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RESEARCH ARTICLE

$\mathbb{A}^1$-connectivity on Chow monoids versus rational equivalence of algebraic cycles

Vladimir Guletskiĭ[1]

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Abstract Let k be a field of characteristic zero, and let X be a projective variety embedded into a projective space over k. For two natural numbers r and d let $C_{r,d}(X)$ be the Chow scheme parametrizing effective cycles of dimension r and degree d on the variety X. Choosing an r-cycle of minimal degree gives rise to a chain of embeddings of Chow schemes, whose colimit is the connective Chow monoid $C_r^\infty(X)$ of r-cycles on X. Let $BC_r^\infty(X)$ be the classifying motivic space of this monoid. In the paper we establish an isomorphism between the Chow group $CH_r(X)_0$ of degree 0 dimension r algebraic cycles modulo rational equivalence on X, and the group of sections of the Nisnevich sheaf of $\mathbb{A}^1$-path connected components of the loop space of $BC_r^\infty(X)$ at $\mathrm{Spec}(k)$. Equivalently, $CH_r(X)_0$ is isomorphic to the group of sections of the stabilized motivic fundamental group $\Pi_1^{S^1\wedge\mathbb{A}^1}(BC_r^\infty(X))$ at $\mathrm{Spec}(k)$.

Keywords Algebraic cycles · Chow monoids · Group completion · Nisnevich sheaves · (Motivic) homotopy category · Bousfield localization · Loop functor · Classifying spaces · $\mathbb{A}^1$-homotopy groups

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✉ Vladimir Guletskiĭ
vladimir.guletskii@liverpool.ac.uk

[1] Department of Mathematical Sciences, University of Liverpool, Peach Street, Liverpool L69 7ZL, England, UK

1 Introduction

Algebraic cycles are linear combinations of closed integral subschemes in algebraic varieties over a field. Two algebraic cycles A and B on a variety X are said to be rationally equivalent if there exists an algebraic cycle Z on $X \times \mathbb{P}^1$, such that, for two fundamental points 0 and ∞ on $\mathbb{P}^1$, the cycle-theoretic fibres $Z(0)$ and $Z(\infty)$ coincide with A and B respectively. Rational equivalence is a fundamental notion in algebraic geometry, which substantially depends on the intersection multiplicities tacitly involved into the definition above. Intersection multiplicities are well controlled in cycles which are cascade intersections of cycles starting from codimension one. This is not always the case, of course. For example, if X is a $K3$-surface, the Chow group of 0-cycles modulo rational equivalence on X is large, in the sense that it cannot be parametrized by an abelian variety over the ground field, [20]. On the other hand, its subgroup generated by divisorial intersections on X is just $\mathbb{Z}$, see [4]. This example tells us that intersection multiplicities are geometrically manageable only for a small fraction of all algebraic cycles appearing in nature.

Another difficulty with algebraic cycles is that they are originally given in terms of groups, i.e. positive and negative multiplicities can appear in a cycle simultaneously. The use of negative numbers was questionable for mathematicians dealing with algebraic equations in sixteenth century. In modern terms, the concern can be expressed by saying that the completion of a monoid is a too formal construction. The problem might seem to be not that funny when passing to the completions of Chow monoids, i.e. gatherings of Chow varieties parametrizing effective cycles on projective varieties embedded into projective spaces. The Chow monoids themselves are geometrically given in terms of Cayley forms, whereas their completions are less visible.

These two things taken together have an effect that, in contrast to rational connectivity, rational equivalence is difficult to deform in a smooth projective family over a base, cf. [15]. As a consequence, the deep conjectures on rational equivalence are hard to approach, and by now they are solved in a small number of cases (see, for example, [26]). The state of things would be possibly better if we could recode rational equivalence into more effective (i.e. positive) data, appropriate for deformation in smooth projective families over a nice base. The purpose of the present paper is to investigate whether the $\mathbb{A}^1$-homotopy type can help in finding such data.

More precisely, let X be a projective variety over a field k, and fix an embedding of X into the projective space $\mathbb{P}^m$. To avoid the troubles with representability of Chow sheaves in positive characteristic, we must assume that k is of characteristic zero. Effective algebraic cycles of dimension r and degree d on X, considered with regard to the embedding $X \subset \mathbb{P}^m$, are represented by the Chow scheme $C_{r,d}(X)$ over k. Let Z_0 be an r-cycle of minimal degree d_0 on X. For example, if $r = 0$ then Z_0 can be a point, and if $r = 1$ then Z_0 can be a line on X. The cycle Z_0 gives rise to a chain of embeddings $C_{r,d}(X) \subset C_{r,d+d_0}(X)$, whose colimit $C_r^\infty(X)$ is the *connective Chow monoid* of effective r-cycles on X. Let $C_r^\infty(X)^+$ be the group completion of $C_r^\infty(X)$ in the category of set-valued simplicial sheaves on the smooth Nisnevich site over k. Let also $\Pi_0^{\mathbb{A}^1}$ be the functor of $\mathbb{A}^1$-connected components and $\Pi_1^{\mathbb{A}^1}$ be the functor of the $\mathbb{A}^1$-fundamental group on simplicial Nisnevich sheaves, see [19] or [2].

Now, consider the Chow group $CH_r(X)_0$ of degree zero r-cycles modulo rational equivalence on X, where the degree of cycle classes is given with regard to the embedding of X into $\mathbb{P}^m$ over k. Any finitely generated field extension K of the ground field k is the function field $k(Y)$ of an irreducible variety Y over k. For a simplicial sheaf $\mathcal{F}$, let $\mathcal{F}(K)$ be the stalk of $\mathcal{F}$ at the generic point $\mathrm{Spec}(K)$ of the variety Y. In the paper we establish a canonical (up to a projective embedding) isomorphism

$$CH_r(X_K)_0 \simeq \Pi_0^{\mathbb{A}^1}\left(C_r^\infty(X)^+\right)(K),$$

for an arbitrary finitely generated field extension K over k (Theorem 5.1).

Let, furthermore, $BC_r^\infty(X)$ be the motivic classifying space of the connective Chow monoid $C_r^\infty(X)$. We also prove that

$$CH_r(X_K)_0 \simeq \Pi_0^{\mathbb{A}^1}\big(\Omega \mathrm{Ex} BC_r^\infty(X)\big)(K),$$

where Ω is right adjoint to the simplicial suspension Σ in the pointed category of simplicial Nisnevich sheaves, and Ex is a fibrant replacement functor for simplicial Nisnevich sheaves (Corollary 5.5). Another reformulation of the main result is in terms of $S^1 \wedge \mathbb{A}^1$-fundamental groups, where S^1 is the simplicial circle. Namely,

$$CH_r(X_K)_0 \simeq \Pi_1^{S^1\wedge\mathbb{A}^1}\big(BC_r^\infty(X)\big)(K),$$

i.e. the Chow group of r-cycles of degree zero modulo rational equivalence on X is isomorphic to the stalk at $\mathrm{Spec}(K)$ of the $S^1 \wedge \mathbb{A}^1$-fundamental group of the motivic classifying space of the Chow monoid $C_r^\infty(X)$ (Corollary 5.7). The smashing by S^1 is a sort of stabilization, and not yet fully understood.

The use of the second isomorphism is that it encodes rational equivalence on r-cycles in terms of $\mathbb{A}^1$-path connectedness on the motivic space $L_{\mathbb{A}^1}\Omega \mathrm{Ex} BC_r^\infty(X)$. The localization functor $L_{\mathbb{A}^1}$ is a transfinite machine, which can be described in terms of sectionwise fibrant replacement, the Godement resolution, homotopy limit of the corresponding cosimplicial simplicial sheaves and the Suslin–Voevodsky's singularization functor. The quadruple operation $L_{\mathbb{A}^1}\Omega \mathrm{Ex}\, B$ is a bigger machine recoding rational equivalence into $\mathbb{A}^1$-path connectivity, at some technical cost, of course.

The proof of the main result (Theorem 5.1) is basically a gathering of known facts in $\mathbb{A}^1$-homotopy theory of schemes and Chow sheaves, collected in the right way. The substantial arguments are Lemma 3.5 and the use of [2, Proposition 6.2.6] by Asok and Morel. In Sect. 2 we introduce the needed tools from simplicial sheaves on a small site and the functor Π_0. Section 3 is devoted to the Bousfield localization of simplicial sheaves by an interval and to proving Lemma 3.5 which says that the group completion commutes with the localized Π_0. In Sect. 4 we pass to Nisnevich sheaves on schemes and recall the theory of Chow sheaves following [25]. The main results appear in Sect. 5, where we prove the existence of the canonical (up to a projective embedding) isomorphisms between the Chow groups and the stalks of the corresponding motivic homotopy groups of $C_r^\infty(X)^+$ and $BC_r^\infty(X)$. In Appendix we collect the needed basics from homotopical algebra, in order to make the text more self-contained.

2 Π_0 and monoids in simplicial sheaves

Let Δ be the simplex category, i.e. the category whose objects are finite sets $[n] = \{0, 1, \ldots, n\}$, for all $n \in \mathbb{N}$, and morphisms $[m] \to [n]$ are order-preserving functions from $[m]$ to $[n]$. Let $\mathcal{S}$ be a Cartesian monoidal category with a terminal object $*$. The category $\Delta^{\mathrm{op}}\mathcal{S}$ of simplicial objects in $\mathcal{S}$ is the category of contravariant functors from Δ to $\mathcal{S}$. Since $[0]$ is the terminal object in Δ, the functor $\Gamma\colon \Delta^{\mathrm{op}}\mathcal{S} \to \mathcal{S}$, sending $\mathcal{X}$ to $\mathcal{X}_0$, is the functor of global sections on simplicial objects in $\mathcal{S}$ considered as presheaves on Δ. The functor Γ admits left adjoint $\mathrm{Const}\colon \mathcal{S} \to \Delta^{\mathrm{op}}\mathcal{S}$ sending an object $\mathcal{X}$ in $\mathcal{S}$ to the constant presheaf on Δ determined by $\mathcal{X}$.

Assume, moreover, that $\mathcal{S}$ is cocomplete. For any object $\mathcal{X}$ in $\Delta^{\mathrm{op}}\mathcal{S}$, let $\Pi_0(\mathcal{X})$ be the coequalizer of the morphisms $\mathcal{X}_1 \rightrightarrows \mathcal{X}_0$ induced by the two morphisms from $\Delta[0]$ to $\Delta[1]$. This gives a functor $\Pi_0\colon \Delta^{\mathrm{op}}\mathcal{S} \to \mathcal{S}$ and the canonical epimorphism $\Psi\colon \Gamma \to \Pi_0$. If $\mathcal{Y}$ is an object in $\mathcal{S}$, and $f\colon \mathcal{X} \to \mathrm{Const}(\mathcal{Y})$ is a morphism in $\Delta^{\mathrm{op}}\mathcal{S}$, the precompositions of $f_0\colon \mathcal{X}_0 \to \mathcal{Y}$ with the two morphisms from $\mathcal{X}_1$ to $\mathcal{X}_0$ coincide. By universality of the coequalizer, we obtain the morphism $f'\colon \Pi_0(\mathcal{X}) \to \mathcal{Y}$. The correspondence $f \mapsto f'$ is one-to-one and natural in $\mathcal{X}$ and $\mathcal{Y}$. In other words, Π_0 is left adjoint to Const. Since products in $\Delta^{\mathrm{op}}\mathcal{S}$ are objectwise, the functor Π_0 preserves finite products. Certainly, if $\mathcal{C}$ is the terminal category, then Π_0 is the usual functor of connected components on simplicial sets.

Let $\mathcal{C}$ be an essentially small category and let τ be a subcanonical topology on it. Assume also that $\mathcal{C}$ contains all finite products and let $*$ be the terminal object in it. Let $\mathcal{P}$ be the category of presheaves of sets on $\mathcal{C}$ and let $\mathcal{S}$ be the full subcategory of set valued sheaves on $\mathcal{C}$ in the topology τ. Since τ is subcanonical, the Yoneda embedding $h\colon \mathcal{C} \to \mathcal{P}$, sending an object X to the representable presheaf $h_X = \mathrm{Hom}_{\mathcal{C}}(-, X)$, and a morphism $f\colon X \to Y$ to the morphism of presheaves $h_f = \mathrm{Hom}_{\mathcal{C}}(-, f)$, takes its values in the category of sheaves $\mathcal{S}$. If $*$ is the terminal object in $\mathcal{C}$, then $h_* = \mathrm{Hom}_{\mathcal{C}}(-, *)$ is the terminal object in $\mathcal{P}$ and $\mathcal{S}$. Limits in $\mathcal{S}$ are limits in $\mathcal{P}$. In particular, we have objectwise finite products in $\mathcal{S}$ and the category $\mathcal{S}$ is Cartesian monoidal.

