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RESEARCH ARTICLE

Decomposition theorem and Riesz basis for axisymmetric potentials in the right half-plane

Slah Chaabi[1] · Stéphane Rigat[1]

Abstract We consider the Weinstein equation, also known as the equation governing generalized axisymmetric potentials (GASP), with complex coefficients $L_m u = \Delta u + (m/x)\partial_x u = 0$, $m \in \mathbb{C}$. We generalize results known for $m \in \mathbb{R}$ to the case $m \in \mathbb{C}$. In particular, explicit expressions of fundamental solutions for Weinstein operators and their estimates near singularities are presented, a Green's formula for GASP in the right half-plane $\mathbb{H}^+$ for Re $m < 1$ is established. We prove a new decomposition theorem for the GASP in annular domains for $m \in \mathbb{C}$, which is in fact a generalization of the Bôcher's decomposition theorem. In particular, using bipolar coordinates, it is proved for annuli that a family of solutions for the GASP equation in terms of associated Legendre functions of first and second kind is complete. This family is shown to be a Riesz basis in some non-concentric circular annuli.

Keywords Axially symmetric solutions · Fundamental solutions · Riesz basis · Elliptic functions and integrals

Mathematics Subject Classification 35B07 · 33E05 · 35J15 · 35E05 · 42C15

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1 Introduction

In this article, we study the Weinstein differential operator

$$L_m = x^{-m} \operatorname{div}(x^m \nabla \cdot) = \Delta + \frac{m}{x} \frac{\partial}{\partial x}, \qquad m \in \mathbb{C},$$

well-defined on the right half-plane $\mathbb{H}^+ = \{(x, y) \in \mathbb{R}^2 : x > 0\} = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ with the convention $x^m = \exp(m \ln x)$. This class of operators is also called operators governing axisymmetric potentials. They have been studied quite extensively in cases $m \in \mathbb{N}$ and $m \in \mathbb{R}$ in [10–13, 16–18, 22–24, 29–33, 36–38, 41, 51–68]. We will focus exclusively on the case $m \in \mathbb{C}$, recalling in the course of the paper some results for integer values of m. The Weinstein equation reads

$$L_m u = 0. \tag{1}$$

The main motivation for which we consider the case $m \in \mathbb{C}$ is that if we complexify the coordinates by writing $z = x + iy$, (1) takes the form

$$\frac{\partial^2 u}{\partial z \partial \overline{z}} + \frac{m/2}{z + \overline{z}} \left(\frac{\partial u}{\partial z} + \frac{\partial u}{\partial \overline{z}} \right) = 0,$$

which is a particular case of the equation

$$\frac{\partial^2 u}{\partial z \partial \overline{z}} + \frac{\alpha}{z + \overline{z}} \frac{\partial u}{\partial z} + \frac{\beta}{z + \overline{z}} \frac{\partial u}{\partial \overline{z}} = 0$$

considered with $\alpha, \beta \in \mathbb{C}$ in [49, Equation (5.7), p. 20].

Equation (1) also appears in physics in the study of the behavior of plasma in a tokamak. The role of tokamak, which has a toroidal geometry, is to control location of the plasma in its chamber by applying magnetic fields on its boundary. It is possible to assume that plasma is axially symmetric what reduces this problem to a plane section in $\mathbb{H}^+$, where the magnetic flux in the vacuum between the plasma and the circular boundary of the chamber satisfies a second-order elliptic nonlinear partial differential equation, the so-called Grad–Shafranov equation, which reduces to the homogeneous equation (1) with $m = -1$ (Fig. 1).

Note that in this instance, (1) takes place in an annular domain rather than in a simply connected domain, see [8, 9, 46]. This fact motivates our decomposition theorem, Theorem 5.9.

In the sequel, the sense in which the solutions are studied will be specified. We will also look at solutions to the equation in the sense of distributions

$$L_m u = \delta_{(x,y)},$$

where $\delta_{(x,y)}$ denotes the Dirac mass at $(x, y) \in \mathbb{H}^+$.
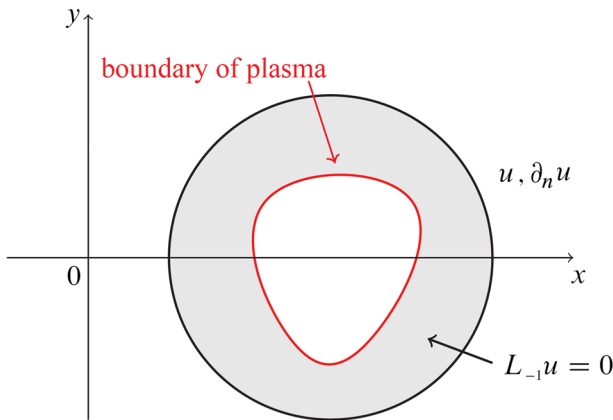
Fig. 1 Section of a Tokamak

The mentioned above class of operators was first considered by Weinstein in [54], where he studied the case $m \in \mathbb{N}^*$. He also established a relation between the axisymmetric potentials for $m \in \mathbb{N}^*$ and harmonic functions on $\mathbb{R}^{m+2}$, see Proposition 2.4.

In [20,58,59], Weinstein and Diaz–Weinstein established the correspondence principle between axisymmetric potentials L_m and L_{2-m}, see Proposition 2.3. They deduced an expression of a fundamental solution (where the singular point is taken on the y-axis) for $m \in \mathbb{R}$ and established a link between the Weinstein equation and Tricomi equations and their fundamental solutions.

Let us return to the book [49]. Studying elliptic equations with analytic coefficients, Vekua provided means to express their fundamental solutions by using the Riemann functions, introduced earlier (see e.g. [28]) in the real hyperbolic context, he also investigated generalized elliptic equations with complex operators ∂_z and $\partial_{\bar{z}}$. In heuristic words, in the same way as a harmonic function is the real part of a holomorphic function, or the sum of a holomorphic and an anti-holomorphic function, Vekua established the fact that solutions to elliptic equations, and therefore GASP, can be written as a sum of two functionals, one applied to an arbitrary holomorphic function and the other applied to an arbitrary anti-holomorphic function. These functionals can be written explicitly in terms of the Riemann function, by using the hypergeometric functions [49] or fractional derivations [16]. In [34], Henrici gave a very interesting introduction to the work of Vekua.

More recently, basing on the work of Vekua, Savina [44] gave a series representation of fundamental solutions for the operator $\widehat{L}u = \Delta u + a\partial_x u + b\partial_y u + cu$ and studied the convergence of this series. She also provided an application of her results to the Helmholtz equation.

In [31], Gilbert studied the non-homogeneous Weinstein equation, i.e. the case $m \geq 0$, and gave an integral formula for this class of equations. In particular, an explicit solution was given when the second member depends only on one variable.

Some Dirichlet problems are considered in [40,41] in a special geometry, the so-called "geometry with separable variable".

Even if some results presented in this paper are known for real values of m, we make a totally self-contained presentation involving elementary technics not necesseraly used in the papers mentioned above. For instance, usual arguments involving estimates of hypergeometric integrals are replaced by arguments using the Lebesgue dominated convergence theorem. Our main result is a decomposition theorem for axisymmetric potentials which is new also for real values of m. We obtain a Liouville-type result for the solutions of Weinstein equation on $\mathbb{H}^+$, with an interesting observation that there is a loss of strict ellipticity of the Weinstein operator on the boundary of $\mathbb{H}^+$. An application of the decomposition theorem is given by showing that an explicit family of axisymmetric potentials constructed by introduction of bipolar coordinates is a Riesz basis in some annuli.

The plan of the paper is the following. In Sect. 2, we recall preliminary information about fundamental solutions for linear partial differential operators with non-constant coefficients. Proposition 2.1 provides with a connection between fundamental solutions for L_m and fundamental solutions for $L_m^\star$, where $L_m^\star$ denotes the formal adjoint of L_m. The Weinstein principle [59], valid for m real and complex, establishes a connection between L_m and L_{2-m}. We state it without proof as Proposition 2.3. Proposition 2.4, valid only for $m \in \mathbb{N}$, is fundamental in the sense that one can compute a fundamental solution for L_m just knowing the usual fundamental solution for the Laplacian in $\mathbb{R}^{m+2}$. The corresponding computations are done for $m \in \mathbb{N}$ and for $m \in \mathbb{Z}$ in Sect. 3.

The extension of formulas for fundamental solutions to the case $m \in \mathbb{C}$ is the core of Sect. 4. First we describe in Proposition 4.2 in an elementary way the behavior of fundamental solutions near their singularities. Next we use the corresponding estimates to establish the main result of the section, Theorem 4.4.

Section 5 is dedicated to the decomposition theorem. First we modify the fundamental solutions built earlier in order to get fundamental solutions which vanish on the boundary of $\mathbb{H}^+$. Next, in Proposition 5.3, we show that if u is a solution for $L_m u = 0$ which vanishes on the boundary of $\mathbb{H}^+$, then $u \equiv 0$ on $\mathbb{H}^+$. Let us emphasize here that, although this statement looks obvious, this is not the case due to the loss of ellipticity of L_m on the boundary of $\mathbb{H}^+$. Let us mention that Proposition 5.3 is a consequence of the maximum principle for pseudo-analytic functions given in a recent paper by Chalendar–Partington [14] for more general function σ than x^m, but in [14] there is an additional assumption on σ, which in our case corresponds to the assumption $|m| \geq 1$. The proof of Proposition 5.3 is quite long, but not difficult, it follows careful estimates of fundamental solutions in some parts of $\mathbb{H}^+$. Finally the decomposition theorem, Theorem 5.9, is proved. Its proof is similar to the proof of the Bôcher's decomposition theorem presented in [4]. We end Sect. 5 with a Poisson formula for axisymmetric potentials in $\mathbb{H}^+$, Proposition 5.10.

In Sect. 6, we consider the case where the annular domain is a kind of annulus. We introduce very classical (in physics) bipolar coordinates, cf. [39], in which the GASP equation has a form presented in Theorem 6.2. Next, applying the method of separation of variables we obtain a basis of solutions in disks and complements of disks in $\mathbb{H}^+$, see Theorem 6.3. In Sect. 7 it is shown that this basis forms a Riesz basis.

2 Notations and preliminaries

Throughout $\mathbb{H}^+ = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ stands for the right half-plane, all scalar functions are assumed to be complex valued. If Ω is an open set in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, let $\mathcal{D}(\Omega)$ designate the space of C^∞-functions compactly supported on Ω, where supp $f = \overline{\{x \in \Omega : f(x) \neq 0\}}$. If K is a compact set in Ω, let $\mathcal{D}_K(\Omega)$ be the set of functions $\varphi \in \mathcal{D}(\Omega)$ such that supp $\varphi \subset K$.

The partial derivatives of a differentiable function u on an open set $\Omega \subset \mathbb{R}^n$ will be denoted by $\partial u / \partial x_i$, $\partial_{x_i} u$ or u_{x_i}, $i \in \{1, \ldots, n\}$. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multi-index, we denote

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$

with $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

It is assumed that the reader is familiar with the terminology of distributions and we refer to [35].

Let L be a linear differential operator on Ω,

$$L = \sum_{|\alpha| \leq N} a_\alpha \partial^\alpha, \qquad N \in \mathbb{N},$$

where the summation runs over the multi-indices α of length $|\alpha| \leq N$, a_α are $C^\infty(\Omega)$-functions. If T is a distribution, then $LT = \sum_{|\alpha| \leq N} a_\alpha \partial^\alpha T$. Denote by $L^\star$ the adjoint operator of L in the sense of distributions, namely,

$$L^\star T = \sum_{|\alpha| \leq N} (-1)^{|\alpha|} \partial^\alpha (a_\alpha T).$$

One can easily check, if $f, g \in \mathcal{D}(\Omega)$, we have

$$\langle Lf, g \rangle = \langle f, L^\star g \rangle.$$

Let $a \in \Omega$ and L be a differential operator on Ω. A *fundamental solution for L on Ω at $a \in \Omega$* is a distribution T_a such that

$$LT_a = \delta_a,$$

where the equality is understood in the sense of distributions on Ω. This equality can be rewritten as

$$\varphi(a) = \langle LT_a, \varphi \rangle = \langle T_a, L^\star \varphi \rangle, \qquad \varphi \in \mathcal{D}(\Omega).$$

In particular, if $a \in \Omega$ and T_a is a fundamental solution to $L^\star$ at a on Ω and if $g \in \mathcal{D}(\Omega)$ is such that $g = L(\varphi)$ with $\varphi \in \mathcal{D}(\Omega)$, then

$$\varphi(a) = \langle T_a, g \rangle, \qquad a \in \Omega.$$

Indeed, we have

$$\varphi(a) = \langle \delta_a, \varphi \rangle = \langle L^\star T_a, \varphi \rangle = \langle T_a, L\varphi \rangle = \langle T_a, g \rangle, \qquad a \in \Omega.$$

These fundamental solutions are therefore a good tool for solving equations $L\varphi = g$ on $\mathcal{D}(\Omega)$ if $g \in \mathcal{D}(\Omega)$.

