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RESEARCH ARTICLE

The constants of Lotka–Volterra derivations

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Abstract The ring of constants of Lotka–Volterra derivations is determined in arbitrary dimension. It is always a polynomial ring.

Keywords Factorisable derivation · Lotka–Volterra derivation · Polynomial constant · Polynomial ring

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1 Introduction

Let $n > 2$ be an integer, $R = K[x_1, x_2, \dots, x_n]$ the n-variable polynomial ring over the field K of characteristic 0. A *derivation* of R is a K-linear map $\delta\colon R \to R$ that satisfies the Leibniz rule $\delta(fg) = f\delta(g) + g\delta(f)$ for every pair of polynomials. By this identity, the values at the generators $\delta(x_i) = g_i \in R$, $i = 1, \dots, n$, determine δ. Another way of expressing this is

$$\delta = g_1 \frac{\partial}{\partial x_1} + g_2 \frac{\partial}{\partial x_2} + \dots + g_n \frac{\partial}{\partial x_n}.$$

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The polynomial $f \in R$ is a *constant* of the derivation δ if $\delta(f) = 0$. The set of constants of R is a subalgebra of R due to the Leibniz rule, we shall denote it R^δ. A central problem concerning derivations is to describe their rings of constants. There is no general procedure for determining R^δ and it may be neither a polynomial ring nor a finitely generated ring (see [5] for more details).

Given parameters $C_i \in K, i = 1, \ldots, n$, the Lotka–Volterra derivation d is defined on the generators as

$$d(x_i) = x_i(x_{i-1} - C_i x_{i+1}),$$

where (and further on) the indexing is circular, that is $n + e$ and e are identified for every integer e. The special case of the Volterra derivation, where all $C_i = 1$, was considered in [2]. In the present paper we shall observe a confirmation why the Volterra case is remarkable. The case $n = 3$ for arbitrary parameters C_i was considered in the paper [4]. Here we assume $n > 3$.

The Lotka–Volterra derivations are a type of factorizable derivations, that is, derivations defined by $d(x_i) = x_i f_i$, where f_i are polynomials of degree 1 for $i = 1, \ldots, n$. We may associate the factorizable derivation with any given derivation of a polynomial ring, this helps to determine constants of arbitrary derivations (see, for example, [3]). Moreover, Lotka–Volterra systems play a significant role in population biology, laser physics and plasma physics (see for instance [1] and references therein).

Before stating the main theorem, we define the generating polynomials. Let $f = \sum_{i=1}^n \left(\prod_{j=1}^{i-1} C_j\right) x_i = x_1 + C_1x_2 + C_1C_2x_3 + \cdots$. Take any nonempty subset $\mathcal{A} \subseteq \mathbb{Z}_n$ of integers mod n closed under $i \mapsto i+2$. If n is odd then $\mathcal{A} = \mathbb{Z}_n$; for n even we have two additional proper subsets $\mathcal{E} = \{2i : i \leqslant n/2\}$ and $\mathcal{O} = \{2i - 1 : i \leqslant n/2\}$. Given $\mathcal{A}$, let $C_i, i \in \mathcal{A}$, be positive rational numbers such that $\prod_{i \in \mathcal{A}} C_i = 1$. Then there exist unique coprime positive integers $\theta_i, i \in \mathcal{A}$, such that $\theta_{i+2} = C_i\theta_i$. (Indeed, the rational vectors $(\tau_i)_{i\in\mathcal{A}}$ form a 1-dimensional subspace because of fixed positive ratios. Hence if we take the given positive rational vectors, then multiplying by the smallest common denominator and dividing by the greatest common divisor of the numerators we obtain $(\theta_i)_{i\in\mathcal{A}}$.) Let us define $g_\mathcal{A} = \prod_{i\in\mathcal{A}} x_i^{\theta_i}$. Let $\mathcal{A}' = \{i + 1 : i \in \mathcal{A}\}$.

Our main results are the following two theorems.

Theorem 1.1 *Let d be the Lotka–Volterra derivation with parameters $C_1, \ldots, C_n$. Then the ring of constants R^d is a polynomial algebra. Assume that not all parameters are equal to 1 and $n > 4$. Then the number of generators is equal to*

- 0 *if $\prod C_i \neq 1$ and no $g_\mathcal{A}$ is defined;*
- 3 *if n is even and both $g_\mathcal{E}$ and $g_\mathcal{O}$ are defined;*
- 2 *if n is odd and $g_{\mathbb{Z}_n}$ is defined, or n is even and $\prod C_i = 1$ but only one of $g_\mathcal{E}$ and $g_\mathcal{O}$ is defined;*
- 1 *if $\prod C_i = 1$ but no $g_\mathcal{A}$ is defined, or n is even and $\prod C_i \neq 1$ but only one of $g_\mathcal{E}$ and $g_\mathcal{O}$ is defined.*

Suppose now that $n = 4$. In this case there is a further quadratic invariant if $C_1C_2C_3C_4 = -1$ and there are two consecutive indices such that both corresponding parameters are equal to 1. Assume $C_1 = C_2 = 1$ and $C_4 = -1/C_3$ (for the other possibilities one has to rotate the indices appropriately), then

$$f_4 = x_1^2 + x_2^2 + x_3^2 + C_3^2 x_4^2 + 2x_1x_2 - 2x_1x_3 \\ - 2C_3x_1x_4 + 2x_2x_3 - 2C_3x_2x_4 + 2C_3x_3x_4.$$

If $C_1 = C_2 = C_3 = 1$ and $C_4 = -1$ then this procedure would give two possibilities for f_4, they differ by $4x_1x_3$. As $4x_1x_3 = 4g_{\mathcal{O}}$, which is defined in this case, one of the two possibilities for f_4 is sufficient.

Theorem 1.2 *Assume $n = 4$ and let d be the Lotka–Volterra derivation with parameters $C_1, \dots, C_4$. Then the ring of constants R^d is a polynomial algebra. If not all parameters are equal to* 1 *then the number of generators is equal to*

- 0 *if $\prod C_i \neq 1$ and none of $g_{\mathcal{O}}, g_{\mathcal{E}}, f_4$ is defined;*
- 3 *if both $g_{\mathcal{E}}$ and $g_{\mathcal{O}}$ are defined;*
- 2 *if $\prod C_i = 1$ but only one of $g_{\mathcal{E}}$ and $g_{\mathcal{O}}$ is defined or one of parameters is equal to -1 and the other three are equal to* 1*;*
- 1 *if $\prod C_i = 1$ but $g_{\mathcal{O}}, g_{\mathcal{E}}$ are not defined, or $\prod C_i = -1$ and only two consecutive parameters are equal to* 1*, or $\prod C_i \neq \pm 1$ but one of $g_{\mathcal{O}}, g_{\mathcal{E}}$ is defined.*

It is not stated explicitly in the theorems but the generators are always those polynomials $g_{\mathcal{A}}$ that are defined together with f if $\prod C_i = 1$ (or together with f_4 if $n = 4$, $C_1C_2C_3C_4 = -1$ and two consecutive parameters are equal to 1). Denote by $\mathcal{H}$ this set of generators. (If n is even and both $g_{\mathcal{E}}$ and $g_{\mathcal{O}}$ are defined, then of course $g_{\mathbb{Z}_n}$ is also defined. But it is superfluous in the generating set because $g_{\mathbb{Z}_n} = g_{\mathcal{E}} g_{\mathcal{O}}$.)

It is routine to check, see Lemma 2.1, that f is a constant if and only if $\prod C_i = 1$, each $g_{\mathcal{A}}$ is constant if defined and f_4 is a constant if $n = 4$, $C_1 = C_2 = 1$, and $C_4 = -1/C_3$.

The statements of theorems are similar to [2, Theorem 1.1] where all parameters are equal to 1 and there are $[n/2] + 1$ free generators of the ring of constants. As it was noted there, the surprising feature is that the generators are independent. This phenomenon might deserve further study.

The outline of proofs of theorems is also similar to the proof of [2, Theorem 1.1] subject to new complications related to arithmetic properties of parameters C_i which require different arguments in different cases.

The case $n = 4$ was also investigated in [6]. The problem is solved there for parameters such that $C_1C_2C_3C_4$ is either 1 or not a root of unity. The last sentence of the statement in [6, Lemma 3.2] is not correct. The assumption should be $(C_1 \cdots C_n)^m \neq 1$ instead of $C_1 \cdots C_n \neq 1$. In this paper we treat the more difficult case, when $C_1C_2C_3C_4 \neq 1$ is a root of unity. See Propositions 2.10 and 2.11.

2 Proofs

We prove Theorems 1.1 and 1.2 simultaneously, indicating the differences along the way. The proof splits into three parts. First, we show that the polynomials are indeed in the kernel of d. Second, that any polynomial in R^d can be expressed as a polynomial of elements of $\mathcal{H}$. And finally, that they are algebraically independent. As the case $n = 4$ requires a special attention, Proposition 2.10 will be partly replaced by Proposition 2.11.

Before embarking on the proof we make a short observation. The polynomial ring has a grading $R = \bigoplus_{a=0}^{\infty} R_a$, where R_a is the K-vectorspace of homogeneous polynomials of degree a. The derivation d admits this grading of R, more precisely, $d(R_a) \subseteq R_{a+1}$. So the ring of constants has a grading $R^d = \bigoplus_{a=0}^{\infty} R_a^d$. In particular, if a polynomial is a constant of d, then so are all its homogeneous components. For a polynomial $g \in R$ let $M(g)$ denote the set of monomials occurring in g with a nonzero coefficient.

Let $m = \prod_{i=1}^{n} x_i^{\alpha_i}$ be a monomial, then

$$d(m) = \sum_{i=1}^{n} \alpha_i (x_{i-1} - C_i x_{i+1}) m = \sum_{i=1}^{n} (\alpha_{i+1} - C_{i-1}\alpha_{i-1}) x_i m.$$

That is, the coefficient of mx_i in $d(m)$ is $\alpha_{i+1} - C_{i-1}\alpha_{i-1}$. So

$$mx_i \in M(d(m)) \iff \alpha_{i+1} \neq C_{i-1}\alpha_{i-1}. \tag{1}$$

Looking at it from the back end, consider the monomial $m' = \prod_{i=1}^{n} x_i^{\beta_i}$. Then

$$m' \in M(d(m'/x_i)) \iff \beta_i > 0 \quad \text{and} \quad \beta_{i+1} \neq C_{i-1}\beta_{i-1}. \tag{2}$$

Now let us proceed with the first part of the proof.