For a presheaf $\mathcal{X}$, let $\mathcal{X}^{\mathrm{a}}$ be the sheaf associated to $\mathcal{X}$ in τ. Since $\mathcal{P}$ is complete, the sheafification of colimits in $\mathcal{P}$ shows that $\mathcal{S}$ is cocomplete too. In order to make a difference between Π_0 in $\Delta^{\mathrm{op}}\mathcal{S}$ and Π_0 in $\Delta^{\mathrm{op}}\mathcal{P}$, we shall denote the latter functor by π_0, so that, for a simplicial sheaf $\mathcal{X}$, one has $\Pi_0(\mathcal{X}) = \pi_0(\mathcal{X})^{\mathrm{a}}$. As the coequalizer π_0 is sectionwise, $\Pi_0(\mathcal{X})$ is the sheafififcation of the presheaf sending U to $\pi_0(\mathcal{X}(U))$.

Let $\Delta^{\mathrm{op}}\mathcal{S}ets$ be the category of simplicial sets. For a natural number n let $\Delta[n]$ be the representable functor $\mathrm{Hom}_{\Delta}(-, [n])$. For any sheaf $\mathcal{F}$ on $\mathcal{C}$ let $\Delta_{\mathcal{F}}[n]$ be the simplicial sheaf defined by the formula

$$\big(\Delta_{\mathcal{F}}[n]\big)_m(U) = \mathcal{F}(U) \times \mathrm{Hom}_{\Delta}\big([m], [n]\big),$$

for any $U \in \mathrm{Ob}(\mathcal{C})$ and any natural number m. This gives the full and faithful embeddings $\Delta_?[n]\colon \mathcal{S} \to \Delta^{\mathrm{op}}\mathcal{S}$ and $\Delta_{\mathcal{F}}[?]\colon \Delta \to \Delta^{\mathrm{op}}\mathcal{S}$. If $\mathcal{F}$ is h_X, for some object X in $\mathcal{C}$, then we write $\Delta_X[n]$ instead of $\Delta_{\mathcal{F}}[n]$, and $\Delta[n]$ instead of $\Delta_*[n]$. To simplify notation further, we shall identify $\mathcal{C}$ with its image in $\Delta^{\mathrm{op}}\mathcal{S}$ under the embedding $\Delta_?[0]$. For example, for any object X in $\mathcal{C}$ it is the same as the corresponding simplicial

sheaf $\mathcal{X} = \Delta_X[0] = \mathrm{Const}(h_X)$, and the same on morphisms in $\mathcal{C}$. The cosimplicial object $\Delta[?]\colon \Delta \to \Delta^{\mathrm{op}}\mathcal{S}$ determines the embedding of simplicial sets into $\Delta^{\mathrm{op}}\mathcal{S}$, so that we may also identify $\Delta^{\mathrm{op}}\mathit{Sets}$ with its image in $\Delta^{\mathrm{op}}\mathcal{S}$. This gives the structure of a simplicial category on $\Delta^{\mathrm{op}}\mathcal{S}$, such that, for any two simplicial sheaves $\mathcal{X}$ and $\mathcal{Y}$,

$$\mathbf{Hom}(\mathcal{X}, \mathcal{Y}) = \mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}\big(\mathcal{X} \times \Delta[?], \mathcal{Y}\big).$$

The corresponding (right) action of $\Delta^{\mathrm{op}}\mathit{Sets}$ on $\Delta^{\mathrm{op}}\mathcal{S}$ is given by the formula

$$(\mathcal{X} \times K)_n(U) = \mathcal{X}_n(U) \times K_n,$$

for any simplicial sheaf $\mathcal{X}$ and simplicial set K.

Looking at $\Delta^{\mathrm{op}}\mathcal{S}$ as a symmetric monoidal category with regard to the categorical product in it, one sees that it is closed symmetric monoidal. The internal Hom, bringing right adjoint to the Cartesian products, is given by the formula

$$\mathit{Hom}(\mathcal{X}, \mathcal{Y})_n(U) = \mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}\big(\mathcal{X} \times \Delta_U[n], \mathcal{Y}\big).$$

Throughout the paper we will be working with monoids in $\Delta^{\mathrm{op}}\mathcal{S}$. All monoids and groups will be commutative by default. If $\mathcal{X}$ is a monoid in $\Delta^{\mathrm{op}}\mathcal{S}$, let $\mathcal{X}^+$ be the group completion of $\mathcal{X}$ in $\Delta^{\mathrm{op}}\mathcal{S}$. The terms of $\mathcal{X}^+$ are the sheaves associated with the sectionwise completions of the terms of $\mathcal{X}$ in $\mathcal{P}$. One has a morphism from $\mathcal{X} \times \mathcal{X}$ to $\mathcal{X}^+$, which is an epimorphism in $\Delta^{\mathrm{op}}\mathcal{S}$. When no confusion is possible, the termwise and section-wise completion of $\mathcal{X}$ in $\Delta^{\mathrm{op}}\mathcal{P}$ will be denoted by the same symbol $\mathcal{X}^+$.

Monoids form a subcategory in $\mathcal{P}$. The corresponding forgetful functor has left adjoint sending presheaves to free monoids with concatenation as monoidal operation. The notion of a cancellation monoid in $\mathcal{P}$ is standard and sectionwise. A free monoid in $\mathcal{P}$ is a cancellation monoid. As limits and colimits in $\Delta^{\mathrm{op}}\mathcal{S}$ are termwise, the functors Γ and Const preserve monoids and groups and $\Gamma(\mathcal{X}^+)$ is the same as $\Gamma(\mathcal{X})^+$. Since Π_0 commutes with finite products, it follows that Π_0 also preserves monoids and groups.

The monoid of natural numbers $\mathbb{N}$ is a simplicial sheaf on $\mathcal{C}$. A pointed monoid in $\Delta^{\mathrm{op}}\mathcal{S}$ is a pair $(\mathcal{X}, \iota)$, where $\mathcal{X}$ is a monoid in $\Delta^{\mathrm{op}}\mathcal{S}$ and ι is a morphism of monoids from $\mathbb{N}$ to $\mathcal{X}$. A graded pointed monoid is a triple $(\mathcal{X}, \iota, \sigma)$, where $(\mathcal{X}, \iota)$ is a pointed monoid and σ is a morphism of monoids from $\mathcal{X}$ to $\mathbb{N}$, such that $\sigma \circ \iota = \mathrm{id}_{\mathbb{N}}$, see [19, p. 126]. Notice that to define a morphism from $\mathbb{N}$ to $\mathcal{X}$ is equivalent to choose an element in $\mathcal{X}_0(*)$.

Let $(\mathcal{X}, \iota, \sigma)$ be a pointed graded monoid in $\Delta^{\mathrm{op}}\mathcal{S}$. Since $\sigma \circ \iota = \mathrm{id}_{\mathbb{N}}$, it follows that, for any natural n and any object U in $\mathcal{C}$, we have two maps $\iota_{U,n}\colon \mathbb{N} \to \mathcal{X}_n(U)$ and $\sigma_{U,n}\colon \mathcal{X}_n(U) \to \mathbb{N}$. It implies that $\mathcal{X}_n(U)$ is the coproduct of the sets $\sigma^{-1}_{U,n}(d)$, for all $d \geq 0$. The sets $\sigma^{-1}_{U,n}(d)$ give rise to the simplicial sheaf which we denote by $\mathcal{X}^d$. Then $\mathcal{X}$ is the coproduct of $\mathcal{X}^d$ for all $d \geq 0$. The addition of $\iota(1)$ in $\mathcal{X}$ induces morphisms of simplicial sheaves $\mathcal{X}^d \to \mathcal{X}^{d+1}$ for all $d \geq 0$. Let $\mathcal{X}^\infty$ be the colimit

$$\mathcal{X}^\infty = \mathrm{colim}\left(\mathcal{X}^0 \to \mathcal{X}^1 \to \mathcal{X}^2 \to \cdots\right)$$

in $\Delta^{\mathrm{op}}\mathcal{S}$. Equivalently, $\mathcal{X}^\infty$ is the coequalizer of the addition of $\iota(1)$ in $\mathcal{X}$ and the identity automorphism of $\mathcal{X}$.

Since now we shall assume that the topos $\mathcal{S}$ has enough points, and the category $\mathcal{C}$ is Noetherian. Since filtered colimits commute with finite products, $\mathcal{X}^\infty$ is the colimit taken in the category of simplicial presheaves, i.e. there is no need to take its sheafification. The commutativity of filtered colimits with finite products also yields the canonical isomorphism between the colimit of the obvious diagram composed by the objects $\mathcal{X}^d \times \mathcal{X}^{d'}$, for all $d, d' \geq 0$, and the product $\mathcal{X}^\infty \times \mathcal{X}^\infty$. Since the colimit of that diagram is the colimit of its diagonal, this gives the canonical morphism from $\mathcal{X}^\infty \times \mathcal{X}^\infty$ to $\mathcal{X}^\infty$. The latter defines the structure of a monoid on $\mathcal{X}^\infty$, such that the canonical morphism

$$\pi\colon\ \mathcal{X} = \coprod_{d\geq 0} \mathcal{X}^d \to \mathcal{X}^\infty$$

is a homomorphism of monoids in $\Delta^{\mathrm{op}}\mathcal{S}$. We call $\mathcal{X}^\infty$ the *connective* monoid associated to the pointed graded monoid $\mathcal{X}$.

Notice that the category of simplicial sheaves is exhaustive. In particular, if all the morphisms $\mathcal{X}^d \to \mathcal{X}^{d+1}$ are monomorphisms, the transfinite compositions $\mathcal{X}^d \to \mathcal{X}^\infty$ are monomorphisms too. This happens if $\mathcal{X}$ is a termwise sectionwise cancellation monoid, in which case $\mathcal{X}^\infty$ is a termwise sectionwise cancellation monoid too.

The above homomorphisms π and σ give the homomorphism (π, σ) from $\mathcal{X}$ to $\mathcal{X}^\infty \times \mathbb{N}$. Passing to completions we obtain the homomorphism (π^+, σ^+) from $\mathcal{X}^+$ to $(\mathcal{X}^\infty)^+ \times \mathbb{Z}$.

Lemma 2.1 *Assume $\mathcal{X}$ is a sectionwise cancellation monoid. Then*

$$\big(\pi^+, \sigma^+\big)\colon\ \mathcal{X}^+ \to \big(\mathcal{X}^\infty\big)^+ \times \mathbb{Z}$$

is an isomorphism.

Proof Since the site $\mathcal{S}$ has enough points, it suffices to prove the lemma sectionwise and termwise. Then, without loss of generality, we may assume that $\mathcal{X}$ is a set-theoretical pointed graded cancellation monoid. Clearly, ι^+ is an injection, π^+ is a surjection, and $\pi^+\iota^+ = 0$. Since $\mathcal{X}$ is a cancellation monoid, $\mathcal{X}^+$ is the quotient-set of the set $\mathcal{X} \times \mathcal{X}$ modulo an equivalence relation

$$\big(x_1, x_2\big) \sim \big(x_1', x_2'\big) \quad \Longleftrightarrow \quad x_1 + x_2' = x_2 + x_1'.$$

For any element (x_1, x_2) in $\mathcal{X} \times \mathcal{X}$ let $[x_1, x_2]$ be the corresponding equivalence class. Since $\mathcal{X}$ is a cancellation monoid, so is the monoid $\mathcal{X}^\infty$. If $\pi^+[x_1, x_2]$ is zero, that is $[\pi(x_1), \pi(x_2)] = [0, 0]$ in $(\mathcal{X}^\infty)^+$, it is equivalent to say that $\pi(x_1) = \pi(x_2)$. The latter equality means that there exists a positive integer n, such that $x_2 = x_1 + n\iota(1)$, i.e. $[x_1, x_2] = [0, n\iota(1)]$ in $\mathcal{X}^+$. The element $[0, n\iota(1)]$ sits in the image of ι^+. □

3 Homotopy completion and localization of Π_0

All the above considerations were categorical. Let us now switch to homotopical algebra and consider the injective model structures on $\Delta^{\mathrm{op}}\mathcal{S}$. Recall that a point P of a topos $\mathcal{T}$ is an adjoint pair of functors, $P^*\colon \mathcal{T} \to \mathit{Sets}$ and $P_*\colon \mathit{Sets} \to \mathcal{T}$, such that P^* is left adjoint to P_* and preserves finite limits in $\mathcal{T}$. If $\mathcal{X}$ is an object of $\mathcal{T}$, then $\mathcal{X}_P = P^*(X)$ is the stalk of $\mathcal{X}$ at the point P. We will assume that the topos $\mathcal{S}$ has enough points. Recall that it means that there exists a set of points $P(\mathcal{S})$ of the topos $\mathcal{S}$, such that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ in $\mathcal{S}$ is an isomorphism in $\mathcal{S}$ if and only if, for any point $P \in P(\mathcal{S})$, the morphism $f_P\colon \mathcal{X}_P \to \mathcal{Y}_P$, induced on stalks, is an isomorphism in the category Sets. Respectively, a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ in $\Delta^{\mathrm{op}}\mathcal{S}$ is an isomorphism in $\Delta^{\mathrm{op}}\mathcal{S}$ if and only if, for each $P \in P(\mathcal{S})$, the morphism $f_P\colon \mathcal{X}_P \to \mathcal{Y}_P$ is an isomorphism in $\Delta^{\mathrm{op}}\mathit{Sets}$.