If $m \in \mathbb{N}^*$, the Laplacian in $\mathbb{R}^m$ will be denoted by Δ_m, or Δ when $m = 2$. For $m \in \mathbb{C}$, L_m denotes the *Weinstein operator*:

$$L_m u(x, y) = \Delta u(x, y) + \frac{m}{x} \frac{\partial u}{\partial x}(x, y), \qquad u \in C^2(\mathbb{H}^+), \quad (x, y) \in \mathbb{H}^+.$$

If $f(x, y) = (f_1(x, y), f_2(x, y))$ is a C^1-vector $\mathbb{C}^2$-valued function on an open set in $\mathbb{R}^2$,

$$\operatorname{div} f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}.$$

Similarly, if $f : \mathbb{R}^2 \to \mathbb{C}$ is a C^1-scalar $\mathbb{C}^2$-valued function on an open set in $\mathbb{R}^2$,

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right).$$

With these notations, the operator L_m can be written as follows:

$$L_m u(x, y) = x^{-m} \operatorname{div}(x^m \nabla u)(x, y), \qquad u \in C^2(\mathbb{H}^+).$$

By the Schwarz rule, if u is a function defined on a connected open set in $\mathbb{H}^+$ such that $\operatorname{div}(\sigma \nabla u) = 0$, where $\sigma : \mathbb{H}^+ \to \mathbb{C}^*$ is a C^1-function, then there is a function v which satisfies the well-known generalized Cauchy–Riemann system of equations

$$\begin{cases} \dfrac{\partial v}{\partial x} = -\sigma \dfrac{\partial u}{\partial y}, \\ \dfrac{\partial v}{\partial y} = \sigma \dfrac{\partial u}{\partial x} \end{cases}$$

and v satisfies the conjugate equation $\operatorname{div}(1/\sigma \cdot \nabla v) = 0$, see for example [7]. This observation justifies the fact that we call L_{-m}, $m \in \mathbb{C}$, the conjugate operator of L_m.

The adjoint operator of L_m is

$$L_m^\star u(x, y) = \Delta u(x, y) - \frac{\partial}{\partial x} \frac{m \, u(x, y)}{x} = \Delta u(x, y) - \frac{m}{x} \frac{\partial u}{\partial x}(x, y) + \frac{m}{x^2} u(x, y),$$

where $u \in C^2(\mathbb{H}^+)$, $(x, y) \in \mathbb{H}^+$. This definition is given on $\mathbb{H}^+$ but it is easily transposed to the case of an open set Ω of $\mathbb{H}^+$.

In the case where the functions involved do not depend only on x and y, we will write $L_{m,x,y}$ instead of L_m, which means that the partial derivatives are related to the variables x and y, and all other variables are considered to be fixed.

If $u \in \mathcal{D}(\mathbb{H}^+)$, we define $S_m u$, $Du \in \mathcal{D}(\mathbb{H}^+)$ as

$$(S_m u)(x, y) = x^{-m} u(x, y), \qquad (Du)(x, y) = \frac{\partial u}{\partial x}(x, y).$$

These operators satisfy the following property.

Proposition 2.1 *The operator S_m conjugates $L_m^\star$ and L_m, D conjugates $L_{-m}^\star$ and L_m, which means that*

$$S_m L_m^\star = L_m S_m, \qquad L_{-m}^\star D = D L_m.$$

Remark 2.2 1. Let $m \in \mathbb{C}$, S_m and $L_m S_m$ are self-adjoint operators, i.e. $S_m = S_m^\star$ and $L_m S_m = (L_m S_m)^\star$.
2. Let $\sigma : \Omega \to \mathbb{C}$ be a C^1-function which does not vanish, consider the operator defined on $C^2(\Omega)$ as follows:

$$P_\sigma u(x, y) = \frac{1}{\sigma(x, y)} \operatorname{div}(\sigma(x, y) \nabla u(x, y)), \qquad u \in C^2(\Omega),$$

where Ω is an open set in $\mathbb{R}^2$. Then

$$P_\sigma^\star = \operatorname{div}\left(\sigma \nabla\left(\frac{\cdot}{\sigma}\right)\right).$$

Indeed, if $u, v \in \mathcal{D}(\Omega)$, then, by using the derivation in the sense of distributions, we have

$$\langle P_\sigma u, v \rangle = \int_\Omega \frac{1}{\sigma(x, y)} \operatorname{div}\big(\sigma(x, y) \nabla u(x, y)\big) v(x, y)\, dx dy$$
$$= -\int_\Omega \sigma \nabla u \cdot \nabla\left(\frac{v}{\sigma}\right) dx dy = \int_\Omega u \operatorname{div}\left(\sigma \nabla\left(\frac{v}{\sigma}\right)\right) dx dy$$
$$= \langle u, P_\sigma^\star v \rangle.$$

Define S_σ as

$$(S_\sigma u)(x, y) = \frac{1}{\sigma(x, y)} u(x, y), \qquad u \in C^2(\Omega).$$

Then, one can easily check that $S_\sigma P_\sigma^\star = P_\sigma S_\sigma$, hence S_σ conjugates P_σ and $P_\sigma^\star$. The operators P_σ and S_σ are a generalization of L_m and S_m with the conjugation relation preserved.

If m is a positive integer, introduce an operator $T_m : u \mapsto v$ defined as follows: T_m maps a function u defined on an open set Ω of $\mathbb{H}^+$ to the function

$$v(x_1, \ldots, x_{m+2}) = u\left(\sqrt{x_1^2 + \cdots + x_{m+1}^2}, x_{m+2}\right).$$

The following two propositions can be found in the Weinstein paper [59] in the case $m \in \mathbb{R}$. They can be checked by a direct computation for m real and complex, so we omit the proofs.

Proposition 2.3 (Weinstein principle [59]) *Let Ω be a relatively compact open set in $\mathbb{H}^+$, if $u \colon \Omega \to \mathbb{C}$ is C^2, then for all $m \in \mathbb{C}$,*

$$L_m u = x^{1-m} L_{2-m}\left[x^{m-1} u\right].$$

Proposition 2.4 ([54]) *Let Ω be a relatively compact open set in $\mathbb{H}^+$. If $u \in C^2(\Omega)$ and $m \in \mathbb{N}$, then $\Delta_{m+2}(T_m u) = T_m(L_m u)$.*

These properties will allow us to calculate fundamental solutions for L_m and $L_m^\star$ for $m \in \mathbb{N}$, and, thereafter, for $m \in \mathbb{Z}$. Finally, estimates of formulas for L_m, $L_m^\star$, $m \in \mathbb{Z}$, will show that these formulas actually provide fundamental solutions for L_m and $L_m^\star$ in the case $m \in \mathbb{C}$.

3 Integral expressions of fundamental solutions for integer values of *m*

Let us recall the definition of the Dirac mass in a point $(x, y) \in \mathbb{R}^2$:

$$\langle \delta_{(x,y)}, \varphi \rangle = \varphi(x, y), \qquad \varphi \in \mathcal{D}(\mathbb{R}^2).$$

Proposition 3.1 (partially in [20,53,54]) *Let $m \in \mathbb{N}^\star$. For $(x, y) \in \mathbb{H}^+$ and $(\xi, \eta) \in \mathbb{H}^+$,*

$$E_m(x, y, \xi, \eta) = -\frac{\xi^m}{2\pi} \int_0^\pi \frac{\sin^{m-1}\theta \, d\theta}{\left[(x - \xi)^2 + 4x\xi \sin^2\theta/2 + (y - \eta)^2\right]^{m/2}}$$

is a fundamental solution on $\mathbb{H}^+$ for the operator $L_{m,\xi,\eta}^\star$ at the fixed point $(x, y) \in \mathbb{H}^+$, which means

$$L_{m,\xi,\eta}^\star E_m(x, y, \xi, \eta) = \delta_{(x,y)}(\xi, \eta)$$

in the sense of distributions. Moreover, if $(\xi, \eta) \in \mathbb{H}^+$ is fixed, then

$$L_{m,x,y} E_m(x, y, \xi, \eta) = \delta_{(\xi,\eta)}(x, y)$$

in the sense of distributions, which means that E_m is a fundamental solution on $\mathbb{H}^+$ for the operator $L_{m,x,y}$ at the fixed point $(\xi, \eta) \in \mathbb{H}^+$.

Proof Let $m \in \mathbb{N}^*$. Recall that

$$E(x) = -\frac{1}{m\,\omega_{m+2}\,\|x\|^m}, \qquad x \in \mathbb{R}^{m+2},$$

is a fundamental solution for the Laplacian on $\mathbb{R}^{m+2}$, i.e. in the sense of distributions $\Delta_{m+2}E = \delta_0$, where ω_{m+2} is the area of the unit sphere in $\mathbb{R}^{m+2}$. Thus, for all $v \in \mathcal{D}(\mathbb{R}^{m+2})$,

$$v(t_1, \ldots, t_{m+2}) = -\frac{1}{m\,\omega_{m+2}} \int_{\mathbb{R}^{m+2}} \frac{\Delta_{m+2}v(\tau)\,d\tau_1 d\tau_2 \ldots d\tau_{m+2}}{\left((\tau_1 - t_1)^2 + \cdots + (\tau_{m+2} - t_{m+2})^2\right)^{m/2}},$$

where $\tau = (\tau_1, \ldots, \tau_{m+2})$.

Applying this relation to the function $v = T_m u$, where $u \in \mathcal{D}(\mathbb{H}^+)$, and, due to Proposition 2.4, for all $(x, y) \in \mathbb{H}^+$ we have

$$u(x, y) = -\frac{1}{m\,\omega_{m+2}} \int_{\mathbb{R}^{m+2}} \frac{(L_m u)\left(\sqrt{\xi_1^2 + \cdots + \xi_{m+1}^2}, \xi_{m+2}\right) d\xi_1 \ldots d\xi_{m+2}}{\left((\xi_1 - x)^2 + \xi_2^2 + \cdots + \xi_{m+1}^2 + (\xi_{m+2} - y)^2\right)^{m/2}}.$$

We will simplify this integral expression. For this, we will consider the following hyper-spherical coordinates:

$$\begin{aligned}
\xi_1 &= \xi \cos\theta_1, \\
\xi_2 &= \xi \sin\theta_1 \cos\theta_2, \\
&\cdots \\
\xi_{m-1} &= \xi \sin\theta_1 \cdots \sin\theta_{m-2} \cos\theta_{m-1}, \\
\xi_m &= \xi \sin\theta_1 \cdots \sin\theta_{m-1} \cos\theta_m, \\
\xi_{m+1} &= \xi \sin\theta_1 \cdots \sin\theta_m,
\end{aligned}$$

where $\xi = \sqrt{\xi_1^2 + \cdots + \xi_{m+1}^2} \geq 0$, $\theta_m \in (-\pi, \pi)$ and $\theta_1, \ldots, \theta_{m-1} \in (0, \pi)$. The absolute value of the determinant of the Jacobian matrix defined by this system of coordinates is

$$\xi^m \sin\theta_{m-1} \sin^2\theta_{m-2} \cdots \sin^{m-1}\theta_1.$$

Then, for all $(x, y) \in \mathbb{H}^+$,

$$u(x, y) = \int_{-\infty}^{\infty} d\eta \int_0^{\infty} L_m(u)(\xi, \eta)\, E_m(x, y, \xi, \eta)\, d\xi \tag{2}$$

with

$$E_m(x, y, \xi, \eta)$$
$$= -\frac{\xi^m}{m\,\omega_{m+2}} \int_{-\pi}^{\pi} d\theta_m \int_{\theta_1,\ldots,\theta_{m-1}=0}^{\pi} \frac{\sin\theta_{m-1}\sin^2\theta_{m-2}\cdots\sin^{m-1}\theta_1\,d\theta_1\ldots d\theta_{m-1}}{\left(\xi^2 - 2x\xi\cos\theta_1 + x^2 + (y-\eta)^2\right)^{m/2}}.$$

