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RESEARCH ARTICLE

Nonrational del Pezzo fibrations admitting an action of the Klein simple group

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Abstract We present a series of del Pezzo fibrations of degree 2 admitting an action of the Klein simple group and prove their nonrationality by the reduction modulo p method of Kollár. This is relevant to the embedding of the Klein simple group into the Cremona group of rank 3.

Keywords Rationality question · Del Pezzo fibration · Klein simple group

Mathematics Subject Classification 14E08 · 14E07

1 Introduction

The Klein simple group G_{K} is a finite simple group $G_{\mathrm{K}} \cong \mathrm{PSL}_2(\mathbb{F}_7)$ of order 168. It is well known that G_{K} is the automorphism group of the Klein quartic curve which is defined in $\mathbb{P}^2$ by the equation $x_0^3 x_1 + x_1^3 x_2 + x_2^3 x_0 = 0$. Let S_{K} be the double cover of $\mathbb{P}^2$ ramified along the Klein quartic curve. Then S_{K} is a nonsingular del Pezzo surface of degree 2 admitting a faithful action of G_{K}. Belousov [3] proved that $\mathbb{P}^2$ and S_{K} are the only del Pezzo surfaces admitting a faithful action of G_{K}. In [1], Ahmadinezhad presented a series of G_{K}-Mori fiber spaces $X_n/\mathbb{P}^1$ over $\mathbb{P}^1$ whose general fibers are isomorphic to S_{K} for $n \geqslant 0$. A G-Mori fiber space, where G is a group, is a G-equivariant version of Mori fiber space (see Definition 2.9). Among the above series

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of varieties, $X_n/\mathbb{P}^1$ is a del Pezzo fibration for $n \geqslant 1$ while $X_0/\mathbb{P}^1 = \mathbb{P}^1 \times S_{\mathrm{K}}/\mathbb{P}^1$ is not (see Sect. 2 for details). We have the following conjectures concerning these varieties.

Conjecture 1.1 (Cheltsov–Shramov [3, Conjecture 1.4]) *The fibrations $\mathbb{P}^1 \times \mathbb{P}^2/\mathbb{P}^1$ and $X_n/\mathbb{P}^1$, for $n \geqslant 0$, are the only G_{K}-Mori fiber spaces over $\mathbb{P}^1$ in dimension 3.*

Conjecture 1.2 (Ahmadinezhad [1, Conjecture 3.5]) *The varieties X_n are non-rational for $n \geqslant 2$.*

Note that X_0 and X_1 are both rational. The main result of this paper is the following theorem which supports Conjecture 1.2.

Theorem 1.3 *For $n \geqslant 5$, a very general X_n is not rational.*

We refer the reader to Sect. 3 for the meaning of very generality. Note that $X_n/\mathbb{P}^1$ is a del Pezzo fibration of degree 2 and it satisfies the so-called K-condition (or K^2-condition) for $n \geqslant 2$. Thus, by the results of Pukhlikov [12] and Grinenko [5–7] on nonsingular del Pezzo fibrations of degree 2, if X_n were nonsingular, then it would be birationally rigid for $n \geqslant 2$, which would imply nonrationality in a strong sense. Unfortunately, X_n is singular and we cannot apply the above results directly. Instead, we apply the Kollár's reduction modulo p method introduced in [8] (see also [9]) to prove nonrationality of X_n.

This is relevant to the study of embeddings of the Klein simple group $G_{\mathrm{K}} = \mathrm{PSL}_2(\mathbb{F}_7)$ into the Cremona group $\mathrm{Cr}_3(\mathbb{C})$ of rank 3. If we are given a finite simple subgroup G of $\mathrm{Cr}_3(\mathbb{C}^3)$, then there is a rational G-Mori fiber space X/S such that the embedding $G \subset \mathrm{Cr}_3(\mathbb{C})$ is given by $G \subset \mathrm{Aut}(X) \subset \mathrm{Bir}(X) \cong \mathrm{Cr}_3(\mathbb{C})$ (see [11, Section 4.2]). Such a G-Mori fiber space X/S is called a *Mori regularization* of $G \subset \mathrm{Cr}_3(\mathbb{C})$. Moreover, two embeddings G_1 and G_2 into $\mathrm{Cr}_3(\mathbb{C})$ of a finite simple subgroup G are *conjugate* if and only if there is a G-equivariant birational map between Mori regularizations X_1/S_1 and X_2/S_2 of $G_1 \subset \mathrm{Cr}_3(\mathbb{C})$ and $G_2 \subset \mathrm{Cr}_3(\mathbb{C})$, respectively. In [4], Cheltsov and Shramov proved that there are at least three non-conjugate embeddings of $\mathrm{PSL}_2(\mathbb{F}_7)$ into $\mathrm{Cr}_3(\mathbb{C})$ and each of them comes from rational (G_{K}-) Fano threefolds. Theorem 1.3 implies that, for $n \geqslant 5$, a very general $X_n/\mathbb{P}^1$ cannot be a Mori regularization of any subgroup of $\mathrm{Cr}_3(\mathbb{C})$ isomorphic to G_{K}. If Conjectures 1.1 and 1.2 are both true, then it follows that there is no embedding of $\mathrm{PSL}_2(\mathbb{F}_7)$ into $\mathrm{Cr}_3(\mathbb{C})$ coming from a G_{K}-Mori fiber space over $\mathbb{P}^1$ other than $\mathbb{P}^1 \times \mathbb{P}^2/\mathbb{P}^1$, $X_0/\mathbb{P}^1$ and $X_1/\mathbb{P}^1$. Note that, by [1, Theorem 3.4], there is a G_{K}-equivariant birational map between X_1 and $\mathbb{P}^1 \times \mathbb{P}^2$.

The paper is organized as follows. In Sect. 2, we give an explicit construction of varieties X_n. They are constructed as hypersurfaces of suitable weighted projective space bundle over $\mathbb{P}^1$. Then we show that $X_n/\mathbb{P}^1$ is indeed a del Pezzo fibration for $n \geqslant 1$. In Sect. 3, we prove the main theorem. The proof will be done by the Kollár's reduction modulo p method, which we briefly recall in Sect. 3.2. The very first reduction step is done in Sect. 3.1. In Sect. 3.3, we work over a field of characteristic 2 and construct a specific big line bundle on some nonsingular model of X_n by making use of the purely inseparable double covering structure. This will complete the proof in view of the non-ruledness criterion given in Lemma 3.2.

