q}_j \subseteq \mathcal{Q}_{j-1} \subseteq \cdots \subseteq \mathcal{Q}_1$ of $\mathcal{D}(T)$. Since $\mathcal{D}(T)$ is equivalent to the template T, this also becomes a realization of T.

Proposition 2.10 *Let $n \geqslant 1$ and let $T = \{(\mathcal{A}_i, \mathcal{S}_i)\}_{i \in I^{\mathrm{op}}}$ be a finite chain template indexed by the opposite of the ordered set $I = \{1 < 2 < \cdots < n\}$. Suppose that for each $1 \leqslant i \leqslant n$,* $\mathrm{Spec}(\mathcal{A}_i, \mathcal{S}_i)$ *is a finite set. We define $\{\mathcal{B}_i\}_{1 \leqslant i \leqslant n}$ inductively by letting $\mathcal{B}_n = \mathcal{A}_n$ and setting*

$$\mathcal{B}_i = \mathcal{A}_i + (\mathcal{B}_{i+1} \div \mathcal{S}_{i+1}) \quad \textit{for all} \quad n > i \geqslant 1. \quad (6)$$

On the other hand, we define $\{\mathcal{T}_i\}_{1\leqslant i\leqslant n}$ *inductively by letting* $\mathcal{T}_1 = \mathcal{S}_1$ *and setting* $\mathcal{T}_{i+1}$ *to be the product of the multiplicatively closed families*

$$\mathcal{T}_{i+1} = \mathcal{M}(\mathcal{A}_i, \mathcal{T}_i)\mathcal{S}_{i+1} \quad \textit{for all} \quad 1 \leqslant i \leqslant n-1. \tag{7}$$

Then, we have:

(a) *A chain* $\mathcal{Q}_n \subseteq \cdots \subseteq \mathcal{Q}_2 \subseteq \mathcal{Q}_1$ *of prime ideals is a realization of the template* $T = \{(\mathcal{A}_i, \mathcal{S}_i)\}_{i\in I^{\mathrm{op}}}$ *if and only if it is also a realization of the template* $\mathcal{D}(T) = \{(\mathcal{B}_i, \mathcal{T}_i)\}_{i\in I^{\mathrm{op}}}$*, i.e., the templates* T *and* $\mathcal{D}(T)$ *are equivalent.*
(b) *Fix any integer* $j \in \{1, 2, \ldots, n\}$*. Then, the template* $T = \{(\mathcal{A}_i, \mathcal{S}_i)\}_{i\in I^{\mathrm{op}}}$ *has a realization if and only if* $\mathcal{B}_j \cap \mathcal{T}_j = \varnothing$.

Proof (a) Let $\mathcal{Q}_n \subseteq \cdots \subseteq \mathcal{Q}_2 \subseteq \mathcal{Q}_1$ be a realization of the template T. From the proof of Proposition 2.9 (a), we know that each $\mathcal{B}_i \subseteq \mathcal{Q}_i$. By definition, we know that $\mathcal{T}_1 = \mathcal{S}_1$ and hence $\mathcal{Q}_1 \cap \mathcal{T}_1 = \varnothing$. We now suppose that $\mathcal{Q}_i \cap \mathcal{T}_i = \varnothing$ for each $1 \leqslant i \leqslant j$ for some given j. We know that $\mathcal{Q}_{j+1}$ realizes $(\mathcal{A}_{j+1}, \mathcal{S}_{j+1})$ and hence $\mathcal{Q}_{j+1} \cap \mathcal{S}_{j+1} = \varnothing$. Further, since $\mathcal{Q}_{j+1} \subseteq \mathcal{Q}_j$ and $\mathcal{Q}_j \cap \mathcal{T}_j = \varnothing$ (i.e., $\mathcal{Q}_j \in \mathrm{Spec}(\mathcal{A}_j, \mathcal{T}_j)$), we have

$$\mathcal{Q}_{j+1} \cap \mathcal{M}(\mathcal{A}_j, \mathcal{T}_j) \subseteq \mathcal{Q}_j \cap \mathcal{M}(\mathcal{A}_j, \mathcal{T}_j) = \varnothing.$$

Since $\mathcal{Q}_{j+1}$ is a prime ideal, $\mathcal{Q}_{j+1} \cap \mathcal{T}_{j+1} = \mathcal{Q}_{j+1} \cap \mathcal{M}(\mathcal{A}_j, \mathcal{T}_j)\mathcal{S}_{j+1} = \varnothing$. Hence, each $\mathcal{Q}_i$ realizes the pair $(\mathcal{B}_j, \mathcal{T}_j)$.

Conversely, let $\mathcal{Q}_n \subseteq \cdots \subseteq \mathcal{Q}_2 \subseteq \mathcal{Q}_1$ be a realization of the template $\mathcal{D}(T)$. Then, for each $1 \leqslant i \leqslant n$, we know that $\mathcal{Q}_i$ realizes the pair $(\mathcal{B}_i, \mathcal{T}_i)$. From (6) and (7), it is clear that each $\mathcal{A}_i \subseteq \mathcal{B}_i$ and $\mathcal{S}_i \subseteq \mathcal{T}_i$. Hence, each $\mathcal{Q}_i$ realizes the pair $(\mathcal{A}_i, \mathcal{S}_i)$.

(b) We fix some $j \in \{1, 2, \ldots, n\}$. From part (a), it is clear that the template T can be realized if and only if the template $\mathcal{D}(T)$ can be realized. In particular, this means that if T has a realization, the pair $(\mathcal{B}_j, \mathcal{T}_j)$ can be realized and we must have $\mathcal{B}_j \cap \mathcal{T}_j = \varnothing$. Conversely, if $\mathcal{B}_j \cap \mathcal{T}_j = \varnothing$, we choose some $\mathcal{Q}_j$ realizing $(\mathcal{B}_j, \mathcal{T}_j)$. Then, we have

$$\mathcal{Q}_j \supseteq \mathcal{B}_j = \mathcal{A}_j + (\mathcal{B}_{j+1} \div \mathcal{S}_{j+1}) \supseteq \mathcal{B}_{j+1} \div \mathcal{S}_{j+1}$$

and it follows from Proposition 2.5 (a) that there exists a prime ideal $\mathcal{Q}_{j+1} \subseteq \mathcal{Q}_j$ realizing $(\mathcal{B}_{j+1}, \mathcal{S}_{j+1})$. Further, since $\mathcal{Q}_{j+1} \subseteq \mathcal{Q}_j$ and $\mathcal{Q}_j$ realizes $(\mathcal{A}_j, \mathcal{T}_j)$, we see that $\mathcal{Q}_{j+1} \cap \mathcal{M}(\mathcal{A}_j, \mathcal{T}_j) = \varnothing$. Accordingly, $\mathcal{Q}_{j+1} \cap \mathcal{T}_{j+1} = \mathcal{Q}_{j+1} \cap \mathcal{M}(\mathcal{A}_j, \mathcal{T}_j)\mathcal{S}_{j+1} = \varnothing$, i.e., $\mathcal{Q}_{j+1}$ realizes $(\mathcal{B}_{j+1}, \mathcal{T}_{j+1})$. On the other hand, from (7), we have

$$\mathcal{T}_j = \mathcal{M}(\mathcal{A}_{j-1}, \mathcal{T}_{j-1})\mathcal{S}_j. \tag{8}$$

Since $\mathcal{S}_{j-1} \subseteq \mathcal{T}_{j-1}$, we see that $\mathrm{Spec}(\mathcal{A}_{j-1}, \mathcal{T}_{j-1}) \subseteq \mathrm{Spec}(A_{j-1}, \mathcal{S}_{j-1})$ is finite. From (8), we see that $\mathcal{Q}_j \cap \mathcal{M}(\mathcal{A}_{j-1}, \mathcal{T}_{j-1}) = \varnothing$ and it follows from Proposition 2.5 (c) that we can choose a prime ideal $\mathcal{Q}_{j-1} \supseteq \mathcal{Q}_j$ realizing $(\mathcal{A}_{j-1}, \mathcal{T}_{j-1})$. Further, since $\mathcal{Q}_j$ realizes $(\mathcal{B}_j, \mathcal{S}_j)$ (as $\mathcal{Q}_j$ realizes $(\mathcal{B}_j, \mathcal{T}_j)$ and $\mathcal{S}_j \subseteq \mathcal{T}_j$) and $\mathcal{Q}_{j-1}$ contains $\mathcal{Q}_j$, it follows from Proposition 2.5 (a) that $\mathcal{Q}_{j-1}$ contains $\mathcal{B}_j \div \mathcal{S}_j$. Consequently, we have

$$\mathcal{Q}_{j-1} \supseteq \mathcal{A}_{j-1} + (\mathcal{B}_j \div \mathcal{S}_j) = \mathcal{B}_{j-1}.$$

Hence, $\mathcal{Q}_{j-1}$ realizes $(\mathcal{B}_{j-1}, \mathcal{T}_{j-1})$. Accordingly, starting from $\mathcal{Q}_j$ we can proceed in both directions to give a realization $\mathcal{Q}_n \subseteq \cdots \subseteq \mathcal{Q}_2 \subseteq \mathcal{Q}_1$ of $\mathcal{D}(T)$. From part (a), it follows that this is also a realization of the template T. □

The next result will explain what kinds of collections of prime ideals may arise as $\mathrm{Spec}(\mathcal{A}, \mathcal{S})$ for some pair $(\mathcal{A}, \mathcal{S})$.

Proposition 2.11 *Let* $(\mathcal{C}, \otimes, 1)$ *be a tensor triangulated category and let* $\mathfrak{X}$ *be a collection of prime ideals of* $(\mathcal{C}, \otimes, 1)$. *Then, the following are equivalent:*

(a) *The family* $\mathfrak{X} = \mathrm{Spec}(\mathcal{A}, \mathcal{S})$ *for some thick tensor ideal* $\mathcal{A}$ *and some multiplicatively closed family of objects* $\mathcal{S}$.
(b) *The family* $\mathfrak{X}$ *satisfies the following property: given a prime ideal* $\mathcal{Q}$ *such that*

$$\bigcap_{\mathcal{P} \in \mathfrak{X}} \mathcal{P} \subseteq \mathcal{Q} \subseteq \bigcup_{\mathcal{P} \in \mathfrak{X}} \mathcal{P} \tag{9}$$

then $\mathcal{Q} \in \mathfrak{X}$.

Proof (a)⇒(b). Since each $\mathcal{P} \in \mathfrak{X}$ realizes the pair $(\mathcal{A}, \mathcal{S})$, we have $\mathcal{A} \subseteq \bigcap_{\mathcal{P} \in \mathfrak{X}} \mathcal{P}$ and $\mathcal{S} \cap \bigcup_{\mathcal{P} \in \mathfrak{X}} \mathcal{P} = \varnothing$. Hence, if a prime ideal $\mathcal{Q}$ satisfies the condition in (9), then $\mathcal{Q}$ realizes $(\mathcal{A}, \mathcal{S})$, i.e., $\mathcal{Q} \in \mathfrak{X} = \mathrm{Spec}(\mathcal{A}, \mathcal{S})$.

(b)⇒(a). Given a collection $\mathfrak{X}$ of prime ideals satisfying the condition in (b), we set

$$\mathcal{A} = \bigcap_{\mathcal{P} \in \mathfrak{X}} \mathcal{P}, \qquad \mathcal{S} = \Big(\bigcup_{\mathcal{P} \in \mathfrak{X}} \mathcal{P}\Big)^{c}.$$

Then, a prime ideal $\mathcal{Q}$ realizes $(\mathcal{A}, \mathcal{S})$ if and only if it satisfies (9). Hence, $\mathfrak{X}$ may be expressed as $\mathrm{Spec}(\mathcal{A}, \mathcal{S})$. □

A collection $\mathfrak{X}$ of prime ideals of $(\mathcal{C}, \otimes, 1)$ satisfying condition (b) in Proposition 2.11 will be referred to as a *convex* set. Under the finiteness conditions from Proposition 2.10, we will now characterize the families of chains of prime ideals that realize a given finite chain template $T = \{(\mathcal{A}_i, \mathcal{S}_i)\}_{i \in I^{\mathrm{op}}}$.

Proposition 2.12 *Let* $(\mathcal{C}, \otimes, 1)$ *be a tensor triangulated category and fix some* $n \geqslant 1$. *We consider decreasing chains of length n consisting of prime ideals of* $\mathcal{C}$

$$\mathcal{P}_n \subseteq \mathcal{P}_{n-1} \subseteq \cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1.$$

Let $\mathfrak{X}$ *be a collection of such chains of prime ideals and for any* $1 \leqslant j \leqslant n$, *we denote by* $\mathfrak{X}_j$ *the collection of prime ideals arising as the* j*-th element of a chain in* $\mathfrak{X}$. *Then, the following are equivalent:*

(a) $\mathfrak{X}$ *is a collection of realizations of a finite chain template* $T = \{(\mathcal{A}_i, \mathcal{S}_i)\}_{i \in I^{\mathrm{op}}}$ *indexed by the opposite* I^{op} *of the ordered set* $I = \{1 < 2 < \cdots < n\}$ *such that each* $\mathrm{Spec}(\mathcal{A}_i, \mathcal{S}_i)$ *is a finite set.*

(b) *For each* $1 \leqslant j \leqslant n$, $\mathfrak{X}_j$ *is a finite convex set of prime ideals of* $(\mathcal{C}, \otimes, 1)$. *Further,* $\mathfrak{X}$ *consists of all chains of prime ideals whose* i*-th element is in* $\mathfrak{X}_i$ *for each* $1 \leqslant i \leqslant n$. *In other words, we have*

$$\mathfrak{X} = \left\{ \mathcal{P}_n \subseteq \mathcal{P}_{n-1} \subseteq \cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1 : \mathcal{P}_i \in \mathfrak{X}_i \textit{ for each } 1 \leqslant i \leqslant n \right\}. \tag{10}$$

Proof (a)$\Rightarrow$(b). Let $\mathfrak{X}$ be a collection of realizations of $T = \{(\mathcal{A}_i, \mathcal{S}_i)\}_{i \in I^{\mathrm{op}}}$. Since each $\mathrm{Spec}(\mathcal{A}_i, \mathcal{S}_i)$ is finite, we can construct the equivalent modified template $\mathcal{D}(T) = \{(\mathcal{B}_i, \mathcal{T}_i)\}_{i \in I^{\mathrm{op}}}$ as defined in Proposition 2.10. Fix some j and choose some prime ideal $\mathcal{P} \in \mathfrak{X}_j$. Then, there exists a realization $\mathcal{P}_n \subseteq \cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1$ of T such that $\mathcal{P}_j = \mathcal{P}$. Since T is equivalent to the template $\mathcal{D}(T)$, we see that $\mathcal{P} = \mathcal{P}_j$ realizes $(\mathcal{B}_j, \mathcal{T}_j)$ and hence $\mathfrak{X}_j \subseteq \mathrm{Spec}(\mathcal{B}_j, \mathcal{T}_j)$. Conversely, given any prime $\mathcal{P} \in \mathrm{Spec}(\mathcal{B}_j, \mathcal{T}_j)$, we can construct as in the proof of Proposition 2.10 a realization $\mathcal{P}_n \subseteq \cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1$ of T such that $\mathcal{P}_j = \mathcal{P}$. Hence, $\mathfrak{X}_j = \mathrm{Spec}(\mathcal{B}_j, \mathcal{T}_j)$ and each $\mathfrak{X}_j$ is a finite convex set. Finally, any chain $\mathcal{P}_n \subseteq \cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1$ such that each $\mathcal{P}_i \in \mathfrak{X}_i = \mathrm{Spec}(\mathcal{B}_i, \mathcal{T}_i)$ is a realization of $\mathcal{D}(T) = \{(\mathcal{B}_i, \mathcal{T}_i)\}_{i \in I^{\mathrm{op}}}$ and hence of T. Therefore, the collection of realizations $\mathfrak{X}$ of T must be given by

$$\mathfrak{X} = \left\{ \mathcal{P}_n \subseteq \mathcal{P}_{n-1} \subseteq \cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1 : \mathcal{P}_i \in \mathfrak{X}_i \text{ for each } 1 \leqslant i \leqslant n \right\}.$$

(b)$\Rightarrow$(a). We define a template $T = \{(\mathcal{A}_i, \mathcal{S}_i)\}_{i \in I^{\mathrm{op}}}$ as follows:

$$\mathcal{A}_i = \bigcap_{\mathcal{P} \in \mathfrak{X}_i} \mathcal{P}, \qquad \mathcal{S}_i = \left(\bigcup_{\mathcal{P} \in \mathfrak{X}_i} \mathcal{P} \right)^{\mathrm{c}}.$$

From the proof of Proposition 2.11, we know that $\mathfrak{X}_i = \mathrm{Spec}(\mathcal{A}_i, \mathcal{S}_i)$. Hence, any chain $\mathcal{P}_n \subseteq \cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1$ of prime ideals drawn from the set

$$\mathfrak{X} = \left\{ \mathcal{P}_n \subseteq \mathcal{P}_{n-1} \subseteq \cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1 : \mathcal{P}_i \in \mathfrak{X}_i \text{ for each } 1 \leqslant i \leqslant n \right\}$$

must be a realization of the template $T = \{(\mathcal{A}_i, \mathcal{S}_i)\}_{i \in I^{\mathrm{op}}}$. Conversely, given any realization $\mathcal{P}_n \subseteq \cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1$ of T, we know that each $\mathcal{P}_i \in \mathrm{Spec}(\mathcal{A}_i, \mathcal{S}_i) = \mathfrak{X}_i$ and hence condition (10) ensures that this chain lies in $\mathfrak{X}$. Finally, since each $\mathfrak{X}_i$ is finite, so is each $\mathrm{Spec}(\mathcal{A}_i, \mathcal{S}_i) = \mathfrak{X}_i$. □

We have described in Proposition 2.11 that the subsets $\mathrm{Spec}(\mathcal{A}, \mathcal{S})$ are exactly the convex subsets of $\mathrm{Spec}(\mathcal{C})$. Following Balmer [1, Section 2], the space $\mathrm{Spec}(\mathcal{C})$ is endowed with a Zariski topology with the closed subsets being given by

$$Z(\mathcal{X}) = \{\mathcal{P} \in \mathrm{Spec}(\mathcal{C}) : \mathcal{P} \cap \mathcal{X} = \varnothing\}$$

for each family $\mathcal{X} \subseteq \mathcal{C}$ of objects of $\mathcal{C}$. Then, open sets of $\mathrm{Spec}(\mathcal{C})$ are of the form $U(\mathcal{X}) = \{\mathcal{P} \in \mathrm{Spec}(\mathcal{C}) : \mathcal{P} \cap \mathcal{X} \neq \varnothing\}$ for $\mathcal{X} \subseteq \mathcal{C}$. Further, we know from [4, Proposition 11] that the spectrum $\mathrm{Spec}(\mathcal{C})$ becomes a spectral space in the sense of Hochster [21]. We will now show that the subsets $\mathrm{Spec}(\mathcal{A}, \mathcal{S})$ (which are the convex subsets of $\mathrm{Spec}(\mathcal{C})$) are related to constructible subsets of the spectral space $\mathrm{Spec}(\mathcal{C})$.