Now, a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ in the category of simplicial sheaves $\Delta^{\mathrm{op}}\mathcal{S}$ is a weak equivalence in $\Delta^{\mathrm{op}}\mathcal{S}$ if and only if, for any point $P^*\colon \mathcal{S} \to \mathit{Sets}$ of the topos $\mathcal{S}$, the induced morphism $\Delta^{\mathrm{op}}P^*(f)$ on stalks is a weak equivalence of simplicial sets. Cofibrations are monomorphisms, and fibrations are defined by the right lifting property in the standard way, see [19, Definition 1.2, p. 48]. The pair $(\mathcal{S}, \mathcal{M})$ is then a model category of simplicial sheaves on $\mathcal{C}$ in τ. Notice that the model structure $\mathcal{M}$ is left proper, see [19, Remark 1.5, p. 49]. One can also show that it is cellular. Let H be the homotopy category $\mathrm{Ho}(\Delta^{\mathrm{op}}\mathcal{S})$ of the category $\Delta^{\mathrm{op}}\mathcal{S}$ with regard to $\mathcal{M}$. For any two simplicial sheaves $\mathcal{X}$ and $\mathcal{Y}$ the set of morphisms from $\mathcal{X}$ to $\mathcal{Y}$ in H will be denoted by $[\mathcal{X}, \mathcal{Y}]$.

The simplicial structure on $\Delta^{\mathrm{op}}\mathcal{S}$ is compatible with the model one, so that $\mathcal{S}$ is a simplicial model category. Since

$$\big[\mathcal{X}, \mathcal{Y}\big] \simeq \pi_0 \mathbf{Hom}\big(\mathcal{X}, \mathcal{Y}\big)$$

and

$$\mathbf{Hom}\big(\Delta_U[0], \mathcal{X}\big) \simeq \mathcal{X}(U),$$

$\Pi_0(\mathcal{X})$ is the sheafififcation of the presheaf

$$\pi_0(\mathcal{X})\colon U \mapsto \pi_0 \mathbf{Hom}\big(\Delta_U[0], \mathcal{X}\big) = \big[\Delta_U[0], \mathcal{X}\big] = \big[\mathrm{Const}(h_U), \mathcal{X}\big]$$

on $\mathcal{C}$ in the topology τ. The multiplication of simplicial sheaves and their morphisms by a simplicial set admits right adjoint, so that it commutes with colimits. In particular, $\Pi_0(\Delta_X[n]) \simeq \Delta_X[0]$.

A pointed simplicial sheaf $(\mathcal{X}, x)$ is a pair consisting of a simplicial sheaf $\mathcal{X}$ and a morphism x from $*$ to $\mathcal{X}$. The definition of a morphism of pointed simplicial sheaves is obvious. Let $\Delta^{\mathrm{op}}\mathcal{S}_*$ be the category of pointed simplicial sheaves. The corresponding forgetful functor has the standard left adjoint sending $\mathcal{X}$ to the coproduct $\mathcal{X}_+$ of $\mathcal{X}$ and $*$. The model structure $\mathcal{M}$ induces the corresponding model structure on $\Delta^{\mathrm{op}}\mathcal{S}_*$, such that the above adjunction is a Quillen adjunction. Having two pointed simplicial sheaves $(\mathcal{X}, x)$ and $(\mathcal{Y}, y)$, their wedge product $(\mathcal{X}, x) \vee (\mathcal{Y}, y)$ is the coproduct in

$\Delta^{\mathrm{op}}\mathcal{S}_*$, and the smash product $(\mathcal{X}, x)\wedge(\mathcal{Y}, y)$ is the contraction of the wedge product in $(\mathcal{X}\times\mathcal{Y}, x\times y)$.

Let now S^1 be the simplical circle $\Delta[1]/\partial\Delta[1]$ pointed by the image of the boundary $\partial\Delta[1]$ in then quotient simplicial set, and let S^1 be its image in $\Delta^{\mathrm{op}}\mathcal{S}_*$. Define the simplicial suspension endofunctor Σ on $\Delta^{\mathrm{op}}\mathcal{S}_*$ sending $(\mathcal{X}, x)$ to $S^1\wedge(\mathcal{X}, x)$. Its left adjoint is the simplicial loop functor Ω sending $(\mathcal{X}, x)$ to $\mathcal{H}om_*(S^1, (\mathcal{X}, x))$, where $\mathcal{H}om_*(-,-)$ is the obvious internal Hom in $\Delta^{\mathrm{op}}\mathcal{S}_*$.

Let $\mathcal{X}$ be a monoid in $\Delta^{\mathrm{op}}\mathcal{S}$. For any object U in $\mathcal{C}$ and any positive integer n let $N(\mathcal{X}_n(U))$ be the nerve of $\mathcal{X}_n(U)$, and let $B\mathcal{X}$ be the diagonal of the bisimplicial sheaf $\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}\to\mathcal{S}$ sending $[m]\times[n]$ to the sheaf $U\mapsto N(\mathcal{X}_n(U))_m$. Then $(B\mathcal{X})_n$ is $\mathcal{X}_n^{\times n}$ for $n>0$ and, by convention, $(B\mathcal{X})_0$ is the terminal object $*$ in $\mathcal{S}$, see [19, p. 123]. If $\mathcal{C}$ is a terminal category, then $B\mathcal{X}$ is the usual classifying space of a simplicial monoid $\mathcal{X}$ (that is, a monoid in the category of simplicial sets $\Delta^{\mathrm{op}}\mathcal{S}ets$). Just as in topology, there exists a canonical morphism from $\mathcal{X}$ to $\Omega B(\mathcal{X})$, which is a weak equivalence if $\mathcal{X}$ is a group, loc.cit.

Following Quillen, [24], we will say that a simplicial monoid $\mathcal{X}$ is good if the morphism $B\mathcal{X}\to B\mathcal{X}^+$, induced by the canonical morphism from $\mathcal{X}$ to $\mathcal{X}^+$, is a weak equivalence in $\Delta^{\mathrm{op}}\mathcal{S}ets$. If X is a set-theoretical monoid, then X is good if the corresponding constant simplicial monoid $\mathcal{X}=\mathrm{Const}(X)$ is good as a simplicial monoid. If X is a free monoid in $\mathcal{S}ets$, then $\mathcal{X}=\mathrm{Const}(X)$ is good in $\Delta^{\mathrm{op}}\mathcal{S}ets$, see [24, Proposition Q.1].

Recall that, for any point P of the topos $\mathcal{S}$, the functor $P^*\colon\mathcal{S}\to\mathcal{S}ets$ preserves finite limits. It follows that, if $\mathcal{X}$ is a simplicial sheaf monoid, then the stalk $(B\mathcal{X})_P$ of the classifying space $B\mathcal{X}$ at P is canonically isomorphic to the classifying space $B(\mathcal{X}_P)$ of the stalk $\mathcal{X}_P$ of the simplicial sheaf $\mathcal{X}$ at P. We will say that a simplicial sheaf monoid $\mathcal{X}$ is *pointwise good*, if the morphism $(B\mathcal{X})_P\to(B\mathcal{X}^+)_P$, is a weak equivalence of simplicial sets for each point P in $P(\mathcal{S})$. This is, of course, equivalent to saying that the morphism $B\mathcal{X}\to B\mathcal{X}^+$ is a weak equivalence in $\Delta^{\mathrm{op}}\mathcal{S}$, with regard to the model structure $\mathcal{M}$.

Now, if $\mathcal{X}$ is a monoid in $\mathcal{S}$, we will say that $\mathcal{X}$ is *pointwise free* if $\mathcal{X}_P$ is a free monoid in $\mathcal{S}ets$ for each point P in $P(\mathcal{S})$. If $\mathcal{X}$ is pointwise free, it does not necessarily mean that $\mathcal{X}$ is a free monoid in the category $\mathcal{S}$. It is important, however, that if $\mathcal{X}_0$ is a pointwise free monoid in $\mathcal{S}$, the corresponding constant simplicial sheaf monoid $\mathcal{X}=\mathrm{Const}(\mathcal{X}_0)$ is pointwise good, which is a straightforward consequence of the first part of [24, Proposition Q.1].

Similarly, we will say that a monoid $\mathcal{X}$ in $\mathcal{S}$ is a *pointwise cancellation* monoid if $\mathcal{X}_P$ is a cancellation monoid in $\mathcal{S}ets$ for each point P in $P(\mathcal{S})$. If $\mathcal{X}_0$ is a pointwise cancellation monoid, then the simplicial sheaf monoid $\mathcal{X}=\mathrm{Const}(\mathcal{X}_0)$ is pointwise good by the second part of Quillen's proposition above.

Let Ex be the fibrant replacement functor $\mathrm{Ex}^{\mathcal{G}}$, for the model structure $\mathcal{M}$, constructed by taking the composition of the sectionwise fibrant replacement of simplicial sets, the Godement resolution and the homotopy limit of the corresponding cosimplicial simplicial sheaf, as in [19, Section 2.1]. Since Ex preserves finite limits, it preserves monoids and groups. For the same reason, Ex commutes with taking the classifying spaces of monoids and groups. The right derived functor of Ω can be computed by precomposing it with Ex. We will need the following variation of [19, Lemma 1.2, p. 123].

Lemma 3.1 *If $\mathcal{X}$ is pointwise good, there is a canonical isomorphism*

$$\mathcal{X}^+ \simeq \Omega \mathrm{Ex}\, B(\mathcal{X})$$

in the homotopy category H.

Proof Since $\mathcal{X}$ is pointwise good, the morphism from $B\mathcal{X}$ to $B\mathcal{X}^+$, induced by the canonical morphism from $\mathcal{X}$ to $\mathcal{X}^+$, is a weak equivalence in $\Delta^{\mathrm{op}}\mathcal{S}$. Applying the right derived functor $\mathrm{R}\Omega$ to the weak equivalence $B\mathcal{X} \to B\mathcal{X}^+$ and reverting the corresponding isomorphism in H, we obtain the canonical isomorphism from $\Omega \mathrm{Ex}\, B(\mathcal{X}^+)$ to $\Omega \mathrm{Ex}\, B(\mathcal{X})$, in the homotopy category H. The composition of the canonical morphism $\Sigma\mathcal{X}^+ \to B\mathcal{X}^+$ with the weak equivalence $B\mathcal{X}^+ \to \mathrm{Ex}\, B\mathcal{X}^+$ corresponds to the morphism $\mathcal{X}^+ \to \Omega \mathrm{Ex}\, B\mathcal{X}^+$ under the adjunction between Σ and Ω. The latter morphism is the composition of the canonical morphism $\mathcal{X}^+ \to \Omega B\mathcal{X}^+$ and the morphism $\Omega B\mathcal{X}^+ \to \Omega \mathrm{Ex}\, B\mathcal{X}^+$. The morphism $\mathcal{X}^+ \to \Omega B\mathcal{X}^+$ is a weak equivalence because $\mathcal{X}^+$ is a group. Since any simplicial sheaf of groups $\mathcal{G}$ can be replaced, up to a weak equivalence, by a fibrant simplicial sheaf of groups, without loss of generality we may assume that $\mathcal{X}^+$ is fibrant (see, for example, [19, Lemma 2.32, p. 83]). Replacing the functor B by the universal cocycle construction $\overline{W}$, we see that B preserves, up to a weak equivalence, fibrant objects by [13, Theorem 31]. Then the morphism $\Omega B\mathcal{X}^+ \to \Omega \mathrm{Ex}\, B\mathcal{X}^+$ is a weak equivalence too. Thus, we obtain an isomorphism from $\mathcal{X}^+$ to $\Omega \mathrm{Ex}\, B\mathcal{X}^+$ in H. □

Next, let A be an object of $\mathcal{C}$, and let $\mathcal{A}$ be the corresponding constant simplicial sheaf $\Delta_A[0] = \mathrm{Const}(h_A)$ in $\Delta^{\mathrm{op}}\mathcal{S}$. As in Appendix below, let

$$S = \left\{ \mathcal{X} \wedge \mathcal{A} \to \mathcal{X} : \mathcal{X} \in \mathrm{dom}(I) \cup \mathrm{codom}(I) \right\}$$

be the set of morphisms induced by the morphism from $\mathcal{A}$ to $*$, where $\mathrm{dom}(I)$ and $\mathrm{codom}(I)$ are the sets of domains and codomains of the generating cofibrations in $\mathcal{M}$ on $\mathcal{S}$. As $\Delta^{\mathrm{op}}\mathcal{S}$ is left proper simplicial cellular model category, there exists the left Bousfield localization of $\mathcal{M}$ by S in the sense of Hirschhorne, see [11]. Denote the localized model structure by $\mathcal{M}_A$, and let L_A be the corresponding S-localization functor, which is a fibrant approximation in $\mathcal{M}_A$ on $\Delta^{\mathrm{op}}\mathcal{S}$, see [11, Section 4.3], and the earlier work [8]. Let

$$l\colon \mathrm{Id}_{\Delta^{\mathrm{op}}\mathcal{S}} \to L_A$$

be the corresponding natural transformation. For any simplicial sheaf $\mathcal{X}$ the morphism $l_{\mathcal{X}}\colon \mathcal{X} \to L_A(\mathcal{X})$ is a weak cofibration and $L_A(\mathcal{X})$ is A-local, i.e. fibrant in $\mathcal{M}_A$. The basics on localization functors see in [11, Section 4.3] and Appendix below.

