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RESEARCH ARTICLE

The covering number of the difference sets in partitions of G-spaces and groups

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Abstract We prove that for every finite partition $G = A_1 \cup \cdots \cup A_n$ of a group G there are a cell A_i of the partition and a subset $F \subset G$ of cardinality $|F| \leqslant n$ such that $G = FA_iA_i^{-1}A_i$. A similar result is proved also for partitions of G-spaces. This gives two partial answers to a problem of Protasov posed in 1995.

Keywords G-space · Difference set · Covering number · Compact right topological semigroup · Minimal measure · Idempotent measure · Quasi-invariant measure

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1 Introduction

This paper was motivated by the following problem posed by Protasov in Kourovka Notebook [7].

Problem 1.1 (Protasov, 1995) Is it true that for any partition $G = A_1 \cup \cdots \cup A_n$ of a group G some cell A_i of the partition has $\mathrm{cov}(A_i A_i^{-1}) \leqslant n$?

Here for a non-empty subset $A \subset G$ by

$$\mathrm{cov}(A) = \min\{|F| : F \subset G,\ G = FA\}$$

we denote the *covering number* of A.

In fact, Protasov's problem can be posed in a more general context of ideal G-spaces. Let us recall that a *G-space* is a set X endowed with an action $G \times X \to X$, $(g, x) \mapsto gx$, of a group G. An *ideal G-space* is a pair $(X, \mathcal{I})$ consisting of a G-space X and a G-invariant Boolean ideal $\mathcal{I} \subset \mathcal{B}(X)$ in the Boolean algebra $\mathcal{B}(X)$ of all subsets of X. A *Boolean ideal* on X is a proper non-empty subfamily $\mathcal{I} \subsetneq \mathcal{B}(X)$ such that for any $A, B \in \mathcal{I}$ any subset $C \subset A \cup B$ belongs to $\mathcal{I}$. A Boolean ideal $\mathcal{I}$ is *G-invariant* if $\{gA : g \in G,\ A \in \mathcal{I}\} \subset \mathcal{I}$. A Boolean ideal $\mathcal{I} \subset \mathcal{B}(G)$ on a group G will be called *invariant* if $\{xAy : x, y \in G,\ A \in \mathcal{I}\} \subset \mathcal{I}$. By $[X]^{<\omega}$ and $[X]^{\leqslant\omega}$ we denote the families of all finite and countable subsets of a set X, respectively. The family $[X]^{<\omega}$ (respectively $[X]^{\leqslant\omega}$) is a Boolean ideal on X if X is infinite (respectively uncountable).

For a subset $A \subset X$ of an ideal G-space $(X, \mathcal{I})$ by

$$\Delta(A) = \{g \in G : gA \cap A \neq \varnothing\} \qquad \text{and} \qquad \Delta_{\mathcal{I}}(A) = \{g \in G : gA \cap A \notin \mathcal{I}\}$$

we denote the *difference set* and *$\mathcal{I}$-difference set* of A, respectively.

Given a Boolean ideal $\mathcal{J}$ on a group G and two subsets $A, B \subset G$, we shall write $A =_{\mathcal{J}} B$ if the symmetric difference $A \triangle B = (A \setminus B) \cup (B \setminus A)$ belongs to the ideal $\mathcal{J}$. For a non-empty subset $A \subset G$ put

$$\mathrm{cov}_{\mathcal{J}}(A) = \min\{|F| : F \subset G,\ FA =_{\mathcal{J}} G\}$$

be the *$\mathcal{J}$-covering number* of A. For the empty subset we put $\mathrm{cov}_{\mathcal{J}}(\varnothing) = \infty$ and assume that ∞ is larger than any cardinal number.

Observe that for the left action of the group G on itself we get $\Delta(A) = AA^{-1}$ for every subset $A \subset G$. That is why Problem 1.1 is a partial case of the following general problem.

Problem 1.2 Is it true that for any partition $X = A_1 \cup \cdots \cup A_n$ of an ideal G-space $(X, \mathcal{I})$ some cell A_i of the partition has $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i)) \leqslant n$?

This problem has an affirmative answer for G-spaces with amenable acting group G, see [2, Theorem 4.3]. The paper [2] gives a survey of available partial solutions of Protasov's Problems 1.1 and 1.2. Here we mention the following result of Banakh, Ravsky and Slobodianiuk [3].

Theorem 1.3 *For any partition $X = A_1 \cup \cdots \cup A_n$ of an ideal G-space $(X, \mathfrak{I})$ some cell A_i of the partition has*

$$\mathrm{cov}(\Delta_{\mathfrak{I}}(A_i)) \leqslant \max_{0<k\leqslant n} \sum_{p=0}^{n-k} k^p \leqslant n!$$

In this paper we shall give another two partial solutions to Protasov's Problems 1.1 and 1.2.