Lemma 2.1 *The elements of $\mathcal{H}$ are in R^d.*

Proof If $\prod C_i = 1$ then

$$\begin{aligned} d(f) &= \sum_{i=1}^{n} \prod_{j=1}^{i-1} C_j d(x_i) = \sum_{i=1}^{n} \prod_{j=1}^{i-1} C_j x_i (x_{i-1} - C_i x_{i+1}) \\ &= \sum_{i=1}^{n-1} \left(\prod_{j=1}^{i} C_j - \prod_{j=1}^{i} C_j \right) x_i x_{i+1} + \left(1 - \prod_{j=1}^{n} C_j \right) x_1 x_n = 0. \end{aligned}$$

If $\prod_{i \in \mathcal{A}} C_i = 1$ and each factor is positive and rational then $g_{\mathcal{A}}$ is defined and

$$\begin{aligned} d(g_{\mathcal{A}}) &= \prod_{i \in \mathcal{A}} x_i^{\theta_i} \cdot \sum_{i \in \mathcal{A}} \theta_i (x_{i-1} - C_i x_{i+1}) \\ &= \prod_{i \in \mathcal{A}} x_i^{\theta_i} \cdot \sum_{i \in \mathcal{A}'} (\theta_{i+1} - C_{i-1}\theta_{i-1}) x_i = 0. \end{aligned}$$

If $n = 4$, $C_1 = C_2 = 1$, $C_4 = -1/C_3$, then f_4 is defined and

$$\begin{aligned} d(f_4) &= 2x_1^2(x_4 - x_2) + 2x_2^2(x_1 - x_3) + 2x_3^2(x_2 - C_3x_4) \\ &\quad + 2C_3^2x_4^2\left(x_3 + \frac{x_1}{C_3}\right) + 2x_1x_2(x_4 - x_2 + x_1 - x_3) \\ &\quad - 2x_1x_3(x_4 - x_2 + x_2 - C_3x_4) - 2C_3x_1x_4\left(x_4 - x_2 + x_3 + \frac{x_1}{C_3}\right) \\ &\quad + 2x_2x_3(x_1 - x_3 + x_2 - C_3x_4) - 2C_3x_2x_4\left(x_1 - x_3 + x_3 + \frac{x_1}{C_3}\right) \\ &\quad + 2C_3x_3x_4\left(x_2 - C_3x_4 + x_3 + \frac{x_1}{C_3}\right) = 0. \end{aligned}$$

□

The following proposition is technical in nature. Its function is to provide a tool for showing that if one monomial is in $M(h)$, $h \in R^d$, then many others are there as well. This to work we have to assume that for indices i (outside the examined area) either $\beta_i = 0$ or $\beta_{i-1} = \beta_{i+1} = 0$ so that by (2) the exponent of x_i might not decrease. It is not powerful enough in every case, so partial extensions are necessary in Propositions 2.10 and 2.11.

Proposition 2.2 *Let $m = \prod x_i^{\beta_i}$ be a monomial in $M(h)$ for some $h \in R^d$ and let $1 \leqslant j \leqslant n$ be an index such that $C_j > 0$. Assume that $\beta_{j-1} = 0 = \beta_{j+3}$ and for every $i < j-1$ or $i > j+2$ one of β_i and β_{i+1} is zero. Further put $r = \beta_{j+2} + \beta_{j+1} - C_j\beta_j$. Then there exists an integer $t_0 \geqslant r$ such that $s_0 = (r - t_0)/C_j$ is also an integer and*

$$m x_j^s x_{j+1}^{t_0 - s - \beta_{j+1}} x_{j+2}^{\beta_{j+1} - t_0} \in M(h), \qquad s_0 \leqslant s \leqslant t_0.$$

Conversely, if

$$m x_j^s x_{j+1}^{t-s-\beta_{j+1}} x_{j+2}^{\beta_{j+1}-t} \in M(h)$$

then $s_0 \leqslant s \leqslant t \leqslant t_0$. If $C_{j+1} \neq 0$, then

$$m x_j^{s_0} x_{j+1}^{-\beta_{j+1}} {x_{j+2}}^{\beta_{j+1} - s_0} \in M(h),$$

too. If $C_{j+1} = 0$ then $r \leqslant \beta_{j+1}$.

Proof Denote by $e_{s,t}$ the coefficient of $m x_j^s x_{j+1}^{t-s-\beta_{j+1}} x_{j+2}^{\beta_{j+1}-t}$ in h. We have $e_{0,\beta_{j+1}} \neq 0$ and would like to conclude that $e_{t,t} \neq 0$ for some $t \geqslant r$.

We observe that the coefficient of $m x_j^s x_{j+1}^{1+t-s-\beta_{j+1}} x_{j+2}^{\beta_{j+1}-t}$ in $d(h)$ is equal to

$$\begin{aligned} 0 &= (1 + t - s)(e_{s-1,t} - C_{j+1}e_{s,t+1}) \\ &\qquad + e_{s,t}(\beta_{j+2} + \beta_{j+1} - t - C_j(\beta_j + s)) \\ &= (1 + t - s)(e_{s-1,t} - C_{j+1}e_{s,t+1}) + e_{s,t}(r - t - C_j s). \end{aligned} \tag{3}$$

Note that no other monomial might contribute to $mx_j^s x_{j+1}^{1+t-s-\beta_{j+1}} x_{j+2}^{\beta_{j+1}-t}$ by our assumption on the exponents and by (2). See the comment preceding the proposition.

Pick the minimal s_0 such that there exists some t for which $e_{s_0,t} \neq 0$. (We have $s_0 \leqslant 0$, since $e_{0,\beta_{j+1}} \neq 0$.) Consider the set $T = \{t \in \mathbb{Z} : e_{s_0,t} \neq 0\}$. Of course, $\max T \leqslant \beta_{j+2} + \beta_{j+1}$ and $\min T \geqslant s_0$. From (3) it follows that if $t \in T$ then $t+1 \in T$ unless possibly for $t = t_0 = r - C_j s_0$. If t_0 is not an integer or if it happens that $\max T > t_0$ then T is not bounded from above, a contradiction. So $\max T = t_0$ must be an integer.

Pick the maximal t_1 such that there exists some s for which $e_{s,t_1} \neq 0$. (By the above, $t_1 \geqslant t_0 \geqslant r$ and, by assumption, $t_1 \geqslant \beta_{j+1}$.) Let $S = \{s \in \mathbb{Z} : e_{s,t_1} \neq 0\}$. Again $\min S \geqslant -\beta_j$ and $\max S \leqslant t_1$. From (3) it follows that if $s-1 \in S$ then $s \in S$ unless possibly for $s = s_1 = (r - t_1)/C_j$ or $s = t_1 + 1$. If s_1 is not an integer or if it happens that $\min S < s_1$ then S is not bounded from below, a contradiction. So $s_1 = \min S$ must be an integer and $S = [s_1; t_1] \cap \mathbb{Z}$.

By minimality, we have $s_0 \leqslant s_1$ so $r - t_0 = C_j s_0 \leqslant C_j s_1 = r - t_1 \leqslant r - t_0$. Hence $s_1 = s_0$ and $t_1 = t_0$. Thus $e_{s,t} \neq 0$ implies $s_0 \leqslant s \leqslant t \leqslant t_0$. The previous paragraph also implies that $e_{s,t_0} \neq 0$ for $s_0 \leqslant s \leqslant t_0$, as claimed.

If $C_{j+1} = 0$ then by (3) we get $0 \neq e_{s-1,\beta_{j+1}} \Leftarrow 0 \neq e_{s,\beta_{j+1}}$ unless possibly for $s = (r - \beta_{j+1})/C_j$. As $e_{0,\beta_{j+1}} \neq 0$ we must have $s = (r - \beta_{j+1})/C_j \leqslant 0$ is an integer, as claimed.

If $C_{j+1} \neq 0$ then by (3) we also have that $t+1 \in T$ implies $t \in T$ unless possibly for $t = s_0 - 1$. So $T = [s_0; t_0] \cap \mathbb{Z}$, so $0 \neq e_{s_0,s_0}$. □

In the following proposition the coefficients are determined. The monomial can be chosen to be $mx_j^{s_0} x_{j+1}^{t_0-s_0-\beta_{j+1}} x_{j+2}^{\beta_{j+1}-t_0}$ from the previous proposition.

Proposition 2.3 *Let $h \in R^d$ and $m = \prod x_i^{\beta_i}$ be a monomial in $M(h)$ with the coefficient equal to A. Let $1 \leqslant j_0 \leqslant n$ be an index such that $mx_{j_0}^p x_{j_0+1}^{-p-q} x_{j_0+2}^q \in M(h)$ implies $p, q \geqslant 0$. Assume that for every $i < j_0$ or $i > j_0 + 2$ one of β_i and β_{i+1} is zero and that $\beta_{j_0+2} - \beta_{j_0} C_{j_0} = 0$.*

(i) *If $C_{j_0} \neq 0$ then $mx_{j_0}^p x_{j_0+1}^{-p}$ has the coefficient equal to*

$$\binom{\beta_{j_0+1} - C_{j_0-1}\beta_{j_0-1}}{p} C_{j_0}^{-p} A.$$

(ii) *If $C_{j_0+1} = 0 = \beta_{j_0+3}$ then $mx_{j_0+1}^{-q} x_{j_0+2}^q$ has the coefficient equal to 0. If $C_{j_0+1} \neq 0$ then $mx_{j_0+1}^{-q} x_{j_0+2}^q$ has the coefficient equal to*

$$\binom{\beta_{j_0+1} - \beta_{j_0+3}/C_{j_0+1}}{q} C_{j_0+1}^q A.$$

(iii) *Assume $C_{j_0} \neq 0$ and $\beta_{j_0-1} = 0 = \beta_{j_0+3}$. Suppose C_{j_0} is not positive rational or $C_{j_0} = a/b$ with $(a, b) = 1$ and $p < a$ or $q < b$. Then the coefficient of $mx_{j_0}^p x_{j_0+1}^{-p-q} x_{j_0+2}^q$ is equal to*

$$\binom{\beta_{j_0+1}}{p+q}\binom{p+q}{q} C_{j_0+1}^{q} C_{j_0}^{-p} A.$$

Proof Let $f_{p,q}$ denote the coefficient of $m x_{j_0}^{p} x_{j_0+1}^{-p-q} x_{j_0+2}^{q}$ in $M(h)$. By assumption, $f_{p,q} = 0$ if $p < 0$ or $q < 0$. As in the previous proof the coefficient of $m x_{j_0}^{p} x_{j_0+1}^{1-p-q} x_{j_0+2}^{q}$ in $d(h)$ is equal to

$$\begin{aligned} 0 = &\big(q + \beta_{j_0+2} - (p+\beta_{j_0}) C_{j_0}\big) f_{p,q} \\ &+ \big(\beta_{j_0+1} + 1 - p - q - \beta_{j_0-1} C_{j_0-1}\big) f_{p-1,q} \\ &+ \big(\beta_{j_0+3} - (\beta_{j_0+1} + 1 - p - q) C_{j_0+1}\big) f_{p,q-1}. \end{aligned} \tag{4}$$