Let

$$\omega_m = \int_{-\pi}^{\pi} d\theta_{m-1} \int_{\theta_1,\ldots,\theta_{m-2}=0}^{\pi} \sin\theta_{m-2}\sin^2\theta_{m-3}\cdots\sin^{m-2}\theta_1\,d\theta_2\ldots d\theta_{m-2}$$
$$= \int_{\mathbb{S}_m} 1\,d\sigma = \frac{2\pi^{m/2}}{\Gamma(m/2)},$$

then E_m can be written as

$$E_m(x, y, \xi, \eta) = -\frac{\omega_m\xi^m}{m\,\omega_{m+2}} \int_0^{\pi} \frac{\sin^{m-1}\theta\,d\theta}{\left(\xi^2 - 2x\xi\cos\theta + x^2 + (y-\eta)^2\right)^{m/2}}$$
$$= -\frac{\xi^m}{2\pi} \int_0^{\pi} \frac{\sin^{m-1}\theta\,d\theta}{\left((x-\xi)^2 + 4x\xi\sin^2\theta/2 + (y-\eta)^2\right)^{m/2}}.$$

Also, due to (2) we have

$$L_{m,\xi,\eta}^* E_m(x, y, \xi, \eta) = \delta_{(x,y)}(\xi, \eta).$$

Moreover, since for all $(x, y), (\xi, \eta) \in \mathbb{H}^+$,

$$E_m(x, y, \xi, \eta) = \left(\frac{x}{\xi}\right)^{-m} E_m(\xi, \eta, x, y),$$

and by Proposition 2.1, S_m conjugates $L_m^\star$ and L_m, we have

$$L_{m,x,y} E_m(x, y, \xi, \eta) = L_{m,x,y}\left(\left(\frac{x}{\xi}\right)^{-m} E_m(\xi, \eta, x, y)\right)$$
$$= \left(\frac{x}{\xi}\right)^{-m} L_{m,x,y}^\star E_m(\xi, \eta, x, y)$$

in the sense of distributions. Hence

$$L_{m,x,y} E_m(x, y, \xi, \eta) = \left(\frac{x}{\xi}\right)^{-m} \delta_{(\xi,\eta)}(x, y) = \delta_{(\xi,\eta)},$$

and the proof is complete. $\qquad\square$

This proposition and the Weinstein principle imply the following result.

Proposition 3.2 (partially in [20,53,54]) *Let* $m \in \mathbb{Z} \setminus \mathbb{N}^*$. *For* (x, y), $(\xi, \eta) \in \mathbb{H}^+$,

$$E_m(x, y, \xi, \eta) = \left(\frac{\xi}{x}\right)^{m-1} E_{2-m}(x, y, \xi, \eta)$$

$$= -\frac{\xi x^{1-m}}{2\pi} \int_0^\pi \frac{\sin^{1-m} \theta \, d\theta}{\left[(x-\xi)^2 + 4x\xi \sin^2 \theta/2 + (y-\eta)^2\right]^{1-m/2}}$$

is a fundamental solution on $\mathbb{H}^+$ *for the operator* $L^\star_{m,\xi,\eta}$ *at the fixed point* $(x, y) \in \mathbb{H}^+$ *and it is also a fundamental solution on* $\mathbb{H}^+$ *for the operator* $L_{m,x,y}$ *at the fixed point* $(\xi, \eta) \in \mathbb{H}^+$.

Proof For all $m \in \mathbb{N}^*$, $u \in \mathcal{D}(\mathbb{H}^+)$ and $(x, y) \in \mathbb{H}^+$ we have

$$u(x, y) = \int_{(\xi,\eta)\in\mathbb{H}^+} (L_m u) E_m(x, y, \xi, \eta) \, d\xi d\eta,$$

and by the Weinstein principle, Proposition 2.3, we have

$$u(x, y) = \int_{\mathbb{H}^+} \xi^{1-m} L_{2-m}(\xi^{m-1} u) E_m(x, y, \xi, \eta) \, d\xi d\eta.$$

Denoting $v(x, y) = x^{m-1} u(x, y)$, we obtain

$$x^{1-m} v(x, y) = \int_{\mathbb{H}^+} \xi^{1-m} (L_{2-m} v) E_m(x, y, \xi, \eta) \, d\xi d\eta,$$

then, for all $m' \in \mathbb{Z} \setminus \mathbb{N}^*$, $v \in \mathcal{D}(\mathbb{H}^+)$ and $(x, y) \in \mathbb{H}^+$, putting $m = 2 - m'$, we have

$$v(x, y) = \int_{\mathbb{H}^+} (L_{m'} v) \left(\frac{\xi}{x}\right)^{m'-1} E_{2-m'}(x, y, \xi, \eta) \, d\xi d\eta.$$

The proof of the second statement is similar. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4 Fundamental solutions for the Weinstein equation with complex coefficients

In this section, we will generalize the result obtained in the previous section for $m \in \mathbb{Z}$ to the case $m \in \mathbb{C}$.

Let $m \in \mathbb{C}$. If $\operatorname{Re} m \geq 1$ put

$$E_m = -\frac{\xi^m}{2\pi} \int_0^\pi \frac{\sin^{m-1} \theta \, d\theta}{\left[(x-\xi)^2 + 4x\xi \sin^2 \theta/2 + (y-\eta)^2\right]^{m/2}}, \qquad (3)$$

and if $\operatorname{Re} m < 1$ put

$$E_m = -\frac{\xi x^{1-m}}{2\pi} \int_0^\pi \frac{\sin^{1-m}\theta \, d\theta}{\left[(x-\xi)^2 + 4x\xi \sin^2\theta/2 + (y-\eta)^2\right]^{1-m/2}}, \qquad (4)$$

here, if $\alpha > 0$ is a real number and μ is a complex number, $\alpha^\mu = \exp(\mu \ln \alpha)$. Both values are well defined as the integrals on the right-hand side converge in the Lebesgue sense.

Proposition 4.1 *For $m \in \mathbb{C}$ and $(\xi, \eta) \in \mathbb{H}^+$ fixed, we have*

$$L_{m,x,y} E_m(x, y, \xi, \eta) = 0, \qquad (x, y) \in \mathbb{H}^+ \setminus \{(\xi, \eta)\},$$

and for $(x, y) \in \mathbb{H}^+$ fixed, we have

$$L^\star_{m,\xi,\eta} E_m(x, y, \xi, \eta) = 0, \qquad (\xi, \eta) \in \mathbb{H}^+ \setminus \{(x, y)\}.$$

Proof For convenience, denote

$$f_m(x, y, \xi, \eta, \theta) = \frac{1}{\left[(x-\xi)^2 + 4x\xi \sin^2\theta/2 + (y-\eta)^2\right]^{m/2}}.$$

To prove the first equality of the proposition, it suffices to show that

$$\int_0^\pi L_{m,x,y} f_m(x, y, \xi, \eta, \theta) \sin^{m-1}\theta \, d\theta = 0.$$

Let us compute the partial derivatives of the function f_m:

$$\partial_x f_m = \frac{-m}{2} \frac{2(x-\xi) + 4\xi \sin^2\theta/2}{\left[(x-\xi)^2 + 4x\xi \sin^2\theta/2 + (y-\eta)^2\right]^{m/2+1}}$$
$$(= -m(x - \xi \cos\theta) f_{m+2}),$$

$$\partial_{xx} f_m = \frac{-m}{\left[(x-\xi)^2 + 4x\xi \sin^2\theta/2 + (y-\eta)^2\right]^{m/2+1}}$$
$$+ \frac{m}{2}\left(\frac{m}{2}+1\right) \frac{(2(x-\xi) + 4\xi \sin^2\theta/2)^2}{\left[(x-\xi)^2 + 4x\xi \sin^2\theta/2 + (y-\eta)^2\right]^{m/2+2}},$$

$$\partial_{yy} f_m = \frac{-m}{\left[(x-\xi)^2 + 4x\xi \sin^2\theta/2 + (y-\eta)^2\right]^{m/2+1}}$$
$$+ \frac{m}{2}\left(\frac{m}{2}+1\right) \frac{4(y-\eta)^2}{\left[(x-\xi)^2 + 4x\xi \sin^2\theta/2 + (y-\eta)^2\right]^{m/2+2}}.$$

We then have

$$\Delta f_m = \frac{-2m}{\left[(x-\xi)^2 + 4x\xi \sin^2 \theta/2 + (y-\eta)^2\right]^{m/2+1}}$$
$$+ \frac{m}{2}\left(\frac{m}{2}+1\right)\frac{\left(2(x-\xi)+4\xi \sin^2 \theta/2\right)^2 + 4(y-\eta)^2}{\left[(x-\xi)^2 + 4x\xi \sin^2 \theta/2 + (y-\eta)^2\right]^{m/2+2}}.$$

Note that

$$\left(2(x-\xi)+4\xi \sin^2 \frac{\theta}{2}\right)^2 + (2(y-\eta))^2$$
$$= 4\left[(x-\xi)^2 + 4x\xi \sin^2 \frac{\theta}{2} + (y-\eta)^2\right] - 4\xi^2 \sin^2 \theta,$$

hence

$$\Delta f_m = \frac{m^2}{\left[(x-\xi)^2 + 4x\xi \sin^2 \theta/2 + (y-\eta)^2\right]^{m/2+1}}$$
$$- \frac{m(m+2)\xi^2 \sin^2 \theta}{\left[(x-\xi)^2 + 4x\xi \sin^2 \theta/2 + (y-\eta)^2\right]^{m/2+2}}.$$

Noting that

$$\frac{\partial f_{m+2}}{\partial \theta} = -(m+2)\frac{x\xi \sin \theta}{\left[(x-\xi)^2 + 4x\xi \sin^2 \theta/2 + (y-\eta)^2\right]^{m/2+2}},$$

we have

$$\Delta f_m = m^2 f_{m+2} + m\frac{\xi}{x}\sin \theta \frac{\partial f_{m+2}}{\partial \theta}.$$

Integrating by parts, we have

$$\int_0^\pi \Delta f_m \sin^{m-1} \theta \, d\theta = m^2 \int_0^\pi f_{m+2} \sin^{m-1} \theta \, d\theta + m\frac{\xi}{x}\int_0^\pi \frac{\partial f_{m+2}}{\partial \theta}\sin^m \theta \, d\theta$$
$$= \frac{m}{x}\int_0^\pi m(x-\xi \cos \theta) f_{m+2} \sin^{m-1} \theta \, d\theta$$
$$= -\frac{m}{x}\int_0^\pi \partial_x f_m \sin^{m-1} \theta \, d\theta,$$

and the result is deduced in the case $\operatorname{Re} m \geq 1$. The same argument is valid for $\operatorname{Re} m < 1$. The second equality of the proposition can be deduced immediately from the fact that S_m conjugates $L_m^\star$ and L_m, see Proposition 2.1. □

In the sequel, we will denote

$$d^2 = (x - \xi)^2 + (y - \eta)^2, \qquad k = \frac{4x\xi}{d^2}.$$

The following proposition describes the behavior of E_m defined by (3) and (4) near its singularity. In particular, we show that the behavior of E_m is close to the behavior of fundamental solutions for the Laplacian. This fact is well known for elliptic operators. But we emphasize here that in our proof the estimates of elliptic integrals are elementary (obtained using the dominated convergence theorem) and we do not use estimates arising from classical estimates of hypergeometric functions.

Proposition 4.2 *Let $m \in \mathbb{C}$. For $(x, y) \in \mathbb{H}^+$ fixed,*

$$E_m(x, y, \xi, \eta) \sim \frac{1}{2\pi} \ln \sqrt{(x - \xi)^2 + (y - \eta)^2} \qquad as \ (\xi, \eta) \to (x, y).$$

Proof We start with $\mathrm{Re}\, m \geq 1$. In this case, we have

$$E_m(x, y, \xi, \eta) = -\frac{\xi^m}{2\pi} \int_0^\pi \frac{\sin^{m-1}\theta \, d\theta}{\left[(x - \xi)^2 + 4x\xi \sin^2\theta/2 + (y - \eta)^2\right]^{m/2}}$$

$$= -\frac{1}{2\pi} \left(\frac{\xi}{d}\right)^m \int_0^\pi \frac{\sin^{m-1}\theta \, d\theta}{(1 + k\sin^2\theta/2)^{m/2}}.$$

Note that when $d \to 0$, k tends to $+\infty$.

