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RESEARCH ARTICLE

Quotients of cubic surfaces

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Abstract Let $\Bbbk$ be any field of characteristic zero, X a smooth cubic surface in $\mathbb{P}^3_{\Bbbk}$ and G a group acting on X. We show that if $X(\Bbbk) \neq \varnothing$ and G is not trivial and not a group of order 3 acting in a special way then the quotient surface X/G is rational over $\Bbbk$. For the group G of order 3 we construct examples of both rational and nonrational quotients of both rational and nonrational G-minimal cubic surfaces over $\Bbbk$.

Keywords Rationality problems · del Pezzo surfaces · Minimal model program · Cremona group

Mathematics Subject Classification 14E07 · 14E08 · 14M20

1 Introduction

Let $\Bbbk$ be any field of characteristic zero. If $\Bbbk$ is algebraically closed then any quotient of a rational surface by an action of a finite group is rational by Castelnuovo's criterion. For del Pezzo surfaces of degree 4 and higher the following theorem holds.

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Theorem 1.1 ([6, Theorem 1.1]) *Let $\Bbbk$ be a field of characteristic zero, X a del Pezzo surface over $\Bbbk$ such that $X(\Bbbk) \neq \varnothing$ and G a finite subgroup of automorphisms of X. If $K_X^2 \geqslant 5$ then the quotient variety X/G is $\Bbbk$-rational. If $K_X^2 = 4$ and the order of G is not equal to $1, 2$ or 4 then X/G is $\Bbbk$-rational.*

Moreover we have the following corollary.

Corollary 1.2 ([6, Corollary 1.2]) *Let $\Bbbk$ be a field of characteristic zero, X a smooth rational surface over $\Bbbk$ such that $X(\Bbbk) \neq \varnothing$ and G a finite subgroup of automorphisms of X. If $K_X^2 \geqslant 5$ then the quotient variety X/G is $\Bbbk$-rational.*

In this paper we find for which finite groups a quotient of cubic surface is $\Bbbk$-rational and for which is not. The main result of this paper is the following.

Theorem 1.3 *Let $\Bbbk$ be a field of characteristic zero, X a del Pezzo surface over $\Bbbk$ of degree 3 such that $X(\Bbbk) \neq \varnothing$ and G a subgroup of $\mathrm{Aut}_{\Bbbk}(X)$. Suppose that G is not trivial and G is not a group of order 3 having no curves of fixed points. Then X/G is $\Bbbk$-rational.*

Note that if G is trivial and X is minimal then X is not $\Bbbk$-rational (see [5, Theorem V.1.1]). This gives us an example of a del Pezzo surface of degree 3 such that its quotient by the trivial group is not $\Bbbk$-rational. For a group G of order 3 acting without curves of fixed points on X we construct examples of quotients of G-minimal cubic surface X such that X is $\Bbbk$-rational and X/G is $\Bbbk$-rational, X is $\Bbbk$-rational and X/G is not $\Bbbk$-rational, X is not $\Bbbk$-rational and X/G is $\Bbbk$-rational, and X is not $\Bbbk$-rational and X/G is not $\Bbbk$-rational.

To prove Theorem 1.3 we consider possibilities for groups G acting on X. Our main method is to find a normal subgroup N in G such that the quotient X/N is G/N-birationally equivalent to a smooth rational surface of degree 5 or more. Therefore $\Bbbk$-rationality of X/G is equivalent to $\Bbbk$-rationality of the quotient of the obtained surface by the group G/N and we can use Corollary 1.2.

The plan of this paper is as follows. In Sect. 2 we recall some facts about minimal rational surfaces, groups, singularities and quotients. In Sect. 3 we consider quotients of cubic surfaces by nontrivial groups of automorphisms and show that all of them except a case are always $\Bbbk$-rational.

In the rest of paper we consider the remaining case of a group G of order 3 acting on a G-minimal surface X without curves of fixed points and construct explicit examples of four possible combinations of $\Bbbk$-rationality and non-$\Bbbk$-rationality of X and X/G. To do this we need to find some conditions on the image of the Galois group $\mathrm{Gal}(\overline{\Bbbk}/\Bbbk)$ in the Weyl group $\mathrm{W}(E_6)$ and then find geometric interpretation of the action of the Galois group $\mathrm{Gal}(\overline{\Bbbk}/\Bbbk)$ on X. Thus Sects. 4 and 5 consist of complicated computations which are required to construct explicit examples in Sect. 6.

In Sect. 4 for non-$\Bbbk$-rational quotients of a $\Bbbk$-rational cubic surfaces X by a group of order 3 we find all possibilities of the image of the Galois group $\mathrm{Gal}(\overline{\Bbbk}/\Bbbk)$ in the Weyl group $\mathrm{W}(E_6)$ acting on the Picard group of X. In Sect. 5 we find an explicit geometric interpretation of the obtained actions of the Galois group in terms of equations of X. In Sect. 6 for a group G of order 3 acting on a G-minimal cubic surface X without curves of fixed points we construct examples of $\Bbbk$-rational and non-$\Bbbk$-rational quotients.

Notation Throughout this paper $\Bbbk$ is any field of characteristic zero, $\overline{\Bbbk}$ is its algebraic closure. For a surface X we denote $X \otimes \overline{\Bbbk}$ by $\overline{X}$. For a surface X we denote the Picard group (respectively G-invariant Picard group) by $\mathrm{Pic}(X)$ (respectively $\mathrm{Pic}(X)^G$). The number $\rho(X) = \mathrm{rk}\,\mathrm{Pic}(X)$ (respectively $\rho(X)^G = \mathrm{rk}\,\mathrm{Pic}(X)^G$) is the Picard number (respectively G-invariant Picard number) of X. If two surfaces X and Y are $\Bbbk$-birationally equivalent then we write $X \approx Y$. If two divisors A and B are linearly equivalent then we write $A \sim B$.

2 Preliminaries

2.1 G-minimal rational surfaces

In this subsection we review main notions and results of G-equivariant minimal model program following the papers [1,2,4]. Throughout this subsection G is a finite group.

Definition 2.1 A *rational variety* X is a variety over $\Bbbk$ such that $\overline{X} = X \otimes \overline{\Bbbk}$ is birationally equivalent to $\mathbb{P}^n_{\overline{\Bbbk}}$. A $\Bbbk$-*rational variety* X is a variety over $\Bbbk$ such that X is birationally equivalent to $\mathbb{P}^n_{\Bbbk}$. A variety X over $\Bbbk$ is a $\Bbbk$-*unirational variety* if there exists a $\Bbbk$-rational variety Y and a dominant rational map $\varphi\colon Y \dashrightarrow X$.

Definition 2.2 A G-*surface* is a pair (X, G) where X is a projective surface over $\Bbbk$ and G is a finite subgroup of $\mathrm{Aut}_{\Bbbk}(X)$. A morphism of G-surfaces $f\colon X \to X'$ is called a G-*morphism* if for each $g \in G$ one has $fg = gf$. A smooth G-surface (X, G) is called G-*minimal* if any birational morphism of smooth G-surfaces $(X, G) \to (X', G)$ is an isomorphism. Let (X, G) be a smooth G-surface. A G-minimal surface (Y, G) is called a *minimal model* of (X, G) or G-*minimal model* of X if there exists a birational G-morphism $X \to Y$.

The following theorem is a classical result about G-equivariant minimal model program.

Theorem 2.3 *Any birational G-morphism of smooth G-surfaces $f\colon X \to Y$ can be factorized in the following way:*

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} X_n = Y,$$

where each f_i is a contraction of a set Σ_i of disjoint (-1)-curves on X_i such that Σ_i is defined over $\Bbbk$ and G-invariant. In particular,

$$K_Y^2 - K_X^2 \geqslant \rho(X)^G - \rho(Y)^G.$$

The classification of G-minimal rational surfaces is well known due to Iskovskikh and Manin (see [2,4]). We introduce some important notions before surveying it.

Definition 2.4 A smooth rational G-surface (X, G) admits a *conic bundle* structure if there exists a G-morphism $\varphi\colon X \to B$ such that any scheme fibre is isomorphic to a reduced conic in $\mathbb{P}^2_{\Bbbk}$ and B is a smooth curve.

Definition 2.5 A *del Pezzo surface* is a smooth projective surface X such that the anticanonical divisor $-K_X$ is ample. A *singular del Pezzo surface* is a normal projective surface X such that the anticanonical divisor $-K_X$ is ample and all singularities of X are du Val singularities. A *weak del Pezzo surface* is a smooth projective surface X such that the anticanonical divisor $-K_X$ is nef and big. The number $d = K_X^2$ is called the *degree* of a (singular) del Pezzo surface X.

A del Pezzo surface X of degree 3 is isomorphic to a smooth cubic surface in $\mathbb{P}^3_{\Bbbk}$. The following theorem classifies G-minimal rational surfaces.

Theorem 2.6 ([2, Theorem 1]) *Let X be a G-minimal rational G-surface. Then either X admits a G-equivariant conic bundle structure with $\mathrm{Pic}(X)^G \cong \mathbb{Z}^2$, or X is a del Pezzo surface with $\mathrm{Pic}(X)^G \cong \mathbb{Z}$.*

Theorem 2.7 (cf. [2, Theorem 4]) *Let X admit a G-equivariant conic bundle structure. Suppose that $K_X^2 = 3, 5, 6$ or X is a blowup of $\mathbb{P}^2_{\Bbbk}$ at a point. Then X is not G-minimal.*

The following theorem is an important criterion of $\Bbbk$-rationality over an arbitrary perfect field $\Bbbk$.

Theorem 2.8 ([3, Chapter 4]) *A minimal rational surface X over a perfect field $\Bbbk$ is $\Bbbk$-rational if and only if the following two conditions are satisfied:*

(i) $X(\Bbbk) \neq \varnothing$;
(ii) $K_X^2 \geqslant 5$.

Corollary 2.9 *Let X be a smooth rational G-surface such that $X(\Bbbk) \neq \varnothing$ and $\rho(X)^G + K_X^2 \geqslant 7$. Then there exists a G-minimal model Y of X such that $K_Y^2 \geqslant 6$. In particular, X is $\Bbbk$-rational.*

Proof By Theorem 2.6, there exists a birational G-morphism $f: X \to Z$ such that $\rho(Z)^G \leqslant 2$. By Theorem 2.3, one has

$$K_Z^2 \geqslant K_X^2 + \rho(X)^G - \rho(Z)^G \geqslant 7 - \rho(Z)^G.$$

If $\rho(Z)^G = 1$ then $K_Z^2 \geqslant 6$. If $\rho(Z)^G = 2$ and $K_Z^2 = 5$ then Z is not G-minimal, by Theorem 2.7. Therefore there exists a G-minimal model Y of Z such that $K_Y^2 \geqslant 6$.

The set $X(\Bbbk)$ is not empty. Thus $Y(\Bbbk) \neq \varnothing$ and $X \approx Y$ is $\Bbbk$-rational, by Theorem 2.8. □

In this paper we use the notation of the following remark.