Let us first show (i). Put $q = 0$ and $t = \beta_{j_0+1} - C_{j_0-1}\beta_{j_0-1}$. Then

$$f_{p,0} = f_{p-1,0} \frac{\beta_{j_0+1} - C_{j_0-1}\beta_{j_0-1} + 1 - p}{-\beta_{j_0+2} + \beta_{j_0} C_{j_0} + p C_{j_0}} = f_{p-1,0} \frac{t+1-p}{p C_{j_0}}.$$

By induction it implies $f_{p,0} = \binom{t}{p} C_{j_0}^{-p} A$. Similarly, if $C_{j_0+1} = 0 = \beta_{j_0+3}$ then $q f_{0,q} = \beta_{j_0+3} f_{0,q-1} = 0$. If $C_{j_0+1} \neq 0$ then put $t = \beta_{j_0+1} - \beta_{j_0+3}/C_{j_0+1}$ and

$$\begin{aligned} f_{0,q} &= f_{0,q-1} \frac{(\beta_{j_0+1} - \beta_{j_0+3}/C_{j_0+1} + 1 - q) C_{j_0+1}}{q + \beta_{j_0+2} - \beta_{j_0} C_{j_0}} \\ &= f_{0,q-1} \frac{(t+1-q) C_{j_0+1}}{q}. \end{aligned}$$

So $f_{0,q} = \binom{t}{q} C_{j_0+1}^{q} A$ as claimed in (ii). To obtain (iii), let $t = \beta_{j_0+1}$. Equality (4) simplifies to

$$0 = (q - p C_{j_0}) f_{p,q} + (t+1-p-q) f_{p-1,q} - (t+1-p-q) C_{j_0+1} f_{p,q-1}.$$

By (i) and (ii), the claim is true for $p = 0$ or $q = 0$ and we may apply induction on $p+q$ as long as $0 < p, q$ but $p < a$ or $q < b$. These conditions imply $0 \neq q - p C_{j_0}$ so from the above and by the inductive assumption, we obtain

$$\begin{aligned} f_{p,q} &= \frac{t+1-p-q}{q - p C_{j_0}} \left(\binom{t}{p+q-1}\binom{p+q-1}{q-1} C_{j_0+1}^{q} C_{j_0}^{-p} \right. \\ &\qquad\qquad \left. - \binom{t}{p+q-1}\binom{p+q-1}{q} C_{j_0+1}^{q} C_{j_0}^{-p+1} \right) \\ &= \frac{t+1-p-q}{q - p C_{j_0}} \frac{t!}{(t-p-q+1)!\, p!\, q!} C_{j_0+1}^{q} C_{j_0}^{-p} (q - p C_{j_0}) \\ &= \binom{t}{p+q}\binom{p+q}{q} C_{j_0+1}^{q} C_{j_0}^{-p}. \end{aligned}$$

□

The second and most difficult part of the proof of Theorems 1.1 and 1.2 is the following.

Lemma 2.4 *If $h \in R$ is such that $d(h) = 0$ then it is a polynomial in the elements of $\mathcal{H}$.*

Proof As the indexing of variables in the definition of the derivation is circular, we have freedom to choose the starting index. Not all of parameters are equal to 1 so without loss of generality we can assume that $C_n \neq 1$. We would also choose $C_n \neq 0$ if possible. That is impossible only if all parameters take values in the set $\{0, 1\}$. In this case either all parameters are equal to 0 or there is one parameter equal to 1 following another parameter equal to 0, then we let $C_n = 0$ and $C_1 = 1$.

Clearly, it is enough to prove the lemma for homogeneous polynomials, so let $h \in R_a$. We use the standard lexicographic ordering on monomials of R_a. That is, $\prod x_i^{\alpha_i} \prec \prod x_i^{\beta_i}$ if $\alpha_i < \beta_i$ at the first index that they are not equal. For a homogenous polynomial $k \in R_a$ the *leading monomial* is the lexicographically largest monomial in $M(k)$.

Assume by induction that h has lexicographically the smallest leading monomial among all homogenous polynomials that are counterexamples to the lemma. We prove that there is a polynomial expression F of the elements of $\mathcal{H}$ that has the same leading monomial (of course, it can have also the same coefficient, we may assume that the coefficient is equal to 1). So $h - F$ has only lexicographically smaller monomials and hence can be expressed by induction. This will finish the proof of the lemma.

Let $m_1 = \prod_{i=1}^{n} x_i^{\alpha_i}$ be the leading monomial of h. The following two propositions describe the exponents of the leading monomial in detail. The first one implies that the even-indexed exponents of m_1 are determined by $\alpha_n = \alpha_0$, namely, for every $n/2 \geqslant i \geqslant 1$ we have $\alpha_{2i} = C_{2i-2}\alpha_{2i-2}$. Indeed, if it fails for some even indices then for the smallest such index $2r \geqslant 2$ all assumptions of the proposition would hold. But the conclusion $m_1 \notin M(h)$ would contradict the original choice of $m_1 \in M(h)$. □

Proposition 2.5 *Suppose $m = \prod_{i=1}^{n} x_i^{\gamma_i}$ is a monomial and r is a positive integer with the following properties:*

(i) $\gamma_n = \alpha_n$;
(ii) $\gamma_{2i-1} = \alpha_{2i-1}$ *for* $1 \leqslant i \leqslant r$;
(iii) $\gamma_{2i} = C_{2i-2}\gamma_{2i-2}$ *for* $1 \leqslant i \leqslant r-1$;
(iv) $\gamma_{2r} \neq C_{2r-2}\gamma_{2r-2}$.

Then $m \notin M(h)$.

Proof Assume by contradiction that $m \in M(h)$. The proof is by induction on r. Let $r = 1$. The coefficient of mx_1 in $d(m)$ is $\gamma_2 - C_n\alpha_n \neq 0$, see the line before (1). In turn, by (2), it may occur only in $M(d(mx_1/x_i))$ for any index i. However, by the assumption, $\gamma_1 = \alpha_1$ so we have $mx_1/x_i \succ m_1$ for $i > 1$. By maximality of m_1, these are not in $M(h)$ so mx_1 cannot be canceled. This contradiction establishes the proposition for $r = 1$ for every monomial.

Assume now $r > 1$ and that for $r' < r$ the proposition holds for every monomial. Suppose the set $\mathcal{S} = \{r' < r : \alpha_{2r'} \neq C_{2r'-2}\alpha_{2r'-2}\}$ is nonempty. Let r' be its smallest element. As assumptions of the proposition hold for $m = m_1$ and $r = r'$ but the conclusion $m_1 \notin M(h)$ does not, we must have $\mathcal{S} = \varnothing$. In other words,

$$\alpha_{2i} = C_{2i-2}\alpha_{2i-2}, \qquad 1 \leqslant i \leqslant r-1. \tag{5}$$

As in the first paragraph, mx_{2r-1} has the nonzero coefficient equal to $\gamma_{2r} - C_{2r-2}\gamma_{2r-2}$ in $d(m)$. Again, by (2), this may occur only in $M(d(mx_{2r-1}/x_i))$ for some index i. If $i > 2r-1$ then, by virtue of (5), $mx_{2r-1}/x_i \succ m_1$ so it cannot be in $M(h)$. If $i < 2r-1$ is odd then, by assumption (iii), the coefficient of mx_{2r-1} is $\gamma_{i+1} - C_{i-1}\gamma_{i-1} = 0$ in $d(mx_{2r-1}/x_i)$ so it cannot help in canceling the coefficient $\gamma_{2r} - C_{2r-2}\gamma_{2r-2}$ of mx_{2r-1} in $d(m)$.

If however $i = 2r' < 2r-1$ is even then mx_{2r-1}/x_i satisfies assumptions of the proposition with r'. So, by induction, $mx_{2r-1}/x_i \notin M(h)$. This proves that if $m \in M(h)$ then $d(h) \neq 0$. This contradiction establishes the proposition. ■

Remark 2.6 In fact, in the proof we used (i') $C_n\gamma_n = C_n\alpha_n$, so the lemma holds if we assume $C_n = 0$ instead of (i). It implies $\alpha_{2i} = 0$ for all $i \leqslant n/2$.

Proposition 2.7 *Suppose* $m = \prod_{i=1}^{n} x_i^{\gamma_i} \in M(h)$ *is a monomial and* $r < n/2$ *is a positive integer with the following properties:*

(i) $\gamma_n = \alpha_n$ *(or* $C_n = 0$*);*
(ii) $\gamma_{2i-1} = \alpha_{2i-1}$ *for* $1 \leqslant i \leqslant r$*;*
(iii) $\gamma_{2i} = \alpha_{2i}$ *for* $1 \leqslant i \leqslant r$*.*

Then there exists a nonnegative integer β_{2r-1} *such that* $C_{2r-1}(\gamma_{2r-1} - \beta_{2r-1}) = \gamma_{2r+1}$ *and* $m' = m(x_{2r}/x_{2r-1})^{\beta_{2r-1}} \in M(h)$*. In particular, there exist nonnegative integers* β'_{2i-1} *such that* $C_{2i-1}(\alpha_{2i-1} - \beta'_{2i-1}) = \alpha_{2i+1}$ *for* $1 \leqslant i < n/2$*.*

Proof The proof is by contradiction. Assume that there exists the smallest positive integer r for which there is no such nonnegative integer β_{2r-1}. Let $m_2 \in M(h)$. Suppose that

$$\begin{aligned} &\text{for } r > 1: \text{the exponent of } x_k \text{ is 0 in } m_2/m \text{ for } k < 2r-1 \\ &\qquad\qquad\quad (\text{and for } k = n \text{ if } C_n \neq 0); \\ &\text{for } r = 1: \text{the exponent of } x_n \text{ is at most } \gamma_n \text{ in } m_2. \end{aligned} \tag{6}$$

Denote by s the exponent of x_{2r-1} and by t the exponent of x_{2r} in m_2/m. We claim that $s + t \leqslant 0$, which we prove by (decreasing) induction on s.