Since $\mathrm{Spec}(\mathcal{C})$ is a spectral space, every constructible subset $Y \subseteq \mathrm{Spec}(\mathcal{C})$ is of the form

$$Y = \bigcup_{i=1}^{r} (U_i \cap Z_i)$$

(see, for instance, [32, Section 2]) with each U_i and each $\mathrm{Spec}(\mathcal{C})\backslash Z_i$ quasi-compact and open in $\mathrm{Spec}(\mathcal{C})$. From [1, Proposition 2.14], we know that each quasi-compact open subset of $\mathrm{Spec}(\mathcal{C})$ is of the form $U(a) = \{\mathcal{P} \in \mathrm{Spec}(\mathcal{C}) : a \in \mathcal{P}\}$ for some $a \in \mathcal{C}$. Further, we note that a thick tensor ideal $\mathcal{A}$ contains a finite set $\{a_1, \ldots, a_k\}$ of objects of $\mathcal{C}$ if and only if it contains the direct sum $a_1 \oplus \cdots \oplus a_k$, i.e., every finitely generated thick tensor ideal is principal.

We will say that a multiplicatively closed family $\mathcal{S}$ is finitely generated if there exists a finite set $\{s_1, s_2, \ldots, s_k\}$ of objects of $\mathcal{C}$ such that $\mathcal{S}$ is the smallest multiplicatively closed family containing all objects in $\{s_1, s_2, \ldots, s_k\}$.

Proposition 2.13 (a) *Given a finitely generated (hence principal) thick tensor ideal $\mathcal{A}$ and a finitely generated multiplicatively closed family $\mathcal{S}$,* $\mathrm{Spec}(\mathcal{A}, \mathcal{S})$ *is a constructible subset of* $\mathrm{Spec}(\mathcal{C})$.

(b) *Every convex subset of* $\mathrm{Spec}(\mathcal{C})$ *is pro-constructible, i.e., it may be expressed as an intersection of a family of constructible subsets of* $\mathrm{Spec}(\mathcal{C})$.

(c) *For any thick tensor ideal $\mathcal{A}$ and any multiplicatively closed family $\mathcal{S}$, the subset* $\mathrm{Spec}(\mathcal{A}, \mathcal{S}) \subseteq \mathrm{Spec}(\mathcal{C})$ *is a spectral space, i.e.,* $\mathrm{Spec}(\mathcal{A}, \mathcal{S})$ *is quasi-compact, quasi-separated, has a basis of quasi-compact open subsets and every non-empty irreducible closed subset has a unique generic point.*

Proof (a) Since $\mathcal{S}$ is finitely generated, we can choose a finite set of objects $\{s_1, s_2, \ldots, s_k\}$ such that $\mathcal{S}$ is the smallest multiplicatively closed family containing all objects in $\{s_1, s_2, \ldots, s_k\}$. Then, it is clear that for any object $s \in \mathcal{S}$, we can choose non-negative integers $e_1, e_2, \ldots, e_k$ such that $s = \bigotimes_{i=1}^{k} s_i^{\otimes e_i}$. We now set $s_0 = \bigotimes_{i=1}^{k} s_i$ and see that $Z(\mathcal{S}) = Z(s_0)$. Then, $U(s_0) = \mathrm{Spec}(\mathcal{C})\backslash Z(s_0)$ is quasi-compact and open in $\mathrm{Spec}(\mathcal{C})$. Further, if $\mathcal{A}$ is generated by the object $a \in \mathcal{C}$, it is clear that we may express $\mathrm{Spec}(\mathcal{A}, \mathcal{S}) = U(a) \cap Z(\mathcal{S}) = U(a) \cap Z(s_0)$. Hence, $\mathrm{Spec}(\mathcal{A}, \mathcal{S})$ is constructible.

(b) From Proposition 2.11, we know that each convex subset of $\mathrm{Spec}(\mathcal{C})$ is of the form $\mathrm{Spec}(\mathcal{A}, \mathcal{S})$ for some thick tensor ideal $\mathcal{A}$ and some multiplicatively closed family $\mathcal{S}$. We now express $\mathrm{Spec}(\mathcal{A}, \mathcal{S}) = \bigcap_{a \in \mathcal{A},\, s \in \mathcal{S}} U(a) \cap Z(s)$. Since each $U(a) \cap Z(s)$ is constructible, it follows that the intersection $\mathrm{Spec}(\mathcal{A}, \mathcal{S})$ is pro-constructible.

(c) follows from the fact that a pro-constructible subspace of a spectral space is always spectral in the induced subspace topology (see, for instance, [32, Section 2]). □

For the final result of this section, we will restrict ourselves to tensor triangulated categories that are topologically noetherian (see [2, Definition 3.13]), i.e., $\mathrm{Spec}(\mathcal{C})$ is a noetherian topological space. This happens, for instance, when X is a topologically noetherian scheme and $\mathcal{C} = \mathrm{D}^{\mathrm{perf}}(X)$, i.e., the derived category of perfect complexes over X (see [1, Corollary 5.6]). We denote by $\mathrm{Spec}(\mathcal{C})_{\mathrm{cons}}$ the space $\mathrm{Spec}(\mathcal{C})$ equipped with the constructible topology. From Proposition 2.13 we know that for any thick tensor ideal $\mathcal{A}$ and any multiplicatively closed family $\mathcal{S}$, $\mathrm{Spec}(\mathcal{A}, \mathcal{S})$ is pro-constructible. Equivalently, since $\mathrm{Spec}(\mathcal{C})$ is a spectral space, $\mathrm{Spec}(\mathcal{A}, \mathcal{S})$ is closed in $\mathrm{Spec}(\mathcal{C})_{\mathrm{cons}}$.

Corollary 2.14 *Let* $(\mathcal{C}, \otimes, 1)$ *be a tensor triangulated category that is topologically noetherian, i.e.,* $\mathrm{Spec}(\mathcal{C})$ *is a noetherian space.*

(a) *Given a finitely generated (hence principal) thick tensor ideal* $\mathcal{A}$ *and a multiplicatively closed family* $\mathcal{S}$, $\mathrm{Spec}(\mathcal{A}, \mathcal{S})$ *is a constructible subset of* $\mathrm{Spec}(\mathcal{C})$. *Further, such subsets form a basis for the constructible topology on* $\mathrm{Spec}(\mathcal{C})$.
(b) *Any closed subset of* $\mathrm{Spec}(\mathcal{C})_{\mathrm{cons}}$ *may be expressed as a union* $\bigcup_{i\in I}\mathrm{Spec}(\mathcal{A}_i, \mathcal{S}_i)$ *with each* $\mathcal{A}_i$ *a thick tensor ideal and each* $\mathcal{S}_i$ *a multiplicatively closed family.*

Proof (a) Since $\mathrm{Spec}(\mathcal{C})$ is noetherian, the open subset $\mathrm{Spec}(\mathcal{C})\backslash Z(\mathcal{S})$ is quasi-compact. If we choose an object $a \in \mathcal{A}$ generating $\mathcal{A}$, we can express $\mathrm{Spec}(\mathcal{A}, \mathcal{S}) = U(a) \cap Z(\mathcal{S})$. Hence, $\mathrm{Spec}(\mathcal{A}, \mathcal{S})$ is constructible. Further, let $\mathcal{X}$ and $\mathcal{Y}$ be families of objects and consider the constructible subset $U(\mathcal{X}) \cap Z(\mathcal{Y})$. From the definition of $U(\mathcal{X})$, it is clear that $U(\mathcal{X}) = \bigcup_{x\in\mathcal{X}} U(x)$ and hence we have

$$U(\mathcal{X}) \cap Z(\mathcal{Y}) = \left(\bigcup_{x\in\mathcal{X}} U(x)\right) \cap Z(\mathcal{Y}) = \bigcup_{x\in\mathcal{X}} U(x) \cap Z(\mathcal{Y}) = \bigcup_{x\in\mathcal{X}} \mathrm{Spec}((x), M(\mathcal{Y})),$$

where (x) is the ideal generated by x and $M(\mathcal{Y})$ is the smallest multiplicatively closed family containing $\mathcal{Y}$. Further, every open in $\mathrm{Spec}(\mathcal{C})$ being quasi-compact, the open sets in the constructible topology on $\mathrm{Spec}(\mathcal{C})$ are simply the unions of constructible subsets. Thus, subsets of the form $\mathrm{Spec}(\mathcal{A}, \mathcal{S})$ with $\mathcal{A}$ finitely generated form a basis for the constructible topology on $\mathrm{Spec}(\mathcal{C})$.

(b) Since $\mathrm{Spec}(\mathcal{C})$ is spectral, a subset of $\mathrm{Spec}(\mathcal{C})_{\mathrm{cons}}$ is closed if and only if it is pro-constructible, i.e., it is the intersection of a family of constructible subsets. Let us consider a family $\mathrm{Spec}(\mathcal{A}_j, \mathcal{S}_j)$, $j \in J$, with each $\mathrm{Spec}(\mathcal{A}_j, \mathcal{S}_j)$ a constructible set. Let $\mathcal{B} = \sum_{j\in J}\mathcal{A}_j$ be the smallest ideal containing each of the ideals $\mathcal{A}_j$ and let $\mathcal{T}$ be the smallest multiplicatively closed family containing each of the families $\mathcal{S}_j$. Then, it is clear that $\bigcap_{j\in J}\mathrm{Spec}(\mathcal{A}_j, \mathcal{S}_j) = \mathrm{Spec}(\mathcal{B}, \mathcal{T})$. Combining with the fact (from part (a)) that any constructible subset may be expressed as a union of constructible sets of the form $\mathrm{Spec}(\mathcal{A}, \mathcal{S})$, we obtain the result. □

For a topologically noetherian scheme, we know that there is a homeomorphism $X \simeq \mathrm{Spec}(\mathrm{D}^{\mathrm{perf}}(X))$ of X with the spectrum of the derived category of perfect complexes (see [1, Corollary 5.6]). Then, Corollary 2.14 gives us an understanding of the constructible topology on such a scheme in terms of the subsets $\mathrm{Spec}(\mathcal{A}, \mathcal{S})$ for the tensor triangulated category $\mathrm{D}^{\mathrm{perf}}(X)$.

3 Oka families and a Prime Ideal Principle for tensor triangulated categories

Let $(\mathcal{C}, \otimes, 1)$ be a tensor triangulated category as before. As mentioned in the introduction, [4, Theorem 14] shows that the association

$$\mathcal{I}\triangleleft\mathcal{C} \mapsto c(\mathcal{I}) = \{\mathcal{P} \in \mathrm{Spec}(\mathcal{C}) : \mathcal{P} \supseteq \mathcal{I}\} \tag{11}$$

gives a one-to-one order reversing correspondence between radical thick tensor ideals in $\mathcal{C}$ and closed subspaces of $\mathrm{Spec}(\mathcal{C})^*$, where $\mathrm{Spec}(\mathcal{C})^*$ denotes the spectrum of $\mathcal{C}$ equipped with the inverse topology. In particular, if X is a topologically noetherian scheme and we take $\mathcal{C} = \mathrm{D}^{\mathrm{perf}}(X)$, the derived category of perfect complexes on X, we know that there is a homeomorphism $X \simeq \mathrm{Spec}(\mathrm{D}^{\mathrm{perf}}(X))$ (see [1, Corollary 5.6]) and hence radical thick tensor ideals in $\mathrm{D}^{\mathrm{perf}}(X)$ correspond to closed subspaces of X in the inverse topology. We have restated [4, Theorem 14] as an order-reversing correspondence with closed subspaces rather than as an order-preserving correspondence with open subspaces in order to make it look more similar to the standard Nullstellensatz. As such, this leads us to think about the irreducible closed subspaces of X in the inverse topology. More generally, let $\mathfrak{F}^*$ be a family of closed subspaces of $\mathrm{Spec}(\mathcal{C})^*$ such that $\varnothing \in \mathfrak{F}^*$ and suppose that $\mathfrak{F}^*$ is closed under finite unions. Then, if we consider a closed subspace $K_0 \subseteq \mathrm{Spec}(\mathcal{C})^*$ such that K_0 is minimal with respect to not being in $\mathfrak{F}^*$, it is clear that K_0 is irreducible. Using the correspondence in (11), we have a radical thick tensor ideal $\mathcal{I}_0$ such that $c(\mathcal{I}_0) = K_0$. Translated in terms of ideals in $\mathcal{C}$, we have a family $\mathfrak{F}$ of radical ideals and $\mathcal{I}_0$ is maximal with respect to being a radical ideal not contained in $\mathfrak{F}$. Then, it follows that if $\mathcal{J}, \mathcal{K}$ are radical ideals such that $\mathcal{I}_0 \supseteq \mathcal{J} \otimes \mathcal{K}$, we must have either $\mathcal{I}_0 \supseteq \mathcal{J}$ or $\mathcal{I}_0 \supseteq \mathcal{K}$. However, we would like to have better results along these lines on families of thick tensor ideals in $(\mathcal{C}, \otimes, 1)$.

Accordingly, we turn to some methods from commutative algebra, where there are several well known results of the kind "maximal implies prime". For example, given a commutative ring R and an R-module M, an ideal I that is maximal among annihilators of non-zero elements of M must be prime (see, for example, [20, Proposition 3.12]). In [25], Lam and Reyes gave a criterion that unifies these results, i.e., conditions on a family $\mathcal{F}$ of ideals in a ring such that any ideal that is maximal with respect to not being contained in $\mathcal{F}$ must be prime. They referred to this as the "Prime Ideal Principle". Using the Prime Ideal Principle, the authors in [25] were also able to uncover several new results of a similar nature (see also further work in Lam and Reyes [26] and Reyes [29,30]). The purpose of this section is to construct an analogous Prime Ideal Principle for thick tensor ideals in $(\mathcal{C}, \otimes, 1)$.

We will need to introduce some notation: if $\mathcal{I}, \mathcal{J}$ are thick tensor ideals in $\mathcal{C}$, we set $(\mathcal{I}, \mathcal{J}) = \mathcal{I} + \mathcal{J}$, i.e., the smallest thick tensor ideal containing both $\mathcal{I}$ and $\mathcal{J}$. We may easily verify that $(\mathcal{I}, \mathcal{J}) = \mathcal{I} + \mathcal{J}$ is the smallest thick tensor ideal containing all direct sums $x \oplus y$ where $x \in \mathcal{I}$ and $y \in \mathcal{J}$. Given an object $a \in \mathcal{C}$, we write $(\mathcal{I}, a)$ for the smallest thick tensor ideal containing both $\mathcal{I}$ and a. For any collection X of objects of $\mathcal{C}$, we set

$$(\mathcal{I}{:}X) = \{a \in \mathcal{C} : a \otimes x \in \mathcal{I} \text{ for each } x \in X\}. \tag{12}$$

It may be easily verified that $(\mathcal{I}{:}X)$ contains 0 and satisfies the three conditions in Definition 2.1, i.e., $(\mathcal{I}{:}X)$ is a thick tensor ideal. When $X = \{x\}$ is a singleton, we will denote $(\mathcal{I}{:}\{x\})$ simply by $(\mathcal{I}{:}x)$. When X happens to be a multiplicatively closed family, we note that $(\mathcal{I}{:}X)$ defined in (12) is not necessarily equal to $\mathcal{I} \div X$ as defined in (2).

Further, given a thick tensor ideal $\mathcal{I}$ and an object $a \in \mathcal{C}$, we will denote by $a \otimes \mathcal{I}$ the smallest thick tensor ideal containing all objects $a \otimes x$, where $x \in \mathcal{I}$. Similarly, given thick tensor ideals $\mathcal{I}_1, \mathcal{I}_2$, we denote by $\mathcal{I}_1 \otimes \mathcal{I}_2$ the smallest thick tensor ideal containing all objects $x_1 \otimes x_2$, where $x_1 \in \mathcal{I}_1$ and $x_2 \in \mathcal{I}_2$. In a manner analogous to [25], we will now define Oka families and Ako families of ideals in $(\mathcal{C}, \otimes, 1)$.

Definition 3.1 Let $\mathfrak{F}$ be a family of thick tensor ideals in $(\mathcal{C}, \otimes, 1)$ such that $\mathcal{C} \in \mathfrak{F}$. In what follows, let $\mathcal{I}, \mathcal{J}$ be thick tensor ideals in $\mathcal{C}$. Then, we will say that

- $\mathfrak{F}$ is a *semifilter* if $\mathcal{I} \subseteq \mathcal{J}$ and $\mathcal{I} \in \mathfrak{F}$ implies that $\mathcal{J} \in \mathfrak{F}$.
- $\mathfrak{F}$ is a *filter* if it is a semifilter and for any ideals $\mathcal{I}, \mathcal{J} \in \mathfrak{F}$, the intersection $\mathcal{I} \cap \mathcal{J} \in \mathfrak{F}$.
- $\mathfrak{F}$ is *monoidal* if $\mathcal{I}, \mathcal{J} \in \mathfrak{F}$ implies that $\mathcal{I} \otimes \mathcal{J} \in \mathfrak{F}$.
- $\mathfrak{F}$ is an *Oka family* (resp. a *strongly Oka family*) if $(\mathcal{I}, a), (\mathcal{I}:a) \in \mathfrak{F}$ for some object $a \in \mathcal{C}$ (resp. $(\mathcal{I}, \mathcal{A}), (\mathcal{I}:\mathcal{A}) \in \mathfrak{F}$ for some ideal $\mathcal{A} \triangleleft \mathcal{C}$) implies that $\mathcal{I} \in \mathfrak{F}$.
- $\mathfrak{F}$ is an *Ako family* (resp. a *strongly Ako family*) if $(\mathcal{I}, a), (\mathcal{I}, b) \in \mathfrak{F}$ for objects $a, b \in \mathcal{C}$ (resp. $(\mathcal{I}, a), (\mathcal{I}, \mathcal{B}) \in \mathfrak{F}$ for some object $a \in \mathcal{C}$ and some ideal $\mathcal{B} \triangleleft \mathcal{C}$) implies that $(\mathcal{I}, a \otimes b) \in \mathfrak{F}$ (resp. $(\mathcal{I}, a \otimes \mathcal{B}) \in \mathfrak{F}$).

We will say that a family $\mathfrak{F}$ of thick tensor ideals satisfies the "Prime Ideal Principle" if any ideal that is maximal with respect to not being in $\mathfrak{F}$ is also prime. We will now prove the main Prime Ideal Principle for ideals in $(\mathcal{C}, \otimes, 1)$.