Remark 2.10 Let X be a smooth cubic surface in $\mathbb{P}^3_{\Bbbk}$. Then $\overline{X}$ can be realized as a blowup $f: \overline{X} \to \mathbb{P}^2_{\overline{\Bbbk}}$ at six points $p_1, \ldots, p_6$ in general position. Put $E_i = f^{-1}(p_i)$ and $L = f^*(l)$, where l is the class of a line on $\mathbb{P}^2_{\overline{\Bbbk}}$. One has

$$-K_{\overline{X}} \sim 3L - \sum_{i=1}^{6} E_i.$$

The (-1)-curves on $\overline{X}$ are E_i, the proper transforms $L_{ij} \sim L - E_i - E_j$ of the lines passing through a pair of points p_i and p_j, and the proper transforms

$$Q_j \sim 2L + E_j - \sum_{i=1}^{6} E_i$$

of the conics passing through five points from the set $\{p_1, p_2, p_3, p_4, p_5, p_6\}$.

In this notation one has

$$\begin{array}{llll} E_i \cdot E_j = 0, & E_i \cdot L_{ij} = 1, & E_i \cdot L_{jk} = 0, & \\ L_{ij} \cdot L_{ik} = 0, & L_{ij} \cdot L_{kl} = 1, & E_i \cdot Q_i = 0, & E_i \cdot Q_j = 1, \\ Q_i \cdot Q_j = 0, & Q_i \cdot L_{ij} = 1, & Q_i \cdot L_{jk} = 0, & \end{array}$$

where i, j and k are different numbers from the set $\{1, 2, 3, 4, 5, 6\}$.

2.2 Groups

In this subsection we collect some results and notation concerning groups used in this paper. We use the following notation:

- $\mathfrak{C}_n$ denotes the cyclic group of order n;
- $\mathfrak{D}_{2n}$ denotes the dihedral group of order $2n$;
- $\mathfrak{S}_n$ denotes the symmetric group of degree n;
- $\mathfrak{A}_n$ denotes the alternating group of degree n;
- $(i_1 i_2 \ldots i_j)$ denotes a cyclic permutation of $i_1, \ldots, i_j$;
- $\mathfrak{V}_4$ denotes the Klein group isomorphic to $\mathfrak{C}_2^2$;
- $\langle g_1, \ldots, g_n \rangle$ denotes a group generated by $g_1, \ldots, g_n$;
- $\mathrm{diag}(a_1, \ldots, a_n)$ denotes the diagonal $n \times n$ matrix with entries $a_1, \ldots, a_n$;
- $\mathrm{i} = \sqrt{-1}$;
- $\xi_n = e^{2\pi \mathrm{i}/n}$;
- $\omega = \xi_3 = e^{2\pi \mathrm{i}/3}$.

The group $\mathfrak{S}_5$ can act on a cubic surface. Therefore it is important to know some facts about subgroups of this group. The following lemma is an easy exercise.

Lemma 2.11 *Any nontrivial subgroup $G \subset \mathfrak{S}_5$ contains a normal subgroup N conjugate in $\mathfrak{S}_5$ to one of the following groups:*

- $\mathfrak{C}_2 \cong \langle (12) \rangle$*;*
- $\mathfrak{C}_2 \cong \langle (12)(34) \rangle$*;*
- $\mathfrak{C}_3 \cong \langle (123) \rangle$*;*
- $\mathfrak{V}_4 \cong \langle (12)(34), (13)(24) \rangle$*;*
- $\mathfrak{C}_5 \cong \langle (12345) \rangle$*;*
- $\mathfrak{A}_5$.

2.3 Singularities

In this subsection we review some results about quotient singularities and their resolutions. All singularities appearing in this paper are toric singularities. These singularities are locally isomorphic to the quotient of $\mathbb{A}^2$ by a cyclic group generated by $\operatorname{diag}(\xi_m, \xi_m^q)$. Such a singularity is denoted by $1/m \cdot (1, q)$. If $\gcd(m, q) > 1$ then the group

$$\mathfrak{C}_m \cong \langle \operatorname{diag}(\xi_m, \xi_m^q) \rangle$$

contains a reflection and the quotient singularity is isomorphic to a quotient singularity with smaller m.

A toric singularity can be resolved by a sequence of weighted blowups. Therefore it is easy to describe numerical properties of a quotient singularity. We list here these properties for singularities appearing in our paper.

Remark 2.12 Let the group $\mathfrak{C}_m$ act on a smooth surface X and $f\colon X \to S$ be a quotient map. Let p be a singular point on S of type $1/m \cdot (1, q)$. Let C and D be curves passing through p such that $f^{-1}(C)$ and $f^{-1}(D)$ are $\mathfrak{C}_m$-invariant and tangent vectors of these curves at the point $f^{-1}(p)$ are eigenvectors of the natural action of $\mathfrak{C}_m$ on $T_{f^{-1}(p)}X$ (the curve C corresponds to the eigenvalue ξ_m and the curve D corresponds to the eigenvalue ξ_m^q).

Let $\pi\colon \widetilde{S} \to S$ be the minimal resolution of the singular point p. Table 1 presents some numerical properties of $\widetilde{S}$ and S for the singularities appearing in this paper.

The exceptional divisor of π is a chain of transversally intersecting exceptional curves E_i whose selfintersection numbers are listed in the last column of Table 1. The curves $\pi_*^{-1}(C)$ and $\pi_*^{-1}(D)$ transversally intersect at a point only the first and the last of these curves respectively and do not intersect other components of the exceptional divisor of π.

2.4 Quotients

In this subsection we collect some additional information about quotients of rational surfaces. We use the following definition for convenience.

Definition 2.13 Let X be a G-surface (respectively surface), $\widetilde{X} \to X$ be its minimal resolution of singularities, and Y be a G-minimal model (respectively minimal model) of $\widetilde{X}$. We call the surface Y a *G-MMP-reduction* (respectively *MMP-reduction*) of X.

Table 1 Resolutions of singularities

m	q	$K_{\widetilde{S}}^2 - K_S^2$	$\pi_*^{-1}(C)^2 - C^2$	$\pi_*^{-1}(D)^2 - D^2$	E_i^2
2	1	0	$-1/2$	$-1/2$	-2
3	1	$-1/3$	$-1/3$	$-1/3$	-3
3	2	0	$-2/3$	$-2/3$	$-2, -2$
5	2	$-2/5$	$-2/5$	$-3/5$	$-3, -2$

Table 2 Elements of prime order acting on a cubic surface

Type	Order	Equation	Action
1	2	$x^3+y^3+z^3+\alpha xyz+t^2(ux+vy+wz)=0$	$(x:y:z:-t)$
2	2	$x^3+y^3+xz(z+\alpha t)+yt(z+\beta t)=0$	$(x:y:-z:-t)$
3	3	$x^3+y^3+z^3+\alpha xyz+t^3=0$	$(x:y:z:\omega t)$
4	3	$x^3+y^3+z^3+t^3=0$	$(x:y:\omega z:\omega t)$
5	3	$x^3+y^3+zt(ux+vy)+z^3+t^3=0$	$(x:y:\omega z:\omega^2 t)$
6	5	$x^2y+y^2z+z^2t+t^2x=0$	$(x:\xi_5 y:\xi_5^4 z:\xi_5^3 t)$

We need some results about quotients of del Pezzo surfaces of degree 4.

Lemma 2.14 ([6, Remark 6.2]) *Let a finite group G act on a del Pezzo surface X of degree 4 and $N\cong\mathfrak{C}_2$ be a normal subgroup in G such that N has no curves of fixed points. Then the surface X/N is G/N-birationally equivalent to a conic bundle Y with $K_Y^2=2$. If there exists a $G\times\mathrm{Gal}(\overline{\mathbb{k}}/\mathbb{k})$-fixed point then Y is not G/N-minimal and there exists a G/N-MMP-reduction Z of Y such that $K_Z^2=8$.*

3 del Pezzo surface of degree 3

In this section we prove Theorem 1.3. We start from cyclic groups of prime order. The following theorem classifies actions of cyclic groups of prime order on smooth cubics.

Theorem 3.1 (cf. [1, Theorem 6.10]) *Let a group $\mathfrak{C}_p$ of prime order p act on a del Pezzo surface of degree 3. Then one can choose homogeneous coordinates x, y, z, t in $\mathbb{P}^3_{\overline{\mathbb{k}}}$ such that the equation of $\overline{X}$ and the action of $\mathfrak{C}_p$ are presented in Table 2, where u, v, w, α and β are coefficients. These actions have different sets of fixed points on $\overline{X}$ and correspond to different conjugacy classes of cyclic subgroups in the Weyl group $\mathrm{W}(E_6)$ acting on $\mathrm{Pic}(\overline{X})$.*

Note that elements of type 3 and 4 of Table 2 have curves of fixed points $t=0$ and $x=y=0$ respectively. Therefore an element of order 3 having no curves of fixed points is of type 5 of Table 2. In the latter case the following lemma holds.

Lemma 3.2 *Let a finite group G act on a del Pezzo surface X of degree 3 and $N\cong\mathfrak{C}_3$ be a normal subgroup in G such that N acts as in type 5 of Table 2. Then the surface X/N is G/N-birationally equivalent to a del Pezzo surface of degree 3.*

Proof Let $\overline{X}$ be given by the equation

$$x^3+y^3+zt(ux+vy)+z^3+t^3=0$$

in $\mathbb{P}^3_{\overline{\mathbb{k}}}$ and N act as

$$(x:y:z:t)\mapsto(x:y:\omega z:\omega^2 t).$$

The fixed points of N lie on the line $z = t = 0$. Thus N has three fixed points q_1, q_2 and q_3. One can easily check that on the tangent spaces of $\overline{X}$ at these points N acts as $\langle \mathrm{diag}(\omega, \omega^2)\rangle$. Denote by C_1 and C_2 invariant curves $z = 0$ and $t = 0$ each passing through the three points q_i.

Let $f : X \to X/N$ be the quotient morphism and

$$\pi : \widetilde{X/N} \to X/N$$

the minimal resolution of singularities. The curves $f(C_1)$ and $f(C_2)$ meet each other at the three singular points of X/N and $f(C_1)\cdot f(C_2) = 1$. Thus two curves $\pi_*^{-1} f(C_j)$ are disjoint. Moreover (see Table 1), one has

$$\pi_*^{-1} f(C_j)^2 = f(C_j)^2 - 3\cdot\frac{2}{3} = \frac{1}{3}C_j^2 - 2 = -1.$$

Therefore we can G/N-equivariantly contract the two (-1)-curves $\pi_*^{-1} f(C_j)$ and get a surface Y with $K_Y^2 = 3$.

The surface X/N has only du Val singularities. Therefore X/N is a singular del Pezzo surface and $\widetilde{X/N}$ is a weak del Pezzo surface containing exactly six curves $\pi^{-1}(q_i)$ whose selfintersection is less than -1. Thus Y does not contain curves with selfintersection less than -1. So Y is a del Pezzo surface of degree 3. □

Remark 3.3 Note that in the notation of Lemma 3.2 there are two points on the surface Y where three (-1)-curves meet each other. These points are images of $\pi_*^{-1} f(C_j)$. Such a point is called an *Eckardt point* (see Definition 5.1 below).

Remark 3.4 Note that in the notation of Lemma 3.2 if $\rho(X)^G > 1$ then X is not G-minimal by Theorem 2.7. Therefore the quotient of X/N is equivalent to a quotient of a del Pezzo surface with degree greater than 3 by a group of order 3. By Theorem 1.1 such a quotient is $\mathbb{k}$-rational.

In Sect. 4 for non-$\mathbb{k}$-rational quotient $X/\mathfrak{C}_3$ of $\mathbb{k}$-rational surface X we find restrictions on the image of the Galois group $\mathrm{Gal}(\overline{\mathbb{k}}/\mathbb{k})$ in the Weyl group $\mathrm{W}(E_6)$ which acts on $\mathrm{Pic}(\overline{X})$. Now we show that in all other cases of Theorem 1.3 the quotient of X is $\mathbb{k}$-rational.