As m_1 is the leading monomial, $s \leqslant 0$. If $s = 0$, then $t \leqslant 0$, indeed. So let $s < 0$. If $t \leqslant 0$, then we are done, so assume $t > 0$. Hence, by (1), the monomial $m_2x_{2r-1} \in M(d(m_2))$ has the coefficient equal to $t + \gamma_{2r} - C_{2r-2}\gamma_{2r-2} = t \neq 0$ if $r > 1$, by Proposition 2.5, assumption (iii) and equation (6). If $r = 1$ and $C_n > 0$ then the coefficient of m_2x_1 in $d(m_2)$ is at least $t + \gamma_2 - C_n\gamma_n = t > 0$, so $m_2x_{2r-1} \in M(d(m_2))$. If $r = 1$ and $C_n \not> 0$ then $\gamma_2 = \alpha_2 = C_n\alpha_n = C_n\gamma_n = 0$ and the coefficient of m_2x_1 in $d(m_2)$ is $t > 0$. So $m_2x_{2r-1} \in M(d(m_2))$ also in this case.

This monomial may also occur only in $M(d(m_2x_{2r-1}/x_i))$ for some indices i. If $i < 2r-1$ is even then, by Proposition 2.5, it does not occur in $M(h)$. (Note that here $r > 1$, so the exponent of x_n is γ_n in m_2x_{2r-1}/x_i, so all assumptions of Proposition 2.5 are satisfied. If $C_n = 0$ then see Remark 2.6.) If $i < 2r-1$ is odd then $\gamma_{i+1} - C_{i-1}\gamma_{i-1} = 0$ is the coefficient of m_2x_{2r-1} in $d(m_2x_{2r-1}/x_i)$. Otherwise $i > 2r-1$. Now we apply induction for m_2x_{2r-1}/x_r which still satisfies the conditions of (6), unless $r > 1$, $i = n$ and $C_n \neq 0$. If induction applies then the exponent of

x_{2r-1} is equal to $s+1$ and the exponent of x_{2r} is equal to t or $t-1$, their sum is at least $s+t$. But, by induction, their sum is at most 0, hence $s+t \leqslant 0$.

If, however, induction does not apply then $m_3 = m_2 x_{2r-1}/x_n \in M(h)$. Hence $m_3 x_1 \in M(d(m_3))$ has the coefficient equal to $\gamma_2 - C_n(\gamma_n - 1) = C_n \neq 0$. Now, for a monomial $m_4 = m_3 x_1/x_l \in M(h)$ to cancel this the exponent of x_1 is $\gamma_1 + 1 = \alpha_1 + 1$ in m_4, so $m_1 \prec m_4$, a contradiction. So in fact, induction must apply and we conclude $s+t \leqslant 0$, as claimed.

We claim next that for every $0 \leqslant l$ the monomial $m_1(x_{2r}/x_{2r-1})^l \in M(h)$. This is proved by induction on l. The case $l = 0$ is given, so let $l > 0$ and put $m = m_1(x_{2r}/x_{2r-1})^{l-1} \in M(h)$. By the assumption on exponents, $mx_{2r} \in M(d(m))$, the coefficient is $\alpha_{2r+1} - C_{2r-1}(\alpha_{2r-1} - l + 1) \neq 0$ as we argue by contradiction.

To cancel this from $d(h)$ there must exist a monomial $m_2 = mx_{2r}/x_i \in M(h)$. If $2 < 2r < i < n$ or $2 = 2r < i$ then we contradict the above claim on $s+t$. (In our situation $s = 1-l$, $t = l$ and the conditions of (6) are satisfied.) If, however, $r > 1$ and $i = n$ then $mx_{2r}/x_n \in M(h)$ so $mx_{2r}x_1/x_n \in M(d(mx_{2r}/x_n))$ has the coefficient $\gamma_2 - C_n(\gamma_n - 1) = C_n \neq 0$. The exponent of x_1 in any $mx_{2r}x_1/(x_n x_j)$ is larger than $\gamma_1 = \alpha_1$, so $m_1 \prec mx_{2r}x_1/(x_n x_j)$ which means that $mx_{2r}x_1/x_n$ cannot be canceled, also a contradiction.

If $i < 2r$ then, by Proposition 2.5, i cannot be even. So i must be odd, but if $i < 2r-1$ then (1) shows that the coefficient of mx_{2r} in $d(m_2)$ is $\gamma_{i+1} - C_{i-1}\gamma_{i-1} = 0$. These imply that only $m_2 = mx_{2r}/x_{2r-1} = m_1(x_{2r}/x_{2r-1})^l$ can cancel it, so it must be in $M(h)$. This proves our claim for every $l \geqslant 0$.

However, $m_1(x_{2r}/x_{2r-1})^l \in R$ only if $l \leqslant \alpha_{2r-1}$ so the claim cannot be true for every l. This contradiction shows that indeed there must exist $\beta_{2r-1} \geqslant 0$ satisfying the proposition. ■

Proposition 2.8 *Suppose $C_n \neq 0$ and $m = \prod x_i^{\gamma_i} \in M(h)$ is such that $\gamma_n = \alpha_n$, $\gamma_1 = \alpha_1$ and $\gamma_2 = \alpha_2 = C_n\alpha_n$. If $mx_n^k/m_2 \in M(h)$, where $x_n \nmid m_2$, then $m_2 = x_1^k$.*

Proof The proof is by induction on the exponent i of x_1 in m_2. Let $m' = mx_n^k/m_2 = \prod_j x_j^{\beta_j}$. Of course, $\beta_j \leqslant \gamma_j$ for $j < n$. We have $\gamma_2 = C_n\gamma_n$, so if $\gamma_n > 0$ then $C_n = \gamma_2/\gamma_n > 0$. If, on the other hand, $\gamma_n = 0$ then $\gamma_2 = \beta_2 = 0$.

Now $x_1 m' \in M(d(m'))$ has the coefficient $\beta_2 - C_n\beta_n = \gamma_2 - C_n\gamma_n - (\gamma_2 - \beta_2) - C_n k = 0 - \cdots \neq 0$. It could be canceled only by some $x_1 m'/x_j$, where $1 < j$. All these are lexicographically larger than m_1 if $i = 0$, a contradiction.

If $i > 0$, then the exponent of x_1 is $i-1$ in $m_2 x_j/x_1$. If $j < n$ then $x_1 m'/x_j = mx_n^k/(m_2 x_j/x_1)$ satisfies the assumption of the proposition, so by induction $m_2 x_j/x_1 = x_1^k$, a contradiction. So we must have $x_j = x_n$ and $x_1 m'/x_n = mx_n^{k-1}/(m_2/x_1)$ satisfies the assumption of the proposition. Here induction gives $m_2/x_1 = x_1^{k-1}$ and $m_2 = x_1^k$, as required. ■

Corollary 2.9 *Suppose $C_n \neq 0$ and $m = \prod x_i^{\gamma_i} \in M(h)$ is such that $\gamma_n = \alpha_n$, $\gamma_1 = \alpha_1$ and $\gamma_2 = \alpha_2 = C_n\alpha_n$. Then $l = \gamma_1 - C_{n-1}\gamma_{n-1}$ is a nonnegative integer and $m' = m(x_n/x_1)^l \in M(h)$. In particular, $\alpha_1 - C_{n-1}\alpha_{n-1}$ is a nonnegative integer.*

Proof We show that otherwise $m(x_n/x_1)^l \in M(h)$ for every $l \geqslant 0$, an obvious contradiction.

For $l = 0$ this holds vacuously. Let $l > 0$ and suppose it holds for l, that is $m' = mx_n^l/x_1^l \in M(h)$. The coefficient of $m'x_n \in M(d(m'))$ is $\gamma_1 - l - C_{n-1}\gamma_{n-1}$. If it is 0 then $\gamma_1 - C_{n-1}\gamma_{n-1} = l$ a nonnegative integer as we claimed. Otherwise, it must be canceled by some $m'x_n/x_j = mx_n^{l+1}/x_jx_1^l \in M(h)$. By Proposition 2.8, $j = 1$ and we get the required monomial for $l+1$, too. ■

Proposition 2.10 *Let $h, g \in R^d$, where g is a monomial. If the leading monomial of h is $m_1 = x_1^k g$ with $k > 0$ then $C_1C_2\cdots C_n$ is a k-th root of unity, in particular, all $C_i \neq 0$. If further $n > 4$ then $C_1C_2\cdots C_n = 1$.*

Proof If $1 \neq g = \prod_i x_i^{\gamma_i}$ (with $\gamma_i = \alpha_i$ for $i \neq 1$) is the monomial constant then

$$0 = d(g) = \prod_i x_i^{\gamma_i} \cdot \sum_i \gamma_i(x_{i-1} - C_i x_{i+1}) = g\sum_i(\gamma_{i+1} - C_{i-1}\gamma_{i-1})x_i.$$

Hence $\gamma_{i+2} = C_i\gamma_i$ for every i. If n is odd then $C_1C_2\cdots C_n = 1$ so the proposition is proved. Exactly the same works if n is even and $\gamma_1 \neq 0 \neq \gamma_2$.

So we have to consider two cases. Either $g = 1$ or $g = g_{\mathcal{A}}^l$ with $l > 0$ and $\mathcal{A} \in \{\mathcal{E}, \mathcal{O}\}$. If $l > 0$ then we will use $C_i \neq 0$ for $i \in \mathcal{A}$ to prove that in certain monomials the exponent of x_i is not smaller than in g. For $l = 0$ this is obvious. The rest of the proof works equally for arbitrary l.

The proof is almost the same for $\mathcal{E}$ and for $\mathcal{O}$. First we assume $\mathcal{A} = \mathcal{O}$ and $l > 0$, so $C_{2i-1} > 0$ for every $1 \leqslant i \leqslant n/2$. We apply Proposition 2.2 for $j = n-1, n-3, \ldots, 1$ consecutively. In the first step $r = k$ and $m = m_1$. By Proposition 2.2, we obtain $m_{n-1} = m_1(x_{n-1}/x_1)^{k_{n-1}} \in M(h)$ with $k_{n-1} \geqslant k$. If $C_n = 0$ then we get $k \leqslant 0$, contrary to our assumption. So $C_n \neq 0$ and we also get $m_1(x_1/x_{n-1})^{(k_{n-1}-k)/C_{n-1}} \in M(h)$. As m_1 is the leading monomial $k = k_{n-1}$.