Theorem 1.4 *For any partition $X = A_1 \cup \cdots \cup A_n$ of an ideal G-space $(X, \mathfrak{I})$ either*

- $\mathrm{cov}(\Delta_{\mathfrak{I}}(A_i)) \leqslant n$ *for all cells A_i or else*
- $\mathrm{cov}_{\mathcal{J}}(\Delta_{\mathfrak{I}}(A_i)) < n$ *for some cell A_i and some G-invariant ideal $\mathcal{J} \not\ni \Delta_{\mathfrak{I}}(A_i)$ on G.*

Corollary 1.5 *For any partition $X = A_1 \cup \cdots \cup A_n$ of an ideal G-space $(X, \mathfrak{I})$ either $\mathrm{cov}(\Delta_{\mathfrak{I}}(A_i)) \leqslant n$ for all cells A_i or else $\mathrm{cov}(\Delta_{\mathfrak{I}}(A_i) \cdot \Delta_{\mathfrak{I}}(A_i)) < n$ for some cell A_i.*

Proof By Theorem 1.4, either $\mathrm{cov}(\Delta_{\mathfrak{I}}(A_i)) \leqslant n$ for all cells A_i or else there is a cell A_i of the partition such that $\mathrm{cov}_{\mathcal{J}}(\Delta_{\mathfrak{I}}(A_i)) < n$ for some G-invariant ideal $\mathcal{J} \not\ni \Delta_{\mathfrak{I}}(A_i)$ on X. In the first case we are done. In the second case we can find a $F \subset G$ of cardinality $|F| < n$ such that $F \cdot \Delta_{\mathfrak{I}}(A_i) =_{\mathcal{J}} G$. It follows that for every $x \in G$ the shift $x\Delta_{\mathfrak{I}}(A_i)$ does not belong to $\mathcal{J}$ and hence intersects the set $F \cdot \Delta_{\mathfrak{I}}(A_i)$. So $x \in F \cdot \Delta_{\mathfrak{I}}(A_i) \cdot \Delta_{\mathfrak{I}}(A_i)^{-1} = F \cdot \Delta_{\mathfrak{I}}(A_i) \cdot \Delta_{\mathfrak{I}}(A_i)$ and $\mathrm{cov}(\Delta_{\mathfrak{I}}(A_i) \cdot \Delta_{\mathfrak{I}}(A_i)) \leqslant |F| \leqslant n$. □

For groups G (considered as G-spaces endowed with the left action of G on itself), we can prove a bit more.

Theorem 1.6 *Let G be a group and $\mathfrak{I}$ an invariant Boolean ideal on G with $[G]^{\leqslant\omega} \not\subset \mathfrak{I}$. For any partition $G = A_1 \cup \cdots \cup A_n$ of G either*

- $\mathrm{cov}(\Delta_{\mathfrak{I}}(A_i)) \leqslant n$ *for all cells A_i or else*
- $\mathrm{cov}_{\mathcal{J}}(\Delta_{\mathfrak{I}}(A_i)) < n$ *for some cell A_i and for some G-invariant Boolean ideal $\mathcal{J} \not\ni A_i^{-1}$ on G.*

Corollary 1.7 *For any partition $G = A_1 \cup \cdots \cup A_n$ of a group G either $\mathrm{cov}(A_i A_i^{-1}) \leqslant n$ for all cells A_i or else $\mathrm{cov}(A_i A_i^{-1} A_i) < n$ for some cell A_i of the partition.*

Proof On the group G consider the trivial ideal $\mathfrak{I} = \{\varnothing\}$. By Theorem 1.6, either $\mathrm{cov}(A_i A_i^{-1}) \leqslant n$ for all cells A_i or else $\mathrm{cov}_{\mathcal{J}}(A_i A_i^{-1}) < n$ for some cell A_i and some G-invariant ideal $\mathcal{J} \not\ni A_i^{-1}$ on G. In the first case we are done. In the second case, choose a finite subset $F \subset G$ of cardinality $|F| < n$ such that the set $FA_i A_i^{-1} =_{\mathcal{J}} G$. Since $A_i^{-1} \notin \mathcal{J}$, for every $x \in G$ the set xA_i^{-1} intersects $FA_i A_i^{-1}$ and thus $x \in FA_i A_i^{-1} A_i$ and $\mathrm{cov}(A_i A_i^{-1} A_i) \leqslant |F| < n$. □

Taking into account that the ideal $\mathcal{J}$ appearing in Theorem 1.6 is G-invariant but not necessarily invariant, we can ask the following question.

Problem 1.8 Is it true that for any partition $G = A_1 \cup \cdots \cup A_n$ of a group G some cell A_i of the partition has $\mathrm{cov}_{\mathcal{J}}(A_i A_i^{-1}) \leqslant n$ for some invariant Boolean ideal $\mathcal{J}$ on G?

2 Minimal measures on G-spaces

Theorems 1.4 and 1.6 will be proved with help of minimal probability measures on X and right quasi-invariant idempotent measures on G.

For a G-space X by $P(X)$ we denote the (compact Hausdorff) space of all finitely additive probability measures on X. The action of the group G on X extends to an action of the convolution semigroup $P(G)$ on $P(X)$: for two measures $\mu \in P(G)$ and $\nu \in P(X)$ their convolution is defined as the measure $\mu*\nu \in P(X)$ assigning to each bounded function $\varphi\colon X \to \mathbb{R}$ the real number

$$\mu*\nu(\varphi) = \int_G \int_X \varphi(g^{-1}x)\, d\nu(x)\, d\mu(g).$$

The convolution map $*\colon P(G)\times P(X) \to P(X)$ is right-continuous in the sense that for any fixed measure $\nu \in P(X)$ the right shift $P(G) \to P(X)$, $\mu \mapsto \mu*\nu$, is continuous. This implies that the $P(G)$-orbit $P(G)*\nu = \{\mu*\nu : \mu \in P(G)\}$ of ν coincides with the closure $\overline{\mathrm{conv}}(G\cdot\nu)$ of the convex hull of the G-orbit $G\cdot\nu$ of ν in $P(X)$.

A measure $\mu \in P(X)$ will be called *minimal* if for any measure $\nu \in P(G)*\mu$ we get $P(G)*\nu = P(G)*\mu$. Zorn's Lemma combined with the compactness of the orbits implies that the orbit $P(G)*\mu$ of each measure $\mu \in P(X)$ contains a minimal measure.