Let H_A be the homotopy category of simplicial sheaves converting weak equivalences in $\mathcal{M}_A$ into isomorphisms. As simplicial sheaves with respect to $\mathcal{M}$ form a simplicial closed Cartesian monoidal model category, so is the category of simplicial sheaves with respect to $\mathcal{M}_A$. All simplicial sheaves are cofibrant, in $\mathcal{M}$ and in $\mathcal{M}_A$. It follows that the canonical functors from simplicial sheaves to H and H_A are monoidal.

See Appendix for more details on all such things. For any two simplicial sheaves $\mathcal{X}$ and $\mathcal{Y}$ let $[\mathcal{X}, \mathcal{Y}]_A$ be the set of morphisms from $\mathcal{X}$ to $\mathcal{Y}$ in H_A.

Recall that an object I of a category $\mathcal{D}$ with a terminal object $*$ is called an interval if there exists a morphism

$$\mu : I \wedge I \to I$$

and two morphisms $i_0, i_1 : * \rightrightarrows I$, such that

$$\mu \circ (\mathrm{id}_I \wedge i_0) = i_0 \circ p \qquad \text{and} \qquad \mu \circ (\mathrm{id}_I \wedge i_1) = \mathrm{id}_I,$$

where p is the unique morphism from I to $*$, and $i_0 \coprod i_1 : * \coprod * \to I$ is a monomorphism in $\mathcal{D}$, see [19]. Certainly, the object A is an interval in $\mathcal{C}$ if and only if the object $\mathcal{A}$ is an interval in $\Delta^{\mathrm{op}}\mathcal{S}$.

Since now we shall assume that A is an interval in $\mathcal{C}$. The monoidal multiplication by $\mathcal{A}$ is a natural cylinder functor on $\Delta^{\mathrm{op}}\mathcal{S}$. If $f, g : \mathcal{X} \rightrightarrows \mathcal{Y}$ are two morphisms from $\mathcal{X}$ to $\mathcal{Y}$ in $\Delta^{\mathrm{op}}\mathcal{S}$, a left A-homotopy from f to g is a morphism $H : \mathcal{X} \times \mathcal{A} \to \mathcal{Y}$, such that $H \circ (\mathrm{id}_{\mathcal{X}} \times i_0) = f$ and $H \circ (\mathrm{id}_{\mathcal{X}} \times i_1) = g$. Since all simplicial sheaves are cofibrant in both model structures $\mathcal{M}$ and $\mathcal{M}_A$, A-homotopy is an equivalence relation on the set $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathcal{X}, \mathcal{Y})$, see [12, Proposition 1.2.5 (iii)]. Let $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathcal{X}, \mathcal{Y})_A$ be the set of equivalence classes modulo this equivalence relation. Whenever $\mathcal{Y}$ is A-local, the set $[\mathcal{X}, \mathcal{Y}]_A$ is in the natural bijection with the set $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathcal{X}, \mathcal{Y})_A$.

A point of a simplicial sheaf $\mathcal{X}$ is, by definition, a morphism from the terminal simplicial sheaf to $\mathcal{X}$. Such morphisms can be identified with the set $\mathcal{X}_0(*)$. Two points on $\mathcal{X}$ are said to be A-path connected if and only if they are left homotopic with respect to $\mathcal{A}$.

Since A is an interval, the A-localizing functor L_A can be chosen to be more explicit than the construction given in [11]. Following [19, p. 88], we consider the cosimplicial sheaf

$$\Delta_A\bullet[0] : \Delta \to \mathcal{S}$$

sending $[n]$ to the n-product

$$\big(\Delta_A[0]\big)^n = \Delta_{A^n}[0]$$

and acting on morphisms as follows. For any morphism $f : [m] \to [n]$ define a morphism of sets

$$f' : \{1, \ldots, n\} \to \{0, 1, \ldots, m+1\}$$

setting

$$f'(i) = \begin{cases} \min\{l \in \{0, \ldots, m\} : f(l) \geq i\} & \text{if this set is nonempty,} \\ m+1 & \text{otherwise.} \end{cases}$$

If now $\mathrm{pr}_k\colon A^n \to A$ is the k-th projection and $p\colon A^n \to *$ the unique morphism to the terminal object, where A^n is the n-fold product of A, then

$$\mathrm{pr}_k \circ \Delta_A\bullet[0](f) = \begin{cases} \mathrm{pr}_{f'(k)} & \text{if} \quad f'(k) \in \{1, \dots, m\}, \\ i_0 \circ p & \text{if} \quad f'(k) = m+1, \\ i_1 \circ p & \text{if} \quad f'(k) = 0. \end{cases}$$

For any $\mathcal{X}$ let $\mathrm{Sing}_A(\mathcal{X})$ be the Suslin–Voevodsky simplicial sheaf

$$[n] \mapsto \mathcal{H}om\big(\Delta_{A^n}[0], \Delta_{\mathcal{X}_n}[0]\big),$$

where the internal $\mathcal{H}om$ is taken in the category of sheaves $\mathcal{S}$. It is functorial in $\mathcal{X}$ and $p\colon A^n \to *$ induces the morphism

$$s\colon \mathrm{Id}_{\Delta^{\mathrm{op}}\mathcal{S}} \to \mathrm{Sing}_A.$$

Notice that, although the virtue of A to be an interval is not explicitly used in the Suslin–Voevodsky's construction above, it is used in proving the numerous nice properties of the functor Sing_A, see [19]. In particular, each morphism $s_{\mathcal{X}}$ from $\mathcal{X}$ to $\mathrm{Sing}_A(\mathcal{X})$ is an A-local weak equivalence, i.e. a weak equivalence with regard to the model structure $\mathcal{M}_A$, see [19, Corollary 3.8, p. 89].

As it is shown in [19], there exists a sufficiently large ordinal ω, such that L_A can be taken to be the composition

$$L_A = (\mathrm{Ex} \circ \mathrm{Sing}_A)^{\omega} \circ \mathrm{Ex},$$

where Ex is the functor $\mathrm{Ex}^{\mathcal{S}}$, i.e. the composition of the sectionwise fibrant replacement, the Godement resolution and the homotopy limit of the corresponding cosimplicial simplicial sheaf (see above). Such constructed localization functor L_A is quite explicit, which gives a clearer picture of what are the functors π_0^A and Π_0^A.

The canonical functor from $\Delta^{\mathrm{op}}\mathcal{S}$ to H preserves products. In other words, if $\mathcal{X} \times \mathcal{Y}$ is the product of two simplicial sheaves, the same object $\mathcal{X} \times \mathcal{Y}$, with the homotopy classes of the same projections, is the product of $\mathcal{X}$ and $\mathcal{Y}$ in H and in H_A (see Appendix). The advantage of the above explicit L_A is that it commutes with finite products, see [19, Theorem 1.66 on pp. 69–70 and the remark on p. 87]. Most likely, the general Hirschhorne's construction (see [11, Section 4.3]) also enjoys this property, but we could not find the proof in the literature.

Remark 3.2 The left derived to any localization functor L_A from $\Delta^{\mathrm{op}}\mathcal{S}$ to A-local objects in $\Delta^{\mathrm{op}}\mathcal{S}$ is left adjoint to the right derived of the forgetful functor in the opposite direction on the homotopy level, see [19, Theorem 2.5, p. 71]. This implies, in particular, that any two localizations L_A and L'_A are weak equivalent to each other. Therefore, in all considerations up to (pre-A-localized) weak equivalence in $\Delta^{\mathrm{op}}\mathcal{S}$ we may freely exchange the localization functor L_A considered in [11] by the concrete Suslin–Voevodsky's one, and vice versa.

Lemma 3.3 *For any simplicial sheaf $\mathcal{X}$ the canonical map*

$$\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(*, \mathcal{X})_A \to \mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}\big(*, L_A(\mathcal{X})\big)_A$$

is surjective.

Proof We know that the natural transformation $l\colon \mathrm{Id} \to L_A$ induces the epimorphism $\Pi_0 \to \Pi_0^A$ by [19, Corollary 3.22]. The morphism $\Psi\colon \Gamma \to \Pi_0$ is an epimorphism too. This gives that the map

$$\mathrm{Hom}_{\mathcal{S}}(*, \mathcal{X}_0) \to \mathrm{Hom}_{\mathcal{S}}(*, \Pi_0(\mathcal{X})) \to \mathrm{Hom}_{\mathcal{S}}(*, \Pi_0^A(\mathcal{X}))$$

is surjective. By adjunction, $\mathrm{Hom}_{\mathcal{S}}(*, \mathcal{X}_0) \simeq \mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(*, \mathcal{X})$, and since $L_A(\mathcal{X})$ is A-local, $\mathrm{Hom}_{\mathcal{S}}(*, \Pi_0^A(\mathcal{X}))$ is isomorphic to $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(*, L_A(\mathcal{X}))_A$. □

Now, define the A-localized functor Π_0^A from $\Delta^{\mathrm{op}}\mathcal{S}$ to $\mathcal{S}$ by setting $\Pi_0^A(\mathcal{X})$ to be the sheaf associated to the presheaf

$$U \mapsto [\mathrm{Const}(h_U), \mathcal{X}]_A.$$

Then $\Pi_0^A(\mathcal{X})$ is canonically isomorphic to $\Pi_0(L_A(\mathcal{X}))$, and the morphism l induces the epimorphism $\Pi_0 \to \Pi_0^A$, see [19, Corollary 3.22, p. 94]. As L_A is monoidal,

$$\begin{aligned}\Pi_0^A(\mathcal{X}\times\mathcal{Y}) &= \Pi_0\big(L_A(\mathcal{X}\times\mathcal{Y})\big) = \Pi_0\big(L_A(\mathcal{X})\times L_A(\mathcal{Y})\big)\\ &= \Pi_0\big(L_A(\mathcal{X})\big)\times\Pi_0\big(L_A(\mathcal{Y})\big) = \Pi_0^A(\mathcal{X})\times\Pi_0^A(\mathcal{Y}).\end{aligned}$$

This gives that Π_0^A preserves monoids and groups.

Lemma 3.4 *For any monoid $\mathcal{X}$ in $\Delta^{\mathrm{op}}\mathcal{S}$, one has a canonical isomorphism*

$$\Pi_0(\mathcal{X})^+ \simeq \Pi_0(\mathcal{X}^+)$$

in $\mathcal{S}$.

Proof Since $\Gamma(\mathcal{X}^+) = \Gamma(\mathcal{X})^+$ and $\Pi_0(\mathcal{X})^+$ are completions, one has the universal morphisms γ from $\Gamma(\mathcal{X}^+)$ to $\Pi_0(\mathcal{X})^+$ and δ from $\Pi_0(\mathcal{X})^+$ to $\Pi_0(\mathcal{X}^+)$. Since $\Gamma(\mathcal{X}) = \mathcal{X}_0$, $\Gamma(\mathcal{X}^+) = \mathcal{X}_0^+$ and $\gamma\circ\Gamma(\nu_{\mathcal{X}}) = \nu_{\Pi_0(\mathcal{X})}\circ\Psi$, where ν stays for the corresponding canonical morphisms from the monoids to their completions, the two compositions $\mathcal{X}_1^+ \rightrightarrows \mathcal{X}_0^+ \xrightarrow{\gamma} \Pi_0(\mathcal{X})^+$ coincide, which gives the universal morphism ε from $\Pi_0(\mathcal{X}^+)$ to $\Pi_0(\mathcal{X})^+$. Since Ψ is an epimorphism, and using the uniqueness of the appropriate universal morphisms, we show that δ and ε are mutually inverse isomorphisms of groups in $\mathcal{S}$. □

Let $\mathrm{CMon}(\Delta^{\mathrm{op}}\mathcal{S})$ be the category of commutative monoids in $\Delta^{\mathrm{op}}\mathcal{S}$. Suppose that all cofibrations in $\Delta^{\mathrm{op}}\mathcal{S}$ are symmetrizable, see [9]. Then the simplicial model structure on $\Delta^{\mathrm{op}}\mathcal{S}$ gives rise to a simplicial model structure on $\mathrm{CMon}(\Delta^{\mathrm{op}}\mathcal{S})$, compatible with Bousfield localizations, see [21–23,28,29]. A morphism in $\mathrm{CMon}(\Delta^{\mathrm{op}}\mathcal{S})$ is a

weak equivalence (respectively, fibration) if and only if it is a weak equivalence (respectively, fibration) in $\Delta^{\mathrm{op}}\mathcal{S}$, loc.cit. The classifying space functor B is then a functor from the model category $\mathrm{CMon}(\Delta^{\mathrm{op}}\mathcal{S})$ to the model category $\Delta^{\mathrm{op}}\mathcal{S}$. [19, Lemma 2.35, p. 85], and the universality of a left localization of a model structure (see [11, Definition 3.1.1 (b), pp. 47–48]), being applied to the functor B, yield a (simplicial) weak equivalence

$$(B \circ L_A)(\mathcal{X}) \simeq (L_{S^1 \wedge A} \circ B)(\mathcal{X}),$$

for any commutative monoid $\mathcal{X}$ in $\Delta^{\mathrm{op}}\mathcal{S}$.