Lemma 3.5 *Let a finite group G act on a del Pezzo surface X of degree 3 and $N \cong \mathfrak{C}_p$ be a normal cyclic subgroup of prime order in G such that N acts not as in type 5 of Table 2. Then there exists a G/N-MMP-reduction Y of X/N such that $K_Y^2 \geqslant 5$.*

Proof Let us consider the possibilities case by case. In types 1 and 3 of Table 2 the group N has a pointwisely fixed hyperplane section $t = 0$. In type 3 there are no other fixed points and in type 1 there is only one other fixed point $(0 : 0 : 0 : 1)$. By the Hurwitz formula we have

$$K_{X/N}^2 = \frac{1}{\operatorname{ord} N}\left(K_X - \sum_{\substack{g\in N\\ g\neq \mathrm{id}}} R_g\right)^2,$$

where R_g is the ramification divisor of an element g. In the both cases for nontrivial elements of the group N the ramification divisor is a hyperplane section and it is equal to $-K_X$. Therefore one has

$$K_{X/\mathfrak{C}_2}^2 = \frac{1}{2}(K_X + K_X)^2 = \frac{1}{2}(2K_X)^2 = 6,$$
$$K_{X/\mathfrak{C}_3}^2 = \frac{1}{3}(K_X + 2K_X)^2 = \frac{1}{3}(3K_X)^2 = 9$$

in types 1 and 3 respectively. The surface X/N has at most du Val singularities. Therefore for the minimal resolution of singularities $\widetilde{X/N} \to X/N$ one has $K_{\widetilde{X/N}}^2 = K_{X/N}^2$. Thus for any G/N-MMP-reduction Y of X/N one has $K_Y^2 \geqslant 6$.

In type 2 of Table 2 the group $\mathfrak{C}_2$ fixes pointwisely the lines $x = y = 0$ and $z = t = 0$. The line $z = t = 0$ intersects X at three $\mathfrak{C}_2$-fixed points and the line $x = y = 0$ is contained in X. This line is defined over $\Bbbk$ since it is unique. Therefore this line and can be G-equivariantly contracted. So the quotient X/N is G/N-birationally equivalent to the quotient of del Pezzo surface of degree 4 by a group of order 2 having four fixed points one of which is $G \times \mathrm{Gal}(\overline{\Bbbk}/\Bbbk)$-fixed. By Lemma 2.14, there exists a G/N-MMP-reduction Y of the latter quotient such that $K_Y^2 = 8$.

In type 4 of Table 2 the group $\mathfrak{C}_3$ fixes pointwisely the lines $x = y = 0$ and $z = t = 0$. These lines intersect $\overline{X}$ given by

$$x^3 + y^3 + z^3 + t^3 = 0$$

at points p_1, p_2, p_3 and q_1, q_2, q_3 respectively. Let C_{ij} be a line in $\mathbb{P}^3_{\overline{\Bbbk}}$ passing through p_i and q_j. Assume that C_{ij} does not lie in $\overline{X}$. For some integer a the involution

$$(x : y : z : t) \mapsto (\omega^{2a} z : t : \omega^a x : y)$$

permutes points p_i and q_j, thus the line C_{ij} is invariant under the action of this involution. Therefore the line C_{ij} cannot be tangent to $\overline{X}$ at any of the points p_i and q_j. Then the third point of intersection of C_{ij} with $\overline{X}$ is $\mathfrak{C}_3$-fixed. Thus there are three $\mathfrak{C}_3$-fixed points on C_{ij} but this contradicts the fact that the action of $\mathfrak{C}_3$ is faithful on C_{ij}. So C_{ij} lies in $\overline{X}$ and $C_{ij}^2 = -1$.

Let $f: X \to X/N$ be the quotient morphism and $\pi: \widetilde{X/N} \to X/N$ the minimal resolution of singularities. Then $f(p_i)$ and $f(q_j)$ are singularities of type $1/3\cdot(1, 1)$. Thus $\pi_*^{-1} f(C_{ij})$ are nine disjoint (-1)-curves (see Table 1). We can contract these curves and get a surface Y. One has

$$K_Y^2 = K^2_{\widetilde{X/N}} + 9 = K^2_{X/N} + 9 - 6 \cdot \frac{1}{3} = \frac{1}{3} K_X^2 + 7 = 8.$$

In type 6 of Table 2 the group $\mathfrak{C}_5$ has two invariant lines $x = z = 0$ and $y = t = 0$ lying in $\overline{X}$ given by the equation

$$x^2 y + y^2 z + z^2 t + t^2 x = 0.$$

One can G-equivariantly contract this pair and get a del Pezzo surface of degree 5. So the quotient X/N is G/N-birationally equivalent to the quotient of del Pezzo surface of degree 5 by a group of order 5. By Theorem 1.1, this quotient is $\Bbbk$-rational so it is G/N-birationally equivalent to a surface Y such that $K_Y^2 \geqslant 5$. □

Corollary 3.6 *Let a finite group G of order* 6 *act on a del Pezzo surface X of degree* 3. *Then the surface X/G is birationally equivalent to a surface Y such that $K_Y^2 \geqslant 5$.*

Proof Let $N \subset G$ be the subgroup of order 3. Then, by Lemmas 3.2 and 3.5, the quotient X/N is G/N-birationally equivalent to a surface Z such that either $K_Z^2 \geqslant 5$ or Z is a del Pezzo surface of degree 3. There exists an MMP-reduction Y of $X/G \approx Z/(G/N)$ such that $K_Y^2 \geqslant 5$, by Theorem 1.1 and Lemma 3.5 respectively. □

Remark 3.7 Note that for an element g of type 3 of Table 2 the quotient $\overline{X}/\langle g\rangle$ is isomorphic to $\mathbb{P}^2_{\overline{\Bbbk}}$. Therefore one has

$$\rho(\overline{X})^{\langle g\rangle} = \rho(\mathbb{P}^2_{\overline{\Bbbk}}) = 1.$$

To prove Theorem 1.3 we need to list all possible automorphism groups of cubic surfaces.

Theorem 3.8 (cf. [1, Section 6.5, Table 4]) *Let $\overline{X}$ be a del Pezzo surface of degree* 3. *Then one can choose homogeneous coordinates x, y, z, t in $\mathbb{P}^3_{\overline{\Bbbk}}$ such that the equation of $\overline{X}$ and the full automorphism group* $\mathrm{Aut}(\overline{X})$ *are presented in Table* 3*, where u, v and α are coefficients, and $H_3(3)$ is a group generated by the transformation*

$$(x : y : z : t) \mapsto (x : \omega y : \omega^2 z : t)$$

and a cyclic permutation of x, y and z.

In the paper [1] there is one more column in this table which contains conditions on the parameters. But we are interested only in the structure of the group and its action on $\mathbb{P}^3_{\overline{\Bbbk}}$ so we omit this column.

Lemma 3.9 *Let a finite group G act on a del Pezzo surface X of degree* 3 *and $N \cong \mathfrak{V}_4$ be a normal subgroup in G such that nontrivial elements of N act as in type* 2 *of* Table 2. *Then there exists a G/N-MMP-reduction Y of X/N such that $K_Y^2 = 6$.*

Table 3 Automorphisms groups of cubic surfaces

Type	Group	Equation
I	$\mathfrak{C}_3^3 \rtimes \mathfrak{S}_4$	$x^3+y^3+z^3+t^3=0$
II	$\mathfrak{S}_5$	$x^2y+y^2z+z^2t+t^2x=0$
III	$H_3(3)\rtimes\mathfrak{C}_4$	$x^3+y^3+z^3+\alpha xyz+t^3=0$
IV	$H_3(3)\rtimes\mathfrak{C}_2$	$x^3+y^3+z^3+\alpha xyz+t^3=0$
V	$\mathfrak{S}_4$	$t(x^2+y^2+z^2)+\alpha xyz+t^3=0$
VI	$\mathfrak{S}_3\times\mathfrak{C}_2$	$x^3+y^3+\alpha zt(x+y)+z^3+t^3=0$
VII	$\mathfrak{C}_8$	$x^3+xy^2+yz^2+zt^2=0$
VIII	$\mathfrak{S}_3$	$x^3+y^3+zt(ux+vy)+z^3+t^3=0$
IX	$\mathfrak{C}_4$	$x^3+\alpha y^3+xy^2+yz^2+zt^2=0$
X	$\mathfrak{C}_2^2$	$x^3+y^3+z^3+\alpha xyz+t^2(x+y+uz)=0$
XI	$\mathfrak{C}_2$	$x^3+y^3+z^3+\alpha xyz+t^2(x+uy+vz)=0$

Proof One can choose coordinates in $\mathbb{P}^3_{\overline{\Bbbk}}$ in which $\overline{X}$ is given by the equation

$$t(x^2+y^2+z^2)+\alpha xyz+t^3=0$$

and nontrivial elements of $\mathfrak{V}_4$ switch signs of t and one other variable. The set of points fixed by nontrivial elements of group $\mathfrak{V}_4$ consists of three pointwisely fixed lines

$$x=t=0, \qquad y=t=0, \qquad z=t=0$$

lying in $\overline{X}$ and six isolated fixed points $(1:0:0:\pm\mathrm{i})$, $(0:1:0:\pm\mathrm{i})$ and $(0:0:1:\pm\mathrm{i})$. Thus the quotient X/N is a singular del Pezzo surface with three A_1 singularities. By the Hurwitz formula,

$$K_{X/N}^2=\frac{1}{4}(2K_X)^2=3.$$

Let q_1, q_2 and q_3 be singular points of X/N. Consider the anticanonical embedding

$$X/N \hookrightarrow \mathbb{P}^3_{\Bbbk}.$$

Denote by C_{ij} a line in $\mathbb{P}^3_{\overline{\Bbbk}}$ passing through q_i and q_j. Such a line contains two singular points on the surface X/N of degree 3, therefore all lines C_{ij} lie in X/N. Moreover, one has

$$K_{X/N}\cdot C_{ij}=-1.$$

Thus we can resolve the singularities of X/N, then G/N-equivariantly contract the proper transforms of C_{ij} and get a surface Y with $K_Y^2=6$. $\square$

Lemma 3.10 *Let a finite group G act on a del Pezzo surface X of degree 3 and $N \cong \mathfrak{A}_5$ be a normal subgroup in G. Then there exists a G/N-MMP-reduction Y of X/N such that $K_Y^2 \geqslant 6$.*

Proof From Theorem 3.8, one can see that there is a unique cubic surface $\overline{X}$ with $\mathfrak{A}_5$ action on it. This cubic surface is called the Clebsch cubic. It is well known that the Clebsch cubic is isomorphic to the $\mathfrak{A}_5$-equivariant blowup at six points on $\mathbb{P}^2_{\overline{\Bbbk}}$. For convenience of the reader we remind this construction.

Consider the group $\mathfrak{A}_5$ acting on $\mathbb{P}^2_{\overline{\Bbbk}}$. By a direct computation one can show that each element g of $\mathfrak{A}_5$ is a composition of two elements h_1 and h_2 of order two. Any element of order 2 in $\mathrm{PGL}_3(\Bbbk)$ has a line of fixed points. Thus each element g of $\mathfrak{A}_5$ has an isolated fixed point p which is the intersection point of the lines fixed by elements h_1 and h_2. The stabilizer group N_p of p acts on the tangent space of $\mathbb{P}^2_{\overline{\Bbbk}}$ at p. Therefore N_p is a subgroup in $\mathfrak{A}_5$ and $\mathrm{GL}_2(\Bbbk)$. Thus this is isomorphic to $\mathfrak{C}_2^2$, $\mathfrak{S}_3$ or $\mathfrak{D}_{10}$ if $\operatorname{ord} g$ is 2, 3 or 5 respectively. The images of these groups in $\mathrm{GL}_2(\Bbbk)$ are generated by reflections.