Proposition 3.2 *Let $(\mathcal{C}, \otimes, 1)$ be a tensor triangulated category and let $\mathfrak{F}$ be a family of thick tensor ideals in $\mathcal{C}$ such that $\mathcal{C} \in \mathfrak{F}$. Let $\mathcal{I}$ be a thick tensor ideal of $(\mathcal{C}, \otimes, 1)$ that is maximal with respect to not being contained in $\mathfrak{F}$. Then:*

(a) *If $\mathfrak{F}$ is an Oka family of ideals, then $\mathcal{I}$ is a prime ideal.*
(b) *If $\mathfrak{F}$ is an Ako family of ideals, then $\mathcal{I}$ is a prime ideal.*
(c) *In other words, if $\mathfrak{F}$ is either an Oka family or an Ako family, $\mathfrak{F}$ satisfies the "Prime Ideal Principle", i.e., any ideal that is maximal with respect to not being in $\mathfrak{F}$ must be prime.*

Proof (a) We know that $\mathfrak{F}$ is an Oka family. Suppose that $\mathcal{I}$ is not a prime ideal, i.e., we can choose $a, b \in \mathcal{C}$ such that $a \otimes b \in \mathcal{I}$ but $a \notin \mathcal{I}$ and $b \notin \mathcal{I}$. Then, we note that $\mathcal{I} \subsetneq (\mathcal{I}:a)$ because the latter contains b and $\mathcal{I} \subsetneq (\mathcal{I}, a)$ because $a \notin \mathcal{I}$. However, since $\mathcal{I}$ is maximal with respect to not being contained in $\mathfrak{F}$, we must have $(\mathcal{I}:a) \in \mathfrak{F}$ and $(\mathcal{I}, a) \in \mathfrak{F}$. Since $\mathfrak{F}$ is an Oka family, we conclude that $\mathcal{I} \in \mathfrak{F}$, which is a contradiction.
(b) Again, we suppose that $\mathcal{I}$ is not prime and choose $a, b \in \mathcal{C}$ such that $a \otimes b \in \mathcal{I}$ but $a \notin \mathcal{I}$ and $b \notin \mathcal{I}$. As in part (a), we see that $\mathcal{I} \subsetneq (\mathcal{I}, a)$ and $\mathcal{I} \subsetneq (\mathcal{I}, b)$ because $a, b \notin \mathcal{I}$. Then, $\mathcal{I}$ being maximal with respect to not being contained in $\mathfrak{F}$, we must have $(\mathcal{I}, a), (\mathcal{I}, b) \in \mathfrak{F}$. Since $\mathfrak{F}$ is an Ako family, this implies that $(\mathcal{I}, a \otimes b) \in \mathfrak{F}$. But since $a \otimes b \in \mathcal{I}$, we have $\mathcal{I} = (\mathcal{I}, a \otimes b) \in \mathfrak{F}$, which is a contradiction. □

In order to proceed further, we will need a more explicit description of the thick tensor ideal generated by a collection X of objects in $\mathcal{C}$. For a collection X of objects of $\mathcal{C}$, we denote by $\widetilde{X}$ the smallest thick tensor ideal containing all objects of X. Note that, by definition, thick tensor ideals are full subcategories and hence it is enough to describe the objects of $\widetilde{X}$. For any collection X of objects of $\mathcal{C}$, we now set

$$\overline{X} = \left\{ x \in \mathcal{C} : \text{ there exist } a \in X, b, c \in \mathcal{C} \text{ such that } x \oplus b = a \otimes c \right\}. \tag{13}$$

From (13), it is clear that $\overline{\overline{X}} = \overline{X}$. For a collection X of objects of $\mathcal{C}$, we also consider $\Delta(X)$, the collection of all objects $a \in \mathcal{C}$ such that there exist $b, c \in X$ with a, b, c

forming a distinguished triangle (in some order). We are now ready to describe the thick tensor ideal $\widetilde{X}$ more explicitly.

Proposition 3.3 *Let X be a collection of objects in* $(\mathcal{C}, \otimes, 1)$. *We set $X_0 = X$ and inductively define*

$$X_{i+1} = \Delta(\overline{X}_i) \quad \text{for all} \quad i \geqslant 0. \tag{14}$$

Then, the smallest thick tensor ideal $\widetilde{X}$ containing all objects in X is given by the union of the increasing chain

$$\overline{X}_0 \subseteq \overline{X}_1 \subseteq \overline{X}_2 \subseteq \cdots$$

Proof From the definitions in (13) and (14), it is clear that each $\overline{X}_i \subseteq \widetilde{X}$. We set

$$X' = \bigcup_{i=0}^{\infty} \overline{X}_i.$$

In order to prove the result, it suffices to show that X' is itself a thick tensor ideal. We now consider some $x \in X'$ and choose $i \geqslant 0$ such that $x \in \overline{X}_i$. Hence, there exist $a \in X_i$ and $b, c \in \mathcal{C}$ such that $x \oplus b = a \otimes c$. Then, for any $y \in \mathcal{C}$, it is clear that $(x \otimes y) \oplus (b \otimes y) = a \otimes (c \otimes y)$ and hence $(x \otimes y) \in \overline{X}_i \subseteq X'$. Further, if x splits as $x = x_1 \oplus x_2$, we have $x_1 \oplus x_2 \oplus b = a \otimes c$ with $a \in X_i$ and hence both $x_1, x_2 \in \overline{X}_i$. Finally, we consider a distinguished triangle

$$a \to b \to c \to \mathrm{T}a$$

and assume for the sake of definiteness that $a, c \in X'$. Then, we can choose $j \geqslant 0$ large enough so that $a, c \in \overline{X}_j$. From (14), it follows that $b \in X_{j+1} = \Delta(\overline{X}_j)$ and hence $b \in X'$. We have shown that X' is a thick tensor ideal. □

We remark here that, in particular, if we apply the explicit description in Proposition 3.3 to the case of a thick tensor ideal generated by a single object, it follows from Definition 3.1 that a strongly Oka family is also an Oka family.

Lemma 3.4 *Let $\mathcal{I}$ (resp. $\mathcal{J}$) be a thick tensor ideal of* $(\mathcal{C}, \otimes, 1)$ *that is generated by $X = \{x_i\}_{i \in I}$ (resp. $Y = \{y_j\}_{j \in J}$). Then, the thick tensor ideal $\mathcal{I} \otimes \mathcal{J}$ is generated by the collection $\{x_i \otimes y_j\}_{i \in I, j \in J}$.*

Proof We consider some thick tensor ideal $\mathcal{K}$ containing all objects of the form $\{x_i \otimes y_j\}_{i \in I, j \in J}$. It suffices to show that $\mathcal{K}$ contains all objects of the form $x \otimes y$ with $x \in \mathcal{I}$ and $y \in \mathcal{J}$. We start by fixing some $j \in J$. Then, we know that $x_i \otimes y_j \in \mathcal{K}$ for each $i \in I$. We now use the notation of Proposition 3.3 and set $X_0 = X$. Then, we know from Proposition 3.3 that the ideal $\mathcal{I}$ may be described as the union

$$\mathcal{I} = \bigcup_{n=0}^{\infty} \overline{X}_n.$$

We now consider some $x' \in \overline{X}_0$. From (13), we know that there exist $x \in X_0 = X$ and $b, c \in \mathcal{C}$ such that $x' \oplus b = x \otimes c$. Then, $(x' \otimes y_j) \oplus (b \otimes y_j) = (x \otimes c \otimes y_j)$. Since $x \otimes y_j \otimes c \in \mathcal{K}$, it follows that $x' \otimes y_j \in \mathcal{K}$ for each $x' \in \overline{X}_0$. We will now proceed by induction. Suppose that $x' \otimes y_j \in \mathcal{K}$ for every $x' \in \overline{X}_n$ and $n \leqslant N$ for some given $N \geqslant 0$. We consider some $x' \in X_{N+1}$. By definition, $X_{N+1} = \Delta(\overline{X}_N)$ and hence x' is a part of a distinguished triangle two of whose objects are already in $\overline{X}_N$. For the sake of definiteness, we assume that we have a distinguished triangle

$$a' \to x' \to c' \to \mathrm{T}a'$$

with $a', c' \in \overline{X}_N$. Since $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is exact in both variables, we have an induced distinguished triangle

$$a' \otimes y_j \to x' \otimes y_j \to c' \otimes y_j \to \mathrm{T}(a' \otimes y_j).$$

Since $a' \otimes y_j, c' \otimes y_j \in \mathcal{K}$, it follows that $x' \otimes y_j \in \mathcal{K}$ for each $x' \in X_{N+1}$. Repeating the reasoning for X_0, we now see that $x' \otimes y_j \in \mathcal{K}$ for each $x' \in \overline{X}_{N+1}$. By induction, it follows that $x \otimes y_j \in \mathcal{K}$ for each $x \in \mathcal{I}$. Now since $\{y_j\}_{j \in J}$ generate $\mathcal{J}$, it follows similarly that $x \otimes y \in \mathcal{K}$ for each $x \in \mathcal{I}$, $y \in \mathcal{J}$. □

We remark here that it follows from the proof of Lemma 3.4 and Definition 3.1 that a strongly Ako family is also an Ako family.

Theorem 3.5 *Let* $(\mathcal{C}, \otimes, 1)$ *be a tensor triangulated category as above and let* $\mathfrak{F}$ *be a family of thick tensor ideals in* $(\mathcal{C}, \otimes, 1)$ *such that* $\mathcal{C} \in \mathfrak{F}$. *Consider the following conditions:*

- (P1) $\mathfrak{F}$ *is a monoidal filter.*
- (P2) $\mathfrak{F}$ *is monoidal and, given thick tensor ideals* $\mathcal{I}, \mathcal{J}$ *with* $\mathcal{J} \in \mathfrak{F}$ *and* $\mathcal{I} \supseteq \mathcal{J} \supseteq \mathcal{I}^2$, *we must have* $\mathcal{I} \in \mathfrak{F}$.
- (P3) *For thick tensor ideals* $\mathcal{I}, \mathcal{A}, \mathcal{B}$ *such that* $(\mathcal{I}, \mathcal{A}), (\mathcal{I}, \mathcal{B}) \in \mathfrak{F}$, *we must have* $(\mathcal{I}, \mathcal{A} \otimes \mathcal{B}) \in \mathfrak{F}$.
- (Q1) $\mathfrak{F}$ *is a monoidal semifilter.*
- (Q2) $\mathfrak{F}$ *is monoidal and, given thick tensor ideals* $\mathcal{I}, \mathcal{J}$ *with* $\mathcal{J} \in \mathfrak{F}$ *and* $\mathcal{I} \supseteq \mathcal{J} \supseteq \mathcal{I}^n$ *for some* $n > 1$, *we must have* $\mathcal{I} \in \mathfrak{F}$.
- (Q3) *If* $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$ *and* $\mathcal{I}$ *is a thick tensor ideal such that* $\mathcal{A} \otimes \mathcal{B} \subseteq \mathcal{I} \subseteq \mathcal{A} \cap \mathcal{B}$, *we must have* $\mathcal{I} \in \mathfrak{F}$.

Then, we have:

(a) *The following chart of implications holds:*

$$\begin{array}{ccccc}
(\mathrm{P1}) & \Longrightarrow & (\mathrm{P2}) & \Longrightarrow & (\mathrm{P3}) \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
(\mathrm{Q1}) & \Longrightarrow & (\mathrm{Q2}) & \Longrightarrow & (\mathrm{Q3}).
\end{array}$$

(b) *We have the following implications:*

$$\begin{array}{ccc} (P3) & \Longrightarrow & \text{strongly Ako} \\ \Downarrow & & \Downarrow \\ \text{strongly Oka} & \Longrightarrow & \text{Oka.} \end{array}$$

In particular, a family $\mathfrak{F}$ *satisfying condition* (P3) *also satisfies the Prime Ideal Principle, i.e., any ideal maximal with respect to not being in* $\mathfrak{F}$ *must be prime.*

Proof (a) For thick tensor ideals $\mathcal{I}, \mathcal{J}$, it is clear that $\mathcal{I} \otimes \mathcal{J} \subseteq \mathcal{I} \cap \mathcal{J}$ and hence it follows from Definition 3.1 that (P1) $\Leftrightarrow$ (Q1). Further, it is obvious that (Q2) $\Rightarrow$ (P2).

(Q1) $\Rightarrow$ (Q2). Since $\mathfrak{F}$ is a semifilter, whenever we have $\mathcal{I} \supseteq \mathcal{J}$ with $\mathcal{J} \in \mathfrak{F}$, we see that $\mathcal{I} \in \mathfrak{F}$.

(P2) $\Rightarrow$ (P3). We consider thick tensor ideals $\mathcal{I}, \mathcal{A}, \mathcal{B}$ with $(\mathcal{I}, \mathcal{A}), (\mathcal{I}, \mathcal{B}) \in \mathfrak{F}$. Since $\mathfrak{F}$ is monoidal, we have $(\mathcal{I}, \mathcal{A}) \otimes (\mathcal{I}, \mathcal{B}) \in \mathfrak{F}$. Further, since $(\mathcal{I}, \mathcal{A}) \supseteq (\mathcal{I}, \mathcal{A} \otimes \mathcal{B})$ and $(\mathcal{I}, \mathcal{B}) \supseteq (\mathcal{I}, \mathcal{A} \otimes \mathcal{B})$, we have

$$(\mathcal{I}, \mathcal{A}) \otimes (\mathcal{I}, \mathcal{B}) \supseteq (\mathcal{I}, \mathcal{A} \otimes \mathcal{B}) \otimes (\mathcal{I}, \mathcal{A} \otimes \mathcal{B}) = (\mathcal{I}, \mathcal{A} \otimes \mathcal{B})^2. \tag{15}$$

On the other hand, we know that $(\mathcal{I}, \mathcal{A})$ (resp. $(\mathcal{I}, \mathcal{B})$) is generated by objects of the form $x \oplus a$ with $x \in \mathcal{I}$, $a \in \mathcal{A}$ (resp. $y \oplus b$ with $y \in \mathcal{I}$, $b \in \mathcal{B}$). From Lemma 3.4, it follows that $(\mathcal{I}, \mathcal{A}) \otimes (\mathcal{I}, \mathcal{B})$ is generated by objects of the form

$$(x \oplus a) \otimes (y \otimes b) = ((x \otimes y) \oplus (x \otimes b) \oplus (a \otimes y)) \oplus (a \otimes b), \tag{16}$$

$x, y \in \mathcal{I}$, $a \in \mathcal{A}$, $b \in \mathcal{B}$. Since any element of the form in (16) is in $(\mathcal{I}, \mathcal{A} \otimes \mathcal{B})$, we have

$$(\mathcal{I}, \mathcal{A} \otimes \mathcal{B}) \supseteq (\mathcal{I}, \mathcal{A}) \otimes (\mathcal{I}, \mathcal{B}). \tag{17}$$

From (15) and (17) and applying condition (P2), we see that $(\mathcal{I}, \mathcal{A} \otimes \mathcal{B}) \in \mathfrak{F}$. This proves (P3).

(P2) $\Rightarrow$ (Q2). We suppose that there exist thick tensor ideals $\mathcal{I}, \mathcal{J}$ with $\mathcal{I} \notin \mathfrak{F}$, $\mathcal{J} \in \mathfrak{F}$ and $\mathcal{I} \supseteq \mathcal{J} \supseteq \mathcal{I}^n$ for some $n > 1$. Then, there is a largest integer $k \geqslant 1$ such that $\mathcal{J} + \mathcal{I}^k \notin \mathfrak{F}$. By Lemma 3.4, $(\mathcal{J} + \mathcal{I}^k)^2$ is generated by objects of the form $(x \oplus y) \otimes (x' \oplus y')$ with $x, x' \in \mathcal{J}$ and $y, y' \in \mathcal{I}^k$. Any such object lies in $\mathcal{J} + \mathcal{I}^{k+1}$ and hence we have

$$\mathcal{J} + \mathcal{I}^k \supseteq \mathcal{J} + \mathcal{I}^{k+1} \supseteq (\mathcal{J} + \mathcal{I}^k)^2.$$

Since $\mathcal{J} + \mathcal{I}^{k+1} \in \mathfrak{F}$, it follows from condition (P2) that $\mathcal{J} + \mathcal{I}^k \in \mathfrak{F}$ which is a contradiction.

(P3) $\Rightarrow$ (Q3). Let $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$ and suppose that $\mathcal{A} \otimes \mathcal{B} \subseteq \mathcal{I} \subseteq \mathcal{A} \cap \mathcal{B}$. Then, $(\mathcal{I}, \mathcal{A}) = \mathcal{A} \in \mathfrak{F}$ and $(\mathcal{I}, \mathcal{B}) = \mathcal{B} \in \mathfrak{F}$ and hence it follows from (P3) that $(\mathcal{I}, \mathcal{A} \otimes \mathcal{B}) \in \mathfrak{F}$. But since $\mathcal{A} \otimes \mathcal{B} \subseteq \mathcal{I}$, we have $\mathcal{I} = (\mathcal{I}, \mathcal{A} \otimes \mathcal{B}) \in \mathfrak{F}$.

(Q3)$\Rightarrow$(P3). Suppose that $(\mathcal{I}, \mathcal{A}), (\mathcal{I}, \mathcal{B}) \in \mathfrak{F}$. It is clear that we have $(\mathcal{I}, \mathcal{A}\otimes\mathcal{B}) \subseteq (\mathcal{I}, \mathcal{A}) \cap (\mathcal{I}, \mathcal{B})$. As in (17), we can show that $(\mathcal{I}, \mathcal{A})\otimes(\mathcal{I}, \mathcal{B}) \subseteq (\mathcal{I}, \mathcal{A}\otimes\mathcal{B})$. From condition (Q3), it now follows that $(\mathcal{I}, \mathcal{A}\otimes\mathcal{B}) \in \mathfrak{F}$.

(b) We assume condition (P3). To show that $\mathfrak{F}$ is strongly Ako, we consider $a \in \mathcal{C}$ and thick tensor ideals $\mathcal{I}, \mathcal{B}$ such that $(\mathcal{I}, a), (\mathcal{I}, \mathcal{B}) \in \mathfrak{F}$. Let $\mathcal{A}$ be the thick tensor ideal generated by a. Then, $(\mathcal{I}, \mathcal{A}) = (\mathcal{I}, a)$ lies in $\mathfrak{F}$ and it follows from (P3) that $(\mathcal{I}, \mathcal{A}\otimes\mathcal{B}) \in \mathfrak{F}$. From Lemma 3.4, it is clear that $a\otimes\mathcal{B} = \mathcal{A}\otimes\mathcal{B}$. Hence, $(\mathcal{I}, a\otimes\mathcal{B}) \in \mathfrak{F}$ and $\mathfrak{F}$ is strongly Ako.

In order to show that $\mathfrak{F}$ is strongly Oka, we consider ideals $\mathcal{I}, \mathcal{A}$ with $(\mathcal{I}, \mathcal{A}), (\mathcal{I}:\mathcal{A}) \in \mathfrak{F}$. We now set $\mathcal{B} = (\mathcal{I}:\mathcal{A})$. Then, it is clear that $(\mathcal{I}, \mathcal{B}) = \mathcal{B} \in \mathfrak{F}$ and it follows from condition (P3) that $(\mathcal{I}, \mathcal{A}\otimes\mathcal{B}) \in \mathfrak{F}$. From the definition in (12), we know that if $a \in \mathcal{A}$ and $b \in \mathcal{B} = (\mathcal{I}:\mathcal{A})$, we must have $a\otimes b \in \mathcal{I}$ and hence $\mathcal{A}\otimes\mathcal{B} \subseteq \mathcal{I}$. It follows that $\mathcal{I} = (\mathcal{I}, \mathcal{A}\otimes\mathcal{B}) \in \mathfrak{F}$ and $\mathfrak{F}$ is strongly Oka. We have also noted before that strongly Oka families are also Oka and hence it follows from Proposition 3.2 that $\mathfrak{F}$ satisfies the Prime Ideal Principle.