Claim 4.3

$$\int_0^\pi \frac{\sin^{m-1}\theta \, d\theta}{(1 + k\sin^2\theta/2)^{m/2}} \sim \frac{2^{m-1}}{k^{m/2}} \ln k \qquad as \ k \to +\infty.$$

Proof Putting $u = \sin\theta/2$, we have

$$\int_0^\pi \frac{\sin^{m-1}\theta \, d\theta}{(1 + k\sin^2\theta/2)^{m/2}} = 2^m \int_0^1 \frac{u^{m-1}(1 - u^2)^{(m-2)/2} du}{(1 + ku^2)^{m/2}}$$

$$= \frac{2^m}{k^{m/2}} \int_0^1 \frac{u^{m-1}(1 - u^2)^{(m-2)/2} du}{(1/k + u^2)^{m/2}}.$$

Note that

$$\int_0^1 \frac{u^{m-1}(1 - u^2)^{(m-2)/2} du}{(1/k + u^2)^{m/2}} - \int_0^1 \frac{u^{m-1} du}{(1/k + u^2)^{m/2}}$$

$$= -\int_0^1 \frac{u^{m-1}}{(1/k + u^2)^{m/2}} \left(1 - (1 - u^2)^{(m-2)/2}\right) du$$

where, due to monotone convergence, the right-hand side tends to

$$-\int_0^1 \frac{u^{m-1}}{(u^2)^{m/2}} \left(1 - (1 - u^2)^{(m-2)/2}\right) du = -\int_0^1 \frac{1 - (1 - u^2)^{(m-2)/2}}{u} \, du$$

as $k \to +\infty$. The change of variable $u = \operatorname{sh} t / \sqrt{k}$ gives us

$$\int_0^1 \frac{u^{m-1} \, du}{(1/k + u^2)^{m/2}} = \int_0^{\operatorname{argsh} \sqrt{k}} \operatorname{th}^{m-1} t \, dt.$$

Since $\operatorname{th}^{m-1} t$ tends to 1 as $t \to +\infty$, we deduce that as $k \to +\infty$,

$$\int_0^{\operatorname{argsh} \sqrt{k}} \operatorname{th}^{m-1} dt \sim \int_0^{\operatorname{argsh} \sqrt{k}} dt = \operatorname{argsh} \sqrt{k} \sim \frac{1}{2} \ln k.$$

The proof is complete.   ∎

Due to Claim 4.3, we have

$$E_m(x, y, \xi, \eta) \sim -\frac{1}{2\pi} \left(\frac{x}{d}\right)^m \frac{2^{m-1}}{k^{m/2}} \ln k \sim \frac{1}{2\pi} \ln d$$

as $d \to 0+$. The case $\operatorname{Re} m < 1$ is analogous.   □

Now, we can prove the main result of this section which shows that E_m are fundamental solutions not only for $m \in \mathbb{N}$ but for all $m \in \mathbb{C}$.

Theorem 4.4 *Let $m \in \mathbb{C}$. For $(x, y), (\xi, \eta) \in \mathbb{H}^+$, E_m defined by (3) and (4) is a fundamental solution on $\mathbb{H}^+$ for the operator $L^{\star}_{m,\xi,\eta}$ at the fixed point $(x, y) \in \mathbb{H}^+$, which means that on $\mathbb{H}^+$*

$$L^{\star}_{m,\xi,\eta} E_m(x, y, \xi, \eta) = \delta_{(x,y)}(\xi, \eta)$$

in the sense of distributions. Moreover, if $(\xi, \eta) \in \mathbb{H}^+$ is fixed, then on $\mathbb{H}^+$

$$L_{m,x,y} E_m(x, y, \xi, \eta) = \delta_{(\xi,\eta)}(x, y)$$

in the sense of distributions, which means that E_m is a fundamental solution on $\mathbb{H}^+$ for the operator $L_{m,x,y}$ at the fixed point $(\xi, \eta) \in \mathbb{H}^+$.

Proof Let $m \in \mathbb{C}$ and $u \in \mathcal{D}(\mathbb{H}^+)$. Let $(x, y) \in \mathbb{H}^+$ and $\varepsilon > 0$ be such that $D((x, y), \varepsilon) \subset \mathbb{H}^+$, where $D((x, y), \varepsilon)$ is a disk with center (x, y) and radius ε. Put

$$I_\varepsilon = \int_{\mathbb{H}^+ \backslash D((x,y),\varepsilon)} L_m(u)(\xi, \eta) \, E_m(x, y, \xi, \eta) \, d\xi d\eta$$

$$= \int_{\mathbb{H}^+ \backslash D((x,y),\varepsilon)} \left(L_m(u)(\xi, \eta) \, E_m(x, y, \xi, \eta) - u(\xi, \eta) \, L^{\star}_m(E_m)(x, y, \xi, \eta)\right) d\xi d\eta.$$

We use the fact that $L_m^\star(E_m) = 0$ on $\mathbb{H}^+ \setminus D((x, y), \varepsilon)$. An elementary computation gives

$$L_m(u)\,E_m - u L_m^\star(E) = \partial_\xi\left((\partial_\xi u)\,E_m - u(\partial_\xi E_m) + \frac{m}{\xi}\,u E_m\right)$$
$$+ \partial_\eta\big((\partial_\eta u)\,E_m - u(\partial_\eta E_m)\big).$$

Let us recall the Green formula: If Ω is an open set in $\mathbb{R}^2$ with a piecewise C^1-differentiable boundary, then

$$\int_\Omega \operatorname{div} X(x, y)\,dxdy = \int_{\partial\Omega} X(x, y)\cdot\mathbf{n}(x, y)\,ds,$$

where $\mathbf{n}$ is the outer unit normal vector to $\partial\Omega$ and ds the arc length element on $\partial\Omega$ (positively oriented), $X = (X_1, X_2)\colon \overline{\Omega} \to \mathbb{C}^2$ is a C^1-vector field.

Applying this formula to the open set $\Omega = U \setminus D((x, y), \varepsilon)$, where U is a regular open set in $\mathbb{H}^+$ containing the support of u, we have

$$I_\varepsilon = -\int_{\substack{t\in[0,2\pi]\\ (\xi,\eta)=(x,y)+\varepsilon(\cos t,\sin t)}} \left(\left((\partial_\xi u)\,E_m - u(\partial_\xi E_m) + \frac{m}{\xi}\,u E_m\right)\cos t\right.$$
$$\left.+ \big((\partial_\eta u)\,E_m - u(\partial_\eta E_m)\big)\sin t\right)\varepsilon\,dt.$$

Proposition 4.2 implies

$$\int_{\substack{t\in[0,2\pi]\\ (\xi,\eta)=(x,y)+\varepsilon(\cos t,\sin t)}} \left(\left((\partial_\xi u) + \frac{m}{\xi}\,u\right)\cos t + (\partial_\eta u)\sin t\right)E_m\,\varepsilon\,dt \to 0$$

as $\varepsilon \to 0+$ because $\lim_{\varepsilon\to 0}\varepsilon\ln\varepsilon = 0$. Then, if we want to prove that $\lim_{\varepsilon\to 0} I_\varepsilon$ exists, we have to prove the existence of

$$\lim_{\varepsilon\to 0}\int_{\substack{t\in[0,2\pi]\\ (\xi,\eta)=(x,y)+\varepsilon(\cos t,\sin t)}} u\big((\partial_\xi E_m)\cos t + (\partial_\eta E_m)\sin t\big)\varepsilon\,dt,$$

and this limit will be equal to the limit of I_ε.

Now, assume that $\mathrm{Re}\, m \geq 1$. Denote by J_ε the integral in the previous expression. A computation gives

$$
\mathrm{J}_\varepsilon = -\frac{m}{2\pi} \underbrace{\int_{\substack{t\in[0,2\pi] \\ (\xi,\eta)=(x,y)+\varepsilon(\cos t,\sin t)}} u\,\frac{\xi^{m-1}}{\varepsilon^m} \int_0^\pi \frac{\sin^{m-1}\theta\,d\theta}{(1+k\sin^2\theta/2)^{m/2}}\, \varepsilon\cos t\,dt}_{\mathrm{J}_{\varepsilon,1}}
$$

$$
+\frac{m}{2\pi} \underbrace{\int_{\substack{t\in[0,2\pi] \\ (\xi,\eta)=(x,y)+\varepsilon(\cos t,\sin t)}} u\,\frac{\xi^m}{\varepsilon^{m+2}} \int_0^\pi \frac{\sin^{m-1}\theta\,d\theta}{(1+k\sin^2\theta/2)^{m/2+1}}\, \varepsilon^2\,dt}_{\mathrm{J}_{\varepsilon,2}}
$$

$$
+\frac{m}{2\pi} \underbrace{\int_{\substack{t\in[0,2\pi] \\ (\xi,\eta)=(x,y)+\varepsilon(\cos t,\sin t)}} u\,\frac{\xi^m}{\varepsilon^{m+2}} \int_0^\pi \frac{2x\sin^2\theta/2\,\sin^{m-1}\theta\,d\theta}{(1+k\sin^2\theta/2)^{m/2+1}}\, \varepsilon\cos t\,dt}_{\mathrm{J}_{\varepsilon,3}},
$$

where $k = 4x\xi/\varepsilon^2$.

Claim 4.5

$$
\int_0^\pi \frac{\sin^2\theta/2\,\sin^{m-1}\theta\,d\theta}{(1+k\sin^2\theta/2)^{m/2+1}} \sim \frac{2^{m-1}}{k^{m/2+1}}\ln k \qquad \text{as}\quad k\to+\infty.
$$

Proof We put $u = \sin\theta/2$, then

$$
\int_0^\pi \frac{\sin^2\theta/2\,\sin^{m-1}\theta\,d\theta}{(1+k\sin^2\theta/2)^{m/2+1}} = 2^m \int_0^1 \frac{u^{m+1}(1-u^2)^{(m-2)/2}du}{(1+ku^2)^{m/2+1}}
$$

$$
= \frac{2^m}{k^{m/2+1}} \int_0^1 \frac{u^{m+1}(1-u^2)^{(m-2)/2}du}{(1/k+u^2)^{m/2+1}}.
$$

Note that

$$
\int_0^1 \frac{u^{m+1}(1-u^2)^{(m-2)/2}du}{(1/k+u^2)^{m/2+1}} - \int_0^1 \frac{u^{m+1}\,du}{(1/k+u^2)^{m/2+1}}
$$

$$
= -\int_0^1 \frac{u^{m+1}}{(1/k+u^2)^{m/2+1}}\left(1-(1-u^2)^{(m-2)/2}\right)du
$$

$$
\to -\int_0^1 \frac{u^{m+1}}{(u^2)^{m/2+1}}\left(1-(1-u^2)^{(m-2)/2}\right)du
$$

$$
= -\int_0^1 \frac{1-(1-u^2)^{(m-2)/2}}{u}\,du
$$

as $k \to +\infty$. The change of variable $u = \mathrm{sh}\, t/\sqrt{k}$ gives us

$$\int_0^1 \frac{u^{m+1}\, du}{(1/k + u^2)^{m/2+1}} = \int_0^{\mathrm{argsh}\sqrt{k}} \mathrm{th}^{m+1} t\, dt.$$

Since $\mathrm{th}^{m+1} t$ tends to 1 as $t \to +\infty$, it follows that as $k \to +\infty$

$$\int_0^{\mathrm{argsh}\sqrt{k}} \mathrm{th}^{m+1} t\, dt \sim \int_0^{\mathrm{argsh}\sqrt{k}} dt = \mathrm{argsh}\sqrt{k} \sim \frac{1}{2}\ln k. \qquad \blacksquare$$

Claim 4.6

$$\int_0^\pi \frac{\sin^{m-1}\theta\, d\theta}{(1 + k\sin^2\theta/2)^{m/2+1}} \sim \frac{2^m}{mk^{m/2}} \qquad \text{as } k \to +\infty.$$

Proof Putting as previously $u = \sin\theta/2$, we have

$$\int_0^\pi \frac{\sin^{m-1}\theta\, d\theta}{(1 + k\sin^2\theta/2)^{m/2+1}} = 2^m \int_0^1 \frac{u^{m-1}(1-u^2)^{(m-2)/2}\, du}{(1+ku^2)^{m/2+1}}$$

$$= \frac{2^m}{k^{m/2+1}} \int_0^1 \frac{u^{m-1}(1-u^2)^{(m-2)/2}\, du}{(1/k + u^2)^{m/2+1}}.$$