Lemma 3.5 *For any pointwise good commutative monoid $\mathcal{X}$ in $\Delta^{\mathrm{op}}\mathcal{S}$, one has a canonical isomorphism*

$$\Pi_0^A(\mathcal{X})^+ \simeq \Pi_0^A(\mathcal{X}^+).$$

in $\mathcal{S}$.

Proof Since $\mathcal{X}$ is pointwise good, one has the isomorphism

$$\mathcal{X}^+ \simeq (\Omega \circ \mathrm{Ex} \circ B)(\mathcal{X})$$

in H by Lemma 3.1, where Ω is the simplicial loop functor and Ex is the (pre-A-localized) fibrant replacement for simplicial sheaves. Applying L_A we get the isomorphism

$$L_A(\mathcal{X}^+) \simeq L_A\big((\Omega \circ \mathrm{Ex} \circ B)(\mathcal{X})\big).$$

By [19, Theorem 2.34, p. 84],

$$L_A\big((\Omega \circ \mathrm{Ex} \circ B)(\mathcal{X})\big) \simeq \big(\Omega \circ \mathrm{Ex} \circ L_{S^1 \wedge A}\big)\big(B(\mathcal{X})\big).$$

Since $B \circ L_A \simeq L_{S^1 \wedge A} \circ B$, we obtain the isomorphism

$$L_A(\mathcal{X}^+) \simeq (\Omega \circ \mathrm{Ex} \circ B)(L_A(\mathcal{X}))$$

in H. Let $\Phi = \Phi_{\mathrm{Mon}}$ be the functor constructed in [19, Lemma 1.1, p. 123], i.e. the cofibrant replacement functor in $\mathrm{CMon}(\Delta^{\mathrm{op}}\mathcal{S})$. Since the morphism $\Phi(L_A(\mathcal{X})) \to L_A(\mathcal{X})$ is a weak equivalence in $\Delta^{\mathrm{op}}\mathcal{S}$, we get the isomorphism

$$L_A\big(\mathcal{X}^+\big) \simeq \big(\Omega \circ \mathrm{Ex} \circ B\big)\big(\Phi(L_A(\mathcal{X}))\big)$$

in H. The monoid $\Phi(L_A(\mathcal{X}))$ is termwise free. Therefore,

$$\big(\Omega \circ \mathrm{Ex} \circ B\big)\big(\Phi(L_A(\mathcal{X}))\big) \simeq \big(\Phi(L_A(\mathcal{X}))\big)^+$$

by [19, Lemma 1.2, p. 123]. Applying Π_0 and using Lemma 3.4, we obtain the isomorphisms

$$\begin{aligned}\Pi_0^A(\mathcal{X}^+) &= \Pi_0\big(L_A(\mathcal{X}^+)\big) = \Pi_0\big((\Phi(L_A(\mathcal{X})))^+\big)\\ &= \Pi_0\big(\Phi(L_A(\mathcal{X}))\big)^+ = \Pi_0\big(L_A(\mathcal{X})\big)^+ = \Pi_0^A(\mathcal{X})^+\end{aligned}$$

in the category of sheaves $\mathcal{S}$. □

4 Chow monoids in Nisnevich sheaves

Now we turn from homotopical algebra to algebraic geometry, and specialize the abstract material of the previous sections to the case when $\mathcal{C}$ is the category of smooth schemes of finite type over a field k, the topology τ is the Nisnevich topology on $\mathcal{C}$, and A is the affine line $\mathbb{A}^1$ over k. Throughout all schemes will be separated by default.

The standard Yoneda construction can be extended to a functor h from the category of Noetherian schemes over k to presheaves on $\mathcal{C}$, sending any Noetherian scheme to the corresponding representable presheaf, and the same on morphisms. Then h is a functor to the category of sheaves in étale topology, and so in the Nisnevich one, see [1, p. 347], i.e. the Nisnevich topology is subcanonical. Composing h with the constant functor $\mathrm{Const} = \Delta_?[0]$ from $\mathcal{S}$ to $\Delta^{\mathrm{op}}\mathcal{S}$ we obtain the embedding of the category of Noetherian schemes over k into $\Delta^{\mathrm{op}}\mathcal{S}$.

The scheme $\mathrm{Spec}(k)$ is the terminal object in $\mathcal{C}$. The affine line $\mathbb{A}^1$ over k is an interval in $\Delta^{\mathrm{op}}\mathcal{S}$ with two obvious morphisms i_0 and i_1 from $\mathrm{Spec}(k)$ to $\mathbb{A}^1$. As above, the interval $\mathbb{A}^1$ gives the natural cylinder and the corresponding notion of left homotopy on morphisms in $\Delta^{\mathrm{op}}\mathcal{S}$. The set of points on a simplicial sheaf $\mathcal{X}$ is the set $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathrm{Spec}(k), \mathcal{X})$ of k-points on $\mathcal{X}$, and it coincides with the set $\mathcal{X}_0(k)$. The set of $\mathbb{A}^1$-path connected components on k-points is the set $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathrm{Spec}(k), \mathcal{X})_{\mathbb{A}^1}$. If $\mathcal{X}$ is fibrant in $\mathcal{M}_{\mathbb{A}^1}$, then $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathrm{Spec}(k), \mathcal{X})_{\mathbb{A}^1}$ can be identified with $[\mathrm{Spec}(k), \mathcal{X}]_{\mathbb{A}^1}$.

Let $\mathcal{X}$ be a monoid in the category $\Delta^{\mathrm{op}}\mathcal{S}$. Its completion $\mathcal{X}^+$ is a group object, so that $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathrm{Spec}(k), \mathcal{X}^+)$ is a group in $\Delta^{\mathrm{op}}\mathcal{S}$. The morphism $\mathcal{X} \to \mathcal{X}^+$ induces a map from $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathrm{Spec}(k), \mathcal{X})$ to $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathrm{Spec}(k), \mathcal{X}^+)$. By the universality of group completion, there exists a unique map from $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathrm{Spec}(k), \mathcal{X})^+$ to $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathrm{Spec}(k), \mathcal{X}^+)$ with the obvious commutativity.

Lemma 4.1 *For a simplicial Nisnevich sheaf monoid $\mathcal{X}$, the canonical map from $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathrm{Spec}(k), \mathcal{X})^+$ to $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathrm{Spec}(k), \mathcal{X}^+)$ is bijective, and, repspectively, the map from $(\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathrm{Spec}(k), \mathcal{X})_{\mathbb{A}^1})^+$ to $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathrm{Spec}(k), \mathcal{X}^+)_{\mathbb{A}^1}$ is a surjection.*

Proof Since $\mathrm{Spec}(k)$ is Henselian, the set $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathrm{Spec}(k), \mathcal{X}^+)$ is the quotient of the Cartesian square $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathrm{Spec}(k), \mathcal{X})^2$. The set $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathrm{Spec}(k), \mathcal{X})^+$ is also the quotient of the same Cartesian square. The maps from Hom-sets to the sets of $\mathbb{A}^1$-path connected components are surjective. □

Next, let K be a field extension of the ground field k, and let $\mathcal{S}_K$ be the category of set-valued Nisnevich sheaves on the category $\mathcal{C}_K$ of smooth schemes over K. Let $\mathcal{M}_K$ be the injective model structure on the category $\Delta^{\mathrm{op}}\mathcal{S}_K$, obtained in the same way as the

model structure $\mathcal{M}$ for the category $\Delta^{\mathrm{op}}\mathcal{S}$ over $\mathrm{Spec}(k)$. Let $f\colon \mathrm{Spec}(K) \to \mathrm{Spec}(k)$ be the morphism induced by the extension $k \subset K$, and let $f^*\colon \Delta^{\mathrm{op}}\mathcal{S} \to \Delta^{\mathrm{op}}\mathcal{S}_K$ be the scalar extension functor induced by sending schemes over k to their fibred products with $\mathrm{Spec}(K)$ over $\mathrm{Spec}(k)$, and then using the fact that any sheaf is a colimit of representable ones. As the morphism f is smooth, there are two standard adjunctions

$$f_\# \dashv f^* \dashv f_*$$

for the functor f^*, see, for example, [18].

Lemma 4.2 *For any field extension K of the ground field k, one can choose the localization functors $L_{\mathbb{A}^1}$ and $L_{\mathbb{A}^1_K}$ in $\Delta^{\mathrm{op}}\mathcal{S}$ and $\Delta^{\mathrm{op}}\mathcal{S}_K$ respectively, to have a canonical isomorphism*

$$f^* L_{\mathbb{A}^1} \simeq L_{\mathbb{A}^1_K} f^*.$$

Proof Let $\mathrm{Ex}^{\mathcal{G}}_K$ be the fibrant replacement in $\Delta^{\mathrm{op}}\mathcal{S}_K$ obtained in the same way as $\mathrm{Ex}^{\mathcal{G}}$ was constructed for $\Delta^{\mathrm{op}}\mathcal{S}$, see [19, p. 70]. Let also Sing_K be the Suslin–Voevodsky endofunctor on $\Delta^{\mathrm{op}}\mathcal{S}_K$. Straightforward verifications show that $f^*\mathrm{Ex}^{\mathcal{G}} \simeq \mathrm{Ex}^{\mathcal{G}}_K f^*$ and $f^* \circ \mathrm{Sing} \simeq \mathrm{Sing}_K \circ f^*$. Choose $L_{\mathbb{A}^1}$ (respectively, $L_{\mathbb{A}^1_K}$) to be the transfinite compositions of the functors $\mathrm{Ex}^{\mathcal{G}}$ and Sing (respectively, $\mathrm{Ex}^{\mathcal{G}}_K$ and Sing_K) in $\Delta^{\mathrm{op}}\mathcal{S}$ (respectively, in $\Delta^{\mathrm{op}}\mathcal{S}_K$). □

We now need to refresh some things from [25]. For a scheme X let $\mathcal{C}(X)$ be the free commutative monoid generated by points of X, and let $\mathcal{Z}(X)$ be the group completion of $\mathcal{C}(X)$. An algebraic cycle ζ is an element in $\mathcal{Z}(X)$. As such, ζ is a finite linear combination $\sum m_i \zeta_i$ of points ζ_i on X with integral coefficients m_i. The cycle ζ is said to be effective if and only if $m_i \geq 0$ for all i. This is equivalent to say that ζ is an element of $\mathcal{C}(X)$.

The support $\mathrm{Supp}(\zeta)$ of ζ is the union of the Zariski closures of the points ζ_i with the induced reduced structures on them. The correspondence between points on X and the reduced irreducible closed subschemes of X allows to consider algebraic cycles as linear combinations $Z = \sum m_i Z_i$, where Z_i is the Zariski closure of the point ζ_i, for each i. Then $\mathrm{Supp}(Z)$ is the same thing as $\mathrm{Supp}(\zeta)$. The points ζ_i, or the corresponding reduced closed subschemes Z_i, are prime cycles on X. The dimension of a point in X is the dimension of its Zariski closure in X. Let then $\mathcal{C}_r(X)$ be the submonoid in $\mathcal{C}(X)$ generated by points of dimension r, and, respectively, let $\mathcal{Z}_r(X)$ be the subgroup in the abelian group $\mathcal{Z}(X)$ generated by points of dimension r in X.

Let S be a Noetherian scheme, let k be a field, and let $P\colon \mathrm{Spec}(k) \to S$ be a k-point of S. Recall that a fat point of S over P is an ordered pair (P_0, P_1) of two morphisms of schemes

$$P_0\colon \mathrm{Spec}(k) \to \mathrm{Spec}(R) \qquad \text{and} \qquad P_1\colon \mathrm{Spec}(R) \to S,$$

where R is a discrete valuation ring with the residue field k, such that

$$P_1 \circ P_0 = P,$$

the image of P_0 is the closed point of $\mathrm{Spec}(R)$, and P_1 sends the generic point $\mathrm{Spec}(R_{(0)})$ into the generic point of S.

Let $X \to S$ be a scheme of finite type over S, and let $Z \to X$ be a closed subscheme in X. Let R be a discrete valuation ring, $D = \mathrm{Spec}(R)$, and let $f\colon D \to S$ be an arbitrary morphism of schemes from D to S. Let also $\eta = \mathrm{Spec}(R_{(0)})$ be the generic point of D,

$$X_D = X \times_S D, \qquad Z_D = Z \times_S D \qquad \text{and} \qquad Z_\eta = Z \times_S \eta.$$

Then there exists a unique closed embedding $Z'_D \to Z_D$, such that its pull-back $Z'_\eta \to Z_\eta$, with respect to the morphism $Z_\eta \to Z_D$, is an isomorphism, and the composition

$$Z'_D \to Z_D \to D$$

is a flat morphism of schemes, see [10, Proposition 2.8.5].

In particular, we have such a "platification" if (P_0, P_1) is a fat point over the k-point P and $f = P_1$. Let then X_P be the fibre of the morphism $X_D \to D$ over the point P_0,

$$Z_P = Z_D \times_{X_D} X_P \qquad \text{and} \qquad Z'_P = Z'_D \times_{Z_D} Z_P.$$

Since the closed subscheme Z'_D of X_D is flat over D, one can define the pull-back $(P_0, P_1)^*(Z)$ of the closed subscheme Z to the fibre X_P of the morphism $X \to S$, with regard to the fat point P, as the cycle associated to the closed embedding $Z'_P \to X_P$ in the standard way (consult [7, 1.5] for what "the standard way" means).