2 Construction of del Pezzo fibrations

We construct del Pezzo fibrations $X_n/\mathbb{P}^1$ as hypersurfaces in suitable weighted projective space bundles over $\mathbb{P}^1$. We refer the reader to [2] for Cox rings (which are also known as homogeneous coordinate rings) of toric varieties. In this section we work over $\mathbb{C}$.

Throughout this paper, we define $f = x_0^3 x_1 + x_1^3 x_2 + x_2^3 x_0$. We see that f is the defining polynomial of the Klein quartic curve whose automorphism group is the Klein simple group. Let P_n be the projective simplicial toric variety with the Cox ring

$$\mathrm{Cox}(P_n) = \mathbb{C}[w_0, w_1, x_0, x_1, x_2, y]$$

which is $\mathbb{Z}^2$-graded as

$$\begin{pmatrix} w_0 & w_1 & x_0 & x_1 & x_2 & y \\ 1 & 1 & 0 & 0 & 0 & -n \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

and with the irrelevant ideal $I = (w_0, w_1) \cap (x_0, x_1, x_2, y)$, that is, P_n is the geometric quotient

$$P_n = (\mathbb{A}^6 \setminus V(I))/(\mathbb{C}^*)^2,$$

where the action of $(\mathbb{C}^*)^2$ on $\mathbb{A}^6 = \mathrm{Spec}\,\mathrm{Cox}(P_n)$ is given by the above matrix. Note that the Weil divisor class group $\mathrm{Cl}(P_n)$ is isomorphic to $\mathbb{Z}^2$. There is a natural morphism $\Pi\colon P \to \mathbb{P}^1$ defined as the projection to coordinates w_0, w_1, and this realizes P as a weighted projective space bundle over $\mathbb{P}^1$ whose fibers are $\mathbb{P}(1, 1, 1, 2)$. For a nonnegative integer n and homogeneous polynomials $a \in \mathbb{C}[w_0, w_1]$ and $f \in \mathbb{C}[x_0, x_1, x_2]$ of degree respectively $2n$ and 4, define

$$X_n = (ay^2 + f = 0) \subset P_n,$$

and let $\pi = \Pi|_{X_n} \colon X_n \to \mathbb{P}^1$. Throughout this paper, we assume that a does not have a multiple component.

Remark 2.1 Let us note that $X_n/\mathbb{P}^1$ constructed as above coincides with the one given in [1, Section 3]. Indeed, choose and fix any pair $b, c \in \mathbb{C}[w_0, w_1]$ of homogeneous polynomials of degree n such that $a = bc$ and define

$$\mathcal{X}'_n = (bt^2 + cf = 0) \subset \mathbb{P}^1_{w_0, w_1} \times \mathbb{P}(1_{x_0}, 1_{x_1}, 1_{x_2}, 2_t).$$

Let $\pi' \colon \mathcal{X}'_n \to \mathbb{P}^1$ be the projection to the coordinates w_0, w_1. Then, $(c = t = f = 0) \subset \mathcal{X}'_n$ is a disjoint union of n-curves $C'_1, \dots, C'_n$ and $\mathcal{X}'_n$ has a singularity of type $\mathbb{C} \times 1/2(1, 1)$ along each C'_i. Blowing up $\mathcal{X}'_n$ along these curves and then contracting the strict transforms of the π'-fibers containing C'_i, we obtain a birational map $\mathcal{X}'_n \dashrightarrow \mathcal{X}_n$ to the del Pezzo fibration $\mathcal{X}_n \to \mathbb{P}^1$ constructed in [1].

Now we have a birational map $\Psi\colon P_n \dashrightarrow \mathbb{P}^1 \times \mathbb{P}(1,1,1,2)$ defined by the correspondence $t = cy$. It is easy to see that Ψ restricts to a birational map $\psi\colon X_n \dashrightarrow \mathcal{X}'_n$. Moreover, it is straightforward to see that $\psi^{-1}\colon \mathcal{X}'_n \dashrightarrow X_n$ is obtained by blowing up $\mathcal{X}'_n$ along $C'_1, \dots, C'_n$ and then contracting the strict transforms of the fibers containing C'_i. This shows $\mathcal{X}_n/\mathbb{P}^1 \cong X_n/\mathbb{P}^1$.

Remark 2.2 Let us explain that both X_0 and X_1 are rational. If $n = 0$, then we have $X_0 \cong \mathbb{P}^1 \times S$, where $S = (y^2 + f = 0) \subset \mathbb{P}(1,1,1,2)$ is a (nonsingular) del Pezzo surface of degree 2, and thus X_0 is clearly rational. Suppose $n = 1$. Then, as explained in Remark 2.1, X_1 is birational to $\mathcal{X}'_1 = (by^2 + cf = 0) \subset \mathbb{P}^1 \times \mathbb{P}(1,1,1,2)$, where $b, c \in \mathbb{C}[w_0, w_1]$ are homogeneous polynomials of degree 1 such that $a = bc$. It is clear that the projection $\mathcal{X}'_1 \dashrightarrow \mathbb{P}(1,1,1,2)$ is birational. Hence $\mathcal{X}'_1$ and X_1 are rational.

In the rest of this section, we show that $\pi : X_n \to \mathbb{P}^1$ is indeed a del Pezzo fibration for $n \geqslant 1$.

Definition 2.3 Let $\pi : X \to \mathbb{P}^1$ be a surjective morphism with connected fibers from a normal projective 3-fold X. We say that $\pi : X \to \mathbb{P}^1$ is a *del Pezzo fibration* over $\mathbb{P}^1$ if the following conditions are satisfied:

- X is $\mathbb{Q}$-factorial and has only terminal singularities.
- $-K_X$ is π-ample.
- $\rho(X) = 2$.

Remark 2.4 We explain the natural affine open subsets of P_n and X_n. We refer the reader to [13] for details. Since we will work over an algebraically closed field of characteristic 2 in the next section, we assume in this remark that the ground field of P_n and X_n is an algebraically closed field $\Bbbk$ of arbitrary characteristic.