It follows from Day's Fixed Point Theorem [8, 1.14] that for a G-space X with amenable acting group G each minimal measure μ on X is G-invariant, which implies that the set $\overline{\mathrm{conv}}(G\cdot\mu)$ coincides with the singleton $\{\mu\}$.

For an ideal G-space $(X, \mathcal{I})$ let $P_{\mathcal{I}}(X) = \{\mu \in P(X) : \mu(A) = 0,\ A \in \mathcal{I}\}$.

Lemma 2.1 *For any ideal G-space $(X, \mathcal{I})$ the set $P_{\mathcal{I}}(X)$ contains some minimal probability measure.*

Proof Let $\mathcal{U}$ be any ultrafilter on X, which contains the filter $\mathcal{F} = \{F \subset X : X\setminus F \in \mathcal{I}\}$. This ultrafilter $\mathcal{U}$ can be identified with the 2-valued measure $\mu_{\mathcal{U}}\colon \mathcal{B}(X) \to \{0, 1\}$ such that $\mu_{\mathcal{U}}^{-1}(1) = \mathcal{U}$. It follows that $\mu_{\mathcal{U}}(A) = 0$ for any subset $A \in \mathcal{I}$. In the $P(G)$-orbit $P(G)*\mu_{\mathcal{U}}$ choose any minimal measure $\mu = \nu*\mu_{\mathcal{U}}$ and observe that for every $A \in \mathcal{I}$ the G-invariance of the ideal $\mathcal{I}$ implies $\mu(A) = \int_G \mu_{\mathcal{U}}(x^{-1}A)\, d\nu(x) = 0$. So, $\mu \in P_{\mathcal{I}}(X)$. □

For a subset A of a group G put

$$\mathsf{ls}_{12}(A) = \inf_{\mu\in P(G)} \sup_{y\in G} \mu(Ay).$$

Lemma 2.2 *If a subset A of a group G has $\mathsf{ls}_{12}(A) = 1$, then $\mathrm{cov}(A^{-1}) < \omega$ and $\mathrm{cov}(G\setminus A) \geqslant \omega$.*

Proof If $\mathrm{cov}(A^{-1}) \geqslant \omega$, then for every non-empty finite subset $T \subset G$ we could find a point $x_T \notin TT^{-1}A^{-1}$ and observe that $x_T^{-1} \notin ATT^{-1}$ and hence $x_T^{-1}T \cap AT = \varnothing$.

Then for the uniformly distributed measure $\mu_T = 1/|T| \cdot \sum_{t\in T} \delta_{x_T^{-1}t}$ on the set $x_T^{-1}T$ we get $\mu_T(AT) = 0$. By the compactness of the space $P(G)$, the net $(\mu_T)_{T\in[G]^{<\omega}}$ has a limit point $\mu_\infty \in P(X)$, which means that for every set $B \subset G$, finite subset $F \subset G$ and $\varepsilon > 0$ there is a finite set $T \supset F$ in G such that $|\mu_T(B) - \mu_\infty(B)| < \varepsilon$. Since $\mathsf{ls}_{12}(A) = 1$, for the measure μ_∞ there is a point $y \in G$ such that $\mu_\infty(Ay) > 1/2$. By the limit property of μ_∞ there is a finite subset $T \ni y$ in G such that $|\mu_T(Ay) - \mu_\infty(Ay)| > 1/2$. Then $0 < \mu_T(Ay) \leqslant \mu_T(AT) = 0$, which is a desired contradiction showing that $\mathrm{cov}(A^{-1}) < \omega$.

To see that $\mathrm{cov}(G\setminus A) \geqslant \omega$, it suffices to check that $G \neq F(G\setminus A)$ for any finite set $F \subset G$. Consider the uniformly distributed measure $\mu = 1/|F| \cdot \sum_{x\in F} \delta_{x^{-1}}$ on the set F^{-1}. Since $\mathsf{ls}_{12}(A) = 1$, for the measure μ there is a point $y \in G$ such that $1 - 1/|F| < \mu(Ay) = 1/|F| \cdot \sum_{x\in F} \delta_{x^{-1}}(Ay)$, which implies that $\mu(Ay) = 1$ and $\mathrm{supp}\mu = F^{-1} \subset Ay$. Then $F^{-1}y^{-1} \cap (G\setminus A) = \varnothing$ and $y^{-1} \notin F(G\setminus A)$. □

Remark 2.3 By [1, Theorem 3.8], for every subset A of a group G we get $\mathsf{ls}_{12}(A) = 1 - \mathsf{is}_{21}(G\setminus A)$ where $\mathsf{is}_{21}(B) = \inf_{\mu\in P_\omega(G)} \sup_{x\in G} \mu(xB)$ for $B \subset G$ and $P_\omega(G)$ denotes the set of finitely supported probability measures on G.

For a probability measure $\mu \in P(X)$ on a G-space X and a subset $A \subset X$ put $\overline{\mu}(A) = \sup_{x\in G} \mu(xA)$.

3 A density version of Theorem 1.4

In this section we shall prove the following density theorem, which will be used in the proof of Theorem 1.4 presented in the next section.