In particular, if Z is a prime cycle on X, then we have the pull-back cycle $(P_0, P_1)^*(\zeta)$ on X_P. Extending by linearity we obtain a pull-back homomorphism

$$(P_0, P_1)^*\colon \mathcal{Z}(X) \to \mathcal{Z}(X_P).$$

Following [25], we say that an algebraic cycle $\zeta = \sum m_i \zeta_i$ on X is a relative cycle on X over S if the images of the points ζ_i under the morphism $X \to S$ are the generic points of the scheme S, and, for any k-point P on S, and for any two fat points extending P, the pull-backs of the cycle $Z = \overline{\zeta}$ to X_P, with regard to these two fat points, coincide, see [25, Definition 3.1.3].

Notice that any cycle, which is flat over S, is a relative cycle for free. But not any relative cycle on X/S is flat. This is why we need the "platification" above.

Let $\mathcal{C}_r(X/S)$ be the abelian submonoid in $\mathcal{C}(X)$ generated by relative cycles of relative dimension r over S. It is important that whenever S is a regular Noetherian scheme and X is of finite type over S, then $\mathcal{C}_r(X/S)$ is a free commutative monoid

generated by closed integral subschemes in X which are equidimensional of dimension r over S, see [25, Corollary 3.4.6, p. 40]. Let also

$$\mathcal{Z}_r(X/S) = \mathcal{C}_r(X/S)^+$$

be the group completion of the monoid $\mathcal{C}_r(X/S)$.

Now, fix a Noetherian reduced scheme T, and let $\mathfrak{N}$ be the category of Noetherian schemes over T. Let $X \to T$ be a scheme of finite type over T. For any object $S \to T$ in $\mathfrak{N}$ let

$$\mathcal{Z}_r(X/T)(S) = \mathcal{Z}_r(X \times_T S/S).$$

If $f\colon S' \to S$ is a flat morphism of Noetherian schemes over T, the induced morphism $\mathrm{id}_X \times_T f\colon X \times_T S' \to X \times_T S$ is also flat, and one has the standard flat pull-back homomorphism

$$(\mathrm{id}_X \times_T f)^*\colon \mathcal{Z}_r(X \times_T S/S) \to \mathcal{Z}_r(X \times_T S'/S').$$

If f is not flat, then the situation is more difficult.

However, if T is a regular scheme, due to the Suslin–Voevodsky definition of relative cycles given above, the correct pull-back exists for any morphism f, see [25, Proposition 3.3.15].

This all aggregates, when T is a regular scheme, into the presheaf $\mathcal{Z}_r(X/T)$, which is nothing else but the sectionwise completion of the presheaf $\mathcal{C}_r(X/T)$ of relative effective cycles on the category $\mathfrak{N}$.

Since now we will assume that T is regular of characteristic 0. Let $X \to T$ be a projective scheme over T, and fix a closed embedding

$$X \to \mathbb{P}^m_T$$

over T. If Z is a relative equidimensional cycle on $X \times_T S/S$, its pullback Z_P to the fibre X_P of the morphism $X \times_T S \to S$ over a point P on S has its degree $\deg(Z_P)$, computed with regard to the induce embedding of $X \times_T S$ into $\mathbb{P}^m_S$ over S. Since Z is a relative cycle, the degree $\deg(Z_P)$ is locally constant on S, see [25, Proposition 4.4.8]. It follows that, if S is connected, then $\deg(Z_P)$ does not depend on P, see [25, Corollary 4.4.9] Therefore, the degree of Z over S is correctly defined, and we may consider a subpresheaf

$$\mathcal{C}_{r,d}(X/T) \subset \mathcal{C}_r(X/T),$$

whose sections on S are relative cycles of degree d on $X \times_T S/S$.

The integer d is non-negative, and there is only one cycle in the set $\mathcal{C}_{r,0}(X/T)(S)$, namely the cycle 0 whose coefficients are all zeros. The grading by degrees gives the obvious structure of a graded monoid on the presheaf $\mathcal{C}_r(X/T)$ whose neutral element is the cycle 0 sitting in $\mathcal{C}_{r,0}(X/T)(S)$.

It follows from the results in [25] (see also [14]) that the presheaves $\mathcal{C}_{r,d}(X/T)$ are representable by a scheme $C_{r,d}(X/T)$, the so-called *Chow scheme* of effective relative cycles of relative dimension r and degree d over T. This Chow scheme is projective over T, i.e. there exist a structural morphism from $C_{r,d}(X/T)$ to T, and a closed embedding of $C_{r,d}(X/T)$ into $\mathbb{P}^N_T$ over T, arising from the representability above.

Notice that the Chow sheaves are representable because T is a regular scheme of characteristic zero. If T would be of positive characteristic, then only h-representability takes place, see [25]. If T is the spectrum of an algebraically closed field of characteristic zero, then $C_{r,d}(X/T)$ is the classical Chow scheme of effective r-cycles of degree d on X.

Since now we will assume that $T = \mathrm{Spec}(k)$, where k is a field of characteristic zero, and systematically drop the symbol $/T$ from the notation. According to our convention to identify schemes and the corresponding representable sheaves, we will write $C_{r,d}(X)$ instead of $h_{C_{r,d}(X)}$. Certainly, the latter sheaf $C_{r,d}(X)$ should be identified with the sheaf $\mathcal{C}_{r,d}(X)$.

Let

$$C_r(X) = \coprod_{d\geq 0} C_{r,d}(X),$$

where the coproduct is taken in the category $\mathcal{S}$, not in $\mathfrak{N}$. Such defined $C_r(X)$ is also a coproduct in $\mathcal{P}$. If we would consider the coproduct of all Chow schemes $C_{r,d}(X)$ in $\mathfrak{N}$ first, and then embed it into $\mathcal{S}$ by the Yoneda embedding, that would be a priori a different sheaf, as Yoneda embedding in general does not commute with coproducts. However, the canonical morphism from the above sheafification to this second sheaf is an isomorphism on the Henselizations of the local rings at points of varieties over k. Therefore, the two constructions are actually isomorphic in $\mathcal{S}$. This also gives that the coproduct of $C_{r,d}(X)$, for all $d \geq 0$, in $\mathfrak{N}$ represents $\mathcal{C}_r(X)$.

Identifying $\mathcal{S}$ with its image in $\Delta^{\mathrm{op}}\mathcal{S}$ under the functor Const, we consider $C_r(X)$ as the graded *Chow monoid* in the category of simplicial sheaves on the smooth Nisnevich site over $\mathrm{Spec}(k)$. The completion $C_r(X)^+$ of $C_r(X)$ in $\mathcal{S}$ is the sheafification of the completion of $C_r(X)$ as a presheaf. The latter is sectionwise.

Let $\mathcal{O}^h_{P,Y}$ be the Henselization of the local ring $\mathcal{O}_{P,Y}$ of a smooth algebraic scheme Y over k at a point $P \in Y$. Since $\mathcal{O}_{P,Y}$ is a regular Noetherian ring, the ring $\mathcal{O}^h_{P,Y}$ is regular and Noetherian too. As we mentioned above, the set

$$\mathcal{C}_r\left(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\mathcal{O}^h_{P,Y})/\mathrm{Spec}(\mathcal{O}^h_{P,Y})\right)$$

is a free commutative monoid generated by closed integral subschemes in the scheme $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\mathcal{O}^h_{P,Y})$, which are equidimensional of dimension r over $\mathrm{Spec}(\mathcal{O}^h_{P,Y})$, by [25, Corollary 3.4.6]. Then we see that the monoid $C_r(X)$ is pointwise free, and hence it is a pointwise cancellation monoid in the category $\mathcal{S}$. It follows also that the Chow monoid $C_r(X)$ is pointwise good in the category $\Delta^{\mathrm{op}}\mathcal{S}$, and the canonical morphism from $C_r(X)$ to $C_r(X)^+$ is a monomorphism.

Let K be a finitely generated field extension of k. Since $\mathrm{Spec}(K)$ is Henselian, $C_r(X)^+(K)$ is the same as the group completion $(C_r(X)(K))^+$. On the other hand, the same group $C_r(X)^+(K)$ can be also identified with the group of morphisms from $\mathrm{Spec}(K)$ to $C_r(X)^+$, in the category of simplicial Nisnevich sheaves $\Delta^{\mathrm{op}}\mathcal{S}$.

Let d_0 be the minimal degree of positive r-cycles on X, where the degree is computed with regard to the fixed embedding of X into $\mathbb{P}^m$. Choose and fix a positive r-cycle Z_0 with $\deg(Z_0) = d_0$. For any natural number d the d-multiple dZ_0 is an effective dimension r degree dd_0 cycle on X. This gives a morphism α from $\mathbb{N}$ to $C_r(X)$ sending 1 to Z_0. Since $C_r(X)$ is the coproduct of $C_{r,dd_0}(X)$, for all $d \geq 0$, we also have the obvious morphism f from $C_r(X)$ to $\mathbb{N}$, such that $f \circ \alpha = \mathrm{id}_{\mathbb{N}}$. In other words, Z_0 gives the structure of a pointed graded monoid on $C_r(X)$. Automatically, we obtain the connective Chow monoid $C_r^\infty(X)$ associated to $C_r(X)$. By Lemma 2.1, we also have the canonical isomorphism of group objects

$$C_r(X)^+ \simeq C_r^\infty(X)^+ \times \mathbb{Z}$$

in $\Delta^{\mathrm{op}}\mathcal{S}$. The sheaf $C_r^\infty(X)$ can be also understood as the ind-scheme arising from the chain of Chow schemes

$$C_{r,0}(X) \subset C_{r,d_0}(X) \subset C_{r,2d_0}(X) \subset \cdots \subset C_{r,dd_0}(X) \subset \cdots$$

induced by the cycle Z_0 of degree d_0 on X.

As the category $\mathcal{C}$ is Noetherian, the category $\mathcal{S}$ is exhaustive. Since $C_r(X)$ is a pointwise cancellation monoid in $\mathcal{S}$, and the latter category is exhaustive, $C_r^\infty(X)$ is a pointwise cancellation monoid in $\mathcal{S}$ too. Then both monoids, $C_r(X)$ and $C_r^\infty(X)$ are pointwise good monoids in the category $\Delta^{\mathrm{op}}\mathcal{S}$. Moreover, the canonical morphism from $C_r^\infty(X)$ to $C_r^\infty(X)^+$ is a monomorphism in $\mathcal{S}$ and in $\Delta^{\mathrm{op}}\mathcal{S}$.

5 Rational equivalence as $\mathbb{A}^1$-path connectivity

For any algebraic scheme X over k let $CH_r(X)$ be the Chow group of r-dimensional algebraic cycles modulo rational equivalence on X. In this section we prove our main theorem and deduce three corollaries, which give something close to the desired effective interpretation of Chow groups in terms of $\mathbb{A}^1$-path connectivity on loop spaces of classifying spaces of the Chow monoid $C_r^\infty(X)$. We leave it for the reader to decide which of the obtained three isomorphisms is more useful for understanding of Chow groups.

Theorem 5.1 *Let X be a projective algebraic variety with a fixed embedding into a projective space over k. For any finitely generated field extension K of the ground field k, there is a canonical isomorphism*

$$CH_r(X_K) \simeq \Pi_0^{\mathbb{A}^1}\big(C_r(X)^+\big)(K).$$

Proof Without loss of generality, we may assume that $d_0 = 1$. Consider the obvious commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathrm{Spec}(k), C_r(X)^+)_{\mathbb{A}^1} & \xrightarrow{(l^+)_*} & \mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}\big(\mathrm{Spec}(k), L_{\mathbb{A}^1}C_r(X)^+\big)_{\mathbb{A}^1} \\ \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \beta} \\ \mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathrm{Spec}(k), C_r(X))_{\mathbb{A}^1}{}^+ & \xrightarrow{(l_*)^+} & \mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}\big(\mathrm{Spec}(k), L_{\mathbb{A}^1}C_r(X)\big)_{\mathbb{A}^1}{}^+ \end{array}$$

where $l = l_{C_r(X)}$. Since $L_{\mathbb{A}^1}C_r(X)^+$ is $\mathbb{A}^1$-local, the group in the top right corner is canonically isomorphic to the group $\Pi_0^{\mathbb{A}^1}(C_r(X)^+)(k)$. By the same reason, the group in the bottom right corner is canonically isomorphic to the group $\Pi_0^{\mathbb{A}^1}(C_r(X))(k)^+$. Since $\mathrm{Spec}(k)$ is Henselian, the latter group is nothing but the group $\Pi_0^{\mathbb{A}^1}(C_r(X))^+(k)$. Then Lemma 3.5 gives that β is an isomorphism.