Denote by U_{w_i,x_j} the open subset $(w_i \neq 0) \cap (x_j \neq 0) \subset P_n$ and by $U_{w_i,y}$ the open subset $(w_i \neq 0) \cap (y \neq 0) \subset P_n$. Then P_n is covered by U_{w_i,x_j} and $U_{w_i,y}$ for $i = 0, 1$ and $j = 0, 1, 2$; and U_{w_0,x_0} is the affine 4-space $\mathbb{A}^4$. The restrictions of w_1, x_1, x_2, y on U_{w_0,x_0} form affine coordinates of U_{w_0,x_0}. Indeed, if we denote by $\widetilde{w}_1 = w_1/w_0$, $\widetilde{x}_i = x_i/x_0$ for $i = 1, 2$ and $\widetilde{y} = y w_0^n / x_0^2$, then U_{w_0,x_0} is an affine 4-space with affine coordinates $\widetilde{w}_0, \widetilde{x}_1, \widetilde{x}_2, \widetilde{y}$. The affine scheme $X_n \cap U_{w_0,x_0}$ is defined by the equation $\widetilde{y} a(1, \widetilde{w}_1) + f(1, \widetilde{x}_1, \widetilde{x}_2) = 0$. The same description applies for the other U_{w_i,x_j}.

We see that $U_{w_0,y}$ is the quotient $\mathbb{A}^4/\boldsymbol{\mu}_2$ of $\mathbb{A}^4$ by the action of $\boldsymbol{\mu}_2 = \operatorname{Spec} \Bbbk[t]/(t^2)$. Indeed, if we denote by $\widetilde{w}_1 = w_1/w_0$, $\widetilde{x}_i = x_i w_0^{n/2} / y^{1/2}$ for $i = 0, 1, 2$, then $U_{w_0,y}$ is the quotient of $\mathbb{A}^4$ with coordinates $\widetilde{w}_1, \widetilde{x}_0, \widetilde{x}_1, \widetilde{x}_2$ under the $\boldsymbol{\mu}_2$-action given by

$$\widetilde{w}_0 \mapsto \widetilde{w}_0, \qquad \widetilde{x}_i \mapsto \widetilde{x}_i \otimes \bar{t},$$

where $\bar{t} \in \Bbbk[t]/(t^2)$. Here, the above operation defines a ring homomorphism $R \to R \otimes \Bbbk[t]/(t^2)$, where $R = \Bbbk[\widetilde{w}_0, \widetilde{x}_0, \widetilde{x}_1, \widetilde{x}_2]$, and $\mathbb{A}^4/\boldsymbol{\mu}_2 = \operatorname{Spec} R^{\boldsymbol{\mu}_2}$. When $\Bbbk = \mathbb{C}$, we can replace $\boldsymbol{\mu}_2$ with $\mathbb{Z}/2\mathbb{Z}$ and the action is given simply by $\widetilde{w}_0 \mapsto \widetilde{w}_0$ and $\widetilde{x}_i \mapsto -\widetilde{x}_i$. The affine scheme $X_n \cap U_{w_0,y}$ is the quotient of the affine scheme $a(1, \widetilde{w}_1) + f(\widetilde{x}_0, \widetilde{x}_1, \widetilde{x}_2) = 0$ defined by the $\boldsymbol{\mu}_2$-action. The same description applies for $U_{w_1,y}$.

Sometimes we will abuse the notation and say that U_{w_0,x_0} is the affine 4-space $\mathbb{A}^4$ with coordinates w_1, x_1, x_2, y and $X_n \cap U_{w_0,x_0}$ is defined by $ya(1, w_1) + f(1, x_1, x_2) = 0$.

Lemma 2.5 *The variety X_n is nonsingular outside $(x_0 = x_1 = x_2 = 0) \cap X_n$ and it has a singular point of type $1/2(1, 1, 1)$ at each point of $(x_0 = x_1 = x_2 = 0) \cap X_n$.*

Proof Set $U = U_{w_0,x_0}$ which is an affine 4-space with affine coordinates w_1, x_1, x_2, y, then $X_n \cap U$ is defined by $y^2 a_0 + f_0 = 0$, where $a_0 = a(1, w_1)$ and $f_0 = f(1, x_1, x_2)$. It is straightforward to see that

$$\begin{aligned}\operatorname{Sing}(X_n \cap U) &= \left(y^2 \frac{\partial a_0}{\partial w_1} = \frac{\partial f_0}{\partial x_1} = \frac{\partial f_0}{\partial x_2} = 2ya_0 = y^2 a_0 + f_0 = 0\right)\\ &\subset \left(\frac{\partial f_0}{\partial x_1} = \frac{\partial f_0}{\partial x_2} = f_0 = 0\right) = \varnothing,\end{aligned}$$

where the last equality holds since $f_0 = f(1, x_1, x_2)$ defines a nonsingular curve in $\mathbb{A}^2$. By symmetry, we conclude that $X \cap U_{w_i,x_j}$ is nonsingular for $i = 0, 1$ and $j = 0, 1, 2$. Since the open subsets U_{w_i,x_j} for $i = 0, 1$ and $j = 0, 1, 2$ cover $P_n \setminus (x_0 = x_1 = x_2 = 0)$, we see that X_n is nonsingular outside $(x_0 = x_1 = x_2 = 0) \cap X_n$.

Let $P \in (x_0 = x_1 = x_2 = 0) \cap X_n$. Then $a(P) = 0$ and we may assume that w_1 vanishes at P after replacing w_0, w_1. We work on $U = U_{w_0,y} \cong \mathbb{A}^4/\boldsymbol{\mu}_2$. We see that $X_n \cap U$ is the quotient of $V = (a(1, w_1) + f = 0) \subset \mathbb{A}^4$ by the $\boldsymbol{\mu}_2$-action and P corresponds to the origin. Since a vanishes at P and it does not have a multiple component, we have $a(1, w_1) = w_1 +$higher order terms, so that x_0, x_1, x_2 form local coordinates of V at the origin. Thus the point P is of type $1/2(1, 1, 1)$. □

For $n \geqslant 1$, we construct a birational morphism $\theta \colon X_n \to V_n$ as follows. Set $\xi_0 = w_0^n$, $\xi_1 = w_0^{n-1} w_1, \dots, \xi_n = w_1^n$ and let

$$\Theta \colon P_n \to \mathbb{P}\left(1_{x_0}, 1_{x_1}, 1_{x_2}, 2_{y_0}, \dots, 2_{y_n}\right)$$

be the toric morphism defined by the correspondence $y_i = y\xi_i$. Then the image of Θ, which we denote by T_n, is defined by $h_1 = \dots = h_N = 0$, where $h_1, \dots, h_N$ are the homogeneous polynomials in $y_0, \dots, y_n$ defining the image of the n-ple Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^n_{y_0,\dots,y_n}$. We see that $\Theta \colon P_n \to T_n$ is a birational morphism contracting the divisor $(y = 0) \cong \mathbb{P}^1 \times \mathbb{P}^2$ to the plane $\Delta = (y_0 = \dots = y_n = 0) \subset T_n$. It follows that T_n is a projective simplicial toric variety with Picard number 1. The image of X_n under Θ is a hypersurface V_n in T_n defined by $q + f = 0$, where $q = q(y_0, \dots, y_n)$ is a quadratic polynomial such that $q(y\xi_0, \dots, y\xi_n) = ay^2$. The morphism $\theta = \Theta|_{X_n} \colon X_n \to V_n$ is a birational morphism contracting the divisor $(y = 0) \cap X_n \cong \mathbb{P}^1 \times C$ to the curve $\Delta \cap V_n \cong C$, where C is the plane curve defined by $f = 0$.