Theorem 3.1 *Let $(X, \mathcal{I})$ be an ideal G-space and $\mu \in P_{\mathcal{I}}(X)$ a minimal measure on X. If some subset $A \subset X$ has $\overline{\mu}(A) > 0$, then the $\mathcal{I}$-difference set $\Delta_{\mathcal{I}}(A)$ has $\mathcal{J}$-covering number $\mathrm{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A)) \leqslant 1/\overline{\mu}(A)$ for some G-invariant ideal $\mathcal{J} \not\ni \Delta_{\mathcal{I}}(A)$ on G.*

Proof By the compactness of $P(G)*\mu = \overline{\mathrm{conv}}(G\cdot\mu)$, there is a measure $\mu' \in P(G)*\mu \subset P_{\mathcal{I}}(X)$ such that $\mu'(A) = \sup\{\nu(A) : \nu \in P(G)*\mu\} = \overline{\mu}(A)$. We can replace the measure μ by μ' and assume that $\mu(A) = \overline{\mu}(A)$. Choose a positive ε such that

$$\left\lfloor \frac{1}{\overline{\mu}(A) - \varepsilon} \right\rfloor = \left\lfloor \frac{1}{\overline{\mu}(A)} \right\rfloor,$$

where $\lfloor r \rfloor = \max\{n \in \mathbb{Z} : n \leqslant r\}$ denotes the integer part of a real number r.

Consider the set $L = \{x \in G : \mu(xA) > \overline{\mu}(A) - \varepsilon\}$ and choose a maximal subset $F \subset L$ such that $\mu(xA \cap yA) = 0$ for any distinct points $x, y \in F$. The additivity of the measure μ implies that $1 \geqslant \sum_{x\in F} \mu(xA) > |F|(\overline{\mu}(A) - \varepsilon)$ and hence $|F| \leqslant \lfloor 1/(\overline{\mu}(A) - \varepsilon) \rfloor = \lfloor 1/(\overline{\mu}(A)) \rfloor \leqslant 1/\overline{\mu}(A)$. By the maximality of F, for every $x \in L$ there is $y \in F$ such that $\mu(xA \cap yA) > 0$. Then $xA \cap yA \notin \mathcal{I}$ and $y^{-1}x \in \Delta_{\mathcal{I}}(A)$. It follows that $x \in y \cdot \Delta_{\mathcal{I}}(A) \subset F \cdot \Delta_{\mathcal{I}}(A)$ and $L \subset F \cdot \Delta_{\mathcal{I}}(A)$.

We claim that $\mathsf{ls}_{12}(L) = 1$. Given any measure $\nu \in P(G)$, consider the measure $\nu^{-1} \in P(G)$ defined by $\nu^{-1}(B) = \nu(B^{-1})$ for every subset $B \subset G$. By the minimality of μ, we can find a measure $\eta \in P(G)$ such that $\eta * \nu^{-1} * \mu = \mu$. Then

$$\begin{aligned}\overline{\mu}(A) = \mu(A) = \eta*\nu^{-1}*\mu(A) = \int_G \mu(x^{-1}A)\,d\eta*\nu^{-1}(x) \\ \leqslant (\overline{\mu}(A)-\varepsilon)\cdot\eta*\nu^{-1}\big(\big\{x \in G : \mu(x^{-1}A) \leqslant \overline{\mu}(A)-\varepsilon\big\}\big) \\ + \overline{\mu}(A)\cdot\eta*\nu^{-1}\big(\big\{x \in G : \mu(x^{-1}A) > \overline{\mu}(A)-\varepsilon\big\}\big) \\ \leqslant (\overline{\mu}(A)-\varepsilon)\cdot\big(1-\eta*\nu^{-1}(L^{-1})\big) + \overline{\mu}(A)\cdot\eta*\nu^{-1}(L^{-1}) \leqslant \overline{\mu}(A)\end{aligned}$$

implies that $\eta*\nu^{-1}(L^{-1}) = 1$. It follows from

$$1 = \eta*\nu^{-1}(L^{-1}) = \int_G \nu^{-1}(y^{-1}L^{-1})\,d\eta(y)$$

that for every $\delta > 0$ there is a point $y \in G$ such that $\nu(Ly) = \nu^{-1}(y^{-1}L^{-1}) > 1-\delta$. So, $\mathsf{ls}_{12}(L) = 1$.

By Lemma 2.2, the family $\mathcal{J} = \{B \subset G : B \subset E(G \setminus L) \text{ for some } E \in [G]^{<\omega}\}$ is a G-invariant ideal on G, which does not contain the set $L \subset F\cdot\Delta_{\mathcal{I}}(A_i)$ and hence does not contain the set $\Delta_{\mathcal{I}}(A_i)$. It follows that $\mathrm{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A_i)) \leqslant |F| \leqslant 1/\overline{\mu}(A)$. □

4 Proof of Theorem 1.4

Let $X = A_1 \cup \cdots \cup A_n$ be a partition of an ideal G-space $(X, \mathcal{I})$. By Lemma 2.1, there exists a minimal probability measure $\mu \in P(X)$ such that $\mathcal{I} \subset \{A \in \mathcal{B}(G) : \mu(A) = 0\}$.

For every $i \in \{1,\dots,n\}$ consider the number $\overline{\mu}(A_i) = \sup_{x\in G}\mu(xA_i)$ and observe that $\sum_{i=1}^n \overline{\mu}(A_i) \geqslant 1$. There are two cases.

