European Journal of Mathematics (2015) 1:717–720
DOI 10.1007/s40879-015-0057-1

RESEARCH ARTICLE

On a paper by Yuri G. Zarhin

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Received: 27 February 2015 / Revised: 6 May 2015 / Accepted: 13 May 2015 /
Published online: 6 August 2015

Abstract In a recent paper, (Math Notes 91(3–4): 508–516, 2012) Zarhin proved that each member of a naturally defined family of linear maps $\mathbb{C}^n \to \mathbb{C}^n$ has co-rank one. We present a direct proof of Zarhin's result about complex polynomials with distinct roots; it is rather similar to that of Appendix by Vik.S. Kulikov to Zarhin's paper but we give explicit constants. We also discuss the case of a polynomial with multiple roots.

Keywords Complex polynomials with distinct roots · Derivate has constant rank · Conjecture on case of coincident roots

Mathematics Subject Classification 30C10 · 15A15

1 Zarhin's result

First we recall the main result of [1]. Let f be a monic complex polynomial of degree n with distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_n$. Define

$$M(f) = \big(f'(\alpha_1), f'(\alpha_2), \ldots, f'(\alpha_n)\big),$$

then the derivative dM_f of M for each such f has rank $n-1$.

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Since the map $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \mapsto f(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$ is a regular ($n!$-sheeted) covering at points α where α_i are distinct, the study of the rank of the derivative of M can be done equivalently at f or α. We do so at α.

The $n \times n$ matrix $T = T(f)$ of the derivative map is given by

$$\begin{aligned} T_{ij} &= \frac{\partial}{\partial \alpha_j} f'(\alpha_i) = \frac{\partial}{\partial \alpha_j}(\alpha_i-\alpha_1)(\alpha_i-\alpha_2)\cdots\widehat{(\alpha_i-\alpha_i)}\cdots(\alpha_i-\alpha_n) \\ &= -(\alpha_i-\alpha_1)(\alpha_i-\alpha_2)\cdots\widehat{(\alpha_i-\alpha_j)}\cdots\widehat{(\alpha_i-\alpha_i)}\cdots(\alpha_i-\alpha_n), \qquad i \neq j, \\ T_{ii} &= \sum_j (\alpha_i-\alpha_1)(\alpha_i-\alpha_2)\cdots\widehat{(\alpha_i-\alpha_j)}\cdots\widehat{(\alpha_i-\alpha_i)}\cdots(\alpha_i-\alpha_n). \end{aligned}$$

Our proof of Zarhin's result shows that the matrix T has some remarkable properties and so it might be of independent interest.

We will simplify the notation by writing

$$\begin{aligned} f_k(x) &= \frac{f(x)}{(x-\alpha_k)} = -\frac{\partial f}{\partial \alpha_k}, \\ f_{k\ell}(x) &= \frac{f(x)}{(x-\alpha_k)(x-\alpha_\ell)}, \qquad k \neq \ell, \\ f_{k\ell m}(x) &= \frac{f(x)}{(x-\alpha_k)(x-\alpha_\ell)(x-\alpha_m)}. \end{aligned}$$

Then, for $i \neq j$, $T_{ij} = f_{ij}(\alpha_i)$.

For $X \subset \{1, 2, \ldots n\}$ with m elements, we let $T[X]$ denote the $(n-m)\times(n-m)$ submatrix of T obtained by omitting the j^{th} row and column for each $j \in X$ and let $D[X] = \det T[X]$; so the principal minor of T is $D[n]$. We also let $\Delta(g)$ denote the discriminant of a polynomial g.

We note that the sum of the columns of T is zero and so rank $T < n$. Since the discriminant of a polynomial with distinct roots is non-zero, the proof will be completed by

Proposition *For each k, $1 \leq k \leq n$,*

$$D[k] = (-1)^{\binom{n-1}{2}}(n-1)!\,\Delta(f_k) = (-1)^{\binom{n-1}{2}}(n-1)! \prod_{\substack{1\leq i<j\leq n \\ i,j\neq n}} (\alpha_i-\alpha_j)^2.$$

In particular,

$$D[n] = (-1)^{\binom{n-1}{2}}(n-1)!\,\Delta(f_n) = (-1)^{\binom{n-1}{2}}(n-1)! \prod_{1\leq i<j<n} (\alpha_i-\alpha_j)^2.$$

Proof We prove the result for $k = n$ (that is, we are considering the principal minor of T) and the proof is, apart from notation, the same for other values of k. Interchanging both the ith and jth rows and the ith and jth columns of T for $1 \leq i < j < n$ interchanges i and j but does not change the determinant $D[n]$ of the principal minor. So $D[n]$ is a symmetric polynomial in $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$. If we set $\alpha_i = \alpha_j$ then the

i^{th} and j^{th} rows of T are equal, so if $1 \le i < j < n$ and $\alpha_i = \alpha_j$ then $D[n] = 0$. Now we recall the well known result:

If $P(x_1, x_2, \dots, x_r)$ is a symmetric polynomial which vanishes when any pair of the x's are equal then p is a multiple of $\prod_{1\le i<j\le r}(x_i - x_j)^2$.

So $D[n]$ is a multiple of $\Delta(f) = \prod_{1\le i<j<n}(\alpha_i - \alpha_j)^2$, which clearly has total degree $(n-1)(n-2)$ but the total degree of each T_{ij} is $n-2$ so $D[n]$ also has total degree $(n-1)(n-2)$. So $D[n] = c\Delta(f_n)$ for some constant c.