Note that

$$\int_0^1 \frac{u^{m-1}(1-u^2)^{(m-2)/2}\, du}{(1/k + u^2)^{m/2+1}} - \int_0^1 \frac{u^{m-1}\, du}{(1/k + u^2)^{m/2+1}}$$

$$= -\int_0^1 \frac{u^{m-1}\left(1 - (1-u^2)^{(m-2)/2}\right)}{(1/k + u^2)^{m/2+1}}\, du.$$

Let us estimate the right-hand side of this equality:

$$\int_0^1 \frac{u^{m-1}\left(1 - (1-u^2)^{(m-2)/2}\right)\, du}{(1/k + u^2)^{m/2+1}} - \int_0^1 \frac{u^{m-1}}{(1/k + u^2)^{m/2+1}}\left(\frac{m-2}{2}u^2\right) du$$

$$= \int_0^1 \frac{u^{m-1}}{(1/k + u^2)^{m/2+1}}\left(1 - \frac{m-2}{2}u^2 - (1-u^2)^{(m-2)/2}\right) du$$

$$\to \int_0^1 \frac{u^{m-1}}{(u^2)^{m/2+1}}\left(1 - \frac{m-2}{2}u^2 - (1-u^2)^{(m-2)/2}\right) du \qquad (\star)$$

$$= \int_0^1 \frac{1 - (m-2)u^2/2 - (1-u^2)^{(m-2)/2}}{u^3}\, du$$

as $k \to +\infty$. As seen in the proof of Claim 4.5, we have

$$\frac{m-2}{2}\int_0^1 \frac{u^{m+1}}{(1/k + u^2)^{m/2+1}}\, du \sim \frac{m-2}{4}\ln k \qquad (\star\star)$$

as $k \to +\infty$. Due to $(\star)$ and $(\star\star)$, we have

$$\int_0^1 \frac{u^{m-1}}{(1/k + u^2)^{m/2+1}} \left(1 - (1 - u^2)^{(m-2)/2}\right) du \sim \frac{m-2}{4} \ln k$$

as $k \to +\infty$. The change of variable $u = \operatorname{sh} t / \sqrt{k}$ gives

$$\int_0^1 \frac{u^{m-1} du}{(1/k + u^2)^{m/2+1}} = k \int_0^{\operatorname{argsh} \sqrt{k}} \frac{\operatorname{th}^{m-1} t}{\operatorname{ch}^2 t} dt = \frac{k}{m} \operatorname{th}^m \left(\operatorname{argsh} \sqrt{k}\right).$$

It follows that as $k \to +\infty$,

$$\int_0^1 \frac{u^{m-1} du}{(1/k + u^2)^{m/2+1}} \sim \frac{k}{m}.$$

Thus

$$\int_0^1 \frac{u^{m-1}(1 - u^2)^{(m-2)/2} du}{(1/k + u^2)^{m/2+1}} \sim \frac{k}{m}$$

and

$$\int_0^\pi \frac{\sin^{m-1} \theta \, d\theta}{(1 + k \sin^2 \theta/2)^{m/2+1}} \sim \frac{2^m}{mk^{m/2}}$$

as $k \to +\infty$. This completes the proof. ∎

Let us return to the proof of Theorem 4.4. Claim 4.3 implies

$$J_{\varepsilon,1} \sim -\frac{m}{2\pi} \int_{\substack{t \in [0,2\pi] \\ (\xi,\eta)=(x,y)+\varepsilon(\cos t,\sin t)}} u \frac{x^{m-1}}{\varepsilon^m} \frac{2^{m-1}}{k^{m/2}} \ln k \, \varepsilon \cos t \, dt$$

$$\sim +\frac{m}{2\pi x} \varepsilon \ln \varepsilon \int_{\substack{t \in [0,2\pi] \\ (\xi,\eta)=(x,y)+\varepsilon(\cos t,\sin t)}} u(x+\varepsilon \cos t, y+\varepsilon \sin t) \cos t \, dt$$

which tends to 0 as $\varepsilon \to 0+$.

Claim 4.5 implies

$$J_{\varepsilon,3} \sim \frac{m}{2\pi} \int_{\substack{t \in [0,2\pi] \\ (\xi,\eta)=(x,y)+\varepsilon(\cos t,\sin t)}} u \frac{x^m}{\varepsilon^{m+2}} 2x \frac{2^{m-1}}{k^{m/2+1}} \ln k \, \varepsilon \cos t \, dt$$

$$\sim -\frac{m}{4\pi x} \varepsilon \ln \varepsilon \int_{\substack{t \in [0,2\pi] \\ (\xi,\eta)=(x,y)+\varepsilon(\cos t,\sin t)}} u(x+\varepsilon \cos t, y+\varepsilon \sin t) \cos t \, dt$$

which tends to 0 as $\varepsilon \to 0+$.

Finally, Claim 4.6 implies

$$
\begin{aligned}
J_{\varepsilon,2} &\sim \frac{m}{2\pi} \int_{\substack{t\in[0,2\pi]\\ (\xi,\eta)=(x,y)+\varepsilon(\cos t,\sin t)}} u\, \frac{x^m}{\varepsilon^{m+2}}\, \frac{2^m}{mk^{m/2}}\, \varepsilon^2\, dt \\
&\sim \frac{1}{2\pi} \int_{\substack{t\in[0,2\pi]\\ (\xi,\eta)=(x,y)+\varepsilon(\cos t,\sin t)}} u(x+\varepsilon\cos t,\, y+\varepsilon\sin t)\, dt
\end{aligned}
$$

which tends to $u(x, y)$ as $\varepsilon \to 0+$.

So we have proved that for all $m \in \mathbb{C}$ such that $\operatorname{Re} m > 0$,

$$
\begin{aligned}
\lim_{\varepsilon\to 0+} \int_{\mathbb{H}^+\setminus D((x,y),\varepsilon)} L_m(u)(\xi,\eta)\, E_m(x,y,\xi,\eta)\, d\xi d\eta \\
= \int_{\mathbb{H}^+} L_m(u)(\xi,\eta)\, E_m(x,y,\xi,\eta)\, d\xi d\eta = u(x,y),
\end{aligned}
$$

therefore E_m indeed is a fundamental solution for the operator $L_m^\star$ for all $m \in \mathbb{C}$ with $\operatorname{Re} m > 0$. The case $m \in \mathbb{C}$ with $\operatorname{Re} m \leq 1$ is similar.

Due to Proposition 2.1, we also have dual assertions for fundamental solutions for the operator L_m. □

The following proposition is roughly a consequence of the previous theorem. Of course, it is a classical statement, but we would like to present its short proof.

Proposition 4.7 *Let $m \in \mathbb{C}$ and let Ω be a relatively compact open set in $\mathbb{H}^+$ whose boundary is piecewise C^1-differentiable. Then, for $(x, y) \in \Omega$ and $u \in C^2(\overline{\Omega})$, we have*

$$
\begin{aligned}
u(x,y) = \int_\Omega L_m(u)\, E_m\, d\xi d\eta \\
- \int_{\partial\Omega} \left[(\partial_\xi u)\, E_m - u(\partial_\xi E_m) + \frac{m}{\xi}\, u E_m,\ (\partial_\eta u)\, E_m - u(\partial_\eta E_m) \right] \cdot \mathbf{n}\, ds,
\end{aligned}
$$

where $u = u(\xi, \eta)$, $E_m = E_m(x, y, \xi, \eta)$, $\mathbf{n}$ is the outer unit normal vector to $\partial\Omega$ and ds is the arc length element on $\partial\Omega$ (positively oriented).

Proof Indeed, if $u \in C^2(\overline{\Omega})$, for $(x, y) \in \Omega$ and $\varepsilon > 0$ such that $\overline{D((x, y), \varepsilon)} \subset \Omega$, we have

$$
\int_{\Omega\setminus D((x,y),\varepsilon)} L_m(u)\, E_m d\xi\, d\eta = \int_{\Omega\setminus D((x,y),\varepsilon)} \left(L_m(u)\, E_m - L_m^\star(E_m)u \right) d\xi d\eta.
$$

By the Green formula, the latter integral is equal to

$$\int_{\partial\Omega}\left[(\partial_\xi u)\,E_m - u(\partial_\xi\,E_m) + \frac{m}{\xi}\,u E_m,\ (\partial_\eta u)\,E_m - u(\partial_\eta\,E_m)\right]\cdot\mathbf{n}\,ds$$

$$-\int_{\substack{t\in[0,2\pi]\\(\xi,\eta)=(x,y)+\varepsilon(\cos t,\sin t)}}\left(\left((\partial_\xi u)\,E_m - u(\partial_\xi\,E_m) + \frac{m}{\xi}\,u E_m\right)\cos t\right.$$

$$\left.+\ \big((\partial_\eta u)\,E_m - u(\partial_\eta\,E_m)\big)\sin t\right)\varepsilon\,dt,$$

and, as we saw in the previous proof, it tends to

$$\int_{\partial\Omega}\left[(\partial_\xi u)\,E_m - u(\partial_\xi\,E_m) + \frac{m}{\xi}\,u E_m,\ (\partial_\eta u)\,E_m - u(\partial_\eta\,E_m)\right]\cdot\mathbf{n}\,ds + u(x,y),$$

as $\varepsilon\to 0$. Due to integrability of E_m near (x,y) we have

$$\lim_{\varepsilon\to 0}\int_{\Omega\setminus D((x,y),\varepsilon)} L_m(u)\,E_m\,d\xi d\eta = \int_\Omega L_m(u)\,E_m\,d\xi d\eta,$$

and the proof is complete.   □

5 Liouville-type result and decomposition theorem for axisymmetric potentials

In the previous section, we have seen that fundamental solutions E_m in the complex case have different expressions depending on whether $\operatorname{Re} m < 1$ or $\operatorname{Re} m \geq 1$. Hence the behavior of E_m will be different in each case.

We will modify fundamental solutions so that they vanish at the boundary of $\mathbb{H}^+$, which means that they tend to zero on the y-axis and at infinity. Expression (4) satisfies this property: $E_m(x, y, \cdot, \cdot)$ tends to 0 as $x \to 0+$ and $\|(x, y)\| \to +\infty$; whereas (3) does not. Consider

$$E_m(x, y, \xi, \eta) - E_m(-x, y, \xi, \eta), \qquad \operatorname{Re} m \geq 1.$$

It is also a fundamental solution on $\mathbb{H}^+$ and it satisfies the required property. Let us put

- for $\operatorname{Re} m < 1$,

$$F_m(x, y, \xi, \eta) = E_m(x, y, \xi, \eta),$$

- for $\operatorname{Re} m \geq 1$,

$$F_m(x, y, \xi, \eta) = E_m(x, y, \xi, \eta) - E_m(-x, y, \xi, \eta).$$

We will need the following definition of convergence on the boundary of $\mathbb{H}^+$.

Definition 5.1 Let $u \colon \mathbb{H}^+ \to \mathbb{R}$ be a function defined on $\mathbb{H}^+$. We write

$$\lim_{\partial \mathbb{H}^+} u = 0$$

if and only if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$ and all $(x, y) \in H^+$, $x \leq 1/n$ or $\|(x, y)\| \geq n$ implies $|u(x, y)| \leq \varepsilon$.

Proposition 5.2 Let $u \colon \mathbb{H}^+ \to \mathbb{C}$. We have $\lim_{\partial \mathbb{H}^+} u = 0$ if and only if

$$\lim_{\|(x,y)\| \to +\infty} u(x, y) = 0 \quad \text{and} \quad \lim_{(0, y)} u = 0, \quad y \in \mathbb{R}.$$

Proof The direct implication is easy. Conversely, assume $\lim_{\|(x,y)\| \to +\infty} u(x, y) = 0$ and $\lim_{(0, y)} u = 0$, $y \in \mathbb{R}$. Let $\varepsilon > 0$, then there is $A > 0$ such that for all $(\xi, \eta) \in \mathbb{H}^+$, $\sqrt{\xi^2 + \eta^2} \geq A$ implies $|u(\xi, \eta)| \leq \varepsilon$. Similarly, for all $y \in \mathbb{R}$, there is $\alpha_y \in (0, 1)$ such that for all $(\xi, \eta) \in \mathbb{H}^+$, $\sqrt{\xi^2 + (\eta - y)^2} < \alpha_y$ implies $|u(\xi, \eta)| \leq \varepsilon$.