Lemma 2.6 *If $n \geqslant 1$, then V_n is a normal projective $\mathbb{Q}$-factorial 3-fold with Picard number 1.*

Proof Note that X_n is $\mathbb{Q}$-factorial since it has only quotient singularities. It follows that V_n is $\mathbb{Q}$-factorial since θ is an extremal contraction (which is not necessarily K_{X_n}-negative). We see that the singularity of T_n along the plane Δ is of type $\mathbb{P}^2 \times 1/n(1, 1)$ and V_n intersects Δ transversally. Moreover, outside the curve $\Delta \cap V_n$, singular points of V_n are of type $1/2(1, 1, 1)$. This implies that V_n is a V-submanifold of T_n and thus, by [2, Proposition 3.5], V_n is quasi-smooth in T_n. Here, we refer the reader to [2, Section 3] for the definitions of V-submanifold and quasi-smoothness. It then follows, from [14, Proposition 4], that $\rho(V_n) = \rho(T_n) = 1$ since V_n is quasi-smooth hypersurface defined by a global section of an ample divisor on T_n. □

Lemma 2.7 *For $n \geqslant 1$, the fibration $\pi : X_n \to \mathbb{P}^1$ is a del Pezzo fibration.*

Proof Assume that $n \geqslant 1$. We see that X has only terminal singularities of type $1/2(1, 1, 1)$ and it is $\mathbb{Q}$-factorial. By Lemma 2.6, we have $\rho(X_n) = \rho(V_n) + 1 = 2$ since $\theta\colon X_n \to V_n$ is a birational morphism contracting a prime divisor. This shows that π is an extremal contraction and thus $X_n/\mathbb{P}^1$ is indeed a del Pezzo fibration. □

Remark 2.8 The above arguments apply to more general cases without any change. Let $g \in \mathbb{C}[x_0, x_1, x_2]$ be a homogeneous polynomial of degree 4 such that the plane curve in $\mathbb{P}^2$ defined by g is nonsingular. Then the hypersurface $X_n = (ay^2 + g = 0) \subset P_n$, where $a \in \mathbb{C}[w_0, w_1]$ is a homogeneous polynomial of degree $2n$ which does not have a multiple component, together with the projection $\pi : X_n \to \mathbb{P}^1$ is a del Pezzo fibration provided that $n \geqslant 1$.

The variety V_n and the birational map $\theta\colon X_n \dashrightarrow V_n$ are constructed in order to prove Lemma 2.7 and we will not use them in what follows. We give a definition of G-Mori fiber space.

Definition 2.9 Let G be a group. A *G-Mori fiber space* is a normal projective variety X, where G acts faithfully on X, together with a G-equivariant morphism $\pi : X \to S$ onto a normal projective variety S with the following properties:

- X is $G\mathbb{Q}$-factorial, that is, every G-invariant Weil divisor on X is $\mathbb{Q}$-Cartier, and X has only terminal singularities.
- $-K_X$ is π-ample.
- $\dim S < \dim X$ and π has connected fibers.
- $\operatorname{rank} \operatorname{Pic}^G(X) - \operatorname{rank} \operatorname{Pic}^G(S) = 1$.

Note that the Klein simple group $G = \mathrm{PSL}_2(\mathbb{F}_7)$ acts on $X_n/\mathbb{P}^1$ along the fibers, so that $X_n/\mathbb{P}^1$ is a G-Mori fiber space for $n \geqslant 1$. For $n = 0$, $X_0/\mathbb{P}^1 \cong S \times \mathbb{P}^1/\mathbb{P}^1$ is not a del Pezzo fibration. Nevertheless, we have $\rho^G(X_0) = 1$, so that $X_0/\mathbb{P}^1$ is a G-Mori fiber space as well.

3 Proof of Theorem 1.3

3.1 Reduction modulo 2

In the following, we drop the subscript n and write $P = P_n$, $X = X_n$. In the previous section, the toric variety P was defined over $\mathbb{C}$. We can define P over an arbitrary field or more generally an arbitrary ring. For a field or a ring K, we denote by P_K the toric variety over Spec K defined by the same fan as that of P. Then, since $f = x_0^3x_1+x_1^3x_2+x_2^3x_0$ is defined over $\mathbb{Z}$, we can define the subscheme $X_K = (ay^2+f = 0) \subset P_K$ for a homogeneous polynomial $a \in K[w_0, w_1]$ of degree $2n$.

Let $a = \alpha_0 w_0^{2n} + \alpha_1 w_0^{2n-1} w_1 + \cdots + \alpha_{2n} w_1^{2n}$, $\alpha_i \in \mathbb{C}$. Assume that $\alpha_0, \ldots, \alpha_{2n}$ are very general so that they are algebraically independent over $\mathbb{Z}$. Then, the ring $\mathbb{Z}[\alpha_0, \ldots, \alpha_{2n}]$ is isomorphic to a polynomial ring of $2n + 1$ variables over $\mathbb{Z}$ and the ideal (2) is a prime ideal. Define

$$R = \mathbb{Z}[\alpha_0, \ldots, \alpha_{2n}]_{(2)}$$

which is a DVR whose residue field is of characteristic 2. Then we can define $X_R = (ay^2 + f = 0) \subset P_R$, which is a scheme over Spec R and whose geometric generic fiber is isomorphic to $X_{\mathbb{C}}$.