Consider such a point p for an element g of order 5. The stabilizer N_p of p is isomorphic to $\mathfrak{D}_{10}$ and g acts in the tangent space of $\mathbb{P}^2_{\overline{\Bbbk}}$ at p as $\mathrm{diag}(\xi_5, \xi_5^4)$. One can easily check that the action of g in the tangent spaces of the two other fixed points is conjugate to $\mathrm{diag}(\xi_5, \xi_5^2)$.

The group $\mathfrak{A}_5$ contains six subgroups isomorphic to $\mathfrak{C}_5$. Let $p_1, \dots, p_6$ be fixed points of these subgroups whose stabilizers are isomorphic to $\mathfrak{D}_{10}$. Consider an $\mathfrak{A}_5$-equivariant blowup $\sigma\colon \overline{X} \to \mathbb{P}^2_{\overline{\Bbbk}}$ of the points $p_1, \dots, p_6$. The surface $\overline{X}$ is a del Pezzo surface of degree 3.

We use the notation of Remark 2.10. Let g_2, g_3 and g_5 be elements in $\mathfrak{A}_5$ of order 2, 3 and 5 respectively.

The stabilizer of any point p_i in $\mathfrak{A}_5$ is isomorphic to $\mathfrak{D}_{10}$. Therefore there are five lines passing through the point p_i that are pointwisely fixed by an element of order 2 in $\mathfrak{A}_5$. But in $\mathfrak{A}_5$ there are only 15 elements of order 2. Thus each element of order 2 fixes pointwisely a line passing through a pair of points p_i and p_j on $\mathbb{P}^2_{\overline{\Bbbk}}$ and fixes pointwisely a (-1)-curve L_{ij} on $\overline{X}$. By the Lefschetz fixed-point formula, the element g_2 has three isolated fixed points. Two of them are $E_i \cap Q_j$ and $Q_i \cap E_j$ and the third is the preimage of the isolated fixed point p of g_2 on $\mathbb{P}^2_{\overline{\Bbbk}}$. In the tangent space of $\overline{X}$ at p the stabilizer group N_p of p acts as $\mathfrak{C}_2^2$ generated by reflections.

An element g_3 does not have any invariant (-1)-curve. Therefore it cannot have curves of fixed points. Thus the action of g_3 on $\mathbb{P}^2_{\overline{\Bbbk}}$ is conjugate to $\mathrm{diag}(1, \omega, \omega^2)$. Hence on the surface $\overline{X}$ the element g_3 has three isolated points and acts in the tangent spaces of $\mathbb{P}^2_{\overline{\Bbbk}}$ at these points as $\mathrm{diag}(\omega, \omega^2)$. These points cannot be points of the blowup since the stabilizer of any point p_i in $\mathfrak{A}_5$ is $\mathfrak{D}_{10}$. Therefore there are three g_3-fixed points on $\overline{X}$ and in the tangent spaces of $\overline{X}$ at these points as $\mathrm{diag}(\omega, \omega^2)$. Two of these points do not lie on (-1)-curves and the third one is a point of intersection of three (-1)-curves.

An element g_5 has three fixed points on $\mathbb{P}^2_{\overline{\Bbbk}}$: namely p_k for some $k \in \{1, 2, 3, 4, 5, 6\}$ and two points in whose tangent space the action of g_5 is conjugate to $\mathrm{diag}(\xi_5, \xi_5^2)$. The

(-1)-curve Q_k is g_5-invariant thus the quadric $\sigma(Q_k)$ passes through two g_5-fixed points different from p_k. Therefore the element g_5 has four fixed points on $\overline{X}$, two of them lie on E_k and two on Q_k. The element g_5 acts in the tangent spaces of $\overline{X}$ at g_5-fixed points as $\mathrm{diag}(\xi_5, \xi_5^2)$.

Let $f\colon X \to X/N$ be the quotient morphism, $\pi\colon \widetilde{X/N} \to X/N$ the minimal resolution of singularities, and put $E = f(E_i)$, $Q = f(Q_j)$. There are four singular points on X/N: two singular points of type $1/5\cdot(1,2)$ lie on the curves E and Q respectively, one singular point of type A_1 is the intersection point $E \cap Q$ and one singular point of type A_2 lies neither on E nor on Q. We have (see Table 1)

$$\begin{aligned}
K^2_{\widetilde{X/N}} &= K^2_{X/N} - \frac{4}{5} = \frac{1}{60}(6K_X)^2 - \frac{4}{5} = 1,\\
\rho(\widetilde{X/N})^{G/N} &\geqslant \rho(X/N)^{G/N} + 4 = \rho(X)^G + 4 \geqslant 5,\\
\pi_*^{-1}(E)^2 &= E^2 - \frac{1}{2} - \frac{2}{5} = \frac{1}{60}\left(\sum_{i=1}^{6} E_i\right)^2 - \frac{9}{10} = -1,\\
\pi_*^{-1}(Q)^2 &= Q^2 - \frac{1}{2} - \frac{2}{5} = \frac{1}{60}\left(\sum_{i=1}^{6} Q_i\right)^2 - \frac{9}{10} = -1.
\end{aligned}$$

Thus we can G/N-equivariantly contract curves $\pi_*^{-1}(E)$ and $\pi_*^{-1}(Q)$. We obtain a surface Z such that $K_Z^2 = 3$ and $\rho(Z)^{G/N} \geqslant 4$. By Corollary 2.9, there exists a G/N-minimal model Y of Z such that $K_Y^2 \geqslant 6$. □

Now we prove Theorem 1.3.

Proof of Theorem 1.3 We consider each case of Theorem 3.8 and show that if the group G is not trivial and is not conjugate to $\mathfrak{C}_3$ acting as in type 5 of Table 2 then there exists a G/N-MMP-reduction Y of X/N such that $K_Y^2 \geqslant 5$.

In case I, the group $\mathrm{Aut}(\overline{X})$ is $\mathfrak{C}_3^3 \rtimes \mathfrak{S}_4$. The group $\mathfrak{C}_3^3$ is a diagonal subgroup of $\mathrm{PGL}_4(\Bbbk)$ and $\mathfrak{S}_4$ permutes coordinates. We consider a normal subgroup $H = G \cap \mathfrak{C}_3^3$. If there is an element of type 3 in this subgroup then we consider a group $N \subset H$ generated by elements of type 3. Then the group N is normal and one of the following possibilities holds:

- If N is generated by one element of type 3 then $N \cong \mathfrak{C}_3$, the quotient X/N is smooth and, by the Hurwitz formula, one has

$$K^2_{X/N} = \frac{1}{3}(3K_X)^2 = 9.$$

- If N is generated by two elements of type 3 then $N \cong \mathfrak{C}_3^2$, the quotient X/N has only one singular point of type $1/3\cdot(1,1)$ and, by the Hurwitz formula, one has

$$K^2_{X/N} = \frac{1}{9}(5K_X)^2 = \frac{25}{3}.$$

- If N is generated by three elements of type 3 then $N \cong \mathfrak{C}_3^3$, the quotient X/N is smooth and, by the Hurwitz formula, one has

$$K_{X/N}^2 = \frac{1}{27}(9K_X)^2 = 9.$$

For any G/N-MMP-reduction Y of X/N one has $K_Y^2 \geqslant 8$.

If the group H does not contain elements of type 3 then either H is trivial or H is isomorphic to $\mathfrak{C}_3$ generated by an element of type 4 or 5 or $\mathfrak{C}_3^2$ generated by the elements $\mathrm{diag}(1, 1, \omega, \omega)$ and $\mathrm{diag}(1, \omega, 1, \omega)$.

In the last case the group G is a subgroup of $\mathfrak{C}_3^2 \rtimes \mathfrak{D}_8$ where $\mathfrak{D}_8 = \langle(1243), (23)\rangle$. If G contains the element $(14)(23)$ then the group $N = \langle(14)(23)\rangle$ is normal in G and there exists a G/N-MMP-reduction Y of X/N such that $K_Y^2 \geqslant 5$, by Lemma 3.5. Otherwise the group G is isomorphic to $\mathfrak{C}_3^2 \rtimes \mathfrak{C}_2$ or $\mathfrak{C}_3^2$.

If G contains a normal subgroup $N \cong \mathfrak{C}_3$ generated by element of type 4 then there exists a G/N-MMP-reduction Y of X/N such that $K_Y^2 \geqslant 5$, by Lemma 3.5. Otherwise G is conjugate to $\mathfrak{C}_3^2 \rtimes \mathfrak{C}_2$ where $\mathfrak{C}_2$ is either $\langle(23)\rangle$ or $\langle(14)\rangle$. In this case G contains a normal subgroup $N \cong \mathfrak{C}_3$ generated by an element of type 5. By Lemma 3.2, the quotient X/N is G/N-birationally equivalent to a del Pezzo surface Z of degree 3 and the group $G/N \cong \mathfrak{S}_3$. The quotient $Z/\mathfrak{S}_3$ is $\mathbb{k}$-birationally equivalent to a surface Y with $K_Y^2 \geqslant 5$, by Corollary 3.6.

If the group $H \cong \mathfrak{C}_3$ is generated by an element of type 4 then there exists a G/H-MMP-reduction Y of X/H such that $K_Y^2 \geqslant 5$, by Lemma 3.5.

If the group $H \cong \mathfrak{C}_3$ is generated by an element of type 5 then $G \subset \mathfrak{S}_3 \times \mathfrak{C}_2$ and the quotient X/H is G/H-birationally equivalent to a del Pezzo surface Z of degree 3, by Lemma 3.2. Thus if G/H is nontrivial then it contains a subgroup N of order 2. There exists a G/N-MMP-reduction Y of X/N such that $K_Y^2 \geqslant 5$, by Lemma 3.5.

If the group H is trivial then G is a subgroup of $\mathfrak{S}_4$. Then the group G contains a normal subgroup N isomorphic to $\mathfrak{C}_2$, $\mathfrak{C}_3$ or $\mathfrak{V}_4$, by Lemma 2.11. If $N \cong \mathfrak{C}_2$ or $N \cong \mathfrak{V}_4$ then there exists a G/N-MMP-reduction Y of X/N such that $K_Y^2 \geqslant 5$, by Lemma 3.5 or Lemma 3.9 respectively. If $N \cong \mathfrak{C}_3$ then either G is generated by an element of type 5 or $G \cong \mathfrak{S}_3$ and there exists an MMP-reduction Y of X/G such that $K_Y^2 \geqslant 5$, by Corollary 3.6.

In case II, the group G contains a normal subgroup N isomorphic to $\mathfrak{C}_2$, $\mathfrak{C}_3$, $\mathfrak{V}_4$, $\mathfrak{C}_5$ or $\mathfrak{A}_5$, by Lemma 2.11. If N is not isomorphic to $\mathfrak{C}_3$ then there exists a G/N-MMP-reduction Y of X/N such that $K_Y^2 \geqslant 5$, by Lemmas 3.5, 3.9 or 3.10. Otherwise $N \cong \mathfrak{C}_3$ and G is a subgroup of $\mathfrak{S}_3 \times \mathfrak{C}_2$. Subgroups of this group are considered in case I.