Case 1. For every $i \in \{1,\dots,n\}$, $\overline{\mu}(A_i) \leqslant 1/n$. In this case for every $x \in G$ we get

$$1 = \sum_{i=1}^n \mu(xA_i) \leqslant \sum_{i=1}^n \overline{\mu}(A_i) \leqslant n\cdot\frac{1}{n} = 1$$

and hence $\mu(xA_i) = 1/n$ for every $i \in \{1,\dots,n\}$. For every $i \in \{1,\dots,n\}$ fix a maximal subset $F_i \subset G$ such that $\mu(xA_i \cap yA_i) = 0$ for any distinct points $x, y \in F_i$. The additivity of the measure μ implies that $1 \geqslant \sum_{x\in F_i}\mu(xA_i) \geqslant |F_i|/n$ and hence $|F_i| \leqslant n$. By the maximality of F_i, for every $x \in G$ there is a point $y \in F_i$ such that $\mu(xA_i \cap yA_i) > 0$ and hence $xA_i \cap yA_i \notin \mathcal{I}$. The G-invariance of the ideal $\mathcal{I}$ implies that $y^{-1}x \in \Delta_{\mathcal{I}}(A_i)$ and so $x \in y\cdot\Delta_{\mathcal{I}}(A_i) \subset F_i\cdot\Delta_{\mathcal{I}}(A_i)$. Finally, we get $G = F_i\cdot\Delta_I(A_i)$ and $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i)) \leqslant |F_i| \leqslant n$.

Case 2. For some i we get $\overline{\mu}(A_i) > 1/n$. In this case Theorem 3.1 guarantees that $\mathrm{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A_i)) \leqslant 1/\overline{\mu}(A_i) < n$ for some G-invariant ideal $\mathcal{J} \not\ni \Delta_{\mathcal{I}}(A_i)$ on G.

5 Applying idempotent quasi-invariant measures

In this section we develop a technique involving idempotent right quasi-invariant measures, which will be used in the proof of Theorem 1.6 presented in the next section.

A measure $\mu \in P(G)$ on a group G will be called *right quasi-invariant* if for any $y \in G$ there is a positive constant $c > 0$ such that $c \cdot \mu(Ay) \leqslant \mu(A)$ for any subset $A \subset G$.

For an ideal G-space $(X, \mathfrak{I})$ and a measure $\mu \in P(X)$ consider the set

$$P_{\mathfrak{I}}(G; \mu) = \big\{\lambda \in P(G) : \lambda * \delta_g * \mu \in P_{\mathfrak{I}}(X) \ \text{ for all } \ g \in G\big\}$$

and observe that it is closed and convex in the compact Hausdorff space $P(G)$.

Lemma 5.1 *Let $(X, \mathfrak{I})$ be an ideal G-space with countable acting group G. If for some measure $\mu \in P(X)$ the set $P_{\mathfrak{I}}(G; \mu)$ is not empty, then it contains a right quasi-invariant idempotent measure $\nu \in P_{\mathfrak{I}}(G; \mu)$.*

Proof Choose any strictly positive function $c\colon G \to (0, 1]$ such that $\sum_{g \in G} c(g) = 1$ and consider the σ-additive probability measure $\lambda = \sum_{g \in G} c(g)\delta_{g^{-1}} \in P(G)$. On the compact Hausdorff space $P(G)$ consider the right shift $\Phi\colon P(G) \to P(G)$, $\Phi\colon \nu \mapsto \nu * \lambda$.

We claim that $\Phi(P_{\mathfrak{I}}(G; \mu)) \subset P_{\mathfrak{I}}(G; \mu)$. Given any measure $\nu \in P_{\mathfrak{I}}(G; \mu)$, we need to check that $\Phi(\nu) = \nu * \lambda \in P_{\mathfrak{I}}(G; \mu)$, which means that $\nu * \lambda * \delta_x * \mu \in P_{\mathfrak{I}}(X)$ for all $x \in G$. It follows from $\nu \in P_{\mathfrak{I}}(G; \mu)$ that $\nu * \delta_{g^{-1}x} * \mu \in P_{\mathfrak{I}}(X)$. Since the set $P_{\mathfrak{I}}(X)$ is closed and convex in $P(X)$, we get

$$\nu * \lambda * \delta_x * \mu = \sum_{g \in G} c(g) \cdot \nu * \delta_{g^{-1}} * \delta_x * \mu = \sum_{g \in G} \nu * \delta_{g^{-1}x} * \mu \ \in \ P_{\mathfrak{I}}(X).$$

So, $\Phi(P_{\mathfrak{I}}(G; \mu)) \subset P_{\mathfrak{I}}(G; \mu)$ and, by the Schauder Fixed Point Theorem, the continuous map Φ on the non-empty compact convex set $P_{\mathfrak{I}}(G; \mu) \subset P(G)$ has a fixed point, which implies that the closed set $S = \{\nu \in P_{\mathfrak{I}}(G; \mu) : \nu * \lambda = \nu\}$ is not empty. It is easy to check that S is a subsemigroup of the convolution semigroup $(P(G), *)$. Being a compact right-topological semigroup, S contains an idempotent $\nu \in S \subset P_{\mathfrak{I}}(G; \mu)$ according to the Ellis Theorem (see [4, Corollary 2.6] or [9, Theorem 4.1]). Since $\nu * \lambda = \nu$, for every $A \subset G$ and $x \in G$ we get

$$\nu(A) = \nu * \lambda(A) = \sum_{g \in G} c(g) \cdot \nu * \delta_{g^{-1}}(A) = \sum_{g \in G} c(g) \cdot \nu(Ag) \geqslant c(x) \cdot \nu(Ax),$$

which means that ν is right quasi-invariant. □

Remark 5.2 Lemma 5.1 does not hold for uncountable groups, in particular for the free group F_α with uncountable set α of generators. This group admits no right quasi-invariant measure. Assuming conversely that some measure $\mu \in P(F_\alpha)$ is right quasi-invariant, fix a generator $a \in \alpha$ and consider the set A of all reduced words $w \in F_\alpha$

that end with a^n for some $n \in \mathbb{Z}\setminus\{0\}$. Observe that $F_\alpha = Aa \cup A$ and hence $\mu(A) > 0$ or $\mu(Aa) > 0$. Since μ is right quasi-invariant both cases imply that $\mu(A) > 0$ and then $\mu(Ab) > 0$ for any generator $b \in \alpha\setminus\{a\}$. But this is impossible since the family $(Ab)_{b\in\alpha\setminus\{a\}}$ is disjoint and uncountable.