By induction, we obtain monomials $m_{2i-1} = m_{2i+1}(x_{2i-1}/x_{2i+1})^{k_{2i-1}} \in M(h)$ with $k_{2i-1} \geqslant k_{2i+1}$. Finally we reach $i = 1$ and obtain a monomial $m = m_3(x_1/x_3)^{k_1}$ in $M(h)$ such that the exponent of x_1 is $\alpha_1 - k_{n-1} + k_1 \geqslant \alpha_1$. As m_1 is the leading monomial we must have $k_1 = k_3 = \cdots = k_{n-1} = k$. So in each of the applications of Proposition 2.2 we have $r = t_0 = k_{n-1} = k$, hence if $g{x_j}^s x_{j+1}^{t-s}{x_{j+2}}^{k-t} \in M(h)$ then $0 \leqslant s \leqslant t \leqslant k$.

Let now $\mathcal{A} = \mathcal{E}$ and $l > 0$, so $C_{2i} > 0$ for every $1 \leqslant i \leqslant n/2$. First we apply Proposition 2.2 for $j = n, m = m_1$. Then $r = k$ and we get for $s = 0, k_n = t_0 \geqslant k$ that $m_n = m_1 x_n^{k_n} x_1^{-k} x_2^{k-k_n} \in M(h)$. Now we proceed as in the odd case and consecutively we obtain monomials $m_{2i} = m_{2i+2}(x_{2i}/x_{2i+2})^{k_{2i}} \in M(h)$ with $k_{2i} \geqslant k_{2i+2}$. Thus we find that in m_2 the exponent of x_n is $\alpha_n + k_n - k_{n-2} \leqslant \alpha_n$ and the exponent of x_2 is $\alpha_2 + k - k_n + k_2 \geqslant \alpha_2 + k$. Now apply Proposition 2.2 for $j = n$, $m = m_3$. Then $r \geqslant k$, so $t_0 \geqslant k$ and $s = s_0 \leqslant 0$ gives $m_3 x_n^s x_1^{t_0-s_0} x_2^{-t_0} \in M(h)$. By m_1 being the leading term, at every application of Proposition 2.2 we have $r = t_0 = k$ and $s_0 = 0$ and all $k_{2i} = k$.

From now on there is no loss of generality in assuming $\mathcal{A} = \mathcal{O}$ or $g = 1$, that is, $\alpha_{2i} = 0$ for every i.

We now determine the coefficients in question. We will apply Proposition 2.3 repeatedly, but first we have to establish that all C_i are nonzero. Without loss of generality we assume that m_1 has a coefficient equal to 1. We prove by induction on i that the coefficient of gx_i^k is $\prod_{j<i} C_j^k$. We would like to apply Proposition 2.3 for $j_0 = i - 1$. If i is even then the coefficient of gx_{i+1}^k is $\prod_{j<i+1} C_j^k$ by considering $q = k$ in Proposition 2.3 (ii). (By Proposition 2.2, $mx_{j_0}^p x_{j_0+1}^{-p-q} x_{j_0+2}^q \in M(h)$ implies $p, q \geqslant 0$.) If i is odd, then $\beta_{j_0+3} > 0$ implies $C_{j_0+1} > 0$ and $k = \beta_{j_0+1} - \beta_{j_0+3}/C_{j_0+1}$. So we can apply Proposition 2.3 (ii) again. Finally, for $i = n + 1$ we get $1 = C_1^k \cdots C_n^k$ so $z = C_1 C_2 \cdots C_n$ is a k-th root of unity, in particular, all $C_i \neq 0$. Let us abbreviate $d_i = \prod_{j<i} C_j^k$. We see from our applications of Proposition 2.3 that for every i, s the coefficient of $gx_i^{k-s} x_{i+1}^s$ is $d_i \binom{k}{s} C_i^s$.

Now let $n > 4$. If $z \neq 1$ then there must be an index j such that C_j is not positive rational. For otherwise every $0 < C_j$ and hence $0 < \prod C_j = z$ implies $z = 1$ as the only such root of unity. Fix this index j.

Case (⋆): j is odd. We use Proposition 2.3 with $j_0 = j$ and $m = gx_{j_0}^k$. All assumptions are satisfied, so the coefficient of $gx_j x_{j+2}^{k-1}$ is equal to $kd_{j+1} C_j^{-1} C_{j+1}^{k-1} = kd_j C_j^{k-1} C_{j+1}^{k-1}$. Also, the coefficient of $gx_j^{k-1} x_{j+2}^1$ is equal to $kd_j C_j C_{j+1}$.

Case (⋆⋆): C_i is positive rational for every odd index i, so j is even. Proposition 2.2 is not sufficient in this case, we have to treat five consecutive indices. This is the point where it is crucial that $n \neq 4$, so $n \geqslant 6$. Let $e_{a,b,c,d}$ denote the coefficient of $gx_{j-1}^a x_j^b x_{j+1}^c x_{j+2}^d x_{j+3}^{k-a-b-c-d}$. Here we rely on the following equation expressing the coefficient of $gx_{j-1}^a x_j^b x_{j+1}^c x_{j+2}^d x_{j+3}^{k-a-b-c-d+1}$ in $d(h) = 0$:

$$\begin{aligned} 0 &= be_{a-1,b,c,d} + (c - C_{j-1}a)e_{a,b-1,c,d} + (d - C_j b)e_{a,b,c-1,d} \\ &\quad + (k - a - b - c - d + 1 - C_{j+1}c)e_{a,b,c,d-1} - C_{j+2}d e_{a,b,c,d}. \end{aligned} \tag{7}$$

As j is even, we have $e_{0,0,0,0} = d_{j+3} \neq 0$. We claim that if $a < 0$ or $c < 0$ or $k - a - b - c - d < 0$ then $e_{a,b,c,d} = 0$. Its proof is similar to the proof of Proposition 2.2.

Let $a_0 \leqslant 0$ be the smallest possible such that there exist b_0, c_0, d_0 with $e_{a_0,b_0,c_0,d_0} \neq 0$. Without loss of generality also assume that c_0 is as small as possible for this a_0. Suppose here $c_0 \neq C_{j-1}a_0$. Then by (7) for a_0, b_0+1, c_0, d_0 we have $e_{a_0,b_0+1,c_0,d_0-1} \neq 0$ or $e_{a_0,b_0+1,c_0,d_0} \neq 0$. In either case we can increase b again. But clearly b is bounded, so we conclude that $c_0 = C_{j-1}a_0$. Mutatis mutandis we get that if $c_1 \leqslant c_0$ is the smallest possible and a_1 is the smallest possible for such c_1 then $c_1 = C_{j-1}a_1$. As $C_{j-1} > 0$ we have $c_1 = c_0$ and $a_1 = a_0$. The same proof shows that $k - a - b - c - d$ is the smallest for $k - a - b - c - d = C_{j+1}c_0$.

Consider now $e_{a_0,b_0,c_0,d_0} \neq 0$ where $c_0 = C_{j-1}a_0$ and $k - a_0 - b_0 - c_0 - d_0 = C_{j+1}c_0$. By assumption, C_j is not positive rational (and nonzero), in particular, for nonnegative integers b, d the expression $d - C_j b = 0$ only if $b = d = 0$. Hence in (7) for $a_0, b_0, c_0 + 1, d_0$ we have $e_{a_0,b_0,c_0+1,d_0-1} \neq 0$ or $e_{a_0,b_0-1,c_0+1,d_0} \neq 0$. We repeat this step $b_0 + d_0$ times to obtain $e_{a_0,0,c_0+b_0+d_0,0} \neq 0$. Suppose $a_0 < 0$ and hence $k - a_0 - b_0 - c_0 - d_0 = C_{j+1}C_{j-1}a_0 < 0$. We now apply Proposition 2.2 for the first three indices, that is for $j-1, j, j+1$. We get $t_0 \geqslant r = c_0 + b_0 + d_0 - C_{j-1}a_0$ resulting in $a_0 + t_0 \geqslant k + a_0 + b_0 + c_0 + d_0 - k - C_{j-1}a_0 > k - C_{j-1}a_0 > k$. That is

$m_{j-1} = gx_{j-1}^{a_0+t_0} x_{j+1}^{c_0+b_0+d_0-t_0} x_{j+3}^{k-a_0-b_0-c_0-d_0} \in M(h)$ the exponent of x_{j-1} is at least $\alpha_{j-1}+k+1$. By induction for $i = j-3, j-5, \ldots, 3$ we obtain that the exponent of x_i in $m_i \in M(h)$ is at least $\alpha_i + k + 1$. Thus we reach $i = 1$ and a monomial in $M(h)$ where the exponent of x_1 is at least $\alpha_1 + 1$, contradicting that m_1 is the leading monomial. So $a_0 = c_0 = k - b_0 - d_0 = 0$.

Now determine $e_{0,s,k-1-s,1}$ using Proposition 2.3 for $j_0 = j$ and $m = gx_{j+1}^k$. The above discussion confirms that the assumptions are satisfied. So

$$e_{0,k-1,0,1} = \frac{kd_{j+1}C_{j+1}}{C_j^{k-1}} \quad \text{and} \quad e_{0,1,0,k-1} = \frac{kd_{j+1}C_{j+1}^{k-1}}{C_j}.$$

As j is even, the assumptions of Proposition 2.3 (i) are satisfied for $j_0 = j-1$, $m = gx_j^{k-1}x_{j+2}$ and $t = \beta_{j_0+1} - C_{j_0-1}\beta_{j_0-1} = k-1$. So $e_{k-1,0,0,1} = kd_{j-1}C_{j-1}C_jC_{j+1}$. We can now apply it again to $j_0 = j+1$, $m = gx_{j-1}^{k-1}x_{j+2}$ and $t = \beta_{j_0+1} - C_{j_0-1}\beta_{j_0-1} = 1$ to get $e_{k-1,0,1,0} = kd_{j-1}C_{j-1}C_j$. Exactly the same way we obtain $e_{1,0,k-1,0} = kd_{j-1}C_{j-1}^{k-1}C_j^{k-1}$.