Lemma 3.1 *Let $\Bbbk$ be an algebraically closed field which is uncountable. If $X_{\Bbbk} = (ay^2 + f = 0) \subset P_{\Bbbk}$ is not ruled for a very general $a \in \Bbbk[w_0, w_1]$, then $X = X_{\mathbb{C}} = (ay^2 + f = 0) \subset P_{\mathbb{C}}$ is not ruled for a very general $a \in \mathbb{C}[w_0, w_1]$.*

Proof Let X' be the geometric special fiber of $X_R \to$ Spec R defined over $\Bbbk$. We can write $X' = (a'y^2 + f = 0) \subset P_{\Bbbk}$ for some $a' \in \Bbbk[w_0, w_1]$ and a' corresponds to a very general element. By the Matsusaka theorem [9, V.1.6 Theorem], if X' is not ruled, then X is not ruled. This completes the proof. □

3.2 Kollár's technique

In this subsection, we briefly recall Kollár's argument of proving non-ruledness of suitable covering spaces in positive characteristic. We apply the following non-ruledness criterion which is a slight generalization of [9, V.5.1 Lemma].

Lemma 3.2 *Let Y be a smooth proper variety defined over an algebraically closed field and $\mathcal{M}$ a big line bundle on Y. If there is an injection $\mathcal{M} \hookrightarrow (\Omega_Y^i)^{\otimes m}$ for some $i > 0$ and $m > 0$, then Y is not separably uniruled.*

Proof Suppose that Y is separably uniruled. Then, there exists a separable dominant map $\varphi\colon \mathbb{P}^1 \times V \dashrightarrow Y$, where V is a normal projective variety. After shrinking V, we may assume that φ is a morphism and V is smooth. The homomorphism $\varphi^*\Omega_Y^1 \hookrightarrow \Omega^1_{V\times\mathbb{P}^1}$ is an isomorphism on a non-empty open subset since φ is separable. This induces an injection $\varphi^*\mathcal{M}^{\otimes k} \hookrightarrow (\Omega^i_{V\times\mathbb{P}^1})^{\otimes mk}$ for any $k \geqslant 1$. The invertible sheaf $\mathcal{M}$ is big so that the global sections of $\varphi^*\mathcal{M}^{\otimes k}$ separate points on a nonempty open subset of $V \times \mathbb{P}^1$ for a sufficiently large k. This is a contradiction since the global sections of $(\Omega^i_{V\times\mathbb{P}^1})^{\otimes mk}$ do not separate points in a fiber. □

Remark 3.3 Our aim is to prove that the variety X_n defined over an algebraically closed field of characteristic 2 is not ruled. In view of Lemma 3.2, it is enough to construct a resolution $r\colon Y \to X_n$ and a big line bundle $\mathcal{M}$ which is a subsheaf of

$(\Omega_Y^i)^{\otimes m}$ for some $m > 0$. As we will see in Sect. 3.3, there is a purely inseparable cover $X_n \to Z_n$ of degree 2 for some normal projective variety Z_n. In the following we explain the Kollár's construction of a big line bundle on a nonsingular model of a suitable cyclic covering space in a general setting.

Let Z be a variety of dimension n defined over an algebraically closed field $\Bbbk$ of characteristic $p > 0$, $\mathcal{L}$ a line bundle on Z and $s \in H^0(Z, \mathcal{L}^{\otimes m})$ a global section of $\mathcal{L}^m$ for some $m > 0$. Let $U = \operatorname{Spec} \bigoplus_{i \geqslant 0} \mathcal{L}^{-i}$ be the total space of the line bundle $\mathcal{L}$ and let $\rho_U : U \to Z$ be the natural morphism. We denote by $y \in H^0(U, \rho_U^* \mathcal{L})$ the zero section and define

$$Z\left[\sqrt[m]{s}\right] = (y^m - s = 0) \subset U.$$

We say that $Z\left[\sqrt[m]{s}\right]$ is the *cyclic covering of Z obtained by taking mth roots of s*. Set $X = Z\left[\sqrt[m]{s}\right]$ and let $\rho = \rho_U|_X : X \to Z$ be the cyclic covering.

From now on we assume that Z is nonsingular and m is divisible by p. We have a natural differential $d : \mathcal{L}^m \to \mathcal{L}^m \otimes \Omega_Z^1$ whose construction is given below. Let τ be a local generator of $\mathcal{L}$ and $t = g\tau^m$ a local section. Let $x_1, \dots, x_n$ be local coordinates of of Z. Then define

$$d(t) = \sum \frac{\partial g}{\partial x_i} \tau^m dx_i.$$

This is independent of the choices of local coordinates and the local generator τ, and thus defines d. For the section $s \in H^0(Z, \mathcal{L}^m)$, we can view $d(s)$ as a sheaf homomorphism $d(s) : \mathcal{O}_Z \to \mathcal{L}^m \otimes \Omega_Z^1$. By taking the tensor product with $\mathcal{L}^{-m}$, we obtain $ds : \mathcal{L}^{-m} \to \Omega_Z^1$.

Definition 3.4 ([9, V.5.8 Definition]) We define $\mathcal{Q}(\mathcal{L}, s) = (\det \operatorname{Coker}(ds))^{\vee\vee}$.

We have $\mathcal{Q}(\mathcal{L}, s) \cong \mathcal{L}^m \otimes \omega_Z$.

Lemma 3.5 ([9, 5.5 Lemma]) *There is an injection $\rho^* \mathcal{Q}(\mathcal{L}, s) \hookrightarrow (\Omega_X^{n-1})^{\vee\vee}$.*

Remark 3.6 Let $x_1, \dots, x_n$ be local coordinates of Z at a point P and $s = g\tau^{\otimes m}$ as before. Then, $\rho^* \mathcal{Q}(\mathcal{L}, s) \subset (\Omega_X^1)^{\vee\vee}$ is generated by the form

$$\eta = (\pm) \frac{dx_2 \wedge \cdots \wedge dx_n}{\partial f / \partial x_1} = (\pm) \frac{dx_1 \wedge dx_3 \wedge \cdots \wedge dx_n}{\partial f / \partial x_2} = (\pm) \frac{dx_1 \wedge \cdots \wedge dx_{n-1}}{\partial f / \partial x_n}.$$

See [9, V.5.9 Lemma] for details.

Let us show that if the singularity of X is mild, then we can lift $\rho^* \mathcal{Q}(\mathcal{L}, s)$ to an invertible subsheaf of Ω_Y^{n-1}, where Y is a suitable nonsingular model of X. For simplicity of description, we assume that $p = 2$ and $n = \dim Z = 3$.