It remains to show that strongly Ako families are also Oka. Let $\mathfrak{F}$ be strongly Ako and suppose that $(\mathcal{I}, a), (\mathcal{I}:a) \in \mathfrak{F}$. We set $\mathcal{B} = (\mathcal{I}:a)$. Again since $(\mathcal{I}, \mathcal{B}) = \mathcal{B} \in \mathfrak{F}$, we see that $(\mathcal{I}, a\otimes\mathcal{B}) \in \mathfrak{F}$. But $a\otimes\mathcal{B} \subseteq \mathcal{I}$ and hence $\mathcal{I} = (\mathcal{I}, a\otimes\mathcal{B}) \in \mathfrak{F}$ and hence $\mathfrak{F}$ is Oka. □

For the rest of this section, we shall construct various families of thick tensor ideals that satisfy the Prime Ideal Principle.

Proposition 3.6 *Let $(\mathcal{C}, \otimes, 1)$ be a tensor triangulated category as above. Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be families of thick tensor ideals in $\mathcal{C}$ such that $\mathfrak{F}_1$ is monoidal and $\mathfrak{F}_2$ is closed under finite intersections. Consider the family*

$$\mathfrak{F} = \{\mathcal{C}\} \cup \left\{\mathcal{I}\triangleleft\mathcal{C} : \textit{there exist } \mathcal{J}_1 \in \mathfrak{F}_1, \mathcal{J}_2 \in \mathfrak{F}_2 \textit{ such that } \mathcal{J}_1 \subseteq \mathcal{I} \subseteq \mathcal{J}_2\right\}. \tag{18}$$

Then, $\mathfrak{F}$ is both a strongly Oka family and a strongly Ako family. In particular, $\mathfrak{F}$ satisfies the Prime Ideal Principle.

Proof We will show that $\mathfrak{F}$ satisfies condition (Q3) in Theorem 3.5. By Theorem 3.5(b), condition (P3) which is equivalent to (Q3) will then imply that $\mathfrak{F}$ is a strongly Oka and a strongly Ako family.

We choose $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$ and consider a thick tensor ideal $\mathcal{I}$ satisfying $\mathcal{A}\otimes\mathcal{B} \subseteq \mathcal{I} \subseteq \mathcal{A}\cap\mathcal{B}$. If either $\mathcal{A} = \mathcal{C}$ or $\mathcal{B} = \mathcal{C}$, the result is obvious. Hence, we suppose that $\mathcal{A} \neq \mathcal{C}$ and $\mathcal{B} \neq \mathcal{C}$ and choose $\mathcal{J}_1, \mathcal{K}_1 \in \mathfrak{F}_1$ and $\mathcal{J}_2, \mathcal{K}_2 \in \mathfrak{F}_2$ such that

$$\mathcal{J}_1 \subseteq \mathcal{A} \subseteq \mathcal{J}_2, \qquad \mathcal{K}_1 \subseteq \mathcal{B} \subseteq \mathcal{K}_2.$$

It now follows that

$$\mathcal{J}_1\otimes\mathcal{K}_1 \subseteq \mathcal{A}\otimes\mathcal{B} \subseteq \mathcal{I} \subseteq \mathcal{A}\cap\mathcal{B} \subseteq \mathcal{J}_2\cap\mathcal{K}_2.$$

Since $\mathfrak{F}_1$ is monoidal and $\mathfrak{F}_2$ is closed under finite intersections, we see that $\mathcal{J}_1\otimes\mathcal{K}_1 \in \mathfrak{F}_1$ and $\mathcal{J}_2\cap\mathcal{K}_2 \in \mathfrak{F}_2$. Hence, it follows from the definition in (18) that $\mathcal{I} \in \mathfrak{F}$. □

Corollary 3.7 (a) *Let $\mathcal{J}$, $\mathcal{K}$ be thick tensor ideals in* $(\mathcal{C}, \otimes, 1)$. *Consider the following family of ideals:*

$$\mathfrak{F} = \{\mathcal{C}\} \cup \{\mathcal{I} \lhd \mathcal{C} : \mathcal{J}^n \subseteq \mathcal{I} \subseteq \mathcal{K} \text{ for some } n \geqslant 1\}.$$

Then, $\mathfrak{F}$ satisfies the Prime Ideal Principle, i.e., any thick tensor ideal that is maximal with respect to not being in $\mathfrak{F}$ must be prime.

(b) *Let $\{\mathcal{J}_j\}_{j\in J}$ be a family of thick tensor ideals in* $(\mathcal{C}, \otimes, 1)$. *Then, any ideal that is maximal with respect to not containing a finite product of $\mathcal{J}_j$ is prime.*

Proof (a) We set $\mathfrak{F}_1 = \{\mathcal{J}^n : n \geqslant 1\}$ and $\mathfrak{F}_2 = \{\mathcal{K}\}$. Then, $\mathfrak{F}_1$ is monoidal and $\mathfrak{F}_2$ is closed under finite intersections. Now applying Proposition 3.6, we see that $\mathfrak{F}$ satisfies the Prime Ideal Principle.

(b) We set $\mathfrak{F}_1 = \{\mathcal{J}_{j_1} \otimes \cdots \otimes \mathcal{J}_{j_k} : j_1, \ldots, j_k \in J\}$ and $\mathfrak{F}_2 = \{\mathcal{C}\}$. It is clear that $\mathfrak{F}_1$ is monoidal and $\mathfrak{F}_2$ is closed under finite intersections. The result now follows from Proposition 3.6. □

Proposition 3.8 *Let* $(\mathcal{C}, \otimes, 1)$ *be a tensor triangulated category and let $\mathcal{S}$ be a multiplicatively closed family of objects of $\mathcal{C}$. Then:*

(a) *A thick tensor ideal that is maximal with respect to being disjoint from $\mathcal{S}$ is also prime.*
(b) *A thick tensor ideal that is maximal among ideals $\mathcal{I}$ satisfying $\bigcap_{i=1}^{\infty} \mathcal{I}^i \cap \mathcal{S} = \varnothing$ is also prime.*
(c) *Let $\mathfrak{F}$ be a monoidal semifilter. Then, a thick tensor ideal that is maximal among ideals $\mathcal{I}$ satisfying $\bigcap_{i=1}^{\infty} \mathcal{I}^i \notin \mathfrak{F}$ is also prime.*

Proof (a) We consider the family

$$\mathfrak{F}_{\mathcal{S}} = \{\mathcal{I} \lhd \mathcal{C} : \mathcal{I} \cap \mathcal{S} \neq \varnothing\}.$$

It is clear that $\mathfrak{F}_{\mathcal{S}}$ is a semifilter. Further, if we choose $x \in \mathcal{I} \cap \mathcal{S}$, $y \in \mathcal{J} \cap \mathcal{S}$ for thick tensor ideals $\mathcal{I}, \mathcal{J} \in \mathfrak{F}_{\mathcal{S}}$, it follows that $(x \otimes y) \in (\mathcal{I} \otimes \mathcal{J}) \cap \mathcal{S}$. Thus, $\mathfrak{F}_{\mathcal{S}}$ is also monoidal and we see that it satisfies condition (Q1) in Theorem 3.5. Hence, $\mathfrak{F}_{\mathcal{S}}$ satisfies the Prime Ideal Principle.

(c) Given the monoidal semifilter $\mathfrak{F}$, we consider

$$\mathfrak{F}_{\infty} = \left\{\mathcal{I} \lhd \mathcal{C} : \bigcap_{i=1}^{\infty} \mathcal{I}^i \in \mathfrak{F}\right\}.$$

Since $\mathfrak{F}$ is a semifilter, it is clear that so is $\mathfrak{F}_{\infty}$. We consider $\mathcal{I}, \mathcal{J} \in \mathfrak{F}_{\infty}$ and note that

$$\bigcap_{i=1}^{\infty} (\mathcal{I} \otimes \mathcal{J})^{\otimes i} \supseteq \bigcap_{i=1}^{\infty} \mathcal{I}^i \otimes \bigcap_{i=1}^{\infty} \mathcal{J}^i \in \mathfrak{F}.$$

Since $\mathfrak{F}$ is a semifilter, it follows that $\bigcap_{i=1}^{\infty} (\mathcal{I} \otimes \mathcal{J})^{\otimes i} \in \mathfrak{F}$ and hence $(\mathcal{I} \otimes \mathcal{J}) \in \mathfrak{F}_{\infty}$. Hence, $\mathfrak{F}_{\infty}$ is a monoidal semifilter and satisfies the Prime Ideal Principle.

(b) follows by applying the result of (c) with $\mathfrak{F} = \mathfrak{F}_{\mathcal{S}}$. □

Remark 3.9 We mention here that the result of part (a) of Proposition 3.8 is already known as a special case of [1, Lemma 2.2].

The next result will deal with thick tensor ideals that are annihilators of objects from categories that are "modules" over the tensor triangulated category $(\mathcal{C}, \otimes, 1)$. For this, we recall that Stevenson [34, Definition 3.2] has introduced module actions for a tensor triangulated category $(\mathcal{C}, \otimes, 1)$ on a triangulated category $\mathcal{M}$. More explicitly, a *module* over $(\mathcal{C}, \otimes, 1)$ consists of a triangulated category $\mathcal{M}$ along with an action

$$* : \mathcal{C} \times \mathcal{M} \to \mathcal{M} \tag{19}$$

that is exact in both variables; in other words, for any $a \in \mathcal{C}$ and $m \in \mathcal{M}$, the functors $a * \cdot : \mathcal{M} \to \mathcal{M}$ and $\cdot * m : \mathcal{C} \to \mathcal{M}$ are exact. Further, the action in (19) satisfies appropriate associative, distributive and unit properties and is well behaved with respect to the translation operator on the triangulated category $\mathcal{M}$ (see [34, Definition 3.2]). For a detailed study on modules over tensor triangulated categories, we refer the reader to [34].

Given an object $m \in \mathcal{M}$, we let the annihilator $\mathrm{Ann}(m)$ be the collection of all objects $a \in \mathcal{C}$ such that $a * m = 0$. Given that the action $*$ in (19) is exact in both variables, it is clear that $\mathrm{Ann}(m) \subseteq \mathcal{C}$ is actually a thick tensor ideal.

Proposition 3.10 *Let* $(\mathcal{C}, \otimes, 1)$ *be a tensor triangulated category and let* $\mathcal{M}$ *be a triangulated category that has the structure of a* $\mathcal{C}$*-module.*

(a) *Let* $\mathcal{S}$ *be a multiplicatively closed family of objects in* $(\mathcal{C}, \otimes, 1)$. *Consider the following family of thick tensor ideals:*

$$\mathfrak{F} = \left\{\mathcal{I} \triangleleft \mathcal{C} : \textit{for any } m \in \mathcal{M},\ \mathcal{I} * m = 0 \Rightarrow s * m = 0 \textit{ for some } s \in \mathcal{S}\right\}. \tag{20}$$

Then, $\mathfrak{F}$ *is a strongly Ako semifilter. In particular, the family* $\mathfrak{F}$ *satisfies the Prime Ideal Principle.*

(b) *A thick tensor ideal of* $(\mathcal{C}, \otimes, 1)$ *that is maximal among the annihilators of non-zero objects of* $\mathcal{M}$ *is also prime.*

Proof (a) It is immediate from (20) that $\mathfrak{F}$ is a semifilter. To show that $\mathfrak{F}$ is strongly Ako, we choose thick tensor ideals $\mathcal{I}, \mathcal{B} \triangleleft \mathcal{C}$ and some object $a \in \mathcal{C}$ such that $(\mathcal{I}, a), (\mathcal{I}, \mathcal{B}) \in \mathfrak{F}$. Suppose that $(\mathcal{I}, a \otimes \mathcal{B}) * m = 0$ for some $m \in \mathcal{M}$. Then, $\mathcal{I}, \mathcal{B} \subseteq \mathrm{Ann}(a * m)$ and hence $(\mathcal{I}, \mathcal{B}) * (a * m) = 0$. Since $(\mathcal{I}, \mathcal{B}) \in \mathfrak{F}$, we conclude that there exists some $s \in \mathcal{S}$ such that $s * a * m = 0$. It follows that $a \in \mathrm{Ann}(s * m)$.

On the other hand, since $\mathcal{I} * m = 0$, we have $\mathcal{I} \subseteq \mathrm{Ann}(m) \subseteq \mathrm{Ann}(s * m)$. Then, $(\mathcal{I}, a) * (s * m) = 0$. Since $(\mathcal{I}, a) \in \mathfrak{F}$, it follows that there exists $s' \in \mathcal{S}$ such that $s' * (s * m) = (s' \otimes s) * m = 0$. Finally, since $\mathcal{S}$ is multiplicatively closed, we know that $s' \otimes s \in \mathcal{S}$. This shows that $(\mathcal{I}, a \otimes \mathcal{B}) \in \mathfrak{F}$ and hence $\mathfrak{F}$ is strongly Ako. In particular, it now follows from Theorem 3.5 (b) that $\mathfrak{F}$ satisfies the Prime Ideal Principle.

(b) In particular, we take $\mathcal{S} = \{1\}$. Then, from the definition in (20), it is clear that

$$\mathcal{I} \notin \mathfrak{F} \iff \mathcal{I} \subseteq \mathrm{Ann}(m) \quad \text{for some } \ 0 \neq m \in \mathcal{M}.$$

In part (a), we have shown that $\mathfrak{F}$ satisfies the Prime Ideal Principle. Hence, if $\mathcal{I}$ is maximal with respect to being contained in some $\mathrm{Ann}(m)$ with $m \neq 0$, it must be prime. Finally, it is clear that an ideal is maximal with respect to being contained in some $\mathrm{Ann}(m)$ with $m \neq 0$ if and only if it is maximal among the annihiliators of non-zero objects of $\mathcal{M}$. □

Analogously to the usual definition in commutative algebra, we will say that a thick tensor ideal $\mathcal{I}$ in $(\mathcal{C}, \otimes, 1)$ is essential if it has non-trivial intersection with every non-zero thick tensor ideal in $(\mathcal{C}, \otimes, 1)$. Further, as in [1, Corollary 2.4], an object $a \in \mathcal{C}$ will be called $\otimes$-nilpotent if there exists an integer $n > 0$ such that $a^{\otimes n} = 0$. As such, we will say that $(\mathcal{C}, \otimes, 1)$ is $\otimes$-reduced if it has no non-zero $\otimes$-nilpotent objects. We now have the following result.

Proposition 3.11 *Let* $(\mathcal{C}, \otimes, 1)$ *be a tensor triangulated category and let* $\mathfrak{F}$ *be the family of essential thick tensor ideals of* $(\mathcal{C}, \otimes, 1)$. *Then, if* $(\mathcal{C}, \otimes, 1)$ *is* $\otimes$*-reduced,* $\mathfrak{F}$ *is a monoidal semifilter. In particular, a thick tensor ideal of* $(\mathcal{C}, \otimes, 1)$ *that is maximal with respect to not being essential must be prime.*

Proof If $\mathcal{I} \in \mathfrak{F}$ is an essential ideal and $\mathcal{J}$ is a thick tensor ideal containing $\mathcal{I}$, it is clear that $\mathcal{J}$ is also essential. Hence, $\mathfrak{F}$ is a semifilter.

We now suppose that $\mathcal{I}_1, \mathcal{I}_2 \in \mathfrak{F}$ and consider some non-zero thick tensor ideal $\mathcal{A}$. Since $\mathcal{I}_1$ is essential, we can choose $0 \neq x \in \mathcal{I}_1 \cap \mathcal{A}$. We now consider the thick tensor ideal (x) generated by x. Now since $\mathcal{I}_2$ is essential, we may choose $0 \neq y \in (x) \cap \mathcal{I}_2$ and consider $y \otimes y$. Since $(\mathcal{C}, \otimes, 1)$ is $\otimes$-reduced, it follows that $0 \neq y \otimes y \in (\mathcal{I}_1 \otimes \mathcal{I}_2) \cap \mathcal{A}$. This shows that $\mathcal{I}_1 \otimes \mathcal{I}_2$ is essential and hence $\mathfrak{F}$ is a monoidal semifilter. It now follows from Theorem 3.5 that any ideal that is maximal with respect to not being essential is also prime. □

Proposition 3.12 *Let* $(\mathcal{C}, \otimes, 1)$ *be a tensor triangulated category. Then, an ideal that is maximal among thick tensor ideals* $\mathcal{I}$ *satisfying* $\mathcal{I}^{\otimes n} \supsetneq \mathcal{I}^{\otimes n+1}$ *for each* $n \geqslant 0$ *must be prime.*

Proof We will show that the following family of thick tensor ideals:

$$\mathfrak{F} = \left\{ \mathcal{I} \triangleleft \mathcal{C} : \text{there exists } n \geqslant 0 \text{ such that } \mathcal{I}^{\otimes n} = \mathcal{I}^{\otimes (n+1)} \right\}$$

is an Oka family by showing that it satisfies condition (Q2) in Theorem 3.5. If $\mathcal{J}, \mathcal{K} \in \mathfrak{F}$, we can choose N large enough so that $\mathcal{J}^{\otimes N} = \mathcal{J}^{\otimes (N+1)}$ and $\mathcal{K}^{\otimes N} = \mathcal{K}^{\otimes (N+1)}$. It follows that we have $(\mathcal{J} \otimes \mathcal{K})^{\otimes N} = (\mathcal{J} \otimes \mathcal{K})^{\otimes (N+1)}$ and hence $\mathfrak{F}$ is monoidal.

We now consider a thick tensor ideal $\mathcal{I}$ such that there exists $n > 1$ with $\mathcal{I} \supseteq \mathcal{J} \supseteq \mathcal{I}^{\otimes n}$ and $\mathcal{J} \in \mathfrak{F}$. We choose $m \geqslant 0$ such that $\mathcal{J}^{\otimes m} = \mathcal{J}^{\otimes (m+1)}$. Then, we have

$$\mathcal{J}^{\otimes m} \supseteq \mathcal{I}^{\otimes mn} \supseteq \mathcal{J}^{\otimes mn} = \mathcal{J}^{\otimes m} \quad \Longrightarrow \quad \mathcal{J}^{\otimes m} = \mathcal{I}^{\otimes mn}. \tag{21}$$

By reasoning similar to (21), we see that $\mathcal{J}^{\otimes (m+1)} = \mathcal{I}^{\otimes (m+1)n}$ and hence $\mathcal{I}^{\otimes mn} = \mathcal{I}^{\otimes (m+1)n}$. Since $n > 1$, we now have

$$\mathcal{I}^{\otimes mn} \supseteq \mathcal{I}^{\otimes (mn+1)} \supseteq \mathcal{I}^{\otimes (m+1)n} \quad \Longrightarrow \quad \mathcal{I}^{\otimes mn} = \mathcal{I}^{\otimes (mn+1)}.$$

Then, $\mathcal{I} \in \mathfrak{F}$ and the family $\mathfrak{F}$ satisfies condition (Q2) in Theorem 3.5. As such, $\mathfrak{F}$ satisfies the Prime Ideal Principle. □

4 Monoidal semifilters of ideals and realizations of pairs

Let $(\mathcal{C}, \otimes, 1)$ be a tensor triangulated category as before. In Sect. 2, we considered pairs $(\mathcal{A}, \mathcal{S})$, where $\mathcal{A}$ is a thick tensor ideal and $\mathcal{S}$ is a multiplicatively closed family of objects of $\mathcal{C}$. In Sect. 3, we have already seen that for such a multiplicatively closed family $\mathcal{S}$, we can define

$$\mathfrak{F}_{\mathcal{S}} = \{\mathcal{I} \triangleleft \mathcal{C} : \mathcal{I} \cap \mathcal{S} \neq \varnothing\}. \tag{22}$$

Then, $\mathfrak{F}_S$ turns out to be a monoidal semifilter (and hence an Oka family) of ideals in $\mathcal{C}$. More generally, in this section, we will consider pairs $(\mathcal{A}, \mathfrak{F})$, where $\mathcal{A}$ is a thick tensor ideal and $\mathfrak{F}$ is a monoidal semifilter. Our purpose is to study realizations of templates of such pairs in a manner similar to Sect. 2.