In cases III and IV, let us consider the group $H = G \cap H_3(3)$. If $\mathrm{ord}\, H > 3$ then H contains a normal subgroup N generated by an element of order 3 acting as in type 3 of Table 2. The group N is normal in G and there exists a G/N-MMP-reduction Y of X/N such that $K_Y^2 \geqslant 5$, by Lemma 3.5.

If H is generated by an element of type 5 and G/H is not trivial then $G \cong \mathfrak{S}_3$ and there exists an MMP-reduction Y of X/G such that $K_Y^2 \geqslant 5$, by Corollary 3.6.

If H is trivial and G is not trivial then $G \subset \mathfrak{C}_4$ contains a normal subgroup N of order 2 and there exists a G/N-MMP-reduction Y of X/N such that $K_Y^2 \geqslant 5$, by Lemma 3.5.

In case VII, if G is not trivial then G contains a normal subgroup N of order 2 and there exists a G/N-MMP-reduction Y of X/N such that $K_Y^2 \geqslant 5$, by Lemma 3.5.

In the other cases the group G is conjugate in $\mathrm{PGL}_4(\overline{\mathbb{k}})$ to a subgroup of $\mathfrak{S}_5$. All these possibilities were considered in case II.

In all cases one has $Y(\mathbb{k}) \neq \varnothing$ since $X(\mathbb{k}) \neq \varnothing$. Thus

$$Y/(G/N) \approx X/G$$

is $\mathbb{k}$-rational, by Corollary 1.2. □

4 Minimality conditions

Let X be a cubic surface in $\mathbb{P}^3_{\mathbb{k}}$ and let a group $G \cong \mathfrak{C}_3$ act on X as in type 5 of Table 2. In this section we find some conditions for the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{k}}/\mathbb{k})$ on the set of (-1)-curves under which the surface X is $\mathbb{k}$-rational and X/G is not $\mathbb{k}$-rational.

Throughout this section we use the notation of Remark 2.10. Let Γ be the image of the Galois group $\mathrm{Gal}(\overline{\mathbb{k}}/\mathbb{k})$ in the Weyl group $\mathrm{W}(E_6)$ acting on $\mathrm{Pic}(\overline{X})$ (see [5, Section IV.3]). The group Γ effectively acts on the set of (-1)-curves on X. The group $\mathrm{W}(E_6)$ contains a subgroup $\mathfrak{S}_6$ acting in the following way: for $\sigma \in \mathfrak{S}_6$ one has

$$\sigma E_i = E_{\sigma(i)}, \qquad \sigma L_{ij} = L_{\sigma(i)\sigma(j)}, \qquad \sigma Q_i = Q_{\sigma(i)}.$$

Lemma 4.1 *The image of the group G in the Weyl group $\mathrm{W}(E_6)$ is conjugate to $\langle(123)(456)\rangle$.*

Proof The order of the Weyl group $\mathrm{W}(E_6)$ is equal to $51840 = 2^7 \cdot 3^4 \cdot 5$. By the Sylow theorem, all groups of order 81 are conjugate in $\mathrm{W}(E_6)$. The group of order 81 acts on the Fermat cubic (see Table 3)

$$x^3 + y^3 + z^3 + t^3 = 0.$$

Thus any element of order 3 in $\mathrm{W}(E_6)$ is of type 3, 4 or 5 from Table 2.

For any element g of type 3 of Table 2 one has $\rho(\overline{X})^{\langle g\rangle} = 1$, by Remark 3.7. In the subgroup $\mathfrak{S}_6 \subset \mathrm{W}(E_6)$ there is an element $(123)(456)$ which is of type 4 or 5 from Table 2 since

$$\rho(\overline{X})^{\langle(123)(456)\rangle} > 1.$$

But an element of type 4 has invariant (-1)-curves (see the proof of Lemma 3.5) and the element $(123)(456)$ does not have invariant (-1)-curves. Therefore the action of the group G is conjugate to $\langle(123)(456)\rangle$ in the group $\mathrm{W}(E_6)$. □

Remark 4.2 An alternative proof of Lemma 4.1 is the following. One can look at Table 1 in [5, Chapter IV, Section 5] and see that conjugacy classes of elements of order 3 in the group $W(E_6)$ correspond to the rows 3, 18 and 22 of this table. But in the eighth column of the table one can see that the element g corresponds to the 18-th row. In the ninth column one can see that g is conjugate to $(123)(456)$ in $W(E_6)$. Also one can see that the order of the centralizer of G in $W(E_6)$ is 108.

From now on we can assume that the group G acts on the set of (-1)-curves on X as $\langle(123)(456)\rangle$. The Galois group $\mathrm{Gal}(\overline{\Bbbk}/\Bbbk)$ commutes with the group G. Therefore to describe possibilities for the group Γ we should find the centralizer of G in $W(E_6)$.

Lemma 4.3 *The centralizer of* $G = \langle(123)(456)\rangle$ *in* $\mathfrak{S}_6$ *is a subgroup* $\mathfrak{C}_3^2 \rtimes \mathfrak{C}_2$ *generated by* $a = (123)$, $b = (456)$ *and* $c = (14)(25)(36)$.

Proof Note that in the group $\mathfrak{S}_6$ there are

$$\frac{6!}{3!\cdot 3!\cdot 2}\cdot 4 = 40$$

elements conjugate to $(123)(456)$. Therefore the order of the centralizer of $G = \langle(123)(456)\rangle$ is equal to 18.

The elements a, b and c obviously commute with the element $(123)(456)$ and the group $\mathfrak{C}_3^2 \rtimes \mathfrak{C}_2 = \langle a, b, c\rangle$ has order 18. Thus the centralizer of $G = \langle(123)(456)\rangle$ in $\mathfrak{S}_6$ is a subgroup $\mathfrak{C}_3^2 \rtimes \mathfrak{C}_2 = \langle a, b, c\rangle$. □

Note that the group G has exactly three orbits that consist of (-1)-curves meeting each other: $\{L_{14}, L_{25}, L_{36}\}$, $\{L_{15}, L_{26}, L_{34}\}$ and $\{L_{16}, L_{24}, L_{35}\}$. The other orbits consist of disjoint (-1)-curves. Therefore the set of nine (-1)-curves L_{ij}, $i \in \{1, 2, 3\}$, $j \in \{4, 5, 6\}$, is invariant under the action of centralizer of G in $W(E_6)$. The group

$$\mathfrak{C}_3^2 \rtimes \mathfrak{C}_2 \cong \langle a, b, c\rangle$$

can realize any permutation of this set of (-1)-curves that preserves the intersection form. Therefore to find the centralizer of G in $W(E_6)$ we should find a subgroup in $W(E_6)$ acting trivially on the set of nine (-1)-curves L_{ij}, where $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$.

Lemma 4.4 *The subgroup of* $W(E_6)$ *fixing each of the nine* (-1)*-curves* L_{ij}*, where* $i \in \{1, 2, 3\}$ *and* $j \in \{4, 5, 6\}$*, is a group* $\mathfrak{S}_3$ *generated by elements* r *and* s *of order* 3 *and* 2 *respectively acting on the set of* (-1)*-curves in the following way:*

$$s(E_i) = Q_i, \qquad s(Q_i) = E_i, \qquad s(L_{ij}) = L_{ij},$$
$$r(E_i) = Q_i \quad \text{if } i \in \{1, 2, 3\}, \qquad r^2(E_i) = Q_i \quad \text{if } i \in \{4, 5, 6\},$$
$$r(E_i) = L_{jk} \quad \text{if } i \in \{4, 5, 6\} \text{ and } j, k \in \{4, 5, 6\} \text{ differ from } i,$$
$$r^2(E_i) = L_{jk} \quad \text{if } i \in \{1, 2, 3\} \text{ and } j, k \in \{1, 2, 3\} \text{ differ from } i.$$

Proof Let us consider the (-1)-curve E_1. Since L_{14}, L_{15} and L_{16} are invariant, the image of E_1 can be only E_1, L_{23} or Q_1. The action of the group on these three (-1)-curves defines the whole action of the group $\langle r, s\rangle$ on the set of (-1)-curves. The group

$\mathfrak{S}_3 = \langle r, s \rangle$ fixes all the nine (-1)-curves L_{ij}, where $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$, and permutes E_1, L_{23} and Q_1 in all possible ways. □

Proposition 4.5 *The centralizer of $G = \langle (123)(456) \rangle$ in $\mathrm{W}(E_6)$ is a subgroup*

$$H \cong (\mathfrak{C}_3^2 \rtimes \mathfrak{C}_2) \times \mathfrak{S}_3$$

where the first factor is generated by a, b and cs, and the second factor is generated by r and s.

Proof By Lemmas 4.3 and 4.4, the centralizer of G in $\mathrm{W}(E_6)$ is generated by the subgroups $\mathfrak{C}_3^2 \rtimes \mathfrak{C}_2 = \langle a, b, c \rangle$ and $\mathfrak{S}_3 = \langle r, s \rangle$. Obviously, the elements a, b, cs, r and s generate this group. One can easily check that a, b, cs and r, s pairwisely commute. □

By Remark 3.4, if the quotient of X is not $\Bbbk$-rational then $\rho(X)^G = 1$. Moreover, if $\rho(X) = 1$ then X is not $\Bbbk$-rational, by Theorem 2.8. Thus to construct non-$\Bbbk$-rational quotient of $\Bbbk$-rational cubic surface we should find all possibilities of the Galois group Γ such that $\rho(X) > 1$ and $\rho(X)^G = 1$.

The group Γ is a subgroup of the group $H \cong (\mathfrak{C}_3^2 \rtimes \mathfrak{C}_2) \times \mathfrak{S}_3$ where the first factor is generated by a, b and cs, and the second factor is generated by r and s. We denote the projection on the first factor

$$H \cong (\mathfrak{C}_3^2 \rtimes \mathfrak{C}_2) \times \mathfrak{S}_3 \twoheadrightarrow \mathfrak{C}_3^2 \rtimes \mathfrak{C}_2$$

by π_1, and the projection on the second factor

$$H \cong (\mathfrak{C}_3^2 \rtimes \mathfrak{C}_2) \times \mathfrak{S}_3 \twoheadrightarrow \mathfrak{S}_3$$

by π_2.

Lemma 4.6 *If $\pi_2(\Gamma)$ is trivial and $\pi_1(\Gamma) \subset \mathfrak{C}_3^2$ or $\pi_2(\Gamma) = \langle s \rangle$ but $s \notin \Gamma$ then $\rho(X)^G > 1$.*

Proof In these cases the group Γ is a subgroup of the group $\langle a, b, c \rangle$. Therefore the groups Γ and $G \cong \mathfrak{C}_3$ preserve $\sum_{i=1}^{6} E_i$ and $\rho(X)^G > 1$. □

Lemma 4.7 *If $\pi_2(\Gamma) = \langle s \rangle$ and $\pi_1(\Gamma) \subset \mathfrak{C}_3^2$ then $\rho(X)^G > 1$.*

Proof In this case the groups Γ and G preserve the divisor $\sum_{i=1}^{3} E_i - \sum_{i=4}^{6} E_i$. Therefore one has $\rho(X)^G > 1$. □

Lemma 4.8 $\rho(X)^{\langle abcs \rangle} = \rho(X)^{\langle a^2b, cs \rangle} = 1$.