In the following lemma for a measure $\mu \in P(X)$ we put $\overline{\mu}(A) = \sup_{x\in G}\mu(xA)$.

Lemma 5.3 *Let $(X, \mathcal{I})$ be an ideal G-space and $\mu \in P(X)$ a measure on X such that the set $P_{\mathcal{I}}(G;\mu)$ contains an idempotent right quasi-invariant measure λ. For a subset $A \subset X$ and numbers $\delta \leqslant \varepsilon < \sup_{x\in G}\lambda*\mu(xA)$ consider the sets $M_\delta = \{x \in G : \mu(xA) > \delta\}$ and $L_\varepsilon = \{x \in G : \lambda*\mu(xA) > \varepsilon\}$. Then:*

- (i) $\lambda(gM_\delta^{-1}) > (\varepsilon-\delta)/(\overline{\mu}(A)-\delta)$ *for any point* $g \in L_\varepsilon$*;*
- (ii) *the set M_δ does not belong to the G-invariant Boolean ideal $\mathcal{J}_\delta \subset \mathcal{P}(G)$ generated by $G\setminus L_\delta$;*
- (iii) $\mathrm{cov}_{\mathcal{J}_\delta}(\Delta_{\mathcal{I}}(A)) < 1/\delta$.

Proof Consider the measure $\nu = \lambda*\mu$ and put $\overline{\nu}(A) = \sup_{x\in G}\nu(xA)$ for a subset $A \subset X$.
(i) Fix a point $g \in L_\varepsilon$ and observe that

$$\begin{aligned}\varepsilon < \lambda*\mu(gA) &= \int_G \mu(x^{-1}gA)\,d\lambda(x)\\ &\leqslant \delta\cdot\lambda\big(\{x\in G : \mu(x^{-1}gA)\leqslant\delta\}\big) + \overline{\mu}(A)\cdot\lambda\big(\{x \in G : \mu(x^{-1}gA) > \delta\}\big)\\ &= \delta\cdot\big(1-\lambda(gM_\delta^{-1})\big) + \overline{\mu}(A)\lambda(gM_\delta^{-1}) = \delta + (\overline{\mu}(A)-\delta)\lambda(gM_\delta^{-1})\end{aligned}$$

which implies $\lambda(gM_\delta^{-1}) > \gamma \overset{\text{def}}{=} (\varepsilon-\delta)/(\overline{\mu}(A)-\delta)$.

(ii) To derive a contradiction, assume that the set M_δ belongs to the G-invariant ideal generated by $G\setminus L_\delta$ and hence $M_\delta \subset E(G\setminus L_\delta)$ for some finite subset $E \subset G$. Then

$$M_\delta \subset E(G\setminus L_\delta) = G\setminus \bigcap_{e\in E} eL_\delta.$$

Choose an increasing number sequence $(\varepsilon_k)_{k=0}^\infty$ such that $\delta \leqslant \varepsilon < \varepsilon_0$ and $\lim_{k\to\infty}\varepsilon_k = \overline{\nu}(A)$. For every $k \in \omega$ fix a point $g_k \in L_{\varepsilon_k}$. The preceding item applied to the measure ν and set L_δ (instead of μ and M_δ) yields the lower bound

$$\lambda(g_kL_\delta^{-1}) > \frac{\varepsilon_k-\delta}{\overline{\nu}(A)-\delta}$$

for every $k \in \omega$. Then $\lim_{k\to\infty}\lambda(g_kL_\delta^{-1}) = 1$ and hence $\lim_{k\to\infty}\lambda(g_kL_\delta^{-1}g) = 1$ for every $g \in G$ by the right quasi-invariance and additivity of the measure λ. Choose k so large that $\lambda(g_kL_\delta^{-1}g^{-1}) > 1-\gamma/|E|$ for all $g \in E$. Then the set $\bigcap_{g\in E} g_kL_\delta^{-1}g^{-1}$ has measure $> 1-\gamma$ and hence it intersects the set $g_kM_\delta^{-1}$ which has measure $\lambda(g_kM_\delta) \geqslant \gamma$. Consequently, the set M_δ^{-1} intersects $\bigcap_{g\in E} L_\delta^{-1}g^{-1}$, and the set M_δ intersects $\bigcap_{g\in E} gL_\delta = G\setminus E(G\setminus L_\delta)$, which contradicts the choice of the set E.

(iii) To show that $\mathrm{cov}_{\mathcal{J}_\delta}(\Delta_{\mathcal{I}}(A)) < 1/\delta$, fix a maximal subset $F \subset L_\delta$ such that $\nu(xA \cap yA) = 0$ for any distinct points $x, y \in F$. The additivity of the measure ν guarantees that $1 \geqslant \sum_{x \in F} \nu(xA) > |F| \cdot \delta$ and hence $|F| < 1/\delta$. On the other hand, the maximality of F guarantees that for any $x \in L_\delta \setminus F$ there is $y \in F$ such that $\nu(xA \cap yA) > 0$ and hence $xA \cap yA \notin \mathcal{I}$ and $y^{-1}x \in \Delta_{\mathcal{I}}(A)$. Then $x \in y \cdot \Delta_{\mathcal{I}}(A) \subset F \cdot \Delta_{\mathcal{I}}(A)$ and hence $L_\delta \subset F \cdot \Delta_{\mathcal{I}}(A)$. The inclusion $G \setminus (F \cdot \Delta_{\mathcal{I}}(A)) \subset G \setminus L_\delta \in \mathcal{J}_\delta$ implies $\mathrm{cov}_{\mathcal{J}_\delta}(F \cdot \Delta_{\mathcal{I}}(A)) \leqslant |F| < 1/\delta$. □