Definition 4.1 Let $(\mathcal{C}, \otimes, 1)$ be a tensor triangulated category as before. We consider a pair $(\mathcal{A}, \mathfrak{F})$ satisfying the following conditions:

- $\mathcal{A}$ is a thick tensor ideal in $(\mathcal{C}, \otimes, 1)$.
- $\mathfrak{F}$ is a family of thick tensor ideals that is a monoidal semifilter.
- Let $\mathfrak{F}^c$ denote the collection of all thick tensor ideals of $(\mathcal{C}, \otimes, 1)$ not contained in $\mathfrak{F}$. Then, every non-empty increasing chain of ideals in $\mathfrak{F}^c$ has an upper bound in $\mathfrak{F}^c$.

Then, we will say that a prime ideal $\mathcal{P}$ is a *realization of the pair* $(\mathcal{A}, \mathfrak{F})$ if $\mathcal{A} \subseteq \mathcal{P}$ and $\mathcal{P} \notin \mathfrak{F}$. We let $\operatorname{Spec}(\mathcal{A}, \mathfrak{F})$ denote the collection of all prime ideals realizing the pair $(\mathcal{A}, \mathfrak{F})$. Further, we let $\mathcal{M}(\mathcal{A}, \mathfrak{F})$ be the multiplicatively closed family given by the complement of $\bigcup_{\mathcal{Q} \in \operatorname{Spec}(\mathcal{A}, \mathfrak{F})} \mathcal{Q}$.

More generally, let $(I, \leqslant)$ be a partially ordered set and let $T = \{(\mathcal{A}_i, \mathfrak{F}_i)\}_{i \in I}$ be a collection of such pairs indexed by I. Then, we will say that a collection $\{\mathcal{P}_i\}_{i \in I}$ of prime ideals in $\mathcal{C}$ is a *realization of the template* T if each $\mathcal{P}_i$ realizes the pair $(\mathcal{A}_i, \mathfrak{F}_i)$ and $\mathcal{P}_i \subseteq \mathcal{P}_j$ for every $i \leqslant j$ in I.

Henceforth, we will only consider pairs $(\mathcal{A}, \mathfrak{F})$ as in Definition 4.1. Further, since $\mathfrak{F}$ is a semifilter, it is clear that if there exists a prime ideal $\mathcal{P}$ realizing a pair $(\mathcal{A}, \mathfrak{F})$, we must have $\mathcal{A} \notin \mathfrak{F}$.

Proposition 4.2 *Let* $(\mathcal{A}, \mathfrak{F})$ *be a pair as in Definition* 4.1. *Suppose that* $\mathcal{A} \notin \mathfrak{F}$. *Then, there always exists a prime ideal in* $(\mathcal{C}, \otimes, 1)$ *that realizes the pair* $(\mathcal{A}, \mathfrak{F})$.

Proof From Theorem 3.5, we know that a thick tensor ideal that is maximal with respect to not being in the monoidal semifilter $\mathfrak{F}$ must be prime. Further, since every increasing chain of ideals in $\mathfrak{F}^c$ has an upper bound in $\mathfrak{F}^c$, given that $\mathcal{A} \notin \mathfrak{F}$, it follows from Zorn's lemma that $\mathcal{A}$ must be contained in a thick tensor ideal $\mathcal{P}$ that is maximal with respect to not being in $\mathfrak{F}$. Then, $\mathcal{P}$ is a prime ideal realizing the pair $(\mathcal{A}, \mathfrak{F})$. □

Remark 4.3 In particular, let $\mathcal{S}$ be a multiplicatively closed family of objects of $\mathcal{C}$ and set $\mathfrak{F} = \mathfrak{F}_{\mathcal{S}} = \{\mathcal{I} \triangleleft \mathcal{C} : \mathcal{I} \cap \mathcal{S} \neq \varnothing\}$ as in (22). It is clear that the union of any chain of ideals in the complement $\mathfrak{F}_{\mathcal{S}}^c$ of $\mathfrak{F}_{\mathcal{S}}$ lies in $\mathfrak{F}_{\mathcal{S}}^c$. Then, if $\mathcal{A}$ is a thick tensor ideal such that $\mathcal{A} \cap \mathcal{S} = \varnothing$, i.e., $\mathcal{A} \notin \mathfrak{F}_{\mathcal{S}}$, it follows from Proposition 4.2 that we can find a prime ideal $\mathcal{P}$ realizing the pair $(\mathcal{A}, \mathfrak{F}_{\mathcal{S}})$. This allows us to recover [1, Lemma 2.2] as a special case.

Lemma 4.4 *Let $\mathcal{A}$ be a thick tensor ideal and let $\mathfrak{F}$ be a monoidal semifilter. Consider the full subcategory $\mathcal{A} \div \mathfrak{F}$ of $\mathcal{C}$ defined by*

$$\mathcal{A} \div \mathfrak{F} = \{a \in \mathcal{C} : \textit{there exists } \mathcal{I} \in \mathfrak{F} \textit{ such that } a \otimes \mathcal{I} \subseteq \mathcal{A}\}. \tag{23}$$

Then, $\mathcal{A} \div \mathfrak{F}$ is a thick tensor ideal in $(\mathcal{C}, \otimes, 1)$.

Proof From (23), it is clear that if $a \in \mathcal{A} \div \mathfrak{F}$, then $a' \otimes a \in \mathcal{A} \div \mathfrak{F}$ for any $a' \in \mathcal{C}$. Now, suppose that $a \in \mathcal{A} \div \mathfrak{F}$ splits as $a = b \oplus c$. We choose $\mathcal{I} \in \mathfrak{F}$ such that $a \otimes \mathcal{I} \subseteq \mathcal{A}$. Then, for any $x \in \mathcal{I}$, we have

$$a \otimes x = (b \otimes x) \oplus (c \otimes x).$$

Since $a \otimes x \in a \otimes \mathcal{I} \subseteq \mathcal{A}$ and $\mathcal{A}$ is thick, it follows that $b \otimes x \in \mathcal{A}$ and $c \otimes x \in \mathcal{A}$ for any $x \in \mathcal{I}$. Hence, $b \otimes \mathcal{I} \subseteq \mathcal{A}$ and $c \otimes \mathcal{I} \subseteq \mathcal{A}$ and we see that $b, c \in \mathcal{A} \div \mathfrak{F}$. Finally, we consider a distinguished triangle

$$a \to b \to c \to \mathrm{T}a,$$

where two out of a, b, c lie in $\mathcal{A} \div \mathfrak{F}$. For the sake of definiteness, suppose that $a, b \in \mathcal{A} \div \mathfrak{F}$. We now choose ideals $\mathcal{I}_1, \mathcal{I}_2 \in \mathfrak{F}$ such that $a \otimes \mathcal{I}_1, b \otimes \mathcal{I}_2 \subseteq \mathcal{A}$. Then, it is clear that $a \otimes (\mathcal{I}_1 \otimes \mathcal{I}_2) \subseteq a \otimes \mathcal{I}_1 \subseteq \mathcal{A}$ and $b \otimes (\mathcal{I}_1 \otimes \mathcal{I}_2) \subseteq b \otimes \mathcal{I}_2 \subseteq \mathcal{A}$. We choose $y \in \mathcal{I}_1 \otimes \mathcal{I}_2$ and consider the induced triangle

$$a \otimes y \to b \otimes y \to c \otimes y \to \mathrm{T}(a \otimes y). \tag{24}$$

It follows from (24) that $c \otimes y \in \mathcal{A}$ for each $y \in \mathcal{I}_1 \otimes \mathcal{I}_2$. Hence, $c \otimes (\mathcal{I}_1 \otimes \mathcal{I}_2) \subseteq \mathcal{A}$. Since $\mathfrak{F}$ is a monoidal family, we know that $\mathcal{I}_1 \otimes \mathcal{I}_2 \in \mathfrak{F}$ and hence $c \in \mathcal{A} \div \mathfrak{F}$. □

Lemma 4.5 *Let $\mathcal{S}$ be a multiplicatively closed family of objects of $\mathcal{C}$ and let $\mathfrak{F}$ be a monoidal semifilter. Suppose that every non-empty increasing chain of ideals in $\mathfrak{F}^c$ has an upper bound in $\mathfrak{F}^c$. Consider the family $\mathcal{S} \otimes \mathfrak{F}$ of ideals of $\mathcal{C}$ defined by*

$$\mathcal{S} \otimes \mathfrak{F} = \left\{\mathcal{I} \triangleleft \mathcal{C} : \mathcal{I} \supseteq s \otimes \mathcal{I}' \textit{ for some } \mathcal{I}' \in \mathfrak{F},\ s \in \mathcal{S}\right\}. \tag{25}$$

Then, $\mathcal{S} \otimes \mathfrak{F}$ is a monoidal semifilter. Further, every non-empty increasing chain of ideals in the complement $(\mathcal{S} \otimes \mathfrak{F})^c$ of $\mathcal{S} \otimes \mathfrak{F}$ has an upper bound in $(\mathcal{S} \otimes \mathfrak{F})^c$.

Proof From the definition in (25), it is clear that if $\mathcal{I} \subseteq \mathcal{J}$ and $\mathcal{I} \in \mathcal{S} \otimes \mathfrak{F}$, then $\mathcal{J} \in \mathcal{S} \otimes \mathfrak{F}$. Further, let $\mathcal{I}_1, \mathcal{I}_2 \in \mathcal{S} \otimes \mathfrak{F}$ and choose $\mathcal{I}'_1, \mathcal{I}'_2 \in \mathfrak{F}$ as well as $s_1, s_2 \in \mathcal{S}$ such that $s_1 \otimes \mathcal{I}'_1 \subseteq \mathcal{I}_1$, $s_2 \otimes \mathcal{I}'_2 \subseteq \mathcal{I}_2$. Since $\mathcal{S}$ is multiplicatively closed, we have $s = s_1 \otimes s_2 \in \mathcal{S}$. Then

$$s \otimes (\mathcal{I}'_1 \otimes \mathcal{I}'_2) = (s_1 \otimes \mathcal{I}'_1) \otimes (s_2 \otimes \mathcal{I}'_2) \subseteq \mathcal{I}_1 \otimes \mathcal{I}_2. \tag{26}$$

Since $\mathfrak{F}$ is a monoidal semifilter, we know that $\mathcal{I}_1'\otimes\mathcal{I}_2' \in \mathfrak{F}$ and it now follows from (26) that $\mathcal{I}_1\otimes\mathcal{I}_2 \in \mathcal{S}\otimes\mathfrak{F}$.

Finally, let $\{\mathcal{J}_n\}_{n\in N}$ be an increasing chain of ideals in $(\mathcal{S}\otimes\mathfrak{F})^c$ indexed by a totally ordered set $(N, \leqslant)$ and consider $\mathcal{J} = \bigcup_{n\in N}\mathcal{J}_n$. Suppose that $\mathcal{J} \notin (\mathcal{S}\otimes\mathfrak{F})^c$, i.e., there exists $s \in \mathcal{S}$ and $\mathcal{I}' \in \mathfrak{F}$ such that $\mathcal{J} \supseteq s\otimes\mathcal{I}'$. We now consider

$$(\mathcal{J}_n{:}s) = \{a \in \mathcal{C} : s\otimes a \in \mathcal{J}_n\}, \quad n \in N, \qquad (\mathcal{J}{:}s) = \{a \in \mathcal{C} : s\otimes a \in \mathcal{J}\}$$

as in (12). We know that each $(\mathcal{J}_n{:}s)$ and $(\mathcal{J}{:}s)$ are thick tensor ideals. Further, it is also clear that

$$(\mathcal{J}{:}s) = \bigcup_{n\in N}(\mathcal{J}_n{:}s).$$

We claim that each ideal $(\mathcal{J}_n{:}s) \in \mathfrak{F}^c$. Indeed if $(\mathcal{J}_n{:}s) \in \mathfrak{F}$ for some $n \in N$, we have $s\otimes(\mathcal{J}_n{:}s) \subseteq \mathcal{J}_n$ and it follows from the definition in (25) that $\mathcal{J}_n \in \mathcal{S}\otimes\mathfrak{F}$, which is a contradiction. Accordingly, $\{(\mathcal{J}_n{:}s)\}_{n\in N}$ is an increasing chain of ideals in $\mathfrak{F}^c$ and therefore we can choose an upper bound $\mathcal{K} \in \mathfrak{F}^c$ for $\{(\mathcal{J}_n{:}s)\}_{n\in N}$. Then, we have

$$(\mathcal{J}{:}s) = \bigcup_{n\in N}(\mathcal{J}_n{:}s) \subseteq \mathcal{K}. \tag{27}$$

Since $\mathfrak{F}$ is a semifilter and $\mathcal{K} \notin \mathfrak{F}$, it follows from (27) that $(\mathcal{J}{:}s) \notin \mathfrak{F}$. We now recall that $\mathcal{I}' \in \mathfrak{F}$ is such that $s\otimes\mathcal{I}' \subseteq \mathcal{J}$ from which it follows that $\mathcal{I}' \subseteq (\mathcal{J}{:}s)$. Since $\mathfrak{F}$ is a semifilter, it now follows that $(\mathcal{J}{:}s) \in \mathfrak{F}$, which is a contradiction. Hence, we must have $\mathcal{J} \in (\mathcal{S}\otimes\mathfrak{F})^c$ and thus $\mathcal{J}$ becomes an upper bound for the system $\{\mathcal{J}_n\}_{n\in N}$ in $(\mathcal{S}\otimes\mathfrak{F})^c$. □

Proposition 4.6 (a) *Let* $(\mathcal{A}, \mathfrak{F})$ *be a pair and let* $\mathcal{Q}$ *be any prime ideal in* $(\mathcal{C}, \otimes, 1)$. *Then,* $\mathcal{Q}$ *contains a prime ideal realizing the pair* $(\mathcal{A}, \mathfrak{F})$ *if and only if* $\mathcal{A}\div\mathfrak{F} \subseteq \mathcal{Q}$.

(b) *Let* $(\mathcal{A}, \mathfrak{F})$ *be a pair such that* $\mathrm{Spec}(\mathcal{A}, \mathfrak{F})$ *is finite. Then, a prime ideal* $\mathcal{P}$ *is contained in a prime ideal realizing* $(\mathcal{A}, \mathfrak{F})$ *if and only if* $\mathcal{P}$ *is disjoint from* $\mathcal{M}(\mathcal{A}, \mathfrak{F})$.

Proof (a) Let $\mathcal{P}$ be a prime ideal realizing the pair $(\mathcal{A}, \mathfrak{F})$. Choose any $a \in \mathcal{A}\div\mathfrak{F}$ and some $\mathcal{I} \in \mathfrak{F}$ such that $a\otimes\mathcal{I} \subseteq \mathcal{A}$. Then, $a\otimes\mathcal{I} \subseteq \mathcal{A} \subseteq \mathcal{P}$. Suppose that $a \notin \mathcal{P}$. Then, for any $x \in \mathcal{I}$, we have $a\otimes x \in a\otimes\mathcal{I} \subseteq \mathcal{P}$ and hence $x \in \mathcal{P}$. Hence, $\mathcal{I} \subseteq \mathcal{P}$. Since $\mathfrak{F}$ is a semifilter and $\mathcal{I} \in \mathfrak{F}$, it now follows that $\mathcal{P} \in \mathfrak{F}$ which is a contradiction. Hence, $\mathcal{A}\div\mathfrak{F}$ is contained in $\mathcal{P}$ and hence in any prime ideal $\mathcal{Q}$ containing $\mathcal{P}$.

Conversely, suppose that $\mathcal{A}\div\mathfrak{F} \subseteq \mathcal{Q}$. We claim that $\mathcal{A} \notin (\mathcal{C} - \mathcal{Q})\otimes\mathfrak{F}$. Otherwise, there exists $\mathcal{I} \in \mathfrak{F}$ and $s \notin \mathcal{Q}$ such that $s\otimes\mathcal{I} \subseteq \mathcal{A}$, i.e., $s \in \mathcal{A}\div\mathfrak{F} \subseteq \mathcal{Q}$ which is a contradiction. Further, from Lemma 4.5, we know that any increasing chain of ideals in $((\mathcal{C} - \mathcal{Q})\otimes\mathfrak{F})^c$ has an upper bound in $((\mathcal{C} - \mathcal{Q})\otimes\mathfrak{F})^c$. Accordingly, we choose a prime ideal $\mathcal{P}$ realizing the pair $(\mathcal{A}, (\mathcal{C} - \mathcal{Q})\otimes\mathfrak{F})$. Now suppose that there exists some $x \in \mathcal{P} \cap (\mathcal{C} - \mathcal{Q})$ and take any $\mathcal{I}' \in \mathfrak{F}$. Then, $x\otimes\mathcal{I}' \subseteq \mathcal{P}$ and hence $\mathcal{P} \in (\mathcal{C} - \mathcal{Q})\otimes\mathfrak{F}$, which is a contradiction. Hence, $\mathcal{P} \subseteq \mathcal{Q}$.

(b) Suppose that we have prime ideals $\mathcal{P} \subseteq \mathcal{Q}$ such that $\mathcal{Q}$ realizes $(\mathcal{A}, \mathfrak{F})$. Then, $\mathcal{P} \subseteq \bigcup_{\mathcal{Q}\in\mathrm{Spec}(\mathcal{A},\mathfrak{F})}\mathcal{Q}$ and hence $\mathcal{P} \cap \mathcal{M}(\mathcal{A}, \mathfrak{F}) = \varnothing$. On the other hand, suppose that $\mathcal{P} \cap \mathcal{M}(\mathcal{A}, \mathfrak{F}) = \varnothing$, i.e., $\mathcal{P} \subseteq \bigcup_{\mathcal{Q}\in\mathrm{Spec}(\mathcal{A},\mathfrak{F})}\mathcal{Q}$. Since $\mathrm{Spec}(\mathcal{A}, \mathfrak{F})$ is finite, it follows from Proposition 2.2 that $\mathcal{P} \subseteq \mathcal{Q}$ for some $\mathcal{Q} \in \mathrm{Spec}(\mathcal{A}, \mathfrak{F})$. □

Definition 4.7 Let $(\mathcal{A}, \mathfrak{F})$ and $(\mathcal{A}', \mathfrak{F}')$ be two pairs as in Definition 4.1. Then, we will say that $(\mathcal{A}, \mathfrak{F}) \preccurlyeq (\mathcal{A}', \mathfrak{F}')$ if every prime ideal $\mathcal{P}'$ realizing the pair $(\mathcal{A}', \mathfrak{F}')$ contains a prime ideal $\mathcal{P}$ that realizes $(\mathcal{A}, \mathfrak{F})$.