Proof One has $(abcs)^4 = ab$ and $(abcs)^3 = cs$. Note that the groups $\mathrm{Pic}(X)^{\langle ab \rangle}$ and $\mathrm{Pic}(X)^{\langle a^2b \rangle}$ are generated by K_X, $\sum_{i=1}^{3} E_i$ and $\sum_{i=4}^{6} E_i$. One can check that cs-invariants in $\mathrm{Pic}(X)^{\langle ab \rangle}$ and $\mathrm{Pic}(X)^{\langle a^2b \rangle}$ are generated by K_X. □

Corollary 4.9 *Suppose that $\pi_1(\Gamma)$ contains the element cs and at least one element of order 3 and $cs \in \Gamma$ then $\rho(X) = 1$.*

Lemma 4.10 $\rho(X)^{\langle abr \rangle} = \rho(X)^{\langle a^2br \rangle} = 1$.

Proof Note that any abr and a^2br orbit of a (-1)-curve consists of three (-1)-curves meeting each other. Therefore these elements are of type 3 from Table 2. Thus, by Remark 3.7, one has $\rho(X)^{\langle abr \rangle} = \rho(X)^{\langle a^2br \rangle} = 1$. □

Corollary 4.11 *If $r \in \pi_2(\Gamma)$ and $\pi_1(\Gamma)$ contains ab or a^2b then $\rho(X) = 1$.*

As a result of all previous lemmas we have the following proposition.

Proposition 4.12 *Let X be a del Pezzo surface of degree 3 and $G \cong \mathfrak{C}_3$ a group acting as in type 5 of Table 2. Let Γ be the image of the Galois group $\mathrm{Gal}(\overline{\Bbbk}/\Bbbk)$ in the Weyl group $\mathrm{W}(E_6)$. If $\rho(X) > 1$ and $\rho(X)^G = 1$ then we have the following possibilities for Γ up to conjugation:*

(i) $\Gamma = \langle cs \rangle \cong \mathfrak{C}_2$;
(ii) $\Gamma = \langle c, s \rangle \cong \mathfrak{C}_2^2$;
(iii) $\Gamma = \langle r \rangle \cong \mathfrak{C}_3$;
(iv) $\Gamma = \langle ar \rangle \cong \mathfrak{C}_3$;
(v) $\Gamma = \langle a, r \rangle \cong \mathfrak{C}_3^2$;
(vi) $\Gamma = \langle cs, r \rangle \cong \mathfrak{C}_6$;
(vii) $\Gamma = \langle r, s \rangle \cong \mathfrak{S}_3$;
(viii) $\Gamma = \langle a, r, s \rangle \cong \mathfrak{C}_3 \times \mathfrak{S}_3$;
(ix) $\Gamma = \langle r, c \rangle \cong \mathfrak{S}_3$;
(x) $\Gamma = \langle r, c, s \rangle \cong \mathfrak{S}_3 \times \mathfrak{C}_2$.

Proof At first we show that in all other cases one has either $\rho(X) = 1$ or $\rho(X)^G > 1$. If $r \in \pi_2(\Gamma)$ then if $\pi_1(\Gamma)$ contains an element ab or a^2b then $\rho(X) = 1$, by Corollary 4.11. Therefore in this case $\pi_1(\Gamma)$ should be trivial or conjugate to $\langle a \rangle$ or $\langle cs \rangle$. These possibilities correspond to cases (iii)–(x) of the proposition.

Now we can assume that $r \notin \pi_2(\Gamma)$. If $\pi_1(\Gamma)$ contains the element cs and at least one element of order 3 then, by Corollary 4.9, one has $\rho(X) = 1$ in all cases except $\Gamma = \langle ab, c \rangle$, $\Gamma = \langle a^2b, c \rangle$ and $\Gamma = \langle a, b, c \rangle$. In the last three cases we have $\rho(X)^G > 1$, by Lemma 4.6. If $cs \notin \pi_1(\Gamma)$ then $\rho(X)^G > 1$, by Lemmas 4.6 and 4.7. Therefore $\pi_1(\Gamma) = \langle cs \rangle$. This possibility corresponds to cases (i) and (ii) of the proposition.

Now we show that in all these cases one has $\rho(X) > 1$ and $\rho(X)^G = 1$. In cases (i), (ii), (vi), (ix) and (x) the (-1)-curves L_{14}, L_{25} and L_{36} are Γ-invariant. Therefore X is not minimal and $\rho(X) > 1$. In cases (iii), (iv), (v), (vii) and (viii) the triple of disjoint (-1)-curves L_{14}, L_{24} and L_{34} is Γ-invariant. Therefore X is not minimal and $\rho(X) > 1$. In cases (i) and (ii) one has $\rho(X)^G = 1$, by Lemma 4.8. In the other cases one has $\rho(X)^G = 1$, by Lemma 4.10. □

Note that if one can contract a (-1)-curve defined over $\Bbbk$ and $\rho(X) = 2$ or $\rho(X) = 3$ then the obtained del Pezzo surface either is not mininal or can be minimal del Pezzo surface of degree 4. So for each case of Proposition 4.12 we should check whether the surface X is $\Bbbk$-rational.

Lemma 4.13 *If the Galois group* Γ *contains the element* cs *then* X *is not* $\Bbbk$*-rational.*

Proof Note that the (-1)-curves L_{14}, L_{25} and L_{36} are cs-invariant and the other (-1)-curves form cs-invariant pairs of (-1)-curves which are not disjoint. The curves L_{14}, L_{25} and L_{36} meet each other. Therefore we can contract no more than one (-1)-curve and X is not $\Bbbk$-rational, by Theorem 2.8. □

Lemma 4.14 *If the Galois group* Γ *contains the elements* c *and* r *then* X *is not* $\Bbbk$*-rational.*

Proof Note that if a $\langle c, r\rangle$-orbit contains E_i then it contains Q_j with $i \neq j$. Therefore we cannot contract any of these orbits. Also we cannot contract $\langle c, r\rangle$-invariant pairs L_{15} and L_{24}, L_{16} and L_{34}, L_{26} and L_{35}. The (-1)-curves L_{14}, L_{25} and L_{36} are $\langle c, r\rangle$-invariant and meet each other. Therefore we cannot contract more than one (-1)-curve and X is not $\Bbbk$-rational, by Theorem 2.8. □

Lemma 4.15 *If the Galois group* Γ *is contained in* $\langle a, r, s\rangle$ *then* X *is* $\Bbbk$*-rational.*

Proof We can contract (-1)-curves E_4, L_{56} and Q_4, and get a del Pezzo surface of degree 6 which is $\Bbbk$-rational, by Theorem 2.8. □

Now we can prove the following theorem.

Theorem 4.16 *Let* X *be a* $\Bbbk$*-rational del Pezzo surface of degree* 3 *and* $G \cong \mathfrak{C}_3$ *a group acting as in type* 5 *of Table* 2. *Let* Γ *be the image of the Galois group* $\mathrm{Gal}(\overline{\Bbbk}/\Bbbk)$ *in the Weyl group* $\mathrm{W}(E_6)$. *If* X/G *is not* $\Bbbk$*-rational then we have the following possibilities for* Γ *up to conjugation:*

(i) $\Gamma = \langle r\rangle \cong \mathfrak{C}_3$;
(ii) $\Gamma = \langle ar\rangle \cong \mathfrak{C}_3$;
(iii) $\Gamma = \langle a, r\rangle \cong \mathfrak{C}_3^2$;
(iv) $\Gamma = \langle r, s\rangle \cong \mathfrak{S}_3$;
(v) $\Gamma = \langle a, r, s\rangle \cong \mathfrak{C}_3 \times \mathfrak{S}_3$.

Proof For cases (i), (ii), (vi), (x) of Proposition 4.12 the Galois group Γ contains the element cs. Therefore X is not $\Bbbk$-rational, by Lemma 4.13. For case (ix) of Proposition 4.12 the surface X is not $\Bbbk$-rational, by Lemma 4.14.

For cases (iii), (iv), (v), (vii), (viii) of Proposition 4.12 the Galois group Γ is contained in $\langle a, r, s\rangle \cong \mathfrak{C}_3 \times \mathfrak{S}_3$. Therefore X is $\Bbbk$-rational, by Lemma 4.15. □

5 Geometric interpretation

In this section we give a geometric interpretation of the actions of elements in the Galois group Γ considered in Sect. 4. For convenience we assume that the field $\Bbbk$ contains ω. Therefore we can choose homogeneous coordinates in $\mathbb{P}^3_{\Bbbk}$ such that the group G acts as

$$(x:y:z:t) \mapsto (x:y:\omega z:\omega^2 t)$$

on the cubic surface X given by the equation

$$P(x:y) + zt(ux + vy) + z^3 + \alpha t^3 = 0 \tag{1}$$

where $P(x:y)$ is a homogeneous polynomial of degree 3, and u, v and α are parameters.

Let us consider the line $x = y = 0$. This line intersects X in three points e_1, e_2 and e_3 given by the equation

$$z^3 + \alpha t^3 = 0. \tag{2}$$

Definition 5.1 A point p on a cubic surface is called an *Eckardt point* if there are three (-1)-curves passing through p.

Lemma 5.2 *The points e_1, e_2 and e_3 are Eckardt points.*

Proof The surface X is given by (1) in $\mathbb{P}^3_{\Bbbk}$. In coordinates x, y, z, t the points e_1, e_2 and e_3 are $(0:0:-\sqrt[3]{\alpha}:1)$, $(0:0:-\omega\sqrt[3]{\alpha}:1)$ and $(0:0:-\omega^2\sqrt[3]{\alpha}:1)$. Consider the tangent plane at the point e_1. Its equation is

$$u\sqrt[3]{\alpha}\,x + v\sqrt[3]{\alpha}\,y = 3\big(\sqrt[3]{\alpha^2}z + \alpha t\big).$$

We have

$$\begin{aligned} zt(ux + vy) + z^3 + \alpha t^3 &= 3zt\big(\sqrt[3]{\alpha}\,z + \sqrt[3]{\alpha^2}t\big) + z^3 + \alpha t^3 \\ &= \big(z + \sqrt[3]{\alpha}\,t\big)^3 = \left(\frac{\sqrt[3]{\alpha^{-1}}ux + \sqrt[3]{\alpha^{-1}}vy}{3}\right)^3. \end{aligned}$$

So that (1) can be rewritten as

$$P(x:y) + \frac{(ux + vy)^3}{27\alpha} = 0. \tag{3}$$

The last equation has three roots $(\lambda_1 : \mu_1)$, $(\lambda_2 : \mu_2)$ and $(\lambda_3 : \mu_3)$. The three (-1)-curves passing through the point e_1 are given by

$$u\sqrt[3]{\alpha}\,x + v\sqrt[3]{\alpha}\,y = 3\big(\sqrt[3]{\alpha^2}z + \alpha t\big)$$

and $\mu_i x = \lambda_i y$. Similarly we can show that the three (-1)-curves passing through e_2 are given by

$$u\sqrt[3]{\alpha}\,x + v\sqrt[3]{\alpha}\,y = 3\big(\omega\sqrt[3]{\alpha^2}z + \omega^2\alpha t\big)$$

and $\mu_i x = \lambda_i y$, and the three (-1)-curves passing through e_3 are given by

$$u\sqrt[3]{\alpha}\,x + v\sqrt[3]{\alpha}\,y = 3\big(\omega^2\sqrt[3]{\alpha^2}z + \omega\alpha t\big)$$

and $\mu_i x = \lambda_i y$. □

Remark 5.3 Applying explicit equations given in the proof of Lemma 5.2, one can see that the G-orbit of any (-1)-curve passing through a point e_i consists of three (-1)-curves meeting each other at a point. The image of G in the Weyl group $\mathrm{W}(E_6)$ is conjugate to $\langle(123)(456)\rangle$, by Lemma 4.1. Therefore nine curves passing through the Eckardt points e_i are L_{ij}, where $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$. We can assume that the curves L_{14}, L_{26} and L_{35} pass through e_1, the curves L_{16}, L_{25} and L_{34} pass through e_2 and the curves L_{15}, L_{24} and L_{36} pass through e_3.