Let $q_0\colon \mathrm{Spec}(k) \to C_r(X)$ and $q_1\colon \mathrm{Spec}(k) \to C_r(X)$ be two k-points on $C_r(X)$, and suppose q_0 is connected to q_1 by an $\mathbb{A}^1$-path $H\colon \mathbb{A}^1 \to L_{\mathbb{A}^1}C_r(X)$ on $L_{\mathbb{A}^1}C_r(X)$. For any d let $(C_r(X))_d$ be the coproduct $\coprod_{i=0}^{d} C_{r,i}(X)$. Then $(C_r(X))_d$ is canonically embedded into the coproduct $(C_r(X))_{d+1}$. Consider the chain of the embeddings

$$\big(C_r(X)\big)_0 \subset \cdots \subset \big(C_r(X)\big)_d \subset \big(C_r(X)\big)_{d+1} \subset \cdots$$

Applying [11, Proposition 4.4.4, p. 77] (see also Remark 3.2 in Sect. 3) we see that $L_{\mathbb{A}^1}C_r(X)$ is canonically isomorphic to the colimit of the chain of the embeddings

$$L_{\mathbb{A}^1}\big((C_r(X))_0\big) \subset \cdots \subset L_{\mathbb{A}^1}\big((C_r(X))_d\big) \subset L_{\mathbb{A}^1}\big((C_r(X))_{d+1}\big) \subset \cdots$$

Since $\mathbb{A}^1$ is a compact object in the category $\Delta^{\mathrm{op}}\mathcal{S}$, it follows that the homotopy H factorizes through $L_{\mathbb{A}^1}((C_r(X))_d)$, for some degree d. If Z_0 is a degree 1 algebraic cycle of dimension r on X, then Z_0 induces the corresponding embeddings

$$C_{r,0}(X) \subset \cdots \subset C_{r,d}(C).$$

This gives the epimorphism from the coproduct $(C_r(X))_d$ onto $C_{r,d}(X)$. Composing the homotopy $H\colon \mathbb{A}^1 \to L_{\mathbb{A}^1}((C_r(X))_d)$ with the induced morphism from $L_{\mathbb{A}^1}((C_r(X))_d)$ to $L_{\mathbb{A}^1}(C_{r,d}(X))$, we obtain the homotopy

$$H\colon \mathbb{A}^1 \to L_{\mathbb{A}^1}\big(C_{r,d}(X)\big).$$

Since X is proper and of finite type over k, [2, Proposition 6.2.6] gives that the points q_0 and q_1 are $\mathbb{A}^1$-chain connected, and so $\mathbb{A}^1$-path connected on $C_{r,d}(X)$. It means that the map

$$l_*\colon \mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathrm{Spec}(k), C_r(X))_{\mathbb{A}^1} \to \mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}\big(\mathrm{Spec}(k), L_{\mathbb{A}^1}C_r(X)\big)_{\mathbb{A}^1}$$

is injective. Since l_* is surjective by Lemma 3.3, it is bijective. Then $(l^+)_*$ is an isomorphism as well.

Since β and $(l^+)_*$ are isomorphisms, and α is an epimorphism by Lemma 4.1, we see that α is an isomorphism, and then all the maps in the commutative square above are isomorphisms.

Let now A and A' be two r-dimensional algebraic cycles on X. If A is rationally equivalent to A' on X, there exists an effective relative cycle Z on the scheme $X \times_{\mathrm{Spec}(k)} \mathbb{A}^1/\mathbb{A}^1$ of relative dimension r, and an effective dimension r algebraic cycle B on X, such that

$$Z(0) = A + B \qquad \text{and} \qquad Z(1) = A' + B$$

on X. Let h_Z and $h_{B\times\mathbb{A}^1}$ be two regular morphisms from $\mathbb{A}^1$ to the Chow scheme $C_r(X)$ over $\mathrm{Spec}(k)$ corresponding to the relative cycles Z and $B \times \mathbb{A}^1$ on $X \times_{\mathrm{Spec}(k)} \mathbb{A}^1/\mathbb{A}^1$ respectively. Let

$$h\colon \mathbb{A}^1 \to C_r(X) \times C_r(X)$$

be the product of h_Z and $h_{B\times\mathbb{A}^1}$ in the category $\Delta^{\mathrm{op}}\mathcal{S}$. Let

$$H\colon \mathbb{A}^1 \to C_r(X) \times C_r(X) \to C_r(X)^+,$$

be the composition of h and the morphism from $C_r(X) \times C_r(X)$ to the completion $C_r(X)^+$, in $\Delta^{\mathrm{op}}\mathcal{S}$. Then $H_0 = A$ and $H_1 = A'$, where H_0 and H_1 are the precompositions of H with i_0 and i_1 respectively. It means that the cycles A and A' are $\mathbb{A}^1$-path connected on $C_r(X)^+$.

Vice versa, suppose we have a morphism

$$H\colon \mathbb{A}^1 \to C_r(X)^+$$

in $\mathcal{S}$, and let H_0 and H_1 be the compositions of H with i_0 and i_1 respectively. Since $\mathrm{Spec}(k)$ is Henselian, H_0 is represented by two morphisms $H_{0,1}$ and $H_{0,2}$ from $\mathrm{Spec}(k)$ to $C_r(X)$. Similarly, H_1 is represented by two morphisms $H_{1,1}$ and $H_{1,2}$ from $\mathrm{Spec}(k)$ to $C_r(X)$. Since α is an isomorphism and H_0 is $\mathbb{A}^1$-path connected to H_1, it follows that there exist two morphisms f and G from $\mathrm{Spec}(k)$ to $C_r(X)$, such that $H_{0,1} + F$ is $\mathbb{A}^1$-path connected to $H_{0,2} + G$ and $H_{1,1} + F$ is $\mathbb{A}^1$-path connected to $H_{1,2} + G$ on $C_r(X)$. In terms of algebraic cycles on X, it means that the effective r-cycle $H_{0,1}+F$ is rationally equivalent to the effective r-cycle $H_{0,2}+G$, and, similarly, the cycle $H_{1,1}+F$ is rationally equivalent to $H_{1,2} + G$ on $C_r(X)$. Then the cycle $H_0 = H_{0,1} - H_{0,2}$ is rationally equivalent to the cycle $H_1 = H_{1,1} - H_{1,2}$ on $C_r(X)$.

Thus, the Chow group $CH_r(X)$ is isomorphic to $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}}(\mathrm{Spec}(k), C_r(X)^+)_{\mathbb{A}^1}$, i.e. the group in the top left corner of the diagram above. Since, moreover, $(l^+)_*$ is an isomorphism, and the group in the top right corner is canonically isomorphic to $\Pi_0^{\mathbb{A}^1}(C_r(X)^+)(k)$, we obtain the required isomorphism in case when L is the ground field k.

To prove the theorem for an arbitrary finitely generated field extension K of the ground field k, we observe that $f^*C_r(X)$ is $C_r(X_K)$, whence

$$f^*L_{\mathbb{A}^1}C_r(X)^+ = L_{\mathbb{A}^1_K}f^*C_r(X)^+ = L_{\mathbb{A}^1_K}C_r(X_K)^+$$

by Lemma 4.2. Therefore,

$$\begin{aligned}\Pi_0^{\mathbb{A}^1}\big(C_r(X)^+\big)(K) &= \Pi_0\big(L_{\mathbb{A}^1}C_r(X)^+\big)(K) = f^*\Pi_0\big(L_{\mathbb{A}^1}C_r(X)^+\big)(K)\\ &= \Pi_0\big(f^*L_{\mathbb{A}^1}C_r(X)^+\big)(K) = \Pi_0\big(L_{\mathbb{A}^1_K}C_r(X_K)^+\big)(K)\\ &= \Pi_0^{\mathbb{A}^1_K}\big(C_r(X_K)^+\big)(K) \simeq CH_r(X_K)_0.\end{aligned}$$

□

Remark 5.2 Lemma 3.5 provides that the monoidal completion in Theorem 5.1 can be taken before or after computing the $\mathbb{A}^1$-connected component functor. Since the monoidal completion is sectionwise on stalks, we obtain the canonical isomorphisms

$$CH_r(X) \simeq \Pi_0^{\mathbb{A}^1}\big(C_r(X)^+\big)(k) \simeq \Pi_0^{\mathbb{A}^1}\big(C_r(X)\big)^+(k) \simeq \Pi_0^{\mathbb{A}^1}\big(C_r(X)\big)(k)^+.$$

The embedding $X \hookrightarrow \mathbb{P}^m$ gives the degree homomorphism from $CH_r(X)$ to $\mathbb{Z}$. Let $CH_r(X)_0$ be its kernel, i.e. the Chow group of degree 0 cycles of dimension r modulo rational equivalence on X. Then,

$$CH_r(X) \simeq CH_r(X)_0 \times \mathbb{Z}.$$

Let Z_0 be a positive r-cycle of minimal degree on X. As we have seen above, this gives the structure of a pointed graded cancellation monoid on $C_r(X)$, and $C_r^\infty(X)$ is a cancellation monoid too.

Corollary 5.3 *In terms above,* $CH_r(X_K)_0 \simeq \Pi_0^{\mathbb{A}^1}(C_r^\infty(X)^+)(K)$.

Proof By Lemma 2.1, $C_r(X)^+ \simeq C_r^\infty(X)^+ \times \mathbb{Z}$. Since the functor $\Pi_0^{\mathbb{A}^1}$ is monoidal and $\Pi_0^{\mathbb{A}^1}(\mathbb{Z}) = \mathbb{Z}$, we get the formula

$$\Pi_0^{\mathbb{A}^1}\big(C_r(X)^+\big) \simeq \Pi_0^{\mathbb{A}^1}\big(C_r^\infty(X)^+\big) \times \mathbb{Z}.$$

Then apply Theorem 5.1 and the isomorphism $CH_r(X) \simeq CH_r(X)_0 \times \mathbb{Z}$. □

Warning 5.4 If $CH_r(X)_0 \simeq \Pi_0^{\mathbb{A}^1}(C_r^\infty(X))(k)^+ = 0$, it does not imply that the monoid $\Pi_0^{\mathbb{A}^1}(C_r^\infty(X))(k)$ vanishes, as this monoid is by no means a pointwise cancellation monoid. One of the reasons for that is that the Chow schemes $C_{r,d}(X)$ can have many components over k.

Corollary 5.5 *In terms above,* $CH_r(X_K)_0 \simeq \Pi_0^{\mathbb{A}^1}(\Omega \mathrm{Ex}BC_r^\infty(X))(K)$.

Proof The Chow monoid $C_r^\infty(X)$ is pointwise good. Lemma 3.1 gives an isomorphism $C_r^\infty(X)^+ \simeq \Omega \mathrm{Ex} BC_r^\infty(X)$ in H, whence

$$\Pi_0^{\mathbb{A}^1}(C_r^\infty(X)^+) \simeq \Pi_0^{\mathbb{A}^1}\big(\Omega \mathrm{Ex} BC_r^\infty(X)\big). \qquad \square$$

Corollary 5.3 completes the proof.

Recall that, for a pointed simplicial Nisnevich sheaf $(\mathcal{X}, x)$, its motivic, i.e. $\mathbb{A}^1$-fundamental group $\Pi_1^{\mathbb{A}^1}(\mathcal{X}, x)$ is, by definition, the Nisnevich sheaf associated to the presheaf sending a smooth scheme U to the set $[S^1 \wedge U_+, (\mathcal{X}, x)]_{\mathbb{A}^1}$, where the symbol $[-,-]_{\mathbb{A}^1}$ stays now for the sets of morphisms in the pointed homotopy category $\mathsf{H}_{\mathbb{A}^1}$, see [19] or [2]. Similarly, one can define, for a pointed simplicial Nisnevich sheaf $(\mathcal{X}, x)$, the fundamental group $\Pi_1^{S^1\wedge\mathbb{A}^1}(\mathcal{X}, x)$, where $\mathbb{A}^1$ is pointed at any k-rational point on it. This is the Nisnevich sheaf associated to the presheaf sending a smooth scheme U to the set

$$\big[S^1 \wedge U_+, (\mathcal{X}, x)\big]_{S^1\wedge\mathbb{A}^1},$$

where the symbol $[-,-]_{S^1\wedge\mathbb{A}^1}$ stays for the sets of morphisms in the pointed homotopy category $\mathsf{H}_{S^1\wedge\mathbb{A}^1}$.