Proposition 4.8 *Let $(\mathcal{A}, \mathfrak{F})$ and $(\mathcal{A}', \mathfrak{F}')$ be two pairs. Then, the following are equivalent:*

(a) *We have $(\mathcal{A}, \mathfrak{F}) \preccurlyeq (\mathcal{A}', \mathfrak{F}')$, i.e., any prime ideal that realizes $(\mathcal{A}', \mathfrak{F}')$ contains a prime ideal that realizes $(\mathcal{A}, \mathfrak{F})$.*
(b) *The radical of $\mathcal{A} \div \mathfrak{F}$ is contained in the radical of $\mathcal{A}' \div \mathfrak{F}'$, i.e., $r(\mathcal{A} \div \mathfrak{F}) \subseteq r(\mathcal{A}' \div \mathfrak{F}')$.*

Proof (b)$\Rightarrow$(a). Suppose that $r(\mathcal{A} \div \mathfrak{F}) \subseteq r(\mathcal{A}' \div \mathfrak{F}')$ and let $\mathcal{P}'$ be a prime ideal realizing $(\mathcal{A}', \mathfrak{F}')$. Then, we see that

$$\mathcal{A} \div \mathfrak{F} \subseteq r(\mathcal{A} \div \mathfrak{F}) \subseteq r(\mathcal{A}' \div \mathfrak{F}') \subseteq \mathcal{P}'$$

and it follows from Proposition 4.6 that $\mathcal{P}'$ contains a prime ideal realizing the pair $(\mathcal{A}, \mathfrak{F})$.

(a)$\Rightarrow$(b). Consider any prime ideal $\mathcal{P}'$ such that $\mathcal{A}' \div \mathfrak{F}' \subseteq \mathcal{P}'$ (if there is no such prime ideal $\mathcal{P}'$, then $\mathcal{A}' \div \mathfrak{F}' = \mathcal{C}$ and we are done). Then, from Proposition 4.6, we know that $\mathcal{P}'$ contains a prime ideal $\mathcal{P}''$ realizing $(\mathcal{A}', \mathfrak{F}')$. By assumption, there exists a prime ideal $\mathcal{P} \subseteq \mathcal{P}''$ such that $\mathcal{P}$ realizes $(\mathcal{A}, \mathfrak{F})$. Hence, $\mathcal{A} \div \mathfrak{F} \subseteq \mathcal{P} \subseteq \mathcal{P}'' \subseteq \mathcal{P}'$ and therefore $r(\mathcal{A} \div \mathfrak{F}) \subseteq \mathcal{P}'$ for any prime ideal $\mathcal{P}'$ containing $\mathcal{A}' \div \mathfrak{F}'$. It now follows that $r(\mathcal{A} \div \mathfrak{F}) \subseteq r(\mathcal{A}' \div \mathfrak{F}')$. □

For the rest of this section, we will say that two templates indexed by the same partially ordered set are *equivalent* if they have the same realizations. As with pairs in Sect. 2, we will now show how to construct realizations of a finite chain template $T = \{(\mathcal{A}_i, \mathfrak{F}_i)\}_{1 \leqslant i \leqslant n}$ by showing that it is equivalent to a template $\mathrm{D}(T) = \{(\mathcal{B}_i, \mathfrak{F}_i)\}_{1 \leqslant i \leqslant n}$ satisfying the additional condition that

$$(\mathcal{B}_n, \mathfrak{F}_n) \preccurlyeq (\mathcal{B}_{n-1}, \mathfrak{F}_{n-1}) \preccurlyeq \cdots \preccurlyeq (\mathcal{B}_2, \mathfrak{F}_2) \preccurlyeq (\mathcal{B}_1, \mathfrak{F}_1). \tag{28}$$

From Proposition 4.8, it is clear how one can construct a realization of a template of the form (28) starting with a realization $\mathcal{P}_1$ of $(\mathcal{B}_1, \mathfrak{F}_1)$.

Proposition 4.9 *Let $n \geqslant 1$ and let $T = \{(\mathcal{A}_i, \mathfrak{F}_i)\}_{i \in I^{\mathrm{op}}}$ be a finite chain template indexed by the opposite I^{op} of the ordered set $I = \{1 < 2 < \cdots < n\}$. We define $\{\mathcal{B}_i\}_{1 \leqslant i \leqslant n}$ inductively by letting $\mathcal{B}_n = \mathcal{A}_n$ and setting*

$$\mathcal{B}_i = \mathcal{A}_i + (\mathcal{B}_{i+1} \div \mathfrak{F}_{i+1}) \quad \text{for all} \quad n > i \geqslant 1. \tag{29}$$

Then, we have:

(a) *A chain $\mathcal{P}_n \subseteq \cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1$ is a realization of the template $T = \{(\mathcal{A}_i, \mathfrak{F}_i)\}_{i \in I^{\mathrm{op}}}$ if and only if it is also a realization of the template $\mathrm{D}(T) = \{(\mathcal{B}_i, \mathfrak{F}_i)\}_{i \in I^{\mathrm{op}}}$.*

(b) *The template* $T = \{(\mathcal{A}_i, \mathfrak{F}_i)\}_{i \in I^{op}}$ *has a realization if and only if* $\mathcal{B}_1 \notin \mathfrak{F}_1$*, i.e.,*

$$(\mathcal{A}_1 + ((\mathcal{A}_2 + ((\cdots(\mathcal{A}_{n-1} + (\mathcal{A}_n \div \mathfrak{F}_n)) \div \mathfrak{F}_{n-1})) \cdots \div \mathfrak{F}_3) \div \mathfrak{F}_2)) \notin \mathfrak{F}_1.$$

Proof (a) Suppose that $\mathcal{P}_n \subseteq \cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1$ is a realization of the template $\mathrm{D}(T) = \{(\mathcal{B}_i, \mathfrak{F}_i)\}_{i \in I^{op}}$. From (29), it is clear that each $\mathcal{A}_i \subseteq \mathcal{B}_i$ and hence $\mathcal{P}_i$ realizes the pair $(\mathcal{A}_i, \mathfrak{F}_i)$. Hence, $\mathcal{P}_n \subseteq \cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1$ is a realization of T.

Conversely, suppose that $\mathcal{P}_n \subseteq \cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1$ is a realization of T. By definition, we know that $\mathcal{B}_n = \mathcal{A}_n$ and hence $\mathcal{B}_n \subseteq \mathcal{P}_n$. Suppose that $\mathcal{B}_i \subseteq \mathcal{P}_i$ for all $n \geqslant i > j$ for some fixed j. Since $\mathcal{P}_j \supseteq \mathcal{P}_{j+1}$ and $\mathcal{P}_{j+1}$ realizes $(\mathcal{B}_{j+1}, \mathfrak{F}_{j+1})$, it follows from Proposition 4.6(a) that $\mathcal{P}_j \supseteq \mathcal{B}_{j+1} \div \mathfrak{F}_{j+1}$. Further, $\mathcal{A}_j \subseteq \mathcal{P}_j$ because $\mathcal{P}_j$ realizes $(\mathcal{A}_j, \mathfrak{F}_j)$. Hence, $\mathcal{A}_j + (\mathcal{B}_{j+1} \div \mathfrak{F}_{j+1}) = \mathcal{B}_j \subseteq \mathcal{P}_j$ and $\mathcal{P}_j$ realizes the pair $(\mathcal{B}_j, \mathfrak{F}_j)$. Hence, $\mathcal{P}_n \subseteq \cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1$ becomes a realization of $\mathrm{D}(T)$.

(b) We claim that $(\mathcal{B}_{i+1}, \mathfrak{F}_{i+1}) \preccurlyeq (\mathcal{B}_i, \mathfrak{F}_i)$ for each $1 \leqslant i < n$. For this, we note that

$$\mathcal{B}_i \div \mathfrak{F}_i = (\mathcal{A}_i + (\mathcal{B}_{i+1} \div \mathfrak{F}_{i+1})) \div \mathfrak{F}_i \supseteq \mathcal{B}_{i+1} \div \mathfrak{F}_{i+1}.$$

Hence, $r(\mathcal{B}_{i+1} \div \mathfrak{F}_{i+1}) \subseteq r(\mathcal{B}_i \div \mathfrak{F}_i)$ and it follows from Proposition 4.8 that $(\mathcal{B}_{i+1}, \mathfrak{F}_{i+1}) \preccurlyeq (\mathcal{B}_i, \mathfrak{F}_i)$. Now suppose that $\mathcal{B}_1 \notin \mathfrak{F}_1$. Then, using Proposition 4.2, we can obtain a prime ideal $\mathcal{P}_1$ realizing $(\mathcal{B}_1, \mathfrak{F}_1)$. Since the template $\mathrm{D}(T) = \{(\mathcal{B}_i, \mathfrak{F}_i)\}_{i \in I^{op}}$, satisfies condition (28), we can now obtain a realization $\mathcal{P}_n \subseteq \cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1$ of $\mathrm{D}(T)$ starting from $\mathcal{P}_1$.

Conversely, suppose that the template $T = \{(\mathcal{A}_i, \mathfrak{F}_i)\}_{i \in I^{op}}$ is realizable. From part (a), it follows that the template $\mathrm{D}(T) = \{(\mathcal{B}_i, \mathfrak{F}_i)\}_{i \in I^{op}}$ is also realizable and in particular this means that the pair $(\mathcal{B}_1, \mathfrak{F}_1)$ is realizable. Hence, we must have $\mathcal{B}_1 \notin \mathfrak{F}_1$. □

Let $T = \{(\mathcal{A}_i, \mathfrak{F}_i)\}_{i \in I^{op}}$ be a template as above. We will now show that under certain finiteness conditions, we may construct a template $\mathcal{D}(T) = \{(\mathcal{B}_i, \mathfrak{G}_i)\}_{i \in I^{op}}$ equivalent to T such that if we start with an arbitrary realization $\mathcal{Q}_j$ of some pair $(\mathcal{B}_j, \mathfrak{G}_j)$, we can expand it in both directions to form a realization $\mathcal{Q}_n \subseteq \cdots \subseteq \mathcal{Q}_j \subseteq \cdots \subseteq \mathcal{Q}_1$ of $\mathcal{D}(T)$. We notice that since $\mathcal{D}(T)$ is equivalent to T, the latter becomes a realization of T.

Proposition 4.10 *Let* $n \geqslant 1$ *and let* $T = \{(\mathcal{A}_i, \mathfrak{F}_i)\}_{i \in I^{op}}$ *be a finite chain template indexed by the opposite* I^{op} *of the ordered set* $I = \{1 < 2 < \cdots < n\}$*. Suppose that for each* $1 \leqslant i \leqslant n$, $\mathrm{Spec}(\mathcal{A}_i, \mathfrak{F}_i)$ *is a finite set. We define* $\{\mathcal{B}_i\}_{1 \leqslant i \leqslant n}$ *inductively by letting* $\mathcal{B}_n = \mathcal{A}_n$ *and setting*

$$\mathcal{B}_i = \mathcal{A}_i + (\mathcal{B}_{i+1} \div \mathfrak{F}_{i+1}) \quad \textit{for all} \quad n > i \geqslant 1. \tag{30}$$

On the other hand, we define $\{\mathfrak{G}_i\}_{1 \leqslant i \leqslant n}$ *inductively by letting* $\mathfrak{G}_1 = \mathfrak{F}_1$ *and setting* $\mathfrak{G}_{i+1}$ *to be*

$$\mathfrak{G}_{i+1} = \mathcal{M}(\mathcal{A}_i, \mathfrak{G}_i) \otimes \mathfrak{F}_{i+1} \quad \textit{for all} \quad 1 \leqslant i \leqslant n-1. \tag{31}$$

Then, we have:

(a) *The templates* T *and* $\mathcal{D}(T)$ *are equivalent.*

(b) *Choose any integer* $j \in \{1, 2, \dots, n\}$. *Then, the template* $T = \{(\mathcal{A}_i, \mathfrak{F}_i)\}_{i \in I^{op}}$ *has a realization if and only if* $\mathcal{B}_j \notin \mathfrak{G}_j$.

Proof (a) Suppose that $\mathcal{Q}_n \subseteq \cdots \subseteq \mathcal{Q}_2 \subseteq \mathcal{Q}_1$ is a realization of $\mathcal{D}(T)$. From (30) and (31), it is clear that each $\mathcal{A}_i \subseteq \mathcal{B}_i$ and $\mathfrak{F}_i \subseteq \mathfrak{G}_i$. Since each $\mathcal{Q}_i$ realizes the pair $(\mathcal{B}_i, \mathfrak{G}_i)$, we see that it also realizes $(\mathcal{A}_i, \mathfrak{F}_i)$.

Conversely, let $\mathcal{Q}_n \subseteq \cdots \subseteq \mathcal{Q}_2 \subseteq \mathcal{Q}_1$ be a realization of T. From the proof of Proposition 4.9 (a), we see that each $\mathcal{B}_i \subseteq \mathcal{Q}_i$. By definition, we know that $\mathfrak{G}_1 = \mathfrak{F}_1$ and hence $\mathcal{Q}_1 \notin \mathfrak{G}_1$. To proceed by induction, we now suppose that $\mathcal{Q}_i \notin \mathfrak{G}_i$ for all $1 \leqslant i \leqslant j$ for some fixed j. Then, since $\mathcal{Q}_{j+1} \subseteq \mathcal{Q}_j$ and $\mathcal{Q}_j \notin \mathfrak{G}_j$, we have

$$\mathcal{Q}_{j+1} \cap \mathcal{M}(\mathcal{A}_j, \mathfrak{G}_j) \subseteq \mathcal{Q}_j \cap \mathcal{M}(\mathcal{A}_j, \mathfrak{G}_j) = \varnothing. \tag{32}$$

Now if $\mathcal{Q}_{j+1} \in \mathfrak{G}_{j+1} = \mathcal{M}(\mathcal{A}_j, \mathfrak{G}_j) \otimes \mathfrak{F}_{j+1}$, it follows from (25) that there exists $s \in \mathcal{M}(\mathcal{A}_j, \mathfrak{G}_j)$ and $\mathcal{J}' \in \mathfrak{F}_{j+1}$ such that $\mathcal{Q}_{j+1} \supseteq s \otimes \mathcal{J}'$. Since $\mathcal{Q}_{j+1}$ is a prime ideal, it now follows from (32) that $\mathcal{Q}_{j+1} \supseteq \mathcal{J}'$. But, $\mathfrak{F}_{j+1}$ being a semifilter, $\mathcal{Q}_{j+1} \supseteq \mathcal{J}' \in \mathfrak{F}_{j+1}$ implies that $\mathcal{Q}_{j+1} \in \mathfrak{F}_{j+1}$, which is a contradiction. Hence, $\mathcal{Q}_{j+1} \notin \mathfrak{G}_{j+1}$ and $\mathcal{Q}_{j+1}$ realizes the pair $(\mathcal{B}_{j+1}, \mathfrak{G}_{j+1})$.

(b) We choose some $j \in \{1, 2, \dots, n\}$. From part (a), we know that if T has a realization, so does $\mathcal{D}(T)$. Hence, the pair $(\mathcal{B}_j, \mathfrak{G}_j)$ can be realized and we must have $\mathcal{B}_j \notin \mathfrak{G}_j$. Conversely, suppose that $\mathcal{B}_j \notin \mathfrak{G}_j$ and choose some prime ideal $\mathcal{Q}_j$ realizing $(\mathcal{B}_j, \mathfrak{G}_j)$. Then, we have

$$\mathcal{Q}_j \supseteq \mathcal{B}_j = \mathcal{A}_j + (\mathcal{B}_{j+1} \div \mathfrak{F}_{j+1}) \supseteq \mathcal{B}_{j+1} \div \mathfrak{F}_{j+1}$$

and it follows from Proposition 4.6 (a) that there exists a prime ideal $\mathcal{Q}_{j+1} \subseteq \mathcal{Q}_j$ realizing $(\mathcal{B}_{j+1}, \mathfrak{F}_{j+1})$. Further, as in part (a), we see that $\mathcal{Q}_{j+1} \cap \mathcal{M}(\mathcal{A}_j, \mathfrak{G}_j) = \varnothing$ and hence $\mathcal{Q}_{j+1} \notin \mathcal{M}(\mathcal{A}_j, \mathfrak{G}_j) \otimes \mathfrak{F}_{j+1} = \mathfrak{G}_{j+1}$, i.e., $\mathcal{Q}_{j+1}$ realizes $(\mathcal{B}_{j+1}, \mathfrak{G}_{j+1})$. On the other hand, if $j > 1$, we have from (31)

$$\mathfrak{G}_j = \mathcal{M}(\mathcal{A}_{j-1}, \mathfrak{G}_{j-1}) \otimes \mathfrak{F}_j. \tag{33}$$

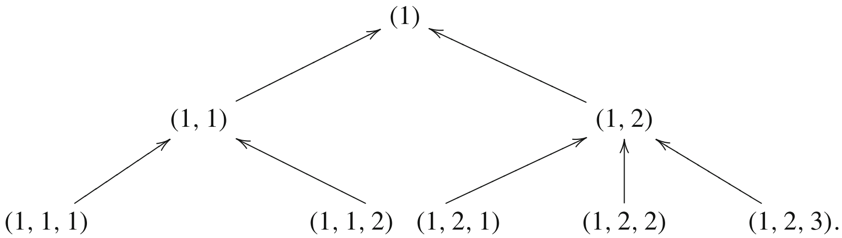
Since $\mathfrak{F}_{j-1} \subseteq \mathfrak{G}_{j-1}$, it follows that $\mathrm{Spec}(\mathcal{A}_{j-1}, \mathfrak{G}_{j-1}) \subseteq \mathrm{Spec}(\mathcal{A}_{j-1}, \mathfrak{F}_{j-1})$ must be finite. From (33), we see that $\mathcal{Q}_j \cap \mathcal{M}(\mathcal{A}_{j-1}, \mathfrak{G}_{j-1}) = \varnothing$ and it follows from Proposition 4.6 (b) that there exists a prime $\mathcal{Q}_{j-1} \supseteq \mathcal{Q}_j$ realizing $(\mathcal{A}_{j-1}, \mathfrak{G}_{j-1})$. Again, since $\mathcal{Q}_j$ realizes $(\mathcal{B}_j, \mathfrak{F}_j)$, Proposition 4.6 (a) implies that $\mathcal{Q}_{j-1} \supseteq \mathcal{Q}_j \supseteq \mathcal{B}_j \div \mathfrak{F}_j$. Hence,

$$\mathcal{Q}_{j-1} \supseteq \mathcal{A}_{j-1} + (\mathcal{B}_j \div \mathfrak{F}_j) = \mathcal{B}_{j-1}$$

and $\mathcal{Q}_{j-1}$ realizes $(\mathcal{B}_{j-1}, \mathfrak{G}_{j-1})$. Accordingly, starting from $\mathcal{Q}_j$ we can proceed in both directions to give a realization $\mathcal{Q}_n \subseteq \cdots \subseteq \mathcal{Q}_2 \subseteq \mathcal{Q}_1$ of the template $\mathcal{D}(T)$ (and hence of T). □

Remark 4.11 In the proof above, we note that the finiteness condition is only used in part (b), i.e., the templates T and $\mathcal{D}(T)$ are always equivalent.