Now we give explicit geometric interpretation of the action of the group $\pi_1(\Gamma)$.

Lemma 5.4 *Let X be a cubic surface given by* (1) *and Γ be the image of the Galois group* $\mathrm{Gal}(\overline{\mathbb{k}}/\mathbb{k})$ *in the Weyl group* $\mathrm{W}(E_6)$*. Let Γ_1 and Γ_2 be the Galois groups of* (2) *and* (3) *respectively. Then in the notation of Sect. 4 one has the following:*

- *if $\pi_1(\Gamma)$ is trivial then Γ_1 and Γ_2 are trivial;*
- *if $\pi_1(\Gamma) = \langle cs\rangle \cong \mathfrak{C}_2$ then Γ_1 is trivial and $\Gamma_2 \cong \mathfrak{C}_2$;*
- *if $\pi_1(\Gamma) = \langle a^2b\rangle \cong \mathfrak{C}_3$ then Γ_1 is trivial and $\Gamma_2 \cong \mathfrak{C}_3$;*
- *if $\pi_1(\Gamma) = \langle ab\rangle \cong \mathfrak{C}_3$ then $\Gamma_1 \cong \mathfrak{C}_3$ and Γ_2 is trivial;*
- *if $\pi_1(\Gamma) = \langle a\rangle \cong \mathfrak{C}_3$ then $\Gamma_1 \cong \mathfrak{C}_3$, $\Gamma_2 \cong \mathfrak{C}_3$ and* (2) *and* (3) *have the same splitting field;*
- *if $\pi_1(\Gamma) = \langle a^2b, cs\rangle \cong \mathfrak{S}_3$ then Γ_1 is trivial and $\Gamma_2 \cong \mathfrak{S}_3$;*
- *if $\pi_1(\Gamma) = \langle ab, cs\rangle \cong \mathfrak{C}_6$ then $\Gamma_1 \cong \mathfrak{C}_3$ and $\Gamma_2 \cong \mathfrak{C}_2$;*
- *if $\pi_1(\Gamma) = \langle a, b\rangle \cong \mathfrak{C}_3^2$ then $\Gamma_1 \cong \mathfrak{C}_3$, $\Gamma_2 \cong \mathfrak{C}_3$ and* (2) *and* (3) *have different splitting fields;*
- *if $\pi_1(\Gamma) = \langle a, b, cs\rangle \cong \mathfrak{C}_3^2 \rtimes \mathfrak{C}_2$ then $\Gamma_1 \cong \mathfrak{C}_3$ and $\Gamma_2 \cong \mathfrak{S}_3$.*

Proof Note that the group Γ_1 permutes the Eckardt points e_1, e_2 and e_3, and the group Γ_2 permutes three (-1)-curves passing through an Eckardt point. In the notation of Remark 5.3 one can see that the elements a^2b and cs of $\mathrm{W}(E_6)$ preserve the Eckardt points e_1, e_2 and e_3, and permute the (-1)-curves passing through each of them. Thus the availability of elements conjugate to a^2b and cs in $\pi_1(\Gamma)$ is equivalent to the availability of elements of order 3 and 2 in Γ_2 respectively.

The element ab permutes three (-1)-curves E_{14}, E_{25} and E_{36}. These curves lie in a plane given by $\mu_i x = \lambda_i y$, where $(\lambda_i : \mu_i)$ is a root of (3). Similarly, the element ab preserves the other roots of (3). Thus the availability of the element ab in $\pi_1(\Gamma)$ is equivalent to the availability of an element of order 3 in Γ_1.

The group $\langle a, b, cs\rangle \cong \mathfrak{C}_3^2 \rtimes \mathfrak{C}_2$ is generated by the elements ab, a^2b and cs. So for any subgroup of $\langle a, b, cs\rangle$ one can obtain the result of this lemma. □

Now we want to find a geometric interpretation of actions of elements r and s. Consider the class L in $\mathrm{Pic}(X)$. We have

$$rL = 4L - \sum_{i=1}^{3} E_i - 2\sum_{i=4}^{6} E_i, \qquad r^2L = 4L - 2\sum_{i=1}^{3} E_i - \sum_{i=4}^{6} E_i,$$

$$sL = 5L - 2\sum_{i=1}^{6} E_i, \qquad srL = 2L - \sum_{i=4}^{6} E_i, \qquad sr^2L = 2L - \sum_{i=1}^{3} E_i.$$

The three fixed points of G on $\overline{X}$ lie on the line $z = t = 0$. We denote these points by q_1, q_2 and q_3 given by the equation

$$P(x:y) = 0. \tag{4}$$

There are two G-invariant hyperplane sections $z = 0$ and $t = 0$ passing through the fixed points of G. We denote these sections by C_1 and C_2.

Let $h\colon \overline{X} \to \mathbb{P}^2_{\overline{\Bbbk}}$ be a G-equivariant blowup of $\mathbb{P}^2_{\overline{\Bbbk}}$ at six points p_1, p_2, p_3, p_4, p_5 and p_6 and l be a class of line on $\mathbb{P}^2_{\overline{\Bbbk}}$. Then G has three fixed points on $\mathbb{P}^2_{\overline{\Bbbk}}$. For each two of these fixed points there is exactly one G-invariant curve passing through these two points that belongs to one of the following six classes: a line, a quadric passing through p_1, p_2 and p_3, a quadric passing through p_4, p_5 and p_6, a quartic passing through p_4, p_5 and p_6 and having nodes at p_1, p_2 and p_3, a quartic passing through p_1, p_2 and p_3 and having nodes at p_4, p_5 and p_6 or a quintic having nodes at p_1, p_2, p_3, p_4, p_5 and p_6. Proper transforms of these curves on X are G-invariant and can be permuted by the group Γ. Denote these curves by R^K_{ij}, where K is a class of curve in $\mathrm{Pic}(\overline{X})$ and i and j are indices of points q_i and q_j, which R^K_{ij} is passing through.

Lemma 5.5 *We can choose notation in such way that the following conditions hold:*

- *the curve C_1 is tangent to the curves $R^L_{12}, R^{rL}_{12}, R^{r^2L}_{12}, R^{sL}_{13}, R^{srL}_{13}, R^{sr^2L}_{13}$ at the point q_1, tangent to the curves $R^L_{23}, R^{rL}_{23}, R^{r^2L}_{23}, R^{sL}_{12}, R^{srL}_{12}, R^{sr^2L}_{12}$ at the point q_2 and tangent to the curves $R^L_{13}, R^{rL}_{13}, R^{r^2L}_{13}, R^{sL}_{23}, R^{srL}_{23}, R^{sr^2L}_{23}$ at the point q_3;*
- *the curve C_2 is tangent to the curves $R^L_{13}, R^{rL}_{13}, R^{r^2L}_{13}, R^{sL}_{12}, R^{srL}_{12}, R^{sr^2L}_{12}$ at the point q_1, tangent to the curves $R^L_{12}, R^{rL}_{12}, R^{r^2L}_{12}, R^{sL}_{23}, R^{srL}_{23}, R^{sr^2L}_{23}$ at the point q_2 and tangent to the curves $R^L_{23}, R^{rL}_{23}, R^{r^2L}_{23}, R^{sL}_{13}, R^{srL}_{13}, R^{sr^2L}_{13}$ at the point q_3.*

Proof One has $C_1 \cdot R^L_{12} = C_2 \cdot R^L_{12} = 3$. Note that the curves C_1, C_2 and R^K_{12} pass through the G-fixed points q_1 and q_2, therefore R^L_{12} cannot meet C_1 and C_2 at any other point since that point should be G-invariant. Therefore R^L_{12} is tangent to C_1 and C_2. The curves C_1 and C_2 have different tangents at the points q_i. Thus we can assume that R^L_{12} is tangent to C_1 at q_1 and tangent to C_2 at q_2.

In the same way we can show that for any class

$$K \in \left\{L, rL, r^2L, sL, srL, sr^2L\right\}$$

the curve R^K_{ij} is tangent to C_1 and C_2 at points q_i and q_j. One has $R^K_{ij} \cdot R^{csrK}_{ij} = 2$ therefore these curves meet each other transversally and have different tangents at points q_i and q_j. Moreover, one has $R^K_{ij} \cdot R^K_{jk} = 1$ therefore these curves meet each other transversally and have different tangents at point q_j. The lemma follows from these two facts. $\square$

Now we give explicit geometric interpretation of the action of the group $\pi_2(\Gamma)$.

Lemma 5.6 *In the notation of Sect. 4 the group $\pi_2(\Gamma)$ contains an element conjugate to s if and only if the Galois group Γ_3 of (4) is of even order.*

Proof Let the group Γ_3 contain an element h such that $h(q_2) = q_3$ and $h(q_3) = q_2$. By Lemma 5.5, the curve R^L_{12} is tangent to C_2 at q_2. Thus the curve $h(R^L_{12})$ is tangent to $h(C_2) = C_2$ at q_3 and passes through q_1. Therefore, by Lemma 5.5, the curve $h(R^L_{12})$ is R^{sL}_{13}, R^{srL}_{13} or $R^{sr^2L}_{13}$. Hence the group $\pi_2(\Gamma)$ contains an element conjugate to s.

Now assume that the group $\pi_2(\Gamma)$ contains an element conjugate to s. If the Galois group Γ_3 is of odd order then this element fixes the points q_1, q_2 and q_3. Therefore the curve R^L_{12} is mapped by s to R^{sL}_{12}. But R^L_{12} is tangent to C_1 at q_1 and R^{sL}_{12} is tangent to C_2 at q_2. This contradiction finishes the proof. □

Lemma 5.7 *Let X be a G-minimal cubic surface given by* (1) *and the Galois group Γ_3 of* (4) *is isomorphic to $\mathfrak{C}_2$. Then the quotient X/G is birationally equivalent to a minimal del Pezzo surface Z of degree* 4. *In particular, X/G is not $\Bbbk$-rational.*

Proof The Galois group Γ_3 of (4) is isomorphic to $\mathfrak{C}_2$. Therefore we can assume that the G-fixed point q_1 is defined over $\Bbbk$ and two other G-fixed points q_2 and q_3 are permuted by Γ_3.

Let $f : X \to X/G$ be the quotient morphism and $\pi : \widetilde{X/G} \to X/G$ the minimal resolution of singularities. There are three singular points of type A_2 on X/G, namely $f(q_1)$, $f(q_2)$ and $f(q_3)$. The curves C_1, C_2 and the point q_1 are defined over $\Bbbk$. Thus the irreducible components of $\pi^{-1} f(q_1)$ are defined over $\Bbbk$. The group $\Gamma_3 \cong \mathfrak{C}_2$ maps the irreducible components of $\pi^{-1} f(q_2)$ to the irreducible components of $\pi^{-1} f(q_3)$. Therefore, one has

$$\rho(\widetilde{X/G}) = \rho(X/G) + 4 = \rho(X)^G + 4 = 5.$$

As in the proof of Lemma 3.2, two curves $\pi_*^{-1} f(C_i)$ are (-1)-curves defined over $\Bbbk$. We can contract this pair and get a del Pezzo surface Y such that $K_Y^2 = 3$ and $\rho(Y) = 3$.