The interval $[-A, A]$ is compact. By the Lebesgue covering lemma, there is $\alpha > 0$ such that for all $y' \in [-A, A]$, the ball $B(y', \alpha)$ is included in one of the balls $B(y, \alpha_y)$ with $y \in [-A, A]$. In particular, if $(\xi, \eta) \in \mathbb{H}^+$ is such that $0 < \xi < \alpha$, then $|u(\xi, \eta)| \leq \varepsilon$. This completes the proof. □

The following proposition is a Liouville-type result for axisymmetric potentials in the right half-plane. As we mentioned in the introduction, this result is not trivial due to the loss of strict ellipticity of the Weinstein operator on the y-axis. Let us mention that in [5, Theorem 7.1] one can find an interesting result on the description of a class of non-strictly elliptic equations with unbounded coefficients.

Proposition 5.3 Let $u \in C^2(\mathbb{H}^+)$ be such that $L_m u = 0$ and $\lim_{\partial \mathbb{H}^+} u = 0$. Then $u \equiv 0$ on $\mathbb{H}^+$.

Proof For $(\xi, \eta) \in \mathbb{H}^+$ and $N \in \mathbb{N}^*$, define

$$\phi_N(\xi, \eta) = \theta_1(N\xi)\theta_2\left(\frac{\xi}{N}\right)\theta_2\left(\frac{\eta}{N}\right),$$

where θ_1 and θ_2 are smooth functions on $\mathbb{R}$, valued on $[0, 1]$ and such that $\theta_1(t) = 1$ for $t \geq 1$, $\theta_1(t) = 0$ for $t \leq 1/2$, $\theta_2(t) = 1$ for $t \in [-1/2, 1/2]$, and $\theta_2(t) = 0$ for $t \in \mathbb{R} \setminus (-1, 1)$. Assume also that all derivatives of θ_1 and θ_2 vanish at $\{-1, -1/2, 1/2, 1\}$ (Fig. 2).

If $u \in C^2(\mathbb{H}^+)$ satisfies $L_m u = 0$, then $u\phi_N \in C^2(\mathbb{H}^+)$ and it is compactly supported on $\mathbb{H}^+$. Throughout the following, we fix $(x, y) \in \mathbb{H}^+$. For N sufficiently large, due to Proposition 4.7 (true if E_m is replaced by F_m), we have

$$u(x, y) = u(x, y)\phi_N(x, y) = \int_{\mathbb{H}^+} L_m(u\phi_N) F_m \, d\xi \, d\eta$$
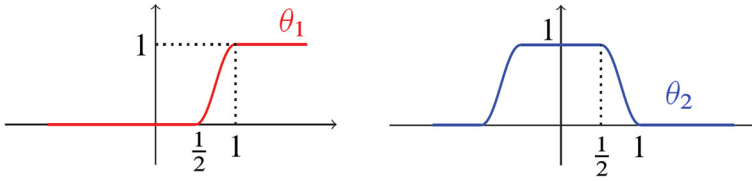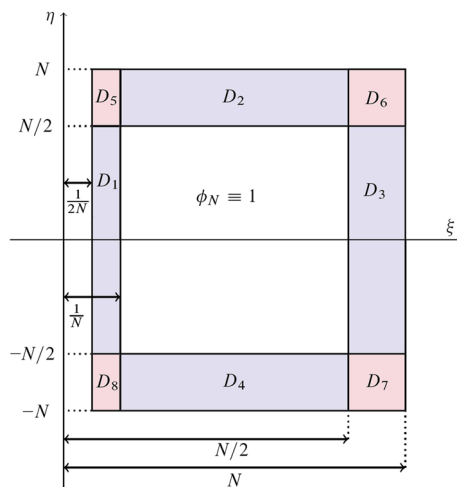
Fig. 2 The functions θ_1 and θ_2

(because the function $L_m(u\phi_N)$ is identically zero in a neighborhood of the singularity of F_m), thus

$$
\begin{aligned}
u(x, y) &= \int_{\mathbb{H}^+} \left[L_m(u)\phi_N + u L_m(\phi_N) + 2\nabla u \cdot \nabla \phi_N \right] F_m \, d\xi \, d\eta \\
&= \int_{\mathbb{H}^+} u \left[L_m(\phi_N) F_m - 2\operatorname{div}(F_m \nabla \phi_N) \right] d\xi \, d\eta \\
&= \int_{D_1 \cup \cdots \cup D_8} u \left[L_m(\phi_N) F_m - 2\operatorname{div}(F_m \nabla \phi_N) \right] d\xi \, d\eta \\
&= -\int_{D_1 \cup \cdots \cup D_8} u \left[L_{-m}(\phi_N) F_m + 2\nabla F_m \cdot \nabla \phi_N \right] d\xi \, d\eta,
\end{aligned}
$$

where $D_1, \ldots, D_8$ are the following domains (which depend on N) (Fig. 3):

$$
D_1 = \left[\frac{1}{2N}, \frac{1}{N} \right] \times \left[-\frac{N}{2}, \frac{N}{2} \right], \qquad D_2 = \left[\frac{1}{N}, \frac{N}{2} \right] \times \left[\frac{N}{2}, N \right],
$$

Fig. 3 Domains D_i

$$D_3 = \left[\frac{N}{2}, N\right] \times \left[-\frac{N}{2}, \frac{N}{2}\right], \qquad D_4 = \left[\frac{1}{N}, \frac{N}{2}\right] \times \left[-N, -\frac{N}{2}\right],$$

$$D_5 = \left[\frac{1}{2N}, \frac{1}{N}\right] \times \left[\frac{N}{2}, N\right], \qquad D_6 = \left[\frac{N}{2}, N\right] \times \left[\frac{N}{2}, N\right],$$

$$D_7 = \left[\frac{N}{2}, N\right] \times \left[-N, -\frac{N}{2}\right], \qquad D_8 = \left[\frac{1}{2N}, \frac{1}{N}\right] \times \left[-N, -\frac{N}{2}\right].$$

Since $\lim_{\partial \mathbb{H}^+} u = 0$,

$$u_N = \sup_{(\xi,\eta) \in D_1 \cup \cdots \cup D_8} |u(\xi, \eta)| \to 0 \quad \text{as} \quad N \to +\infty.$$

We will estimate integrals over sets $D_1, \ldots, D_8$ separately, see auxiliary lemmas below. Recall that, if $(u_N)_N$ and $(v_N)_N$ are complex sequences, $u_N = \mathrm{O}(v_N)$ means that there exists a constant M such that, for every N sufficiently large, $|u_N| \leq M|v_N|$; $u_N = \mathrm{o}(v_N)$ means that for every $\varepsilon > 0$, for every N sufficiently large, $|u_N| \leq \varepsilon|v_N|$.

Lemma 5.4 *On D_1, we have*

$$\sup\left|\frac{\partial \phi_N}{\partial \xi}\right| = \mathrm{O}(N) \quad \text{and} \quad \sup\left|\frac{\partial \phi_N}{\partial \eta}\right| = 0.$$

On $D_2 \cup D_4$, we have

$$\sup\left|\frac{\partial \phi_N}{\partial \xi}\right| = 0 \quad \text{and} \quad \sup\left|\frac{\partial \phi_N}{\partial \eta}\right| = \mathrm{O}\left(\frac{1}{N}\right).$$

On D_3, we have

$$\sup\left|\frac{\partial \phi_N}{\partial \xi}\right| = \mathrm{O}\left(\frac{1}{N}\right) \quad \text{and} \quad \sup\left|\frac{\partial \phi_N}{\partial \eta}\right| = 0.$$

On $D_5 \cup D_8$, we have

$$\sup\left|\frac{\partial \phi_N}{\partial \xi}\right| = \mathrm{O}(N) \quad \text{and} \quad \sup\left|\frac{\partial \phi_N}{\partial \eta}\right| = \mathrm{O}\left(\frac{1}{N}\right).$$

On $D_6 \cup D_7$, we have

$$\sup\left|\frac{\partial \phi_N}{\partial \xi}\right| = \mathrm{O}\left(\frac{1}{N}\right) \quad \text{and} \quad \sup\left|\frac{\partial \phi_N}{\partial \eta}\right| = \mathrm{O}\left(\frac{1}{N}\right).$$

On $D_1 \cup D_5 \cup D_8$, we have

$$\sup |L_{-m}(\phi_N)| = \mathrm{O}(N^2).$$

On $D_2 \cup D_3 \cup D_4 \cup D_6 \cup D_7$, we have

$$\sup |L_{-m}(\phi_N)| = O\left(\frac{1}{N^2}\right).$$

Proof For $(\xi, \eta) \in D_1$, $\phi_N(\xi, \eta) = \theta_1(N\xi)$ and thus

$$\frac{\partial \phi_N}{\partial \xi}(\xi, \eta) = N\theta_1'(N\xi), \qquad \frac{\partial \phi_N}{\partial \eta}(\xi, \eta) = 0,$$

$$L_{-m}\phi_N(\xi, \eta) = N^2\theta_1''(N\xi) - \frac{m\,N}{\xi}\theta_1'(N\xi),$$

which give us

$$\sup_{D_1}\left|\frac{\partial \phi_N}{\partial \xi}\right| = O(N), \qquad \sup_{D_1}\left|\frac{\partial \phi_N}{\partial \eta}\right| = 0, \qquad \sup_{D_1}|L_{-m}(\phi_N)| = O(N^2)$$

since the derivatives of θ_1 are bounded and for $(\xi, \eta) \in D_1$ one gets $\xi \geq 1/(2N)$.

For $(\xi, \eta) \in D_2$, $\phi_N(\xi, \eta) = \theta_2(\eta/N)$ and thus

$$\frac{\partial \phi_N}{\partial \xi}(\xi, \eta) = 0, \qquad \frac{\partial \phi_N}{\partial \eta}(\xi, \eta) = \frac{1}{N}\theta_2'\left(\frac{\eta}{N}\right),$$

$$L_{-m}\phi_N(\xi, \eta) = \frac{1}{N^2}\theta_2''\left(\frac{\eta}{N}\right),$$

which give us

$$\sup_{D_2}\left|\frac{\partial \phi_N}{\partial \xi}\right| = 0, \qquad \sup_{D_2}\left|\frac{\partial \phi_N}{\partial \eta}\right| = O\left(\frac{1}{N}\right), \qquad \sup_{D_2}|L_{-m}(\phi_N)| = O\left(\frac{1}{N^2}\right).$$

The same works for D_4.

For $(\xi, \eta) \in D_3$, $\phi_N(\xi, \eta) = \theta_2(\xi/N)$ and thus

$$\frac{\partial \phi_N}{\partial \xi}(\xi, \eta) = \frac{1}{N}\theta_2'\left(\frac{\xi}{N}\right), \qquad \frac{\partial \phi_N}{\partial \eta}(\xi, \eta) = 0,$$

$$L_{-m}\phi_N(\xi, \eta) = \frac{1}{N^2}\theta_2''\left(\frac{\xi}{N}\right) - \frac{1}{N}\frac{m}{\xi}\theta_2'\left(\frac{\xi}{N}\right),$$

which give us

$$\sup_{D_3}\left|\frac{\partial \phi_N}{\partial \xi}\right| = O\left(\frac{1}{N}\right), \qquad \sup_{D_3}\left|\frac{\partial \phi_N}{\partial \eta}\right| = 0, \qquad \sup_{D_3}|L_{-m}(\phi_N)| = O\left(\frac{1}{N^2}\right).$$

For $(\xi, \eta) \in D_5$, $\phi_N(\xi, \eta) = \theta_1(N\xi)\theta_2(\eta/N)$ and thus

$$\frac{\partial \phi_N}{\partial \xi}(\xi, \eta) = N\theta_1'(N\xi)\theta_2\left(\frac{\eta}{N}\right), \qquad \frac{\partial \phi_N}{\partial \eta}(\xi, \eta) = \frac{1}{N}\theta_1(N\xi)\theta_2'\left(\frac{\eta}{N}\right),$$

$$L_{-m}\phi_N(\xi, \eta) = N^2\theta_1''(N\xi)\theta_2\left(\frac{\eta}{N}\right) + \frac{1}{N^2}\theta_1(N\xi)\theta_2''\left(\frac{\eta}{N}\right)$$
$$- \frac{m}{\xi}N\theta_1'(N\xi)\theta_2\left(\frac{\eta}{N}\right),$$

which give us

$$\sup_{D_5}\left|\frac{\partial \phi_N}{\partial \xi}\right| = \mathrm{O}(N), \qquad \sup_{D_5}\left|\frac{\partial \phi_N}{\partial \eta}\right| = \mathrm{O}\left(\frac{1}{N}\right), \qquad \sup_{D_5}|L_{-m}(\phi_N)| = \mathrm{O}(N^2).$$

The same works for D_8.