For the final result of this section, we will consider finite descending trees. Given a finite descending tree $\mathbb{T}$, a node at the k-th level of $\mathbb{T}$ will be denoted by a multi-index $I = (i_1, \ldots, i_k)$ with k-coordinates. If this node I has $n(I)$ branches, these branch nodes will be denoted by multi-indices $(I, l) = (i_1, \ldots, i_k, l)$, with l varying from 1 to $n(I)$. The root node appearing at the top will be denoted by (1). Then, if there are $n(1)$ branches leading out of the root node, these branches will be labelled as (1, 1), (1, 2), ..., (1, $n(1)$). The following diagram illustrates our labeling scheme for the nodes of such a tree $\mathbb{T}$.



We make $\mathbb{T}$ partially ordered by setting the branch nodes $(I, i) \leqslant I$, $1 \leqslant i \leqslant n(I)$, for each node I and consider templates of pairs indexed by $\mathbb{T}$.

Proposition 4.12 *Let $\{(\mathcal{A}_I, \mathfrak{F}_I)\}_{I \in \mathbb{T}}$ be a template indexed by a finite descending tree $\mathbb{T}$. For each node I of the tree that has no further branches, we set $\mathcal{B}_I = \mathcal{A}_I$. Then, we define $\mathcal{B}_I$ for each multi-index $I \in \mathbb{T}$ inductively by setting*

$$\mathcal{B}_I = \mathcal{A}_I + \sum_{i=1}^{n(I)} (\mathcal{B}_{(I,i)} \div \mathfrak{F}_{(I,i)}), \tag{34}$$

where the sum is taken over all nodes $(I, 1), (I, 2), \ldots, (I, n(I))$ immediately below the node I. Then, the template $\{(\mathcal{A}_I, \mathfrak{F}_I)\}_{I \in \mathbb{T}}$ has a realization if and only if $\mathcal{B}_{(1)} \notin \mathfrak{F}_{(1)}$.

Proof "If part": we claim that starting with any prime ideal realizing $(\mathcal{B}_{(1)}, \mathfrak{F}_{(1)})$, we can obtain a realization of the entire template $\{(\mathcal{A}_I, \mathfrak{F}_I)\}_{I \in \mathbb{T}}$. We prove this by induction on $|\mathbb{T}|$, the number of nodes in the tree $\mathbb{T}$. This is obvious if $|\mathbb{T}| = 1$ and we assume that it holds for all trees with fewer than $|\mathbb{T}|$ nodes. Since $\mathcal{B}_{(1)} \notin \mathfrak{F}_{(1)}$, we can choose a prime $\mathcal{P}_{(1)}$ realizing $(\mathcal{B}_{(1)}, \mathfrak{F}_{(1)})$. From the definition in (34), we know that $\mathcal{A}_{(1)} \subseteq \mathcal{B}_{(1)}$ and each $\mathcal{B}_{(1,i)} \div \mathfrak{F}_{(1,i)} \subseteq \mathcal{B}_{(1)}$ for each node (I, i) immediately below the root node. Hence, $\mathcal{P}_{(1)}$ realizes $(\mathcal{A}_{(1)}, \mathfrak{F}_{(1)})$ and contains prime ideals $\mathcal{P}_{(1,i)}$ realizing $(\mathcal{B}_{(1,i)}, \mathfrak{F}_{(1,i)})$ for each $1 \leqslant i \leqslant n(1)$. Now, $n(1)$ different subtrees $\mathbb{T}_1$, ..., $\mathbb{T}_{n(1)}$ obtained by cutting off the root node all have strictly less than $|\mathbb{T}|$ nodes. By the induction assumption, it follows that starting from each $\mathcal{P}_{(1,i)}$, we may obtain a realization of the subtree $\mathbb{T}_i$. This proves the result.

For the "only if" part, we can reverse our arguments and prove by induction the claim that if $\{\mathcal{P}_I\}_{I \in \mathbb{T}}$ is a realization of $\{(\mathcal{A}_I, \mathfrak{F}_I)\}_{I \in \mathbb{T}}$, $\mathcal{P}_{(1)}$ must be a realization of $(\mathcal{B}_{(1)}, \mathfrak{F}_{(1)})$. Hence, $\mathcal{B}_{(1)} \notin \mathfrak{F}_{(1)}$. □

5 Monoidal families and the Prime Ideal Principle

In this final section of the paper, we shall assume that the tensor triangulated category $(\mathcal{C}, \otimes, 1)$ has the additional property that all thick tensor ideals are radical, i.e., for any thick tensor ideal $\mathcal{I}$ in $(\mathcal{C}, \otimes, 1)$, we have $r(\mathcal{I}) = \mathcal{I}$. This additional assumption is equivalent (see [1, Proposition 4.4]) to the assumption that for any object $a \in \mathcal{C}$, a lies in the ideal generated by the object $a \otimes a$. In fact, it is very frequent for all thick tensor ideals to be radical (see [1, Remark 4.3] and [22, Lemma A.2.6]). In particular, this assumption holds in rigid tensor triangulated categories (see [34, Section 2]). For us, the key consequence of this assumption is the following fact.

Proposition 5.1 (a) *Let* $(\mathcal{C}, \otimes, 1)$ *be a tensor triangulated category such that every thick tensor ideal is a radical ideal. Then, for any thick tensor ideals* $\mathcal{I}$ *and* $\mathcal{J}$*, we have* $\mathcal{I} \otimes \mathcal{J} = \mathcal{I} \cap \mathcal{J}$.

(b) *Let* $\mathfrak{F}$ *be a family of thick tensor ideals such that* $\mathcal{C} \in \mathfrak{F}$*. Then, if* $\mathfrak{F}$ *is monoidal, the family* $\mathfrak{F}$ *is a strongly Oka and a strongly Ako family. In particular,* $\mathfrak{F}$ *satisfies the Prime Ideal Principle.*

Proof (a) It is clear that $\mathcal{I} \otimes \mathcal{J} \subseteq \mathcal{I} \cap \mathcal{J}$. We choose some object $a \in \mathcal{I} \cap \mathcal{J}$. Then, $a \otimes a \in \mathcal{I} \otimes \mathcal{J}$. Since all thick tensor ideals in $(\mathcal{C}, \otimes, 1)$ are radical, it follows that a lies in the ideal generated by $a \otimes a$. We conclude that $a \in \mathcal{I} \otimes \mathcal{J}$ and hence $\mathcal{I} \otimes \mathcal{J} = \mathcal{I} \cap \mathcal{J}$.

(b) We will show that $\mathfrak{F}$ is strongly Oka and strongly Ako by showing that it satisfies condition (Q3) in Theorem 3.5. For this, we consider thick tensor ideals $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$ and some ideal $\mathcal{I}$ such that $\mathcal{A} \otimes \mathcal{B} \subseteq \mathcal{I} \subseteq \mathcal{A} \cap \mathcal{B}$. From part (a), it follows that $\mathcal{A} \otimes \mathcal{B} = \mathcal{I} = \mathcal{A} \cap \mathcal{B}$. By assumption, $\mathfrak{F}$ is monoidal and hence $\mathcal{I} = \mathcal{A} \otimes \mathcal{B} \in \mathfrak{F}$. Hence, the family $\mathfrak{F}$ satisfies condition (Q3) in Theorem 3.5. □

Remark 5.2 Proposition 5.1 (b) may also be proved as follows: all thick tensor ideals being radical, [4, Theorem 14] now gives us a correspondence between closed subspaces of $\mathrm{Spec}(\mathcal{C})^*$ and all ideals in $\mathcal{C}$. As such, if $\mathfrak{F}$ is a monoidal family, then $\mathfrak{F}^* = \{c(\mathcal{I}) : \mathcal{I} \in \mathfrak{F}\}$ is closed under finite unions and hence any closed subspace of $\mathrm{Spec}(\mathcal{C})^*$ that is minimal with respect to not being in $\mathfrak{F}^*$ must be irreducible.

Proposition 5.3 *Let* $(\mathcal{C}, \otimes, 1)$ *be a tensor triangulated category such that every thick tensor ideal is a radical ideal. Then:*

(a) *Let* $\mathcal{I}$ *be a thick tensor ideal that is maximal with respect to being non-principal. Then* $\mathcal{I}$ *is prime.*

(b) *Let* α *be an infinite cardinal and let* $\mathfrak{F}_\alpha$ *denote the family of thick tensor ideals* $\mathcal{I}$ *having a generating set* $G_{\mathcal{I}}$ *of cardinality* $|G_{\mathcal{I}}| \leqslant \alpha$*. Then, any ideal that is maximal with respect to not being in* $\mathfrak{F}_\alpha$ *is prime.*

Proof (a) Using Proposition 5.1 (b), it is enough to show that the collection of principal ideals is monoidal. If we have principal ideals $\mathcal{I} = (x)$ and $\mathcal{J} = (y)$ generated by objects $x, y \in \mathcal{C}$ respectively, it follows from Lemma 3.4 that $\mathcal{I} \otimes \mathcal{J}$ is the principal ideal generated by $x \otimes y$. Hence, the family of principal ideals is monoidal and satisfies the Prime Ideal Principle. The result of part (b) follows similarly. □

Using the same approach as in Proposition 5.3, we can prove the following more general result.

Proposition 5.4 *Let* $(\mathcal{C}, \otimes, 1)$ *be a tensor triangulated category such that every thick tensor ideal is radical. Let* $\mathcal{S}$ *be a multiplicatively closed family of objects of* $\mathcal{C}$ *containing* 1 *and also possibly* 0. *For any infinite cardinal* α, *let* $\mathfrak{G}^{\mathcal{S}}_{\leqslant\alpha}$ *(resp.* $\mathfrak{G}^{\mathcal{S}}_{<\alpha}$*) denote the family of thick tensor ideals* $\mathcal{I}$ *having a generating set* $G_{\mathcal{I}} \subseteq \mathcal{S}$ *such that* $|G_{\mathcal{I}}| \leqslant \alpha$ *(resp.* $|G_{\mathcal{I}}| < \alpha$*). Then, any ideal that is maximal with respect to not being in* $\mathfrak{G}^{\mathcal{S}}_{\leqslant\alpha}$ *(resp.* $\mathfrak{G}^{\mathcal{S}}_{<\alpha}$*) is prime.*

Proof We consider thick tensor ideals $\mathcal{I}, \mathcal{J} \in \mathfrak{G}^{\mathcal{S}}_{\leqslant\alpha}$ with respective generating sets $G_{\mathcal{I}} = \{x_i\}_{i\in I} \subseteq \mathcal{S}$, $G_{\mathcal{J}} = \{y_j\}_{j\in J} \subseteq \mathcal{S}$ of cardinality $\leqslant \alpha$. Since $\mathcal{S}$ is multiplicatively closed, we see that each $x_i \otimes y_j \in \mathcal{S}$. It then follows from Lemma 3.4 that $\mathcal{I}\otimes\mathcal{J}$ may be generated by the set $\{x_i \otimes y_j\}_{i\in I, j\in J}$ of cardinality $|G_{\mathcal{I}} \times G_{\mathcal{J}}| \leqslant \alpha$. Hence, $\mathfrak{G}^{\mathcal{S}}_{\leqslant\alpha}$ is a monoidal family and satisfies the Prime Ideal Principle. The case of $\mathfrak{G}^{\mathcal{S}}_{<\alpha}$ follows similarly. □

Proposition 5.5 *Let* $(\mathcal{C}, \otimes, 1)$ *be a tensor triangulated category such that every thick tensor ideal is radical. Then:*

(a) *Let* $\mathfrak{F}$ *be a monoidal semifilter. Then, the family* $\mathrm{Div}(\mathfrak{F})$ *defined by*

$$\mathrm{Div}(\mathfrak{F}) = \{\mathcal{A} \triangleleft \mathcal{C} : \mathcal{A} \div \mathfrak{F} = \mathcal{A}\}$$

is a strongly Oka family and satisfies the Prime Ideal Principle.

(b) *Let* $\mathcal{S}$ *be a multiplicatively closed family of objects of* $\mathcal{C}$ *such that* $1 \in \mathcal{S}$ *and* $0 \notin \mathcal{S}$. *Then, the family* $\mathrm{Div}(\mathcal{S}) = \{\mathcal{A} \triangleleft \mathcal{C} : \mathcal{A} \div \mathcal{S} = \mathcal{A}\}$ *is a strongly Oka family and satisfies the Prime Ideal Principle.*

Proof (a) From Proposition 5.1, it suffices to show that $\mathrm{Div}(\mathfrak{F})$ is monoidal. Given any thick tensor ideals $\mathcal{A}, \mathcal{B}$, it is clear from the definition in (23) that $(\mathcal{A}\otimes\mathcal{B}) \div \mathfrak{F} = (\mathcal{A}\cap\mathcal{B}) \div \mathfrak{F} \subseteq (\mathcal{A} \div \mathfrak{F}) \cap (\mathcal{B} \div \mathfrak{F}) = (\mathcal{A} \div \mathfrak{F}) \otimes (\mathcal{B} \div \mathfrak{F})$. Conversely, choose an object $x \in (\mathcal{A} \div \mathfrak{F}) \cap (\mathcal{B} \div \mathfrak{F}) = (\mathcal{A} \div \mathfrak{F}) \otimes (\mathcal{B} \div \mathfrak{F})$. Then, there exist $\mathcal{I}, \mathcal{J} \in \mathfrak{F}$ such that $x \otimes \mathcal{I} \subseteq \mathcal{A}$ and $x \otimes \mathcal{J} \subseteq \mathcal{B}$. It follows that $(x \otimes x) \otimes (\mathcal{I} \otimes \mathcal{J}) \subseteq \mathcal{A} \otimes \mathcal{B}$. Since $\mathfrak{F}$ is monoidal, $\mathcal{I} \otimes \mathcal{J} \in \mathfrak{F}$ and hence $(x \otimes x) \in (\mathcal{A}\otimes\mathcal{B}) \div \mathfrak{F}$. Since all ideals in $(\mathcal{C}, \otimes, 1)$ are radical, it now follows that $x \in (\mathcal{A}\otimes\mathcal{B}) \div \mathfrak{F}$. Hence,

$$(\mathcal{A} \div \mathfrak{F}) \otimes (\mathcal{B} \div \mathfrak{F}) = (\mathcal{A}\otimes\mathcal{B}) \div \mathfrak{F}. \tag{35}$$

In particular, if $\mathcal{A}, \mathcal{B} \in \mathrm{Div}(\mathfrak{F})$, (35) reduces to $\mathcal{A}\otimes\mathcal{B} = (\mathcal{A}\otimes\mathcal{B}) \div \mathfrak{F}$. Hence, $\mathcal{A}\otimes\mathcal{B} \in \mathrm{Div}(\mathfrak{F})$ and $\mathrm{Div}(\mathfrak{F})$ is monoidal. Finally, part (b) follows from part (a) by setting $\mathfrak{F} = \mathfrak{F}_{\mathcal{S}}$ where $\mathfrak{F}_{\mathcal{S}} = \{\mathcal{I} \triangleleft \mathcal{C} : \mathcal{I} \cap \mathcal{S} \neq \varnothing\}$. □

Proposition 5.6 *Let* $(\mathcal{C}, \otimes, 1)$ *be a tensor triangulated category such that every thick tensor ideal is a radical ideal. Let* $\mathcal{I}$ *be a thick tensor ideal. Then:*

(a) *The family* $\mathfrak{F}_{\mathcal{I}}$ *of thick tensor ideals containing* $\mathcal{I}$ *is a monoidal semifilter. In particular, any ideal of* $(\mathcal{C}, \otimes, 1)$ *that is maximal with respect to not containing* $\mathcal{I}$ *must be prime.*

(b) *Suppose that $\mathcal{I}$ is a finitely generated (hence principal) ideal. Then, any non-empty increasing chain of ideals in the complement $\mathfrak{F}_{\mathcal{I}}^{c}$ of $\mathfrak{F}_{\mathcal{I}}$ has an upper bound in $\mathfrak{F}_{\mathcal{I}}^{c}$.*

Proof (a) It is clear that $\mathfrak{F}_{\mathcal{I}}$ is a semifilter. Further, if $\mathcal{I} \subseteq \mathcal{A}$ and $\mathcal{I} \subseteq \mathcal{B}$ for some $\mathcal{A}, \mathcal{B} \in \mathfrak{F}_{\mathcal{I}}$, we have $\mathcal{I} \subseteq \mathcal{A} \cap \mathcal{B} = \mathcal{A} \otimes \mathcal{B}$. Hence, $\mathfrak{F}_{\mathcal{I}}$ is also monoidal.

(b) Suppose that $\mathcal{I}$ is generated by an object x. We consider an increasing chain $\{\mathcal{J}_j\}_{j \in N}$ of ideals in $\mathfrak{F}_{\mathcal{I}}^{c}$ indexed by a totally ordered set $(N, \leqslant)$ and the union $\mathcal{J} = \bigcup_{j \in N} \mathcal{J}_j$. Now, if $\mathcal{J} \in \mathfrak{F}_{\mathcal{I}}$, there exists $n \in N$ large enough such that $x \in \mathcal{J}_n$. Hence, $\mathcal{I} \subseteq \mathcal{J}_n$ and $\mathcal{J}_n \in \mathfrak{F}_{\mathcal{I}}$, which is a contradiction. We conclude that $\mathcal{J} \notin \mathfrak{F}_{\mathcal{I}}$. □

We now consider templates that are infinite descending chains. For this we consider the ordered infinite set $I = \{1 < 2 < \cdots\}$ and a collection of principal thick tensor ideals $\mathcal{I}_i$, $i \geqslant 1$. We consider a template $T = \{(\mathcal{A}_i, \mathfrak{F}_i)\}_{i \in I^{\text{op}}}$ indexed by I^{op}, where each $\mathfrak{F}_i$ is given by

$$\mathfrak{F}_i = \mathfrak{F}_{\mathcal{I}_i} = \{\mathcal{J} \triangleleft \mathcal{C} : \mathcal{J} \supseteq \mathcal{I}_i\} \tag{36}$$

and $\mathcal{A}_i$ is any thick tensor ideal. From Proposition 5.6, we know that each $\mathfrak{F}_i$ is a monoidal semifilter and any non-empty increasing chain of ideals in the complement $\mathfrak{F}_i^{c}$ of $\mathfrak{F}_i$ has an upper bound in $\mathfrak{F}_i^{c}$. For any positive integers $m \leqslant n$, we let $T_n^m = \{(\mathcal{A}_i, \mathfrak{F}_i)\}_{n \geqslant i \geqslant m}$ be the template obtained by truncating T. Then, the truncated template T_n^m is indexed by the opposite of the finite ordered set $\{m < m+1 < \cdots < n\}$. We now set $\mathcal{B}_n^n = \mathcal{A}_n$ and define $\mathcal{B}_n^m$ by inductively setting

$$\mathcal{B}_n^m = \mathcal{A}_m + (\mathcal{B}_n^{m+1} \div \mathfrak{F}_{m+1}) \qquad \text{for all} \quad n > m \geqslant 1. \tag{37}$$

For any given m, it is clear that we have an increasing chain $\mathcal{B}_m^m \subseteq \mathcal{B}_{m+1}^m \subseteq \mathcal{B}_{m+2}^m \subseteq \cdots$ and we set $\mathcal{B}_\infty^m = \bigcup_{n \geqslant m}^{\infty} \mathcal{B}_n^m$. We now let $\mathrm{D}(T) = \{(\mathcal{B}_\infty^i, \mathfrak{F}_i)\}_{i \in I^{\text{op}}}$ be the infinite decreasing chain template indexed by the opposite I^{op} of the ordered infinite set $I = \{1 < 2 < \cdots\}$.