To determine the value of c, we consider each T_{ij} as an element of the polynomial ring $R[\alpha_1]$ where $R = \mathbb{C}[\alpha_2, \alpha_3, \dots, \alpha_n]$. We use induction on n; it starts trivially at $n = 2$.

The degrees of the various T_{ij} as polynomials in α_1 are given by

Index	Degree T_{ij}
$i > 1, j > 1$	1
$i > 1, j = 1$	0
$i = 1$	$n-2$

Moreover, the coefficient of α_1^{n-2} in T_{1j} is -1 for $j > 1$ and, since the sum of the columns of T is zero, the coefficient of α_1^{n-2} in T_{11} is $n-1$. There are no occurrences of α_1 in the first column (except for T_{11}) and so when calculating the determinant by using the first row, the terms that contribute $\alpha_1^{2(n-2)}$ to $D[n]$ all come from the product $T_{11}D[1,n]$. (Note that the highest degree term involving α_1 in T_{11} is α_1^{n-2} and that in each entry of $T[1,n]$ it is α_1.) The coefficient of α_1 in the entries of $T[1,n]$ (since these entries are all linear in α_1) are given by

$$\frac{\partial}{\partial\alpha_1} T_{ij} = \frac{\partial}{\partial\alpha_1} f_{ij}(\alpha_i) = -f_{1ij}(\alpha_i), \qquad i \ne j, \quad i, j > 1.$$

But these are precisely the negatives of the off-diagonal entries of the matrix $T(f_1)$ that one obtains from the polynomial $f_1(x) = (x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n)$. The entries of the first column of T do not involve α_1 and since the sum of the columns of T is 0, α_1 does not appear in the sum of the columns of $T[n]$. But the sum of the columns of $T(f_1)$ is also 0 and so we conclude that the matrix $T[1,n] - \alpha_1 T(f_1)$ is independent of α_1. Hence, by applying the induction hypothesis to f_1, the term in $D[n]$ involving α_1^{2n-2} is

$$(n-1)\alpha_1^{n-2}\det(-I_{n-2})\alpha_1^{\binom{n-2}{2}}(n-2)!\,\Delta(f_{n1}) \\ = (-1)^{\binom{n-1}{2}}(n-1)!\,\Delta(f_{1n})\alpha_1^{2(n-2)}$$

proving the proposition. □

2 Multiple roots

Now we consider a monic polynomial $f(x) \in \mathbb{C}[x]$ of degree n with multiple roots. Let $R(f) = \{\alpha_1, \dots, \alpha_r\}$ denote the set of all its roots, $\# R(f) = r$, $r < n$, and

$R_k(f)$ the set of roots of f that have multiplicity exactly k. We order the roots so that their multiplicities are in decreasing order and suppose that $\# R_1(f) = s$; clearly $s < n$. The first $r - s$ rows of the $r \times r$ matrix M are zero, so rank $M \leq s$. Somewhat tentatively, we make the following conjecture and sketch some of the calculations that support it.

Conjecture *The rank of M is s.*

Consider an $s \times s$ submatrix N of M formed from a set of s columns and the last s rows of M. We find that if all the determinants det N are zero then a pair of roots of the polynomial are equal. In particular, calculations that we have carried out suggest that det N is always of the form

$$\pm c \prod (a - b)^t g$$

where the product is over a nonempty set of pairs of distinct roots a, b of f and g is a polynomial in $\{\alpha_1, \ldots, \alpha_r\}$. In various cases, we describe the powers t, the constant c and the polynomials g:

- Let $f(x)$ have only one root, say, α_r of multiplicity 1, then the principal minor N is 1×1 and it is easy to calculate that

$$N = -k_1 \frac{f_r(\alpha_r)}{(\alpha_r - \alpha_1)}$$

  where k_1 is the multiplicity of the root α_1 and (as in Sect. 1) $f_r(x)$ is $f(x)$ with the factor $x - \alpha_r$ omitted.
- Let $f(x)$ have the root α_1 with multiplicity $k > 1$ and the other roots be $\alpha_2, \ldots, \alpha_r$ all of multiplicity 1 then, the principal minor, det N has factors $\alpha_1 - \alpha_\ell$ with index $t = k - 1$ and the factors $\alpha_m - \alpha_\ell$, $\ell, m > 1$, with index $t = 2$ and c is $\pm k(k+1) \cdots (k+r-1)$.
- Let $f(x)$ have the root α_1 with multiplicity $k > 1$, the root α_2 with multiplicity $\ell > 1$ and the other roots of multiplicity 1. Then, when $r = 5$, the determinant of the minors has the form indicated but with a non-trivial factor g. The principal minor has $g = \alpha_1 + 2\alpha_2 - 3\alpha_3$ and one of the other minors has $g = \alpha_1 + 2\alpha_2 - 3\alpha_4$. If both these g vanish then we have that $\alpha_3 = \alpha_4$ and if some other factor of the determinant vanishes then, again two of the α's are equal which contradicts our hypothesis.

This final calculation seems to indicate that it may be difficult to verify the conjecture by a direct calculation.

Acknowledgments The author thanks the editor and the referee for a very careful reading which found several slips.

Reference

1. Zarhin, Yu.G.: Polynomials in one variable and ranks of certain tangent maps. Math. Notes **91**(3–4), 508–516 (2012)