Corollary 5.4 *Let $(X, \mathcal{I})$ be an ideal G-space with countable acting group G and $\mu \in P(X)$ a measure on X such that the set $P_{\mathcal{I}}(G; \mu)$ is not empty. For any partition $X = A_1 \cup \cdots \cup A_n$ of X either:*

- (i) $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i)) \leqslant n$ *for all cells A_i or else*
- (ii) $\mathrm{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A_i)) < n$ *for some cell A_i and some G-invariant Boolean ideal $\mathcal{J} \subset \mathcal{P}(G)$ such that $\{x \in G : \mu(xA) > 1/n\} \notin \mathcal{J}$.*

Proof By Lemma 5.1, the set $P_{\mathcal{I}}(G; \mu)$ contains an idempotent right quasi-invariant measure λ. Then for the measure $\nu = \lambda * \mu \in P_{\mathcal{I}}(X)$ two cases are possible:

(i) Every cell A_i of the partition has $\overline{\nu}(A_i) = \sup_{x \in G} \nu(xA_i) \leqslant 1/n$. In this case we can proceed as in the proof of Theorem 1.4 and prove that $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i)) \leqslant n$ for all cells A_i of the partition.

(ii) Some cell A_i of the partition has $\overline{\nu}(A_i) > 1/n$. In this case Lemma 5.3 guarantees that $\mathrm{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A_i)) < n$ for the G-invariant Boolean ideal $\mathcal{J} \subset \mathcal{P}(G)$ generated by the set $\{x \in G : \nu(xA_i) \leqslant 1/n\}$, and the set $M = \{x \in G : \mu(xA_i) > 1/n\}$ does not belong to the ideal $\mathcal{J}$. □

Next, we extend Corollary 5.4 to G-spaces with arbitrary (not necessarily countable) acting group G. Given a G-space X, denote by $\mathcal{H}$ the family of all countable subgroups of the acting group G. A subfamily $\mathcal{F} \subset \mathcal{H}$ will be called

- *closed* if for each increasing sequence of countable subgroups $\{H_n\}_{n \in \omega} \subset \mathcal{F}$ the union $\bigcup_{n \in \omega} H_n$ belongs to $\mathcal{F}$;
- *dominating* if each countable subgroup $H \in \mathcal{H}$ is contained in some subgroup $H' \in \mathcal{F}$;
- *stationary* if $\mathcal{F} \cap \mathcal{C} \neq \varnothing$ for every closed dominating subset $\mathcal{C} \subset \mathcal{H}$.

It is known (see [6, 4.3]) that the intersection $\bigcap_{n \in \omega} \mathcal{C}_n$ of any countable family of closed dominating sets $\mathcal{C}_n \subset \mathcal{H}$, $n \in \omega$, is closed and dominating in $\mathcal{H}$.

For a measure $\mu \in P(X)$ and a subgroup $H \in \mathcal{H}$ let

$$P_{\mathcal{I}}(H; \mu) = \left\{\lambda \in P(H) : \lambda * \delta_x * \mu \in P_{\mathcal{I}}(X) \ \text{ for all } \ x \in H\right\}.$$

Theorem 5.5 *Let $(X, \mathcal{I})$ be an ideal G-space and $\mu \in P(X)$ a measure on X such that the set $\mathcal{H}_{\mathcal{I}} = \{H \in \mathcal{H} : P_{\mathcal{I}}(H; \mu) \neq \varnothing\}$ is stationary in $\mathcal{H}$. For any partition $X = A_1 \cup \cdots \cup A_n$ of X either:*

- (i) $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i)) \leqslant n$ *for all cells A_i or else*
- (ii) $\mathrm{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A_i)) < n$ *for some cell A_i and some G-invariant Boolean ideal $\mathcal{J} \subset \mathcal{P}(G)$ such that $\{x \in G : \mu(xA_i) > 1/n\} \notin \mathcal{J}$.*

Proof Let $\mathcal{H}_\forall = \{H \in \mathcal{H}_\mathcal{J} : \mathrm{cov}(H \cap \Delta_\mathcal{J}(A_i)) \leqslant n \text{ for all } i \leqslant n\}$ and $\mathcal{H}_\exists = \mathcal{H}_\mathcal{J} \setminus \mathcal{H}_\forall$. It follows that for every $H \in \mathcal{H}_\forall$ and $i \in \{1, \dots, n\}$ we can find a subset $f_i(H) \subset H$ of cardinality $|f_i(H)| \leqslant n$ such that $H \subset f_i(H) \cdot \Delta_\mathcal{J}(A_i)$. The assignment $f_i : H \mapsto f_i(H)$ determines a function $f_i : \mathcal{H}_\forall \to [G]^{<\omega}$ to the family of all finite subsets of G. The function f_i is regressive in the sense that $f_i(H) \subset H$ for every subgroup $H \in \mathcal{H}_\forall$.

By Corollary 5.4, for every subgroup $H \in \mathcal{H}_\exists$, there are an index $i_H \in \{1, \dots, n\}$ and a finite subset $f(H) \subset H$ of cardinality $|f(H)| < n$ such that the set $J_H = H \setminus \big(f(H) \cdot (H \cap \Delta_\mathcal{J}(A_{i_H}))\big)$ generates the H-invariant ideal $\mathcal{J}_H \subset \mathcal{P}(H)$ which does not contain the set $M_H = \{x \in H : \mu(xA_{i_H}) > 1/n\}$.