Lemma 5.6 *Let $\mathcal{X}$ be a pointwise good simplicial sheaf monoid. Then, for a scheme U,*

$$\Pi_0^{\mathbb{A}^1}(\mathcal{X}^+)(U) \simeq \Pi_1^{S^1\wedge\mathbb{A}^1}(B\mathcal{X})(U).$$

Proof Since $\mathcal{X}$ is pointwise good, there is an isomorphism between $\mathcal{X}^+$ and $\Omega \mathrm{Ex} B\mathcal{X}$ in the homotopy category H, by Lemma 3.1. Since the classifying space $B\mathcal{X}$ is pointed connected, the canonical morphism

$$L_{\mathbb{A}^1}\Omega \mathrm{Ex}\, B\mathcal{X} \to \Omega \mathrm{Ex}\, L_{S^1\wedge\mathbb{A}^1} B\mathcal{X}$$

is a simplicial (i.e. pre-$\mathbb{A}^1$-localized) weak equivalence and

$$\Omega \mathrm{Ex}\, L_{S^1\wedge\mathbb{A}^1} B\mathcal{X}$$

is $\mathbb{A}^1$-local by [19, Theorem 2.34, p. 84]. This allows us to make the following identifications:

$$\begin{aligned}
\Pi_0^{\mathbb{A}^1}(\mathcal{X}^+)(U) &\simeq \big[U, \mathcal{X}^+\big]_{\mathbb{A}^1} \simeq \big[U, \Omega \mathrm{Ex}\, B\mathcal{X}\big]_{\mathbb{A}^1} \\
&\simeq \big[U, L_{\mathbb{A}^1}\Omega \mathrm{Ex}\, B\mathcal{X}\big]_{\mathbb{A}^1} \simeq \big[U, \Omega \mathrm{Ex}\, L_{S^1\wedge\mathbb{A}^1} B\mathcal{X}\big]_{\mathbb{A}^1} \\
&\simeq \big[U, \Omega \mathrm{Ex}\, L_{S^1\wedge\mathbb{A}^1} B\mathcal{X}\big] \simeq \big[S^1 \wedge U_+, L_{S^1\wedge\mathbb{A}^1} B\mathcal{X}\big] \\
&\simeq \big[S^1 \wedge U_+, B\mathcal{X}\big]_{S^1\wedge\mathbb{A}^1} \simeq \Pi_1^{S^1\wedge\mathbb{A}^1}(B\mathcal{X})(U).
\end{aligned}$$

$\square$

A straightforward consequence of Corollary 5.3 and Lemma 5.6 is

Corollary 5.7 *In terms above,* $CH_r(X_K)_0 \simeq \Pi_1^{S^1\wedge\mathbb{A}^1}(BC_r^\infty(X))(K)$.

Example 5.8 Let X be a nonsingular projective surface over k, where k is algebraically closed of characteristic zero. Assume that X is of general type and has no transcendental second cohomology group, i.e. the cycle class map from $CH^1(X)$ to the second Weil cohomology group $H^2(X)$ is surjective. In that case the irregularity of X is zero. Bloch's conjecture predicts that $CH_0(X) = \mathbb{Z}$. In other words, any two closed points on X are rationally equivalent to each other. Fixing a point on X gives the Chow monoid $C_0^\infty(X)$, which is nothing else but the the infinite symmetric power $\mathrm{Sym}^\infty(X)$ of the smooth projective surface X. By Corollary 5.5, Bloch's conjecture holds for X if and only if all k-points on the motivic space $L_{\mathbb{A}^1}\Omega\mathrm{Ex}\,B\mathrm{Sym}^\infty(X)$ are $\mathbb{A}^1$-path connected. Bloch's conjecture holds, for example, for the classical Godeaux surfaces, [26], and for the Catanese and Barlow surfaces, see [3,27].

The above vision of Chow groups should be compared with the results of Friedlander, Lawson, Lima-Filho, Mazur and others, who considered topological (i.e. not motivic) homotopy completions of Chow monoids working over $\mathbb{C}$, see [6,17]. A nice survey of this topic, containing many useful references, is the article [16]. The topological homotopy completions of Chow monoids are helpful to understand algebraic cycles modulo algebraic equivalence relation, i.e. the groups $A_r(X)$ of algebraically trivial r-cycles cannot be catched by the topological methods. In contrast, the motivic, i.e. $\mathbb{A}^1$-homotopy completions of Chow monoids, considered above, can give the description of $A_r(X)$, working over an arbitrary ground field of characteristic zero, as the previous examples show. Theorem 5.1 also suggests that the *motivic Lawson homology* groups can be defined by the formula

$$L_rH_n^{\mathcal{M}}(X) = \Pi_{n-2r}^{\mathbb{A}^1}(C_r(X)^+)(k).$$

6 Appendix: homotopical algebra

For the convenience of the reader, we collect here the needed extractions from homotopical algebra. Let first $\mathcal{C}$ be a symmetric monoidal category with product $\otimes$ and unit $\mathbb{1}$. The monoidal product $\otimes$ is called to be closed, and the category $\mathcal{C}$ is called closed symmetric monoidal, if the product $\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ is the so-called adjunction of two variables, i.e. there is bifunctor Hom and two functorial in X, Y, Z bijections

$$\mathrm{Hom}_{\mathcal{C}}\big(X, \mathit{Hom}(Y, Z)\big) \simeq \mathrm{Hom}_{\mathcal{C}}(X\otimes Y, Z) \simeq \mathrm{Hom}_{\mathcal{C}}\big(Y, \mathit{Hom}(X, Z)\big).$$

If $\mathcal{C}$ has a model structure $\mathcal{M}$ in it, an adjunction of two variables on $\mathcal{C}$ is called the Quillen adjunction of two variables, or Quillen bifunctor, if, for any two cofibrations $f : X \to Y$ and $f' : X' \to Y'$ in $\mathcal{M}$ the push-out product

$$f \square f' \colon (X \otimes Y') \coprod_{X \otimes X'} (Y \otimes X') \to Y \wedge Y'$$

is also a cofibration in $\mathcal{M}$, which is trivial if either f and f' is. The model category $(\mathcal{C}, \mathcal{M})$ is called the closed symmetric monoidal model category if $\otimes$ is a Quillen bifunctor and the following extra axiom holds. If $q \colon Q\mathbb{1} \to \mathbb{1}$ is a cofibrant replacement for the unit object $\mathbb{1}$, then the morphisms $q \wedge \mathrm{id} \colon Q\mathbb{1} \wedge X \to \mathbb{1} \wedge X$ and $\mathrm{id} \wedge q \colon X \wedge Q\mathbb{1} \to X \wedge \mathbb{1}$ are weak equivalences for all cofibrant objects X. If we consider the Cartesian product $\mathcal{M} \times \mathcal{M}$ of the model structure $\mathcal{M}$ as a model structure on the Cartesian product $\mathcal{C} \times \mathcal{C}$, then $\otimes$ and Hom induce left derived functor $\otimes^L$ from $\mathrm{Ho}(\mathcal{C} \times \mathcal{C})$ to $\mathrm{Ho}(\mathcal{C})$, and right derived functor $R\mathit{Hom}$ from $\mathrm{Ho}(\mathcal{C} \times \mathcal{C})$ to $\mathrm{Ho}(\mathcal{C})$. It is well known that passing to localization commutes with products of categories, so that we have the equivalence between $\mathrm{Ho}(\mathcal{C} \times \mathcal{C})$ and $\mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C})$. This gives the left derived functor

$$\otimes^L \colon \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$$

and the right derived functor

$$R\mathit{Hom} \colon \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C}).$$

As it was shown in [12], the left derived $\otimes^L$ and the right derived $R\mathit{Hom}$ give the structure of a closed symmetric monoidal category on the homotopy category $\mathrm{Ho}(\mathcal{C})$. Since we assume that all objects in $\mathcal{C}$ are cofibrant in $\mathcal{M}$, it is easy to see that the canonical functor from $\mathcal{C}$ to $\mathrm{Ho}(\mathcal{C})$ is monoidal.

An important particular case is when the symmetric monoidal product $\otimes$ is given by the categorical product in $\mathcal{C}$, i.e. when $\mathcal{C}$ is the Cartesian symmetric monoidal category. Since $\mathrm{Ho}(\mathcal{C})$ admits products, and products in $\mathcal{C}$ are preserved in $\mathrm{Ho}(\mathcal{C})$, for any three objects X, Y and Z in $\mathcal{C}$ one has the canonical isomorphism

$$[X, Y] \times [X, Z] \simeq [X, Y \times Z].$$

Let now $\mathcal{C}$ be a left proper cellular simplicial model category with the model structure $\mathcal{M} = (W, C, F)$ in it, let I and J be the sets of, respectively, generating cofibrations and generating trivial cofibrations in $\mathcal{C}$, and let S be a set of morphisms in $\mathcal{C}$. For simplicity we will also be assuming that all objects in $\mathcal{C}$ are cofibrant, which will always be the case in applications. An object Z in $\mathcal{C}$ is called S-local if it is fibrant, in the sense of the model structure $\mathcal{M}$, and for any morphism $g \colon A \to B$ between cofibrant objects in S the induced morphism from $\mathbf{Hom}(B, Z)$ to $\mathbf{Hom}(A, Z)$ is a weak equivalence in $\Delta^{\mathrm{op}}\mathit{Sets}$. A morphism $f \colon X \to Y$ in $\mathcal{C}$ is an S-local equivalence if the induced morphism from $\mathbf{Hom}(Y, Z)$ to $\mathbf{Hom}(X, Z)$ is a weak equivalence in $\Delta^{\mathrm{op}}\mathit{Sets}$ for any S-local object Z in $\mathcal{C}$. Then there exists a new left proper cellular model structure $\mathcal{M}_S = (W_S, C_S, F_S)$ on the same category $\mathcal{C}$, such that $C_S = C$, W_S consists of S-local equivalences in $\mathcal{C}$, so contains W, and F_S is standardly defined by the right lifting property and so is contained in F. The model structure $\mathcal{M}_S$ is again left proper and cellular with the same set of generating cofibrations I and the new set

of generating trivial cofibrations J_S. The model category $(\mathcal{C}, \mathcal{M}_S)$ is called the (left) Bousfield localization of $(\mathcal{C}, \mathcal{M}_S)$ with respect to S. This all can be found in [11].

Notice that the identity adjunction on $\mathcal{C}$ is a Quillen adjunction and induces the derived adjunction $L\mathrm{Id}\colon \mathrm{Ho}(\mathcal{C}) \dashv \mathrm{Ho}(\mathcal{C}_S)\colon R\mathrm{Id}$, where $\mathrm{Ho}(\mathcal{C}_S)$ is the homotopy category of $\mathcal{C}$ with respect to the model structure $\mathcal{M}_S$. Since cofibrations remain the same and, according to our assumption, all objects are cofibrant, the functor $L\mathrm{Id}$ is the identity on objects and surjective on Hom-sets. To describe $R\mathrm{Id}$ we observe the following. Since F_S is smaller than F, the fibre replacement functor in $(\mathcal{C}, \mathcal{M})$ is different from the fibre replacement functor in $(\mathcal{C}, \mathcal{M}_S)$. Taking into account that $\mathcal{C}$ is left proper and cellular, one can show that there exists a fibrant replacement $\mathrm{Id}_{\mathcal{C}} \to L_S$ in $\mathcal{M}_S$, such that, if X is already fibrant in $\mathcal{M}$, then $L_S(X)$ can be more or less visibly constructed from X and S, see [11, Section 4.3] (or less abstract presentation in [5]). The right derived functor $R\mathrm{Id}$, being the composition of $\mathrm{Ho}(L_S)$ and the functor induced by the embedding of S-local, i.e. cofibrant in $\mathcal{M}_S$, objects into $\mathcal{C}$, identifies $\mathrm{Ho}(\mathcal{C}_S)$ with the full subcategory in $\mathrm{Ho}(\mathcal{C})$ generated by S-local objects of $\mathcal{C}$.

Since $(\mathcal{C}, \mathcal{M})$ is a simplicial model category, then so is $(\mathcal{C}, \mathcal{M}_S)$, see [11, Theorem 4.1.1 (4)]. Suppose that $\mathcal{C}$ is, moreover, closed symmetric monoidal with product $\otimes$, and that the monoidal structure is compatible with the model one in the standard sense, i.e. $\mathcal{C}$ is a symmetric monoidal model category (see above). The new model category $(\mathcal{C}, \mathcal{M}_S)$ is monoidal model, i.e. the model structure $\mathcal{M}_S$ is compatible with the existing monoidal product $\otimes$, if and only if for each f in S and any object X in the union of the domains $\mathrm{dom}(I)$ and codomains $\mathrm{codom}(I)$ of generating cofibrations I in $\mathcal{C}$ the product $\mathrm{id}_X \otimes f$ is in W_S.

This is exactly the case when the set S is generated by a morphism $p\colon A \to \mathbb{1}$, where A is an object in $\mathcal{C}$ and $\mathbb{1}$ is the unit object for the monoidal product $\otimes$, i.e.

$$S = \left\{ X \wedge A \xrightarrow{\mathrm{id}_X \wedge p} X : X \in \mathrm{dom}(I) \cup \mathrm{codom}(I) \right\}.$$

In that case the model structure $\mathcal{M}_S$ is compatible with the monoidal one, so that $(\mathcal{C}, \mathcal{M}_S)$ is a simplicial closed symmetric monoidal model category, which is left proper cellular.

We write $\mathcal{M}_A$ and L_A instead of, respectively, $\mathcal{M}_S$ and L_S when S is generated by A in the above sense. One of the fundamental properties of the localization functor L_A is that, for any two objects X and Y in $\mathcal{C}$, the object $L_A(X \times Y)$ is weak equivalent to the object $L_A(X) \times L_A(Y)$, in the sense of the model structure $\mathcal{M}_A$. The proof of this fact in topology is given on p. 36 of the book [5], and it can be verbally transported to abstract setting. All we need is the Quillen adjunction in two variables in $\mathcal{C}$, and the fact saying that if Y is S-local, then $\mathcal{H}om(X, Y)$ is S-local for any X in $\mathcal{C}$, which is also the consequence of adjunction.

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