Definition 3.7 ([9, V.5.6 Definition], see also [9, V.5.7 Exercise]) We say that $s \in H^0(Z, \mathcal{L}^m)$ has a *critical point* at $P \in Z$ if $d(s) \in H^0(Z, \mathcal{L}^m \otimes \Omega_Z^1)$ vanishes at P.

Denote by $\mathrm{Crit}(s) \subset Z$ the set of critical points of s. We say that s has an *almost nondegenerate critical point* at P if in suitable choice of local coordinates x_1, x_2, x_3 we can write

$$g = \alpha x_1^2 + x_2 x_3 + x_1^3 + h,$$

where $\alpha \in \mathbb{k}$, $s = g\tau^m$ for a local generator τ of $\mathcal{L}$ at P, $h = h(x_1, x_2, x_3)$ consists of monomials of degree at least 3 and it does not involve x_1^3.

Lemma 3.8 ([9, V.5.10 Proposition]) *Suppose that s has only almost nondegenerate critical points. Then the singularities of X are isolated singularities and they can be resolved by blowing up each singular point of X. Moreover, if we denote by $r \colon Y \to X$ the blowup of each singular point of X, then $r^* \rho^* \mathcal{Q}(\mathcal{L}, s) \hookrightarrow \Omega_Y^2$.*

3.3 Construction of a big line bundle

Throughout this subsection, we work over an algebraically closed field $\mathbb{k}$ of characteristic 2 which is uncountable. We write $P = P_{\mathbb{k}}$ and $X = X_{\mathbb{k}}$. We do not assume $n \geqslant 5$ for the moment. Let $P^\circ = P \setminus (x_0 = x_1 = x_2 = 0)$ and $X^\circ = X \cap P^\circ$. Note that P° is the nonsingular locus of P. Define

$$Q = \begin{pmatrix} w_0 & w_1 & x_0 & x_1 & x_2 & z \\ 1 & 1 & 0 & 0 & 0 & -2n \\ 0 & 0 & 1 & 1 & 1 & 4 \end{pmatrix}$$

and set Z to be the hypersurface in Q defined by $za + f = 0$. Let $\rho \colon X \to Z$ be the morphism which is defined by the correspondence $z = y^2$, which is a purely inseparable finite morphism of degree 2.

Lemma 3.9 *Let $a \in \mathbb{k}[w_0, w_1]$ be a general homogeneous polynomial of degree $2n$. Then the set*

$$\mathrm{Crit}(a) = \left(\frac{\partial a}{\partial w_0} = \frac{\partial a}{\partial w_1} = 0 \right) \subset \mathbb{P}^1_{w_0, w_1}$$

consists of finitely many points and $\mathrm{Crit}(a) \cap (a = 0) = \emptyset$. *Moreover, for each $P \in \mathrm{Crit}(a)$, we can choose a local coordinate w of $\mathbb{P}^1$ at P such that*

$$a = \alpha + \beta w^2 + w^3 + \textit{higher order terms}$$

for some $\alpha, \beta \in \mathbb{k}$ with $\alpha \neq 0$.

Proof The set $\mathrm{Crit}(a)$ is clearly a finite set of points. As a generality of a, we in particular require that a does not have a multiple component. It is then clear that $\mathrm{Crit}(a) \cap (a = 0) = \emptyset$. The last assertion follows by counting dimension. Let $P \in \mathbb{P}^1$ be a point and w a local coordinate of $\mathbb{P}^1$ at P. We can write $a = \sum \alpha_i w^i$, $\alpha_i \in \mathbb{k}$. We say that a has a bad critical point at P if $\alpha_1 = \alpha_3 = 0$. Two conditions $\alpha_1 = \alpha_3 = 0$

are imposed for a to have a bad critical point at a given $P \in \mathbb{P}^1$. Since P runs through $\mathbb{P}^1$, we see that homogeneous polynomials a which have a bad critical point at some point $P \in \mathbb{P}^1$ form at most $2 - 1 = 1$ codimensional subfamily in the space of all $a \in \Bbbk[w_0, w_1]$. Thus, a general a does not have a bad critical point at all and the proof is completed. □

Lemma 3.10 *The set*

$$\operatorname{Crit}(f) = \left(\frac{\partial f}{\partial x_0} = \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0\right) \subset \mathbb{P}^2_{x_0,x_1,x_2}$$

consists of finitely many closed points and $\operatorname{Crit}(f) \cap (f = 0) = \varnothing$. *Moreover, for each* $P \in \operatorname{Crit}(f)$, *we can choose local coordinates* t_1, t_2 *of* $\mathbb{P}^2$ *at* P *such that*

$$f = \gamma + t_1 t_2 + \textit{higher order terms}$$

for some $\gamma \neq 0$.

Proof We have

$$\frac{\partial f}{\partial x_0} = x_0^2 x_1 + x_2^3, \qquad \frac{\partial f}{\partial x_1} = x_0^3 + x_1^2 x_2, \qquad \frac{\partial f}{\partial x_2} = x_1^3 + x_2^2 x_0.$$

By a straightforward computation, we have

$$\operatorname{Crit}(f) = \left\{(1 : \zeta^{3i} : \zeta^i) : 0 \leqslant i \leqslant 6\right\},$$

where $\zeta \in \Bbbk$ is a primitive 7th root of unity. It is also straightforward to see that $f(P) \neq 0$ for $P \in \operatorname{Crit}(f)$. For the last assertion, we work on the affine open subset $U = (x_0 \neq 0) \subset \mathbb{P}^2$. Note that $\operatorname{Crit}(f) \subset U$. By setting $x_0 = 1$, we think of x_1, x_2 as affine coordinates of $U \cong \mathbb{A}^2$. We have $f = x_1 + x_1^3 x_2 + x_2^3$ on U. For the verification of the last assertion, it is enough to show that the Hessian of f at $P = (1 : \zeta^{3i} : \zeta^i) \in \operatorname{Crit}(f)$ is nonzero. We have $\partial^2 f/\partial x_1^2 = 1$, $\partial^2 f/\partial x_1 \partial x_2 = x_1^2$ and $\partial^2/\partial x_2^2 = 0$, so that we can compute the Hessian as

$$\begin{vmatrix} 1 & x_1^2 \\ x_1^2 & 0 \end{vmatrix} (P) = \zeta^{12i} \neq 0.$$

Therefore, the last assertion is proved. □

Set $Q^\circ = Q \setminus (x_0 = x_1 = x_2 = 0)$ and $Z^\circ = Z \cap Q^\circ$.