Since $\mathcal{H}_\mathcal{J} = \mathcal{H}_\forall \cup \mathcal{H}_\exists$ is stationary in $\mathcal{H}$, one of the sets $\mathcal{H}_\forall$ or $\mathcal{H}_\exists$ is stationary in $\mathcal{H}$.

If the set $\mathcal{H}_\forall$ is stationary in $\mathcal{H}$, then by Jech's generalization [5], [6, 4.4] of Fodor's Lemma, the stationary set $\mathcal{H}_\forall$ contains another stationary subset $\mathcal{S} \subset \mathcal{H}_\forall$ such that for every $i \in \{1, \dots, n\}$ the restriction $f_i|\mathcal{S}$ is a constant function and hence $f_i(\mathcal{S}) = \{F_i\}$ for some finite set $F_i \subset G$ of cardinality $|F_i| \leqslant n$. We claim that $G = F_i \cdot \Delta_\mathcal{J}(A_i)$. Indeed, given any element $g \in G$, by the stationarity of $\mathcal{S}$ there is a subgroup $H \subset \mathcal{S}$ such that $g \in H$. Then $g \in H \subset f_i(H) \cdot \Delta_\mathcal{J}(A_i) = F_i \cdot \Delta_\mathcal{J}(A_i)$ and hence $\mathrm{cov}(\Delta_\mathcal{J}(A_i)) \leqslant |F_i| \leqslant n$ for all i.

Now assume that the family $\mathcal{H}_\exists$ is stationary in $\mathcal{H}$. In this case for some $i \in \{1, \dots, n\}$ the set $\mathcal{H}_i = \{H \in \mathcal{H}_\exists : i_H = i\}$ is stationary in $\mathcal{H}_\exists$. Since the function $f : \mathcal{H}_\exists \to [G]^{<\omega}$ is regressive, by Jech's generalization [5], [6, 4.4] of Fodor's Lemma, the stationary set $\mathcal{H}_i$ contains another stationary subset $\mathcal{S} \subset \mathcal{H}_i$ such that the restriction $f|\mathcal{S}$ is a constant function and hence $f(\mathcal{S}) = \{F\}$ for some finite set $F \subset G$ of cardinality $|F| < n$. We claim that the set $J = G \setminus (F \cdot \Delta_\mathcal{J}(A_i))$ generates a G-invariant ideal $\mathcal{J}$, which does not contain the set $M = \{x \in G : \mu(xA_i) > 1/n\}$. Assume conversely that $M \in \mathcal{J}$ and hence $M \subset EJ$ for some finite subset $E \subset G$. By the stationarity of the set $\mathcal{S}$, there is a subgroup $H \in \mathcal{S}$ such that $E \subset H$. It follows $H \cap J = H \setminus \big(F \cdot (H \cap \Delta_\mathcal{J}(A_i))\big) = H \setminus \big(f(H) \cdot (H \cap \Delta_\mathcal{J}(A_{i_H}))\big) = J_H$ and

$$M_H = \left\{x \in H : \mu(xA_i) > \frac{1}{n}\right\} = H \cap M \subset H \cap EJ = EJ_H \in \mathcal{J}_H,$$

which contradicts the choice of the ideal $\mathcal{J}_H$. □

6 Proof of Theorem 1.6

Theorem 1.6 is a simple corollary of Theorem 5.5. Indeed, assume that $G = A_1 \cup \dots \cup A_n$ is a partition of a group and $\mathcal{I} \subset \mathcal{P}(G)$ is an invariant ideal on G which does not contain some countable subset and hence does not contain some countable subgroup $H_0 \subset G$. Let $\mathcal{H}$ be the family of all countable subgroups of G and $\mu = \delta_1$ be the Dirac measure supported by the unit 1_G of the group G. We claim that for every subgroup $H \in \mathcal{H}$ containing H_0 the set $P_\mathcal{I}(H; \mu)$ is not empty. It follows from $H_0 \notin \mathcal{I}$ that the family $\mathcal{I}_H = \{H \cap A : A \in \mathcal{I}\}$ is an invariant Boolean ideal on the group H. Then the family $\{H \setminus A : A \in \mathcal{I}\}$ is a filter on H, which can be enlarged to an ultrafilter

$\mathcal{U}_H$. The ultrafilter $\mathcal{U}_H$ determines a 2-valued measure $\mu_H : \mathcal{P}(H) \to \{0, 1\}$ such that $\mu_H^{-1}(1) = \mathcal{U}_H$. By the right invariance of the ideal $\mathcal{I}$, for every $A \in \mathcal{I}$ and $x \in H$ we get $\mu_H * \delta_x * \mu(A) = \mu_H(Ax) = 0$, which means that $\mu_H \in P_{\mathcal{I}}(H; \mu)$. So, the set $\mathcal{H}_{\mathcal{I}} = \{H \in \mathcal{H} : P_{\mathcal{I}}(H; \mu) \neq \varnothing\} \supset \{H \in \mathcal{H} : H \supset H_0\}$ is stationary in $\mathcal{H}$.

Then, by Theorem 5.5, either

- $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i)) \leqslant n$ for all cells A_i or else
- $\mathrm{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A_i)) < n$ for some cell A_i and some G-invariant Boolean ideal $\mathcal{J} \subset \mathcal{P}(G)$ such that $A_i^{-1} = \{x \in G : \delta_1(xA_i) > 1/n\} \notin \mathcal{J}$.

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