For $(\xi, \eta) \in D_6$, $\phi_N(\xi, \eta) = \theta_2(\xi/N)\theta_2(\eta/N)$ and thus

$$\frac{\partial \phi_N}{\partial \xi}(\xi, \eta) = \frac{1}{N}\theta_2'\left(\frac{\xi}{N}\right)\theta_2\left(\frac{\eta}{N}\right), \qquad \frac{\partial \phi_N}{\partial \eta}(\xi, \eta) = \frac{1}{N}\theta_2\left(\frac{\xi}{N}\right)\theta_2'\left(\frac{\eta}{N}\right),$$

$$L_{-m}\phi_N(\xi, \eta) = \frac{1}{N^2}\theta_2''\left(\frac{\xi}{N}\right)\theta_2\left(\frac{\eta}{N}\right) + \frac{1}{N^2}\theta_2\left(\frac{\xi}{N}\right)\theta_2''\left(\frac{\eta}{N}\right)$$
$$- \frac{m}{N\xi}\theta_2'\left(\frac{\xi}{N}\right)\theta_2\left(\frac{\eta}{N}\right),$$

which give us

$$\sup_{D_6}\left|\frac{\partial \phi_N}{\partial \xi}\right| = \mathrm{O}\left(\frac{1}{N}\right), \qquad \sup_{D_6}\left|\frac{\partial \phi_N}{\partial \eta}\right| = \mathrm{O}\left(\frac{1}{N}\right), \qquad \sup_{D_6}|L_{-m}(\phi_N)| = \mathrm{O}\left(\frac{1}{N^2}\right).$$

The same works for D_7. ∎

We now estimate the following quantities for $i \in \{1, \ldots, 8\}$:

$$\int_{D_i}|F_m|\,d\xi d\eta, \qquad \int_{D_i}|\partial_\xi F_m|\,d\xi d\eta \qquad \int_{D_i}|\partial_\eta F_m|\,d\xi d\eta.$$

Lemma 5.5 *For* $\operatorname{Re} m < 1$, *we have:*

- *for* $i = 1$,

$$\int_{D_i}|F_m|\,d\xi d\eta = \mathrm{O}\left(\frac{1}{N^2}\right), \qquad \int_{D_i}\left|\frac{\partial F_m}{\partial \xi}\right|\,d\xi d\eta = \mathrm{O}\left(\frac{1}{N}\right);$$

- *for* $i = 2, 4$,

$$\int_{D_i}|F_m|\,d\xi d\eta = \mathrm{O}(N^2), \qquad \int_{D_i}\left|\frac{\partial F_m}{\partial \eta}\right|\,d\xi d\eta = \mathrm{O}(N);$$

- *for $i = 3$,*

$$\int_{D_i} |F_m| \, d\xi d\eta = O(N^2), \qquad \int_{D_i} \left| \frac{\partial F_m}{\partial \xi} \right| d\xi d\eta = O(N);$$

- *for $i = 5, 8$,*

$$\int_{D_i} |F_m| \, d\xi d\eta = O\left(\frac{1}{N^2}\right), \qquad \int_{D_i} \left| \frac{\partial F_m}{\partial \xi} \right| d\xi d\eta = O\left(\frac{1}{N}\right),$$

$$\int_{D_i} \left| \frac{\partial F_m}{\partial \eta} \right| d\xi d\eta = O\left(\frac{1}{N^2}\right);$$

- *for $i = 6, 7$,*

$$\int_{D_i} |F_m| \, d\xi d\eta = O(N^2), \qquad \int_{D_i} \left| \frac{\partial F_m}{\partial \xi} \right| d\xi d\eta = O(N),$$

$$\int_{D_i} \left| \frac{\partial F_m}{\partial \eta} \right| d\xi d\eta = O(N).$$

Proof By definition, for $\operatorname{Re} m < 1$,

$$F_m(\xi, \eta) = -\frac{\xi x^{1-m}}{2\pi} \int_0^\pi \frac{\sin^{1-m} \theta \, d\theta}{\left[(x - \xi)^2 + 4x\xi \sin^2 \theta/2 + (y - \eta)^2 \right]^{1-m/2}}.$$

Therefore there is a constant C_1 such that for all $(\xi, \eta) \in \mathbb{H}^+$, we have

$$|F_m(\xi, \eta)| \leq \frac{C_1 \xi}{\left[(x - \xi)^2 + (\eta - y)^2 \right]^{1 - \operatorname{Re} m/2}}. \tag{5}$$

Similarly, we have

$$\frac{\partial F_m}{\partial \xi} = \frac{F_m}{\xi} - \frac{\xi x^{1-m}}{2\pi} (m - 2) \int_0^\pi \frac{\left[(\xi - x) + 2x \sin^2 \theta/2 \right] \sin^{1-m} \theta \, d\theta}{\left[(x - \xi)^2 + 4x\xi \sin^2 \theta/2 + (y - \eta)^2 \right]^{2-m/2}},$$

and as before, as for all $\theta \in [0, \pi]$,

$$\left| \frac{\left[(\xi - x) + 2x \sin^2 \theta/2 \right] \sin^{1-m} \theta}{\left[(x - \xi)^2 + 4x\xi \sin^2 \theta/2 + (y - \eta)^2 \right]^{2-m/2}} \right| \leq \frac{\left| (\xi - x) + 2x \sin^2 \theta/2 \right|}{\left[(x - \xi)^2 + (\eta - y)^2 \right]^{2 - \operatorname{Re} m/2}}$$

$$= \frac{|\xi - x \cos \theta|}{\left[(x - \xi)^2 + (\eta - y)^2 \right]^{2 - \operatorname{Re} m/2}} \leq \frac{\xi + x}{\left[(x - \xi)^2 + (\eta - y)^2 \right]^{2 - \operatorname{Re} m/2}},$$

there exists a constant C_2 such that for all N large enough and all $(\xi, \eta) \in \mathbb{H}^+$, we have

$$
\left| \frac{\partial F_m}{\partial \xi} \right| \leq C_2 \left[\frac{1}{\left[(x - \xi)^2 + (\eta - y)^2 \right]^{1 - \operatorname{Re} m/2}} \right.
$$
$$
\left. + \frac{\xi(x + \xi)}{\left[(x - \xi)^2 + (\eta - y)^2 \right]^{2 - \operatorname{Re} m/2}} \right]. \tag{6}
$$

Finally, as

$$
\frac{\partial F_m}{\partial \eta} = (2 - m)(\eta - y) \frac{\xi x^{1-m}}{2\pi} \int_0^\pi \frac{\sin^{1-m} \theta \, d\theta}{\left[(x - \xi)^2 + 4x\xi \sin^2 \theta/2 + (y - \eta)^2 \right]^{2 - m/2}},
$$

there exists a constant C_3 such that for all N large enough and all $(\xi, \eta) \in \mathbb{H}^+$, we have

$$
\left| \frac{\partial F_m}{\partial \eta} \right| \leq \frac{C_3 \xi}{|\eta - y|^{3 - \operatorname{Re} m}}. \tag{7}
$$

Using these inequalities, we estimate integrals on domains D_i.

On D_1: Inequality (5) implies

$$
\int_{D_1} |F_m| \, d\xi \, d\eta = \mathrm{O}(1) \int_{1/2N}^{1/N} d\xi \int_{-N/2}^{N/2} \frac{\xi \, d\eta}{\left[(x - \xi)^2 + (\eta - y)^2 \right]^{1 - \operatorname{Re} m/2}}
$$
$$
= \mathrm{O}\left(\frac{1}{N^2} \right) \int_{-N/2}^{N/2} \frac{d\eta}{\left[(x - 1/N)^2 + (\eta - y)^2 \right]^{1 - \operatorname{Re} m/2}} = \mathrm{O}\left(\frac{1}{N^2} \right).
$$

Then, thanks to (6), we have

$$
\int_{D_1} \left| \frac{\partial F_m}{\partial \xi} \right| d\xi \, d\eta = \mathrm{O}(1) \int_{1/2N}^{1/N} d\xi \int_{-N/2}^{N/2} \left[\frac{1}{\left[(x - \xi)^2 + (\eta - y)^2 \right]^{1 - \operatorname{Re} m/2}} \right.
$$
$$
\left. + \frac{\xi(x + \xi)}{\left[(x - \xi)^2 + (\eta - y)^2 \right]^{2 - \operatorname{Re} m/2}} \right] d\eta
$$
$$
= \mathrm{O}\left(\frac{1}{N} \right) \int_{-N/2}^{N/2} \frac{d\eta}{\left[(x - 1/N)^2 + (\eta - y)^2 \right]^{1 - \operatorname{Re} m/2}} + \mathrm{O}\left(\frac{1}{N^2} \right)
$$
$$
= \mathrm{O}\left(\frac{1}{N} \right).
$$

On D_2: Due to inequality (5), we have

$$
\begin{aligned}
\int_{D_2} |F_m| \, d\xi d\eta &= \mathrm{O}(1) \int_{1/N}^{N/2} d\xi \int_{N/2}^{N} \frac{\xi \, d\eta}{\left[(x-\xi)^2 + (\eta-y)^2\right]^{1-\mathrm{Re}\, m/2}} \\
&= \mathrm{O}(1) \int_{1/N}^{N/2} d\xi \int_{N/2}^{N} \frac{\xi \, d\eta}{|\eta-y|^{2-\mathrm{Re}\, m}} \\
&= \mathrm{O}(N^2) \int_{N/2}^{N} \frac{d\eta}{|\eta-y|^{2-\mathrm{Re}\, m}} \\
&= \mathrm{O}(N^2) \left[\frac{1}{(N-y)^{1-\mathrm{Re}\, m}} - \frac{1}{(N/2-y)^{1-\mathrm{Re}\, m}} \right] = \mathrm{O}\big(N^{\mathrm{Re}\, m+1}\big).
\end{aligned}
$$

Then, thanks to (7), we have

$$
\begin{aligned}
\int_{D_2} \left| \frac{\partial F_m}{\partial \eta} \right| d\xi d\eta &= \mathrm{O}(1) \int_{1/N}^{N/2} d\xi \int_{N/2}^{N} \frac{\xi \, d\eta}{|\eta-y|^{3-\mathrm{Re}\, m}} \\
&= \mathrm{O}(N^2) \int_{N/2}^{N} \frac{d\eta}{|\eta-y|^{3-\mathrm{Re}\, m}} = \mathrm{O}\big(N^{\mathrm{Re}\, m}\big).
\end{aligned}
$$

On D_3: Due to inequality (5), we have

$$
\begin{aligned}
\int_{D_3} |F_m| \, d\xi d\eta &= \mathrm{O}(1) \int_{N/2}^{N} d\xi \int_{-N/2}^{N/2} \frac{\xi \, d\eta}{\left[(x-\xi)^2 + (\eta-y)^2\right]^{1-\mathrm{Re}\, m/2}} \\
&= \mathrm{O}(1) \int_{N/2}^{N} d\xi \int_{-N/2}^{N/2} \frac{\xi \, d\eta}{\left[(x-N/2)^2 + (\eta-y)^2\right]^{1-\mathrm{Re}\, m/2}} \\
&= \mathrm{O}(N^2) \int_{-N/2}^{N/2} \frac{d\eta}{\left[(x-N/2)^2 + (\eta-y)^2\right]^{1-\mathrm{Re}\, m/2}} \\
&= \mathrm{O}(N^2) \int_{-N/2}^{N/2} \frac{d\eta}{[1 + (\eta-y)^2]^{1-\mathrm{Re}\, m/2}} = \mathrm{O}(N^2).
\end{aligned}
$$

Then, thanks to (6), we have

$$
\begin{aligned}
\int_{D_3} \left| \frac{\partial F_m}{\partial \xi} \right| d\xi d\eta &= \mathrm{O}(1) \int_{N/2}^{N} d\xi \int_{-N/2}^{N/2} \left[\frac{1}{\left[(x-\xi)^2 + (\eta-y)^2\right]^{1-\mathrm{Re}\, m/2}} \right. \\
&\qquad\qquad \left. + \frac{\xi(x+\xi)}{\left[(x-\xi)^2 + (\eta-y)^2\right]^{2-\mathrm{Re}\, m/2}} \right] d\eta \\
&= \mathrm{O}(N) \int_{-N/2}^{N/2} \frac{d\eta}{\left[(x-N/2)^2 + (\eta-y)^2\right]^{1-\mathrm{Re}\, m/2}}
\end{aligned}
$$

$$+ O(N^3) \int_{-N/2}^{N/2} \frac{d\eta}{\left[(x - N/2)^2 + (\eta - y)^2\right]^{2-\operatorname{Re} m/2}}$$

$$= O(N) + O(N^3) \int_{-N/2}^{N/2} \frac{d\eta}{(x - N/2)^{4-\operatorname{Re} m}}$$

$$= O(N) + O(N^{\operatorname{Re} m}) = O(N).$$

On D_4: This case is analogous to the case D_2.