Lemma 3.11 *The quasi projective variety* Z° *is nonsingular.*

Proof We work on the open subset $U = U_{w_0,x_0} \subset Q$ which is an affine 4-space with coordinates w_1, x_1, x_2, z. We see that $Z \cap U$ is defined by $a_0 z + f_0 = 0$, where $a_0 = a(1, w_1)$ and $f_0 = f(1, x_1, x_2)$. We have

$$\operatorname{Sing}(Z \cap U) = \left(z \frac{a_0}{\partial w_1} = \frac{\partial f_0}{\partial x_1} = \frac{\partial f_0}{\partial x_1} = a_0 = f_0 = 0\right) = \varnothing,$$

where the last equality follows since the curve $f_0 = 0$ in $\mathbb{A}^2$ is nonsingular. By symmetry, $Z \cap U_{w_i,x_j}$ is nonsingular for $i = 0, 1$ and $j = 0, 1, 2$. Since Z° is covered by U_{w_i,x_j} for $i = 0, 1$ and $j = 0, 1, 2$, the proof is completed. □

Let H_Q and F_Q be divisor classes on Q which correspond to the weight ${}^t(0\ 1)$ and ${}^t(1\ 0)$, respectively, that is, F_Q is the fiber class of the projection $Q \to \mathbb{P}^1$ and $H_Q|_{F_Q} \in |\mathcal{O}_{\mathbb{P}(1,1,1,4)}(1)|$. We set $H_Z = H_Q|_Z$ and $F_Z = F_Q|_Z$. Define $\mathcal{L}$ to be the sheaf $\mathcal{O}_Z(2H_Z - nF_Z)$ whose restriction on Z° is an invertible sheaf. Note that we have $z \in H^0(Z, \mathcal{L}^2)$. It is clear that $X \cong Z[\sqrt{z}]$. In the following we choose and fix a general $a \in \mathbb{k}[w_0, w_1]$ so that the assertions of Lemma 3.9 hold.

Lemma 3.12 *The section $z \in H^0(Z^\circ, \mathcal{L}^2)$ has only almost nondegenerate critical points on Z°.*

Proof Let $\mathrm{Crit}(z) \subset Z^\circ$ be the set of critical points of z. Since

$$\frac{\partial(az + f)}{\partial z} = a,$$

z can be chosen as a part of local coordinates at every point $P \in Z^\circ$ such that $a(P) = 0$. It follows that z does not have a critical point at any point $P \in X \cap (a = 0)$. We work on an open set $U \subset Z^\circ$ on which $a \neq 0$ and prove that $z|_U$ has only almost nondegenerate critical points on U. Since $z = -f/a$ on U and a is a unit on U, it is enough to show that $-a^2 z = af$ has only almost nondegenerate critical points on U. Let $P \in U$ be a critical point of z. We have

$$\frac{\partial(af)}{\partial w_i} = \frac{\partial a}{\partial w_i} f, \qquad \frac{\partial(af)}{\partial x_j} = a\frac{\partial f}{\partial x_j},$$

for $i = 0, 1$ and $j = 0, 1, 2$. Since $a(P) \neq 0$, we have $(\partial f/\partial x_j)(P) = 0$ for $j = 0, 1, 2$. By Lemma 3.10, we have $f(P) \neq 0$, which implies $(\partial a/\partial w_i)(P) = 0$ for $i = 0, 1$. By Lemmas 3.9 and 3.10, we can choose local coordinates w, t_1, t_2 of Z at P such that

$$\begin{aligned} af &= (\alpha + \beta w^2 + w^3 + \cdots)(\gamma + t_1 t_2 + \cdots) \\ &= \alpha\gamma + \beta\gamma w^2 + \alpha t_1 t_2 + \gamma w_1^3 + h, \end{aligned}$$

where $\alpha, \beta, \gamma \in \mathbb{k}$ with $\alpha, \gamma \neq 0$, $h = h(w, t_1, t_2)$ consists of monomials of degree at least 3 and it does not involve w^3. This shows that z has only almost nondegenerate critical points on Z°. □

Define $\mathcal{Q}^\circ = \mathcal{Q}(\mathcal{L}, z)|_{Z^\circ}$ which is an invertible sheaf on Z°. By Lemma 3.5, we have $\rho^*\mathcal{Q}^\circ \hookrightarrow (\Omega^2_{X^\circ})^{\vee\vee}$, where $\rho\colon X^\circ = Z^\circ[\sqrt{z}] \to Z^\circ$. By adjunction, we have $\omega_Z \cong \mathcal{O}_Z(-3H_Z + (2n-2)F_Z)$, hence $\mathcal{Q}^\circ \cong \mathcal{O}_{Z^\circ}(H_Z - 2F_Z)$. Let H_P and F_P be the divisors on P which correspond to ${}^t(0\ 1)$ and ${}^t(1\ 0)$, respectively, so that F_P is the fiber class of $\Pi\colon P \to \mathbb{P}^1$ and $H_P|_{F_P} \in |\mathcal{O}_{\mathbb{P}(1,1,1,2)}(1)|$. We set $H = H_P|_X$ and

$F = F_P|_X$. We have $H = \rho^* H_Z$ and $F = \rho^* F_Z$, hence $\rho^*\mathcal{Q}^\circ \cong \mathcal{O}_{X^\circ}(H - 2F)$. Let $\iota\colon X^\circ \hookrightarrow X$ be the open immersion. The sheaf $\iota_*\rho^*\mathcal{Q}^\circ \cong \mathcal{O}_X(H - 2F)$ is a reflexive sheaf of rank 1 but is not invertible at each singular point of type $1/2(1,1,1)$. We define $\mathcal{M} = \iota_*\rho^*\mathcal{Q}^{\circ 2} \cong \mathcal{O}_X(2H - 4F)$ which is an invertible sheaf on X and we have an injection $\mathcal{M} \hookrightarrow ((\Omega^2_X)^{\otimes 2})^{\vee\vee}$.

Note that X has two kinds of singularities both of which are isolated: one of them are the singular points on X° corresponding to the critical points of z and the other ones are singular points of type $1/2(1,1,1)$. Let $r\colon Y \to X$ be the blowup of X at each singular point. By Lemmas 3.8 and 3.12, Y is nonsingular and we have an injection $r^*\mathcal{M}|_{Y^\circ} \hookrightarrow (\Omega^2_{Y^\circ})^{\otimes 2}$ on the open subset $Y^\circ = r^{-1}(X^\circ)$. We will show that there is an injection $r^*\mathcal{M} \hookrightarrow (\Omega^2_Y)^{\otimes 2}$.