The Galois group Γ_3 acts on the set of 27 (-1)-curves on Y. One cannot contract more than four (-1)-curves on Y since $\rho(Y) = 3$. But in Table 1 in [5, Chapter IV,Section 5] there is only one class of elements of order 2 satisfying this property. This class corresponds to the 11-th row of the table. For this class one cannot contract more than one (-1)-curve on Y (see the second column of the table). Therefore one can contract this curve on Y and get a minimal del Pezzo surface Z of degree 4 with $\rho(Z) = 2$ admitting a structure of conic bundle. The surface $Z \approx X/G$ is not $\Bbbk$-rational, by Theorem 2.8. □

Assume that the G-fixed point q_1 on the cubic surface X is defined over $\Bbbk$. Then after the change of coordinates this cubic is given by the equation

$$\beta x(x^2 - \lambda y^2) + zt(ux + vy) + z^3 + \alpha t^3 = 0,$$

where u, v, α, β and λ are parameters. For this cubic surface the following lemma holds.

Lemma 5.8 *In the notation of Sect. 4 the group $\pi_2(\Gamma)$ contains r if and only if there is an element of order 3 in the Galois group Γ_4 of the equation*

$$4\alpha\theta^3 - \left(u^2 - \frac{v^2}{\lambda}\right)\theta^2 - 2u\beta\theta - \beta^2 = 0 \tag{5}$$

considered as a cubic equation in θ.

Proof Note that the divisors $R_{23}^{L} + R_{23}^{sL}$, $R_{23}^{rL} + R_{23}^{sr^2L}$ and $R_{23}^{r^2L} + R_{23}^{srL}$ are linearly equivalent to $-2K_X$. Therefore these G-invariant pairs of curves passing through q_2 and q_3 are cut from X by the quadric surfaces of the following form:

$$\theta(x^2 - \lambda y^2) = zt.$$

Let us find reducible members in these family of curves. One has

$$\beta x(x^2 - \lambda y^2)t^3 + \theta(ux + vy)(x^2 - \lambda y^2)t^3 + \theta^3(x^2 - \lambda y^2)^3 + \alpha t^6 = 0.$$

If the polynomial in the left hand side for the latter equation is reducible over $\Bbbk(x, y, t)$ then it factorizes in the following way:

$$\big(A\big(x - y\sqrt{\lambda}\big)(x^2 - \lambda y^2) + \sqrt{\alpha}\,t^3\big)\big(B\big(x + y\sqrt{\lambda}\big)(x^2 - \lambda y^2) + \sqrt{\alpha}\,t^3\big) = 0$$

and therefore we have $AB = \theta^3$ and

$$A\big(x - y\sqrt{\lambda}\big)\sqrt{\alpha} + B\big(x + y\sqrt{\lambda}\big)\sqrt{\alpha} = \theta(ux + vy) + \beta x.$$

Therefore the following system of equations holds:

$$\begin{cases} (A + B)\sqrt{\alpha} = \theta u + \beta, \\ (B - A)\sqrt{\lambda\alpha} = \theta v. \end{cases}$$

Solving this system one has

$$A = \frac{(\theta u + \beta)\sqrt{\lambda} - \theta v}{2\sqrt{\lambda\alpha}}, \qquad B = \frac{(\theta u + \beta)\sqrt{\lambda} + \theta v}{2\sqrt{\lambda\alpha}}.$$

Since $AB = \theta^3$, the reducible members of the linear system $|-2K_X|$ passing through q_2 and q_3 are given by (5).

The roots of this equation correspond to the pairs of curves R_{23}^{L} and R_{23}^{sL}, R_{23}^{rL} and $R_{23}^{sr^2L}$, $R_{23}^{r^2L}$ and R_{23}^{srL} which are cyclically permuted by Γ if and only if the group $\pi_2(\Gamma)$ contains r. $\square$

Remark 5.9 At the beginning of this section we assumed that the field $\mathbb{k}$ contains ω. For any field $\mathbb{k}$ the action of a generator of G can be written as

$$(x:y:z:t) \mapsto (y:z:x:t).$$

One can remake the computations (which is much more complicated) for this action. Then Lemmas 5.2, 5.4, 5.5 and 5.8 hold. But Lemma 5.6 does not hold since the curves C_1 and C_2 are not defined over $\mathbb{k}$.

Remark 5.10 Note that in Sects. 4 and 5 we can omit the condition $\operatorname{char}\mathbb{k} = 0$. Therefore the obtained results can be useful to study cubic surfaces over fields such that $\operatorname{char}\mathbb{k} \geqslant 5$.

6 Examples

In this section we construct explicit examples of quotients of del Pezzo surfaces of degree 3 by a group $G \cong \mathfrak{C}_3$ acting as in type 5 of Table 2. We use the notation of Sect. 5.

Lemma 6.1 *Let X be a cubic surface given by (1). Suppose that the Galois groups $\Gamma_1, \Gamma_2, \Gamma_3$ of (2), (3), (4) are trivial and the Galois group Γ_4 of (5) contains an element of order 3. Then the surface X is G-minimal and $\mathbb{k}$-rational, and the quotient X/G is also $\mathbb{k}$-rational.*

Proof The group Γ_1 is trivial. Therefore $X(\mathbb{k})$ contains the points e_1, e_2 and e_3. By Lemmas 5.4, 5.6 and 5.8, the group Γ is conjugate to $\langle r\rangle$. Therefore one can Galois equivariantly contract the curves E_1, L_{23} and Q_1 and get a del Pezzo surface of degree 6 which is $\mathbb{k}$-rational, by Theorem 2.8.

The image of the group G in the Weyl group $\mathrm{W}(E_6)$ is $\langle ab\rangle$ thus X is G-minimal, by Corollary 4.11. Let $f: X \to X/G$ be the quotient morphism and $\pi: \widetilde{X/G} \to X/G$ the minimal resolution of singularities. The group Γ_3 is trivial. Therefore the points q_1, q_2 and q_3 are defined over $\mathbb{k}$. Hence all six irreducible components of the curves $\pi^{-1}f(q_1)$, $\pi^{-1}f(q_2)$ and $\pi^{-1}f(q_3)$ are defined over $\mathbb{k}$. Thus $\rho(\widetilde{X/G}) = 7$, and $\widetilde{X/G}$ and X/G are $\mathbb{k}$-rational, by Corollary 2.9. $\square$

Example 6.2 If the field $\mathbb{k}$ contains ω and an element ν such that $\sqrt[3]{\nu} \notin \mathbb{k}$ then the cubic surface given by the equation

$$2\nu x(x^2 - y^2) + z^3 + t^3 = 0$$

satisfies the conditions of Lemma 6.1.

Lemma 6.3 *Let X be a cubic surface given by (1). Suppose that the Galois groups Γ_1, Γ_2 of (2), (3) are trivial, the Galois group Γ_3 of (4) is isomorphic to $\mathfrak{C}_2$ and the Galois group Γ_4 of (5) contains an element of order 3. Then the surface X is G-minimal and $\mathbb{k}$-rational, and the quotient X/G is not $\mathbb{k}$-rational.*

Proof The group Γ_1 is trivial. Therefore $X(\Bbbk)$ contains the points e_1, e_2 and e_3. By Lemmas 5.4, 5.6 and 5.8, the group Γ is conjugate to $\langle r, s\rangle$. Therefore one can Galois equivariantly contract the curves E_1, L_{23} and Q_1 and get a del Pezzo surface of degree 6 which is $\Bbbk$-rational, by Theorem 2.8. The image of the group G in the Weyl group $\mathrm{W}(E_6)$ is $\langle ab\rangle$ thus X is G-minimal, by Corollary 4.11. The quotient X/G is not $\Bbbk$-rational by Lemma 5.7. □

Example 6.4 Suppose that the field $\Bbbk$ contains ω and does not contain $\sqrt{2}$ and any root of the equation

$$4\theta^3 - 9\theta^2 - 6\theta - 1 = 0.$$

Then the cubic surface given by the equation

$$x(x^2 - 2y^2) + 3xzt + z^3 + t^3 = 0$$

satisfies the conditions of Lemma 6.3.

Lemma 6.5 *Let X be a cubic surface given by (1). Suppose that the Galois groups Γ_1 and Γ_3 of (2) and (4) are trivial, the Galois group Γ_2 of (3) is isomorphic to $\mathfrak{C}_2$ and the Galois group Γ_4 of (5) contains an element of order 3. Then the surface X is G-minimal and not $\Bbbk$-rational, and the quotient X/G is $\Bbbk$-rational.*

Proof The group Γ_1 is trivial. Therefore $X(\Bbbk)$ contains the points e_1, e_2 and e_3. By Lemmas 5.4, 5.6 and 5.8, the group Γ is conjugate to $\langle c, r\rangle$. Therefore X is not $\Bbbk$-rational, by Lemma 4.14.

The image of the group G in the Weyl group $\mathrm{W}(E_6)$ is $\langle ab\rangle$ thus X is G-minimal, by Corollary 4.11. Let $f\colon X \to X/G$ be the quotient morphism and $\pi\colon \widetilde{X/G} \to X/G$ the minimal resolution of singularities. The group Γ_3 is trivial. Therefore the points q_1, q_2 and q_3 are defined over $\Bbbk$. Hence all six irreducible components of the curves $\pi^{-1}f(q_1)$, $\pi^{-1}f(q_2)$ and $\pi^{-1}f(q_3)$ are defined over $\Bbbk$. Thus $\rho(\widetilde{X/G}) = 7$, and $\widetilde{X/G}$ and X/G are $\Bbbk$-rational, by Corollary 2.9. □

Example 6.6 In the assumptions of Example 6.4 the cubic surface given by the equation

$$x(x^2 - y^2) + 3xzt + z^3 + t^3 = 0$$

satisfies the conditions of Lemma 6.5.

Lemma 6.7 *Let X be a cubic surface given by (1). Suppose that the Galois group Γ_1 of (2) is trivial, the Galois group Γ_2 of (3) is isomorphic to $\mathfrak{C}_2$ and the Galois group Γ_3 of (4) is isomorphic to $\mathfrak{C}_2$. Then the surface X is G-minimal and not $\Bbbk$-rational, and the quotient X/G is also not $\Bbbk$-rational.*

Proof The group Γ_1 is trivial. Therefore $X(\Bbbk)$ contains the points e_1, e_2 and e_3. By Lemmas 5.4 and 5.6, the group Γ contains a subgroup $\langle cs\rangle \cong \mathfrak{C}_2$. Therefore X is not

$\Bbbk$-rational, by Lemma 4.14. The image of the group G in the Weyl group $\mathrm{W}(E_6)$ is $\langle ab\rangle$ thus X is G-minimal, by Corollary 4.9. The quotient X/G is not $\Bbbk$-rational, by Lemma 5.7. □

Example 6.8 If the field $\Bbbk$ contains ω and an element λ such that $\sqrt{\lambda} \notin \Bbbk$ then the cubic surface given by the equation

$$x(x^2 - \lambda y^2) + z^3 + t^3 = 0$$

satisfies the conditions of Lemma 6.7.

Remark 6.9 Note that the conditions of Examples 6.2, 6.4, 6.6, 6.8 hold for $\Bbbk = \mathbb{Q}(\omega)$.

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