So in both cases ($\star$) and ($\star\star$) we find an odd index j such that the coefficient of $gx_jx_{j+2}^{k-1}$ is $kd_jC_j^{k-1}C_{j+1}^{k-1}$, while the coefficient of $gx_j^{k-1}x_{j+2}$ is $kd_jC_jC_{j+1}$. We shift the indices by 2 in the second monomial using Proposition 2.3 again first for $j_0 = j+2$ and then for $j_0 = j$. The coefficient of $gx_j^{k-1}x_{j+4}$ is $kd_jC_jC_{j+1}C_{j+2}C_{j+3}$. Then the coefficient of $gx_{j+2}^{k-1}x_{j+4}$ is $kd_{j+2}C_{j+2}C_{j+3}$.

If $n = 5$ then we cannot do this as $j+4$ and j are adjacent. But $n = 5$ is odd, so $g = 1$ and there is no worry that the exponent of x_i becomes smaller than in g (see the remark at the beginning of the proof). So we use Proposition 2.3 to compute the coefficients in a different way, we increase the indices one-by-one. Now the coefficient of $x_j^{k-1}x_{j+3}$ is $kd_jC_jC_{j+1}C_{j+2}$ and using Proposition 2.3 for $j_0 = j-1$ the coefficient of $x_{j+1}^{k-1}x_{j+3}$ is $kd_{j+1}C_{j+1}C_{j+2}$. Doing this again, the coefficient of $x_{j+1}^{k-1}x_{j+4}$ is equal to $kd_{j+1}C_{j+1}C_{j+2}C_{j+3}$ and the coefficient of $x_{j+2}^{k-1}x_{j+4}$ is equal to $kd_{j+2}C_{j+2}C_{j+3}$, the same as above.

By a final induction we prove that the coefficient of $gx_{j+2}^{k-1}x_{j+s}$ is equal to $kd_{j+2}\prod_{i=2}^{s-1}C_{j+i}$. The details are omitted. The case $s = n-2$ implies that the coefficient of $gx_jx_{j+2}^{k-1}$ is $kd_{j+2}C_{j+2}C_{j+3}\cdots C_{j-1}$. However, we have already determined that to be $kd_jC_j^{k-1}C_{j+1}^{k-1}$. So

$$0 = kd_{j+2}C_{j+2}C_{j+3}\cdots C_{j-1} - kd_jC_j^{k-1}C_{j+1}^{k-1} = kd_jC_j^{k-1}C_{j+1}^{k-1}(z-1).$$

Hence $z = 1$ as required. ∎

Proposition 2.11 *Let $n = 4$ and $h, g \in R^d$, where g is a monomial. Assume either every C_i is positive rational, or C_4 is not. If the leading monomial of h is $m_1 = x_1^k g$ with $k > 0$ then $C_1C_2C_3C_4 = \pm 1$. If $C_1C_2C_3C_4 = -1$ then $C_2 = 1$ and at least one of $C_1 = 1$ and $C_3 = 1$ also holds.*

Proof By Proposition 2.10, we have $z = C_1C_2C_3C_4$ is a k-th root of 1. If all parameters are positive rational numbers then the product must be 1. From now on assume C_4

is not positive rational. First we prove that if $z \neq 1$ then $z = -C_2$. In the next step we verify that $z = -1$ so k is even. Then we suppose by contradiction that $C_1 \neq 1 \neq C_3$ from which we derive $C_1C_3 = 1$ and subsequently we reach a contradiction.

As C_4 is not positive rational, $g_{\mathcal{E}}$ is not defined, so $g = g_{\mathcal{O}}^l$ or $g = 1$. Let $f_{a,b,c}$ denote the coefficient of $gx_1^{k-a-b-c}x_2^a x_3^b x_4^c$ in h. By assumption, without loss of generality $f_{0,0,0} = 1$. We observe that the coefficient of $gx_1^{k+1-a-b-c}x_2^a x_3^b x_4^c$ in $d(h)$ is equal to

$$\begin{aligned}0 = (a - C_4c)\, f_{a,b,c} + (b - C_1(k+1-a-b-c))\, f_{a-1,b,c}\\ + (c - C_2a)\, f_{a,b-1,c} + (k+1-a-b-c-C_3b)\, f_{a,b,c-1}.\end{aligned} \tag{8}$$

We have to prove again that exponents cannot become negative, so $a, b, c, k-(a+b+c) \geqslant 0$ if $f_{a,b,c} \neq 0$. Of course, this holds if $g = 1$.

Suppose b_0 is the smallest such that there exist a_0, c_0 such that $f_{a_0,b_0,c_0} \neq 0$. If a_0, c_0 are not both 0 then by (8) one of them might be decreased by 1. (Using here that $a_0 - C_4b_0 \neq 0$). Repeating this until both a_0, c_0 become 0 we conclude $f_{0,b_0,0} \neq 0$. This is the coefficient of $gx_1^{k-b_0}x_3^{b_0}$, so by m_1 being the leading monomial, $b_0 \geqslant 0$. By a similar argument, if $a_1+b_1+c_1$ is the maximal possible such that $f_{a_1,b_1,c_1} \neq 0$, then we conclude that $f_{0,a_1+b_1+c_1,0} \neq 0$. If $a_1+b_1+c_1 > k$ then apply Proposition 2.2 for $j = 1$. We have $t \geqslant r \geqslant a_1+b_1+c_1 - C_1(k-a_1-b_1-c_1) > a_1+b_1+c_1$ (as $C_1 = 1$ if $g \neq 1$). By Proposition 2.2, $f_{0,a_1+b_1+c_1-t,0} \neq 0$ contradicting $b_0 \geqslant 0$. Thus indeed $k-(a+b+c) \geqslant 0$, if $f_{a,b,c} \neq 0$.

We have from the proof of Proposition 2.10 that

$$\begin{aligned} f_{s,0,0} &= \binom{k}{s} C_1^s, & f_{k-s,s,0} &= \binom{k}{s} C_1^k C_2^s,\\ f_{0,k-s,s} &= \binom{k}{s} C_1^k C_2^k C_3^s, & f_{0,0,k-s} &= \binom{k}{s} C_1^k C_2^k C_3^k C_4^s. \end{aligned} \tag{9}$$

We apply Proposition 2.3 for $j_0 = 1$, $m = m_1$ to obtain

$$f_{t,0,s} = \binom{k}{s+t}\binom{s+t}{t} C_1^t C_4^{-s}, \tag{10}$$

as C_4 is not positive rational.

Next we claim that

$$f_{k-1-s,1,s} = k\,\frac{C_1^{k-s}}{C_4^s}\left((z-1)\sum_{i=0}^{s-1}\frac{z^i}{C_2^i}\binom{k-1}{s-1-i} + \binom{k-1}{s}C_2\right).$$

The proof is by induction on s, for $s = 0$ we know it from (9). By (8) for $(k-s, 1, s)$ we get

$$0 = f_{k-1-s,1,s} + (s-(k-s)C_2)\, f_{k-s,0,s} - C_3\, f_{k-s,1,s-1}$$

$$= f_{k-1-s,1,s} + (s-(k-s)C_2)\binom{k}{s}\frac{C_1^{k-s}}{C_4^s}$$
$$-\frac{kC_1^{k-s+1}C_3}{C_4^{s-1}}(z-1)\sum_{i=0}^{s-2}\frac{z^i}{C_2^i}\binom{k-1}{s-2-i}-\binom{k-1}{s-1}\frac{kC_2C_3C_1^{k-s+1}}{C_4^{s-1}}$$
$$= f_{k-1-s,1,s} + k\binom{k-1}{s-1}\frac{C_1^{k-s}}{C_4^s} - C_2\binom{k-1}{s}\frac{kC_1^{k-s}}{C_4^s}$$
$$-\frac{kC_1^{k-s}}{C_4^s}(z-1)\sum_{i=0}^{s-2}\frac{z^{i+1}}{C_2^{i+1}}\binom{k-1}{s-1-(i+1)} - z\binom{k-1}{s-1}\frac{kC_1^{k-s}}{C_4^s},$$

proving the claim for s.

We compare it for $s = k-1$ to (9):

$$kC_1^kC_2^kC_3^{k-1} = f_{0,1,k-1} = \frac{kC_1}{C_4^{k-1}}\left((z-1)\sum_{i=0}^{k-2}\frac{z^i}{C_2^i}\binom{k-1}{k-2-i}+C_2\right). \tag{11}$$

The sum is equal to $((z/C_2+1)^{k-1}-1)/(z/C_2)$ so in the parentheses above we have

$$\frac{C_2}{z}\left((z-1)\left(\left(\frac{z}{C_2}+1\right)^{k-1}-1\right)+z\right) = \frac{C_2}{z}\left((z-1)\left(\frac{z}{C_2}+1\right)^{k-1}+1\right).$$

Dividing both sides of (11) by kC_1C_2 and multiplying by zC_4^{k-1} we get

$$1 = z^k = (z-1)\left(\frac{z}{C_2}+1\right)^{k-1}+1.$$

If $z \neq 1$ then $z = -C_2$, as claimed. From now on we assume $z = -C_2 \neq 1$. In particular, $C_2 = 1$ implies $z = -1$ and $k \geqslant 2$ is even. Otherwise, C_2 is not rational.

If C_2 is not rational then, using Proposition 2.3 (ii) for $j_0 = 2$, we get

$$f_{s,k-t-s,t} = \binom{k}{s+t}\binom{s+t}{t}C_1^kC_2^kC_3^tC_2^{-s}.$$

Comparing the case $s = k-1$, $t = 1$ to (10) for $t = k-1$, $s = 1$, we have

$$kC_1^kC_2C_3 = f_{k-1,0,1} = k\frac{C_1^{k-1}}{C_4} = \frac{kC_1^kC_2C_3}{z},$$

implying $z = 1$, as claimed.