Lemma 5.7 *In the notation above, for any $m \geqslant 1$, we have $(\mathcal{B}_\infty^{m+1}, \mathfrak{F}_{m+1}) \preccurlyeq (\mathcal{B}_\infty^m, \mathfrak{F}_m)$.*

Proof By assumption, each $\mathfrak{F}_i = \mathfrak{F}_{\mathcal{I}_i} = \{\mathcal{J} \triangleleft \mathcal{C} : \mathcal{J} \supseteq \mathcal{I}_i\}$ for the principal ideal $\mathcal{I}_i$. In particular, suppose that $\mathcal{I}_m$ is generated by x_m and consider some object $a \in \mathcal{B}_\infty^m \div \mathfrak{F}_m$. Then, there exists an ideal $\mathcal{J} \in \mathfrak{F}_m$ such that $a \otimes \mathcal{J} \subseteq \mathcal{B}_\infty^m$. Since $\mathcal{J} \supseteq \mathcal{I}_m$, it follows that $a \otimes x_m \in \mathcal{B}_\infty^m$. Accordingly, there exists N large enough so that $a \otimes x_m \in \mathcal{B}_N^m$. In other words, $a \otimes \mathcal{I}_m \subseteq \mathcal{B}_N^m$ and we see that $a \in \mathcal{B}_N^m \div \mathfrak{F}_m$. Therefore, we have $\mathcal{B}_\infty^m \div \mathfrak{F}_m \subseteq \bigcup_{n \geqslant m}^{\infty} (\mathcal{B}_n^m \div \mathfrak{F}_m)$. On the other hand, it is clear that $\mathcal{B}_\infty^m \div \mathfrak{F}_m \supseteq \bigcup_{n \geqslant m}^{\infty} (\mathcal{B}_n^m \div \mathfrak{F}_m)$ and hence $\mathcal{B}_\infty^m \div \mathfrak{F}_m = \bigcup_{n \geqslant m}^{\infty} (\mathcal{B}_n^m \div \mathfrak{F}_m)$. Combining with (37), we have

$$\begin{aligned}
\mathcal{B}_\infty^m \div \mathfrak{F}_m = \bigcup_{n \geqslant m}^{\infty} (\mathcal{B}_n^m \div \mathfrak{F}_m) &\supseteq \bigcup_{n > m}^{\infty} (\mathcal{A}_m + (\mathcal{B}_n^{m+1} \div \mathfrak{F}_{m+1})) \div \mathfrak{F}_m \\
&\supseteq \bigcup_{n \geqslant m+1}^{\infty} (\mathcal{B}_n^{m+1} \div \mathfrak{F}_{m+1}).
\end{aligned}$$

As before, we must have $\mathcal{B}_\infty^{m+1} \div \mathfrak{F}_{m+1} = \bigcup_{n\geqslant m+1}^{\infty}(\mathcal{B}_n^{m+1} \div \mathfrak{F}_{m+1})$ and hence $r(\mathcal{B}_\infty^m \div \mathfrak{F}_m) = \mathcal{B}_\infty^m \div \mathfrak{F}_m \supseteq \mathcal{B}_\infty^{m+1} \div \mathfrak{F}_{m+1} = r(\mathcal{B}_\infty^{m+1} \div \mathfrak{F}_{m+1})$. The result now follows from Proposition 4.8. □

Proposition 5.8 *Let* $(\mathcal{C}, \otimes, 1)$ *be a tensor triangulated category such that all thick tensor ideals are also radical. Let* $T = \{(\mathcal{A}_i, \mathfrak{F}_i)\}_{i\in I^{\mathrm{op}}}$ *be a template as in* (36) *indexed by the opposite* I^{op} *of the ordered infinite set* $I = \{1 < 2 < \cdots\}$ *and let* $\mathrm{D}(T) = \{(\mathcal{B}_\infty^i, \mathfrak{F}_i)\}_{i\in I^{\mathrm{op}}}$ *be as defined above. Then:*

(a) *A chain* $\cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1$ *of thick prime ideals is a realization of the template* T *if and only if it is also a realization of the template* $\mathrm{D}(T)$.
(b) *The template* T *has a realization if and only if* $\mathcal{B}_\infty^1 \notin \mathfrak{F}_1$.
(c) *The template* T *is realizable if and only if each of the truncated templates* T_n^m *is realizable for* $1 \leqslant m \leqslant n$.

Proof (a) The "if part" is clear because each $\mathcal{B}_\infty^m \supseteq \mathcal{A}_m$. For the "only if" part, consider a realization $\cdots \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1$ of the template $T = \{(\mathcal{A}_i, \mathfrak{F}_i)\}_{i\in I^{\mathrm{op}}}$. Then, for any $1 \leqslant m \leqslant n$, $\mathcal{P}_n \subseteq \mathcal{P}_{n-1} \subseteq \cdots \subseteq \mathcal{P}_m$ must be a realization of the truncated template T_n^m. Looking at the expression for $\mathcal{B}_n^m$ in (37), it follows from Proposition 4.9 that $\mathcal{P}_m$ realizes the pair $(\mathcal{B}_n^m, \mathfrak{F}_m)$. Hence, for each $n \geqslant m$, we have $\mathcal{P}_m \supseteq \mathcal{B}_n^m$ and $\mathcal{P}_m \notin \mathfrak{F}_m$. It follows that

$$\mathcal{P}_m \supseteq \bigcup_{n\geqslant m}^{\infty} \mathcal{B}_n^m = \mathcal{B}_\infty^m, \qquad \mathcal{P}_m \notin \mathfrak{F}_m,$$

whence $\mathcal{P}_m$ realizes the pair $(\mathcal{B}_\infty^m, \mathfrak{F}_m)$.

(b) If T is realizable, it follows from part (a) that so is $\mathrm{D}(T) = \{(\mathcal{B}_\infty^i, \mathfrak{F}_i)\}_{i\in I^{\mathrm{op}}}$ and hence in particular $\mathcal{B}_\infty^1 \notin \mathfrak{F}_1$. Conversely, if $\mathcal{B}_\infty^1 \notin \mathfrak{F}_1$, we can choose a prime ideal $\mathcal{P}_1$ realizing $(\mathcal{B}_\infty^1, \mathfrak{F}_1)$. From Lemma 5.7, we know that $(\mathcal{B}_\infty^{m+1}, \mathfrak{F}_{m+1}) \preccurlyeq (\mathcal{B}_\infty^m, \mathfrak{F}_m)$ for each $m \geqslant 1$. It follows that we can choose a prime ideal $\mathcal{P}_2 \subseteq \mathcal{P}_1$ realizing $(\mathcal{B}_\infty^2, \mathfrak{F}_2)$ and so on to obtain a realization of $\mathrm{D}(T) = \{(\mathcal{B}_\infty^i, \mathfrak{F}_i)\}_{i\in I^{\mathrm{op}}}$. This gives a realization of T.

(c) The "only if" part is obvious. For the "if part", we suppose that each truncated template T_n^m is realizable for $1 \leqslant m \leqslant n$. In particular, the truncated template T_n^1 is realizable for each $n \geqslant 1$. Hence, $\mathcal{B}_n^1 \notin \mathfrak{F}_1$ for each $n \geqslant 1$. We know that $\mathcal{B}_\infty^1 = \bigcup_{n\geqslant 1}^{\infty} \mathcal{B}_n^1$. From Proposition 5.6(b), it follows that the increasing chain $\mathcal{B}_1^1 \subseteq \mathcal{B}_2^1 \subseteq \mathcal{B}_3^1 \subseteq \cdots$ of ideals in $\mathfrak{F}_1^c$ must have some upper bound in $\mathfrak{F}_1^c$, say $\mathcal{B}$. But then, $\mathcal{B} \supseteq \mathcal{B}_\infty^1$. Since $\mathfrak{F}_1$ is a semifilter and $\mathcal{B} \notin \mathfrak{F}_1$, we must have $\mathcal{B}_\infty^1 \notin \mathfrak{F}_1$. From part (b), it now follows that the template T is realizable. □

Finally, we will translate our reasoning in terms of irreducible closed subspaces of the inverse topology on $\mathrm{Spec}(\mathcal{C})$. We continue to assume that all thick tensor ideals in $\mathcal{C}$ are radical. As in Sect. 3, we denote by $\mathrm{Spec}(\mathcal{C})^*$ the spectral space $\mathrm{Spec}(\mathcal{C})$ equipped with the inverse topology. As such, following [4, Theorem 14], for every thick tensor ideal $\mathcal{I} \triangleleft \mathcal{C}$, there is a closed subspace

$$c(\mathcal{I}) = \{\mathcal{P} \in \mathrm{Spec}(\mathcal{C}) : \mathcal{P} \supseteq \mathcal{I}\} \subseteq \mathrm{Spec}(\mathcal{C})^* \tag{38}$$

in inverse topology giving a one-to-one order reversing correspondence between thick tensor ideals in $(\mathcal{C}, \otimes, 1)$ and closed subspaces of $Spec(\mathcal{C})^*$. From (38), it is clear that for ideals $\mathcal{I}, \mathcal{J} \triangleleft \mathcal{C}$, we must have $c(\mathcal{I}) \cup c(\mathcal{J}) = c(\mathcal{I} \otimes \mathcal{J})$ and $\bigcap_{i \in I} c(\mathcal{I}_i) = c\big(\sum_{i \in I} \mathcal{I}_i\big)$ for any family $\{\mathcal{I}_i\}_{i \in I}$ of thick tensor ideals.

Proposition 5.9 *Let* $(\mathcal{C}, \otimes, 1)$ *be a tensor triangulated category such that all thick tensor ideals are radical and let* $\{c_i\}_{i \in I}$ *be a basis of closed subspaces for* $Spec(\mathcal{C})^*$. *Let* $\mathfrak{F}^*$ *be a family of closed subspaces of* $Spec(\mathcal{C})^*$ *having the following properties:*

(a) $\varnothing \in \mathfrak{F}^*$.
(b) *For any closed* $K \subseteq Spec(\mathcal{C})^*$ *and any* $i, j \in I$ *such that* $K \cap c_i, K \cap c_j \in \mathfrak{F}^*$, *we must have* $K \cap (c_i \cup c_j) \in \mathfrak{F}^*$.

Then, if $K_0 \subseteq Spec(\mathcal{C})^*$ *is a closed subspace that is minimal with respect to not being in* $\mathfrak{F}^*$, K_0 *is an irreducible closed subspace of* $Spec(\mathcal{C})^*$.

Proof Using the correspondence stated above, we must have ideals $\mathcal{I}_i \triangleleft \mathcal{C}$, $i \in I$, such that $c_i = c(\mathcal{I}_i)$. We now consider the family $\mathfrak{F} = \{\mathcal{I} \triangleleft \mathcal{C} : c(\mathcal{I}) \in \mathfrak{F}^*\}$ of thick tensor ideals. Since $\varnothing \in \mathfrak{F}^*$, we have $\mathcal{C} \in \mathfrak{F}$. Further, condition (b) above corresponds to the following condition on $\mathfrak{F}$: given $\mathcal{I} \triangleleft \mathcal{C}$ and $i, j \in I$

$$\mathcal{I} + \mathcal{I}_i \in \mathfrak{F} \text{ and } \mathcal{I} + \mathcal{I}_j \in \mathfrak{F} \implies (\mathcal{I} + \mathcal{I}_i) \otimes (\mathcal{I} + \mathcal{I}_j) \in \mathfrak{F}. \tag{39}$$

Then, a closed subspace $K_0 \subseteq Spec(\mathcal{C})^*$ that is minimal with respect to not being in $\mathfrak{F}^*$ corresponds to a thick tensor ideal $\mathcal{I}_0$ that is maximal with respect to not being in $\mathfrak{F}$. Now, suppose that there exist $\mathcal{J}, \mathcal{K} \triangleleft \mathcal{C}$ such that $\mathcal{I}_0 \supseteq \mathcal{J} \otimes \mathcal{K}$ but $\mathcal{I}_0 \not\supseteq \mathcal{J}$ and $\mathcal{I}_0 \not\supseteq \mathcal{K}$. Since $\{c(\mathcal{I}_i)\}_{i \in I}$ is a basis for closed subspaces of $Spec(\mathcal{C})^*$, there exist $G_{\mathcal{J}}, G_{\mathcal{K}} \subseteq I$ such that $\mathcal{J} = \sum_{j \in G_{\mathcal{J}}} \mathcal{I}_j$ and $\mathcal{K} = \sum_{k \in G_{\mathcal{K}}} \mathcal{I}_k$. Then, we can choose $j' \in G_{\mathcal{J}}$ and $k' \in G_{\mathcal{K}}$ such that $\mathcal{I}_0 \not\supseteq \mathcal{I}_{j'}$ and $\mathcal{I}_0 \not\supseteq \mathcal{I}_{k'}$. Since $\mathcal{I}_0$ is maximal with respect to not being in $\mathfrak{F}$, we have $\mathcal{I}_0 + \mathcal{I}_{j'} \in \mathfrak{F}$ and $\mathcal{I}_0 + \mathcal{I}_{k'} \in \mathfrak{F}$. From (39), it follows that $(\mathcal{I}_0 + \mathcal{I}_{j'}) \otimes (\mathcal{I}_0 + \mathcal{I}_{k'}) \in \mathfrak{F}$. As in (17), we now have $(\mathcal{I}_0 + \mathcal{I}_{j'}) \otimes (\mathcal{I}_0 + \mathcal{I}_{k'}) \subseteq \mathcal{I}_0 + (\mathcal{I}_{j'} \otimes \mathcal{I}_{k'}) \subseteq \mathcal{I}_0 + (\mathcal{J} \otimes \mathcal{K}) \subseteq \mathcal{I}_0$. On the other hand, we have $\mathcal{I}_0 = \mathcal{I}_0 \otimes \mathcal{I}_0 \subseteq (\mathcal{I}_0 + \mathcal{I}_{j'}) \otimes (\mathcal{I}_0 + \mathcal{I}_{k'})$ and hence $\mathcal{I}_0 = (\mathcal{I}_0 + \mathcal{I}_{j'}) \otimes (\mathcal{I}_0 + \mathcal{I}_{k'}) \in \mathfrak{F}$, which is a contradiction. Hence, $\mathcal{I}_0$ is prime and $K_0 = c(\mathcal{I}_0)$ is an irreducible closed subspace of $Spec(\mathcal{C})^*$. □

Clearly, any family $\mathfrak{F}^*$ of closed subspaces of $Spec(\mathcal{C})^*$ containing $\varnothing$ and closed under finite unions satisfies the conditions in Proposition 5.9. In that case, the corresponding family $\mathfrak{F} = \{\mathcal{I} \triangleleft \mathcal{C} : c(\mathcal{I}) \in \mathfrak{F}^*\}$ of thick tensor ideals in $(\mathcal{C}, \otimes, 1)$ is simply a monoidal family. We will conclude by showing how to construct a family $\mathfrak{F}^*$ that is not closed under finite unions but still satisfies the conditions in Proposition 5.9. This will be done with the help of Ako families of thick tensor ideals in $(\mathcal{C}, \otimes, 1)$.

We consider the basis $\{c(a) = \{\mathcal{P} \in Spec(\mathcal{C}) : a \in \mathcal{P}\}\}_{a \in \mathcal{C}}$ of closed sets for $Spec(\mathcal{C})^*$. Then, any closed subspace $c(\mathcal{I}) \subseteq Spec(\mathcal{C})^*$ can be expressed as the intersection $c(\mathcal{I}) = \bigcap_{a \in \mathcal{I}} c(a)$. Suppose that $0 \neq x \in \mathcal{C}$ is a zero-divisor, i.e., $Ann(x) = \{y \in \mathcal{C} : x \otimes y = 0\} \neq 0$. Further, suppose that we can choose a non-principal thick tensor ideal $\mathcal{J} \subseteq Ann(x)$. Hence, $x \otimes \mathcal{J} = 0$. We now set

$$\mathfrak{F} = \{\mathcal{K} \triangleleft \mathcal{C} : (\mathcal{J}, x) \subseteq \mathcal{K}\} \cup \{(x)\} \cup \{\mathcal{J}\} \tag{40}$$

and consider the corresponding family $\mathfrak{F}^* = \{c(\mathcal{I}) : \mathcal{I} \in \mathfrak{F}\}$ of closed subspaces of $\mathrm{Spec}(\mathcal{C})^*$. Clearly, since $x \otimes \mathcal{J} = 0 \neq \mathfrak{F}$, the family $\mathfrak{F}$ is not monoidal and hence the family $\mathfrak{F}^*$ of closed subspaces of $\mathrm{Spec}(\mathcal{C})^*$ is not closed under finite unions. Since $\mathcal{C} \in \mathfrak{F}$, we know that $\varnothing \in \mathfrak{F}^*$. We claim that $\mathfrak{F}^*$ also satisfies condition (b) in Proposition 5.9. Otherwise, there exist basis elements $c(a), c(b)$ with $a, b \in \mathcal{C}$ and some closed subspace $K \subseteq \mathrm{Spec}(\mathcal{C})^*$ such that $K \cap c(a) \in \mathfrak{F}^*$ and $K \cap c(b) \in \mathfrak{F}^*$ but $K \cap (c(a) \cup c(b)) \notin \mathfrak{F}^*$. Since K is closed in $\mathrm{Spec}(\mathcal{C})^*$, we can find a thick tensor ideal $\mathcal{L}$ such that $K = c(\mathcal{L})$. Then, we see that $(\mathcal{L}, a) \in \mathfrak{F}$ and $(\mathcal{L}, b) \in \mathfrak{F}$, but $(\mathcal{L}, a \otimes b) = (\mathcal{L}, a) \otimes (\mathcal{L}, b) \notin \mathfrak{F}$. From the expression for $\mathfrak{F}$ in (40), this is possible only if one of the two ideals $(\mathcal{L}, a)$ and $(\mathcal{L}, b)$ is equal to (x) and the other is equal to $\mathcal{J}$. Accordingly, we have

$$\mathcal{L} = \mathcal{L} \otimes \mathcal{L} \subseteq (\mathcal{L}, a) \otimes (\mathcal{L}, b) = (x) \otimes \mathcal{J} = 0$$

and hence $\mathcal{L} = 0$. But then, either $\mathcal{J} = (\mathcal{L}, a) = (a)$ or $\mathcal{J} = (\mathcal{L}, b) = (b)$ and $\mathcal{J}$ is principal, which is a contradiction.

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