Lemma 3.13 *There is an injection $r^*\mathcal{M} \hookrightarrow (\Omega^2_Y)^{\otimes 2}$.*

Proof Let P be a singular point of type $1/2(1,1,1)$. Since we know that $r^*\mathcal{M} \hookrightarrow (\Omega^2_Y)^{\otimes 2}$ on the open subset $Y^\circ = r^{-1}(X^\circ)$, it is enough to show that $r^*\mathcal{M} \hookrightarrow (\Omega^2_Y)^{\otimes 2}$ locally around the exceptional divisor of $r\colon Y \to X$ over P. We can write $\mathcal{M}_P = \mathcal{O}_{X,P}\cdot\eta$ for some local section η of $((\Omega^2_X)^{\otimes 2})^{\vee\vee}$ since $\mathcal{M} \subset ((\Omega^2_X)^{\otimes 2})^{\vee\vee}$ is an invertible sheaf. We will show that

$$\eta = g\left(\frac{dh_1\wedge dh_2}{h_1h_2}\right)^{\otimes 2},$$

for some $g, h_1, h_2 \in \mathcal{O}_{X,P}$, and then we will show that $r^*\eta$ does not have a pole along the exceptional divisor over P.

After replacing w_0, w_1, we assume that w_1 vanishes at P (so that w_0 does not vanish at P). We work on an open subset U of $U_{w_0,x_0} \subset P$. Shrinking U, we assume $a \neq 0$ on U. Then $z = -f/a \in \mathcal{O}_U$. Let $\widetilde{w}_1 = w_1/w_0$, $\widetilde{x}_1 = x_1/x_0$, $\widetilde{x}_2 = x_2/x_0$ be the restrictions of w_1, x_1, x_2 to U_{w_0,x_0}. Then, in view of Remark 3.6, after further shrinking U, we see that $\mathcal{M}|_U$ is generated by

$$\left(\frac{d\widetilde{x}_1\wedge d\widetilde{x}_2}{\partial(-f/a)/\partial x_2}\right)^{\otimes 2}.$$

In particular,

$$\mathcal{M}\otimes K(X) = K(X)\cdot(d\widetilde{x}_1\wedge d\widetilde{x}_2)^{\otimes 2} \subset \left(\Omega^2_{K(X)}\right)^{\otimes 2},$$

where $K(X)$ is the function field of X.

Set $\xi_i = x_i/y^{1/2}$ for $i = 0, 1, 2$. Then ξ_0, ξ_1, ξ_2 can be chosen as local coordinates of the orbifold chart of (X, P). Now we have $\widetilde{x}_i = \xi_1/\xi_0$ for $i = 1, 2$, hence

$$d\widetilde{x}_1\wedge d\widetilde{x}_2 = d(\xi_1/\xi_0)\wedge d(\xi_2/\xi_0) = \frac{d(\xi_1\xi_0^3)\wedge d(\xi_2\xi_0^3)}{\xi_0^6}.$$

Here, since the ground field is of characteristic 2 and $\xi_0^2 \in \mathcal{O}_{X,P}$, we have the equality

$$d(\xi_i \xi_0^3) = d\left((\xi_0^2)^2 \frac{\xi_i}{\xi_0}\right) = \xi_0^4 d\left(\frac{\xi_i}{\xi_0}\right)$$

for $i = 1, 2$. Thus $\mathcal{M}_P \otimes K(X) = K(X) \cdot (dh_1 \wedge dh_2)^{\otimes 2}$, where $h_i = \xi_i \xi_0^3$. It follows that

$$\eta = g\left(\frac{dh_1 \wedge dh_2}{h_1 h_2}\right)^{\otimes 2}$$

for some rational function g. By [10, Lemma 5.3], we see that $g = \xi_0^2 \xi_1^2 \xi_2^2 h$ for some $h \in \mathcal{O}_{X,P}$.

Now, by shrinking X, we assume that $r\colon Y \to X$ is the blowup (more precisely, the weighted blowup with weight $1/2(1, 1, 1)$) at P. Then the order of the pole of the rational 2-form

$$r^*\left(\frac{dh_1 \wedge dh_2}{h_1 h_2}\right)^{\otimes 2}$$

along the exceptional divisor E is at most 2 (in fact, an explicit computation shows that the above form does not have a pole along E but we do not need this strong estimate). It is clear that $r^*\xi_i^2$ vanishes along E to order 1 so that r^*g vanishes along E to order at least 3. Therefore, $r^*\eta$ does not have a pole along E and we have an injection $r^*\mathcal{M} \hookrightarrow (\Omega_Y^2)^{\otimes 2}$. □

Lemma 3.14 *If $n \geqslant 5$, then the invertible sheaf $\mathcal{M}$ is big.*

Proof Let m be a positive integer such that $m > n/(n-4)$. We show that the complete linear system of $\mathcal{M}^m \cong \mathcal{O}_X(2mH - 4mF)$ defines a birational map. Set $k = (n-4)m$ and $l = (n-4)m - n$ which are positive integers. Then

$$\begin{aligned}&\left\{y^m w_0^k,\ y^m w_0^{k-1} w_1,\ \ldots,\ y^m w_1^k\right\}\\ &\quad \cup \left\{y^{m-1} w_i^l x_{j_1} x_{j_2} : 0 \leqslant i \leqslant 1,\ 0 \leqslant j_1,\ j_2 \leqslant 2\right\}\end{aligned}$$

is a set of sections of $\mathcal{M}^m$ and they define a generically finite map. Indeed, the restriction of sections $y^m w_0^k$, $y^m w_0^{k-1} w_1$ and $y^{m-1} w_0^l x_{j_1} x_{j_2}$ for $0 \leqslant j_1, j_2 \leqslant 2$ on $X \cap U_{w_0,y}$ are 1, w_1 and x_i^2 for $0 \leqslant j_1, j_2 \leqslant 2$ and they clearly define a generically finite map (in fact an isomorphism). It follows that the complete linear system of $\mathcal{M}^m$ defines a generically finite map and thus $\mathcal{M}$ is big. □

Proof of Theorem 1.3 Assume that $n \geqslant 5$. By Lemmas 3.13, 3.14 and 3.2, a very general X_n defined over $\Bbbk$ is not separably uniruled. In particular, it is not ruled. Then a very general X_n defined over $\mathbb{C}$ is not ruled, by Lemma 3.1, and the proof is completed. □

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