From now on we assume that k is even, $C_2 = 1$, $z = C_1C_3C_4 = -1$ and we also suppose $C_1 \neq 1 \neq C_3$ to derive contradiction. Next we claim

$$f_{k-t-s,t,s} = \binom{k}{s+t}\binom{s+t}{t} C_1^k(-C_3)^s \prod_{i=1}^{s} \frac{k+1-2t-2i}{k+1-2i}. \tag{12}$$

It is true for $s = 0$ and for $t = 0$ so we use induction for $t + s$ assuming $s, t > 0$. From (8) for $(k + 1 - t - s, t, s)$ we obtain

$$f_{k-t-s,t,s} = \frac{k+1-t-2s}{t} f_{k-(t-1)-s,t-1,s} - (-C_3) f_{k-t-(s-1),t,s-1}. \tag{13}$$

We use the inductive assumption on the right hand side. We find identical factors $F = C_1^k(-C_3)^s\binom{k}{s+t-1}\prod_{i=1}^{s-1}(k+1-2t-2i)/(k+1-2i)$ at both terms. Now we have

$$\begin{aligned}
&\frac{1}{F}\left(\frac{k+1-t-2s}{t} f_{k-(t-1)-s,t-1,s} - (-C_3) f_{k-t-(s-1),t,s-1}\right)\\
&\quad = \frac{k+1-t-2s}{t}\binom{s+t-1}{t-1}\frac{k+1-2t}{k+1-2s} - \binom{s+t-1}{t}\\
&\quad = \binom{s+t}{t}\left(\frac{k+1-t-2s}{s+t}\,\frac{k+1-2t}{k+1-2s} - \frac{s}{s+t}\,\frac{k+1-2s}{k+1-2s}\right)\\
&\quad = \binom{s+t}{t}\frac{(k+1)^2-(k+1)(3t+3s)+2t(2s+t)+2s^2}{(s+t)(k+1-2s)}\\
&\quad = \frac{k+1-s-t}{s+t}\binom{s+t}{t}\frac{k+1-2t-2s}{k+1-2s}.
\end{aligned}$$

If we multiply by F, using $\binom{k}{s+t} = (k+1-s-t)/(s+t)\cdot\binom{k}{s+t-1}$, then we get that (13) implies (12).

Now we go on to prove that $C_1C_3 = 1$. We use (8) for $(k - s - 1, 1, s)$:

$$\begin{aligned}
0 = (1-C_1) f_{k-s-2,1,s} &+ (1-C_3) f_{k-s-1,1,s-1}\\
&+ (2s-k+1) f_{k-s-1,0,s} + (k-1-s-sC_4) f_{k-s-1,1,s}
\end{aligned}$$

Let us multiply this equation by $(1-C_3)^{-s}(1-C_1)^s(-1)^s$ and sum them up through $s = 0, \ldots, k-1$. In the resulting sum

$$\sum_{s=0}^{k-1} f_{k-s-2,1,s}\frac{(1-C_1)^{s+1}}{(1-C_3)^s}(-1)^s + \sum_{s=0}^{k-1} f_{k-s-1,1,s-1}\frac{(1-C_1)^s}{(1-C_3)^{s+1}}(-1)^s = 0$$

because each term appears with opposite signs. For the rest we apply (10) and (12) to get

$$0 = kC_1^{k-1} \sum_{s=0}^{k-1} \left(\frac{1-C_1}{1-C_3}\right)^s C_3^s \binom{k-1}{s} \cdot (2s-(k-1))\left(1-C_1+\frac{C_1 s(C_4+1)}{k-1}\right). \tag{14}$$

We divide by the nonzero kC_1^{k-1}. We first split

$$(2s-(k-1))\left(1-C_1+\frac{C_1 s(C_4+1)}{k-1}\right) = A_0 + sA_1 + s^2 A_2,$$

where

$$A_0 = (k-1)(C_1-1), \qquad A_1 = 2-3C_1-C_1C_4, \qquad A_2 = \frac{2(1+C_4)C_1}{k-1}.$$

Now we determine sums of the three distinct series. For this we abbreviate $y = C_3(1-C_1)/(1-C_3)$, so $y+1 = (1-C_1C_3)/(1-C_3)$, $C_1C_3(1+C_4) = C_1C_3 - 1$ and $yC_1(1+C_4) = (y+1)(C_1-1)$. We have

$$\begin{aligned}
\sum_{s=0}^{k-1} y^s \binom{k-1}{s} A_0 &= (k-1)(C_1-1)(y+1)^{k-1},\\
\sum_{s=1}^{k-1} y^s \binom{k-1}{s} sA_1 &= (2-3C_1-C_1C_4)(k-1)(y+1)^{k-2}y,\\
\sum_{s=1}^{k-1} y^s \binom{k-1}{s} s^2 A_2 &= \sum_{s=0}^{k-2} y^s \binom{k-2}{s} (s+1)2(y+1)(C_1-1)\\
&= 2(y+1)(C_1-1)\big((k-2)(y+1)^{k-3}y + (y+1)^{k-2}\big)\\
&= 2(C_1-1)(y+1)^{k-2}((k-1)y+1).
\end{aligned}$$

(Note that the third conclusion is correct even for $k=2$.)

Summing them up we get

$$\begin{aligned}
0 &= (y+1)^{k-2}(C_1-1)\left\{(k-1)\left[(y+1)+\frac{(2-3C_1-C_1C_4)}{C_1-1}y+2y\right]+2\right\}\\
&= (y+1)^{k-2}(C_1-1)\left\{(k-1)\left[y+1-\frac{C_1(1+C_4)y}{C_1-1}\right]+2\right\}\\
&= 2(y+1)^{k-2}(C_1-1).
\end{aligned}$$

Hence $y+1=0$, that is $C_1C_3 = 1 = -C_4$.

Now we derive the final contradiction. Use (8) for $(1, k-1, 0)$ and for $(0, k-1, 1)$ to get

$$0 = (k-1-C_1) f_{0,k-1,0} - f_{1,k-2,0} + f_{1,k-1,0},$$
$$0 = (1-(k-1)C_3) f_{0,k-1,0} + f_{0,k-2,1} - C_4 f_{0,k-1,1}.$$

The last terms in each of these equations can be obtained from (9)

$$f_{1,k-1,0} = kC_1^k, \qquad f_{0,k-1,1} = kC_1^k C_3 = kC_1^{k-1}.$$

We multiply the first equation by C_3 and add to the second, thus canceling the coefficient of $f_{0,k-1,0}$

$$0 = f_{0,k-2,1} - C_3 f_{1,k-2,0} + 2kC_1^{k-1}.$$

On the other hand, using (8) for $(1, k-2, 1)$, we get

$$\begin{aligned} 0 &= (k-2-C_1) f_{0,k-2,1} + (1-(k-2)C_3) f_{1,k-2,0} + (1-C_4) f_{1,k-2,1} \\ &= (k-2-C_1)(f_{0,k-2,1} - C_3 f_{1,k-2,0}) + 2k(k-1)(-1)C_1^k C_3 \frac{3-k}{k-1} \\ &= (k-2-C_1)(-2)kC_1^{k-1} - 2k(3-k)C_1^{k-1} = -2kC_1^{k-1}(1-C_1). \end{aligned}$$

This implies $C_1 = 1$ contradicting our assumptions. This finishes the proof of the proposition. ∎

We are now ready to finish the proof of Lemma 2.4. We have $h \in R^d$ of positive degree and $m_1 \in M(h)$, the leading monomial of h with respect to the lexicographic ordering of monomials. Recall that we assumed $C_n \neq 1$. We also assumed that if $C_n = 0$ then every $C_i \in \{0, 1\}$ and either all $C_i = 0$ or $C_1 = 1$. Of course, by Proposition 2.5, every exponent with even index $\alpha_{2i} = 0$ in this case.

If all $C_i = 0$ then, by Propositions 2.5 and 2.7, all $\alpha_i = 0$ save possibly $\alpha_1 > 0$. But then Proposition 2.10 implies that all $C_i \neq 0$. This contradiction shows that $\alpha_1 = 0$ and $R^d = K$ in this case.

If $C_n = 0$ but $C_1 = 1$ then $\alpha_1 \geqslant \alpha_3 \geqslant \cdots$, by Proposition 2.7. If all α_i are equal and n is even then all coefficients with odd indices $C_{2i+1} = 1$ and $m = g_{\mathcal{O}}^{\alpha_1}$ as we wanted. Otherwise, $\alpha_{n-1} < \alpha_1$ (this clearly holds if n is odd, because then $\alpha_{n-1} = 0$) so $x_n m_1 \in M(d(m))$. But this can be canceled only by $x_n m_1 / x_{n-1}$ and this can be repeated α_{n-1} steps until no division by x_{n-1} is possible. (No other decrease is ever possible.) This contradiction shows that $R^d = K$ also in this case.

So from now on we assume $C_n \neq 0$. If $C_1 = 0$ then all $\alpha_{2i+1} = 0$ for $i > 0$. If n is odd then it implies $\alpha_n = 0$ and hence all $\alpha_i = 0$ save possibly $\alpha_1 > 0$. But then Proposition 2.10 is a contradiction and $R^d = K$ again. If n is even then $g_{\mathcal{E}}$ is defined and $m_1 = g_{\mathcal{E}}^l x_1^{\alpha_1}$. If $\alpha_1 > 0$ then Proposition 2.10 implies that $C_1 \neq 0$, a contradiction. So $\alpha_1 = 0$ and $m_1 = g_{\mathcal{E}}^l$ is a constant and the proof is done by induction for $h - m_1$.

From now on we assume $C_1 \neq 0$. For $n = 4$ also assume that either every parameter is positive rational, or that C_4 is not.

We are going to derive a contradiction, so $R^d = K$ unless the following two conditions hold:

(A) In Proposition 2.7 we have $C_{2i-1}\alpha_{2i-1} = \alpha_{2i+1}$ for every $1 < i < n/2$.
(B) In Proposition 2.7 and Corollary 2.9 the two integers coincide: $C_1(\alpha_1 - \beta_1) = \alpha_3$ and $\alpha'_1 = \alpha_1 - \beta_1 = C_{n-1}\alpha_{n-1}$. In other words, $C_{n-1}\alpha_{n-1}$ is a nonnegative integer and $C_1 C_{n-1}\alpha_{n-1} = \alpha_3$.

Claim 2.12 *If* (A) *and* (B) *hold then h is a polynomial in the elements of $\mathcal{H}$.*

Proof Let $\alpha'_i = \alpha_i$ for $i > 1$. The equalities in (A) and (B) together with Proposition 2.5 imply that $g = \prod_{i=1}^n x_i^{\alpha'_i}$ is a constant of d. If $\alpha'_i \neq 0$ then $\alpha'_{i-2}C_{i-2} \neq 0$ so by backward induction we get that all C_{i-2l} are positive rational. Take $\mathcal{A} \subseteq \mathbb{Z}_n$ minimal closed under $j \mapsto j + 2$ such that $i \in \mathcal{A}$. Then by the above, $\alpha'_i = \alpha'_i \prod_{j\in\mathcal{A}} C_j$ so $\prod_{j\in\mathcal{A}} C_j = 1$, hence $g_{\mathcal{A}}$ is defined and $\alpha'_j = e\theta_j$ for every $j \in \mathcal{A}$ with $e = \gcd(\alpha'_j)_{j\in\mathcal{A}}$. So g can be expressed using the defined ones of $g_{\mathbb{Z}_n}, g_{\mathcal{E}}, g_{\mathcal{O}}$. If $\alpha'_1 = \alpha_1$ then g is the leading monomial of h so $h - g$ is lexicographically smaller so the proof is done by induction.

Suppose that $\alpha'_1 < \alpha_1$ and first assume $n > 4$. Then Proposition 2.10 applies and it follows that f is defined. Hence $f^{\alpha_1-\alpha'_1}g$ has the same lexicographically largest monomial as h, we may cancel it and the proof is done by induction.

Finally, if $n = 4$ then Proposition 2.11 applies and it follows that f is defined or $\alpha_1 - \alpha'_1$ is even and f_4 is defined. So either f is defined and $f^{\alpha_1-\alpha'_1}g$ has the same lexicographically largest monomial as h, or f_4 is defined and $f_4^{(\alpha_1-\alpha'_1)/2}g$ has the same lexicographically largest monomial as h. In both cases we may cancel it and the proof is done by induction. ∎

For the rest we assume that at least one of (A) and (B) does not hold. We first construct a monomial $m = \prod x_i^{\gamma_i} \in M(h)$ such that

$$\gamma_n = \alpha_n, \qquad \gamma_1 = \alpha_1, \qquad \gamma_2 = \alpha_2, \qquad C_1 C_{n-1}\gamma_{n-1} \neq \gamma_3. \tag{15}$$

If (B) does not hold, then $m = m_1$ suffices. If (B) holds and $C_{n-1} = 0$ or $\alpha_{n-1} = 0$ then (A) also holds with $\alpha_{2i+1} = 0$ for $i > 0$.

Suppose (B) holds but (A) does not, and let i be the smallest index such that $C_{2i-1}\alpha_{2i-1} \neq \alpha_{2i+1}$. We apply Proposition 2.7 for decreasing indices $r = i, i-1, \ldots, 2$. Suppose we have reached the step r, that is, given $m = \prod x_i^{\gamma_i} \in M(h)$ with $\gamma_i = \alpha_i$ if $i < 2r + 1$ and for $i = n$. By Proposition 2.7, there exists a nonnegative integer β_{2r-1} such that $C_{2r-1}(\gamma_{2r-1} - \beta_{2r-1}) = \gamma_{2r+1}$ and $m' = m(x_{2r}/x_{2r-1})^{\beta_{2r-1}} \in M(h)$. We replace m for this m'. In the step r the exponent of γ_{2r-1} decreases, so finally $\gamma_3 < \alpha_3 = C_1 C_{n-1}\alpha_{n-1}$. In every step γ_{n-1} is constant, only in step $r = (n-1)/2$ may it grow. As $C_1 C_{n-1} \neq 0$ and $0 < \alpha_{n-1} \leqslant \gamma_{n-1}$ so $\gamma_3 = C_1 C_{n-1}\gamma_{n-1}$ implies that $C_1 C_{n-1} > 0$ is rational. But for $C_1 C_{n-1} > 0$ we have $\gamma_3 < \alpha_3 = C_1 C_{n-1}\alpha_{n-1} \leqslant C_1 C_{n-1}\gamma_{n-1}$. So in any case $C_1 C_{n-1}\gamma_{n-1} \neq \gamma_3$. And we have constructed $m \in M(h)$ with properties (15).

Claim 2.13 *There is a monomial* $m_2 = \prod_i x_i^{\alpha_i+\beta_i} \in M(h)$ *such that*

$$\beta_n + \beta_1 + \beta_2 > 0. \tag{16}$$

We may choose to further assume one of $\beta_2 \leqslant 1$ *and* $\beta_n \leqslant 1$.

Proof We use Corollary 2.9 to obtain $m = m_1(x_n/x_1)^l$ so $\beta_n = l = -\beta_1$ and $C_{n-1}\gamma_{n-1} = \gamma_1 + \beta_1 = \alpha_1 - l$. By (15), $C_1(\gamma_1 - l) \neq \gamma_3$ so $mx_2 \in M(d(m))$, hence it should be canceled by some $mx_2/x_i \in M(h)$. By the previous equality, $i \neq n$ so either $i > 2$ and we have the desired $\beta_n + \beta_1 + \beta_2 = l - l + 1 > 0$ with $\beta_2 = 1$, or $i = 1$. If $i = 1$ then $C_{n-1}\gamma_{n-1} \neq \gamma_1 - l - 1$ so $mx_2x_n/x_1 \in M(d(mx_2/x_1))$. Again, it has to be canceled by mx_2x_n/x_1^2 or mx_n/x_1, for otherwise we would get the desired $\beta_n, \beta_1, \beta_2$ with $\beta_2 \leqslant 1$. This is repeated a number of times, each time increasing β_n by 1 and decreasing one of β_1 and β_2 by 1. We cannot go on infinitely because $\beta_1 \geqslant -\gamma_1$ and $\beta_2 \geqslant -\gamma_2$.

Similarly, if we desire $\beta_n \leqslant 1$, we use Proposition 2.7 to obtain $m = m_1(x_2/x_1)^l$ so $\beta_2 = l = -\beta_1$ and $C_1(\gamma_1 - l) = \gamma_3$. By (15), $C_{n-1}\gamma_{n-1} \neq \gamma_1 - l$ so $mx_n \in M(d(m))$, hence it should be canceled by some $mx_n/x_i \in M(h)$. By the previous equality, $i \neq 2$ so either $i > 2$ and we have the desired $\beta_n + \beta_1 + \beta_2 = 1 - l + l > 0$ with $\beta_n = 1$, or $i = 1$. If $i = 1$ then $C_1(\gamma_1 - l - 1) = \gamma_3 - C_1 \neq \gamma_3$ so $mx_nx_2/x_1 \in M(d(mx_n/x_1))$. Again, it has to be canceled by mx_nx_2/x_1^2 or mx_2/x_1, for otherwise we would get the desired $\beta_n, \beta_1, \beta_2$ with $\beta_n \leqslant 1$. This is repeated a number of times but at most until we reach $\beta_1 = -\gamma_1$, $\beta_n = -\gamma_n$, it cannot be continued further. ■

We use this claim in the following way: If

$$C_n\beta_n \neq \beta_2, \tag{17}$$

then $C_n(\alpha_n + \beta_n) \neq \alpha_2 + \beta_2$ so $m_2x_1 \in M(d(m_2))$ and it must be canceled by $d(m_3)$ for some $m_3 = \prod x_i^{\alpha_i+\beta_i'} \in M(h)$. For this $m_3/m_2 = x_1/x_i$ so $\beta_1' = \beta_1 + 1$ and $\beta_n' + \beta_1' + \beta_2' \geqslant \beta_n + \beta_1 + \beta_2 > 0$. We may replace m_2 by this m_3 and (16) still holds with larger β_1. We can continue as long as (17) holds. Reaching some m_2 for which $\beta_1 > 0$ or $\beta_1 = 0 < \beta_2$ would imply that m_2 is lexicographically larger than m_1. This is our aim.

If $|C_n| > 1$ then we use m_2 from Claim 2.13 such that $\beta_n + \beta_1 + \beta_2 > 0$ and $\beta_2 \leqslant 1$, consequently, β_n is not negative. Now $\beta_n = \beta_1 = 0$ and $\beta_2 = 1$ mean that m_2 is larger than m_1, a contradiction. If $\beta_n \geqslant |\beta_2|$ then $|C_n\beta_n| > \beta_n \geqslant |\beta_2|$ implying $C_n\beta_n - \beta_2 \neq 0$. So (17) is satisfied and we obtain a monomial that is larger than m_2. If, however, $\beta_n < |\beta_2|$, then $\beta_2 = 1$ and $\beta_n = 0$ and still $C_n\beta_n - \beta_2 \neq 0$ and we obtain a monomial that is larger than m_2. We can continue and finally we get a monomial that is larger than m_1, a contradiction.

If $0 < |C_n| < 1$ then we use m_2 from Claim 2.13 such that $\beta_n + \beta_1 + \beta_2 > 0$ and $\beta_n \leqslant 1$. Proceeding exactly as in the previous case (but using $\beta_2 \geqslant |\beta_n|$ or $\beta_2 = 0$, $\beta_n = 1$), we obtain a contradiction.

If C_n is not real then we use m_2 from Claim 2.13 such that $\beta_n + \beta_1 + \beta_2 > 0$. We can proceed the same way as $C_n\beta_n = \beta_2$ would imply $\beta_n = \beta_2 = 0$ so $\beta_1 > 0$, which is a contradiction.

Finally, if $C_n = -1$ then we use m_2 from Claim 2.13 such that $\beta_n + \beta_1 + \beta_2 > 0$. We can proceed the same way as $C_n\beta_n = \beta_2$ would imply $\beta_n + \beta_2 = 0$ so $\beta_1 > 0$ which is a contradiction.

This finishes the proof of Lemma 2.4. □

For the third part we use the following Jacobi criterion for the algebraic independence of polynomials: *Let $F_1, \ldots, F_k$ be polynomials in R. Let $J(F_1, \ldots, F_k)$ denote the $k \times n$ matrix, whose (i, j)-entry is $\partial F_i / \partial x_j$. Then $F_1, \ldots, F_k$ are algebraically independent if and only if $J(F_1, \ldots, F_k)$ has rank k.*

Lemma 2.14 *The polynomials in $\mathcal{H}$ are algebraically independent.*

Proof Clearly, no two of the polynomials can be dependent, as they involve different variables. So assume all three are defined, that is n is even and $\prod_{i\in\mathcal{O}} C_i = 1 = \prod_{i\in\mathcal{E}} C_i$. The Jacobi matrix $J(f, g_{\mathcal{O}}, g_{\mathcal{E}})$ truncated after the first three columns is

$$\begin{pmatrix} 1 & C_1 & C_1C_2 \\ \theta_1 x_1^{-1} g_{\mathcal{O}} & 0 & \theta_3 x_3^{-1} g_{\mathcal{O}} \\ 0 & \theta_2 x_2^{-1} g_{\mathcal{E}} & 0 \end{pmatrix}.$$

Its rows are independent as $x_1^{-1} g_{\mathcal{O}}$ and $x_3^{-1} g_{\mathcal{O}}$ are not constant multiples of each other. In the irregular case for $n = 4$, only at most one of $g_{\mathcal{O}}$ and $g_{\mathcal{E}}$ might be defined so this case does not need more attention. This finishes the proof of the lemma and of Theorems 1.1 and 1.2. □

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