On D_5: Due to inequality (5), we have

$$\int_{D_5} |F_m| \, d\xi \, d\eta = O(1) \int_{1/2N}^{1/N} d\xi \int_{N/2}^{N} \frac{\xi \, d\eta}{\left[(x - \xi)^2 + (\eta - y)^2\right]^{1-\operatorname{Re} m/2}}$$

$$= O\left(\frac{1}{N^2}\right) \int_{N/2}^{N} \frac{d\eta}{(\eta - y)^{2-\operatorname{Re} m}}$$

$$= O\left(\frac{1}{N^2}\right) \left[\frac{1}{(N - y)^{1-\operatorname{Re} m}} - \frac{1}{(N/2 - y)^{1-\operatorname{Re} m}}\right] = O\left(\frac{1}{N^{3-\operatorname{Re} m}}\right).$$

Then, thanks to (6), we have

$$\int_{D_5} \left|\frac{\partial F_m}{\partial \xi}\right| d\xi \, d\eta = O(1) \int_{1/2N}^{1/N} d\xi \int_{N/2}^{N} \left[\frac{1}{\left[(x - \xi)^2 + (\eta - y)^2\right]^{1-\operatorname{Re} m/2}}\right.$$

$$\left. + \frac{\xi(x + \xi)}{\left[(x - \xi)^2 + (\eta - y)^2\right]^{2-\operatorname{Re} m/2}}\right] d\eta$$

$$= O(1) \int_{1/2N}^{1/N} d\xi \int_{N/2}^{N} \left[\frac{1}{\left[(x - 1/N)^2 + (\eta - y)^2\right]^{1-\operatorname{Re} m/2}}\right.$$

$$\left. + \frac{\xi(x + \xi)}{\left[(x - 1/N)^2 + (\eta - y)^2\right]^{2-\operatorname{Re} m/2}}\right] d\eta$$

$$= O\left(\frac{1}{N}\right).$$

Estimate (7) gives

$$\int_{D_5} \left|\frac{\partial F_m}{\partial \eta}\right| d\xi \, d\eta = O(1) \int_{1/2N}^{1/N} d\xi \int_{N/2}^{N} \frac{\xi \, d\eta}{|\eta - y|^{3-\operatorname{Re} m}} = O\left(\frac{1}{N^2}\right).$$

On D_6: Due to (5), we have

$$\int_{D_6} |F_m| \, d\xi d\eta = O(1) \int_{N/2}^{N} d\xi \int_{N/2}^{N} \frac{\xi \, d\eta}{\left[(x-\xi)^2 + (\eta-y)^2\right]^{1-\operatorname{Re} m/2}}$$

$$= O(N^2) \int_{N/2}^{N} \frac{d\eta}{(\eta-y)^{2-\operatorname{Re} m}}$$

$$= O(N^2) \left[\frac{1}{(N-y)^{1-\operatorname{Re} m}} - \frac{1}{(N/2-y)^{1-\operatorname{Re} m}} \right] = O\left(N^{1+\operatorname{Re} m}\right).$$

Then, thanks to (6), we have

$$\int_{D_6} \left| \frac{\partial F_m}{\partial \xi} \right| \, d\xi d\eta = O(1) \int_{N/2}^{N} d\xi \int_{N/2}^{N} \left[\frac{1}{\left[(x-\xi)^2 + (\eta-y)^2\right]^{1-\operatorname{Re} m/2}} \right.$$

$$\left. + \frac{\xi(x+\xi)}{\left[(x-\xi)^2 + (\eta-y)^2\right]^{2-\operatorname{Re} m/2}} \right] d\eta$$

$$= O(1) \int_{N/2}^{N} d\xi \int_{N/2}^{N} \left[\frac{1}{(\eta-y)^{2-\operatorname{Re} m}} + \frac{\xi(x+\xi)}{(\eta-y)^{4-\operatorname{Re} m}} \right] d\eta$$

$$= O(N) + O(N^3) \int_{N/2}^{N} \frac{d\eta}{(\eta-y)^{4-\operatorname{Re} m}} = O(N) + O\left(N^{\operatorname{Re} m}\right)$$

$$= O(N).$$

Estimate (7) gives

$$\int_{D_6} \left| \frac{\partial F_m}{\partial \eta} \right| \, d\xi d\eta = O(1) \int_{N/2}^{N} d\xi \int_{N/2}^{N} \frac{\xi \, d\eta}{|\eta-y|^{3-\operatorname{Re} m}}$$

$$= O(N^2) \int_{N/2}^{N} \frac{d\eta}{|\eta-y|^{3-\operatorname{Re} m}} = O\left(N^{\operatorname{Re} m}\right).$$

On D_7, D_8: These cases are analogous to the cases D_6 and D_5, respectively. ■

Lemma 5.6 *Lemma 5.5 remains true for* $\operatorname{Re} m \geq 1$.

Proof For $\operatorname{Re} m \geq 1$, we have

$$F_m(x, y, \xi, \eta) = -\frac{\xi^m}{2\pi} \int_0^{\pi} \sin^{m-1}\theta \left(\frac{1}{\left[(x-\xi)^2 + 4x\xi \sin^2\theta/2 + (y-\eta)^2\right]^{m/2}} \right.$$

$$\left. - \frac{1}{\left[(x+\xi)^2 - 4x\xi \sin^2\theta/2 + (y-\eta)^2\right]^{m/2}} \right) d\theta.$$

Since for all $(\xi, \eta) \in \mathbb{H}^+$, we have

$$\left| \left[(x+\xi)^2 - 4x\xi \sin^2 \frac{\theta}{2} + (y-\eta)^2 \right]^{m/2} \right| = \left| [x^2 + \xi^2 + 2x\xi \cos\theta + (y-\eta)^2]^{m/2} \right|,$$

then

$$\left| \left[(x+\xi)^2 - 4x\xi \sin^2 \frac{\theta}{2} + (y-\eta)^2 \right]^{m/2} \right| \geq \left[(x-\xi)^2 + (y-\eta)^2 \right]^{\operatorname{Re} m/2} \qquad (8)$$

and there is a constant C_1' such that for all $(\xi, \eta) \in \mathbb{H}^+$, we have

$$|F_m| \leq \frac{C_1' \xi^{\operatorname{Re} m}}{\left[(x-\xi)^2 + (y-\eta)^2 \right]^{\operatorname{Re} m/2}}. \qquad (9)$$

This inequality does not suffice to estimate integrals over D_1. We shall improve it as follows. Rewrite F_m as

$$F_m(x, y, \xi, \eta) = -\frac{\xi^m}{2\pi} \int_0^\pi \sin^{m-1}\theta \, K_m(x, y, \xi, \eta, \theta) \, d\theta,$$

where

$$K_m(x, y, \xi, \eta, \theta) = \frac{1}{\left[(x-\xi)^2 + 4x\xi \sin^2\theta/2 + (y-\eta)^2 \right]^{m/2}}$$
$$- \frac{1}{\left[(x+\xi)^2 - 4x\xi \sin^2\theta/2 + (y-\eta)^2 \right]^{m/2}}.$$

For $(x, y) \in \mathbb{H}^+$, $\theta \in [0, \pi]$ and $\eta \in \mathbb{R}$ fixed, define a function g_m on $[-1/N, 1/N]$, with $1/N < x$, by

$$g_m(\xi) = \frac{1}{\left[(x-\xi)^2 + 4x\xi \sin^2\theta/2 + (y-\eta)^2 \right]^{m/2}}.$$

This function is well defined because

$$(x-\xi)^2 + 4x\xi \sin^2\frac{\theta}{2} + (y-\eta)^2 = x^2 + \xi^2 - 2x\xi \cos\theta + (y-\eta)^2$$
$$\geq (x - |\xi|)^2 + (y-\eta)^2$$

and the last term is greater than $(x - 1/N)^2 > 0$.

We have

$$K_m(x, y, \xi, \eta, \theta) = g_m(\xi) - g_m(-\xi)$$

thus

$$\left| K_m(x, y, \xi, \eta, \theta) \right| \leq 2\xi \sup_{[-\xi, \xi]} |g_m'| \leq 2|m|\xi \frac{|\xi - x| + 2x}{\left[(x - \xi)^2 + (y - \eta)^2 \right]^{1 + \mathrm{Re}\,m/2}},$$

which implies that there exists a constant c_1' such that for all $(\xi, \eta) \in D_1$,

$$|F_m| \leq c_1' \frac{\xi^{\mathrm{Re}\,m+1}}{\left[(x - \xi)^2 + (\eta - y)^2 \right]^{1 + \mathrm{Re}\,m/2}}. \tag{10}$$

Similarly, we have

$$\frac{\partial F_m}{\partial \xi} = \frac{m F_m}{\xi}$$
$$+ \frac{m\xi^m}{2\pi} \int_0^\pi \sin^{m-1}\theta \left(\frac{(\xi - x) + 2x \sin^2\theta/2}{\left[(x - \xi)^2 + 4x\xi \sin^2\theta/2 + (y - \eta)^2 \right]^{m/2+1}} \right. \tag{11}$$
$$\left. - \frac{(\xi + x) - 2x \sin^2\theta/2}{\left[(x + \xi)^2 - 4x\xi \sin^2\theta/2 + (y - \eta)^2 \right]^{m/2+1}} \right) d\theta.$$

and as before, for all $\theta \in [0, \pi]$,

$$\left| \frac{\left[(\xi - x) + 2x \sin^2\theta/2 \right] \sin^{m-1}\theta}{\left[(x - \xi)^2 + 4x\xi \sin^2\theta/2 + (y - \eta)^2 \right]^{m/2+1}} \right| \leq \frac{\left| (\xi - x) + 2x \sin^2\theta/2 \right|}{\left[(x - \xi)^2 + (y - \eta)^2 \right]^{\mathrm{Re}\,m/2+1}}$$
$$= \frac{|\xi - x\cos\theta|}{\left[(x - \xi)^2 + (y - \eta)^2 \right]^{\mathrm{Re}\,m/2+1}} \leq \frac{\xi + x}{\left[(x - \xi)^2 + (y - \eta)^2 \right]^{\mathrm{Re}\,m/2+1}}$$

and thanks to (8), for all $\theta \in [0, \pi]$,

$$\left| \frac{\left[(\xi + x) - 2x \sin^2\theta/2 \right] \sin^{m-1}\theta}{\left[(x + \xi)^2 - 4x\xi \sin^2\theta/2 + (y - \eta)^2 \right]^{m/2+1}} \right| \leq \frac{\left| (\xi + x) - 2x \sin^2\theta/2 \right|}{\left[(x - \xi)^2 + (y - \eta)^2 \right]^{\mathrm{Re}\,m/2+1}}$$
$$= \frac{|\xi + x\cos\theta|}{\left[(x - \xi)^2 + (y - \eta)^2 \right]^{\mathrm{Re}\,m/2+1}} \leq \frac{\xi + x}{\left[(x - \xi)^2 + (y - \eta)^2 \right]^{\mathrm{Re}\,m/2+1}}.$$

These estimates, (11) and (9) show that there is a constant C_2' such that for large enough N and all $(\xi, \eta) \in \mathbb{H}^+$, we have

$$\left| \frac{\partial F_m}{\partial \xi} \right| \leq C_2' \left(\frac{\xi^{\mathrm{Re}\,m-1}}{\left[(x - \xi)^2 + (y - \eta)^2 \right]^{\mathrm{Re}\,m/2}} \right.$$
$$\left. + \frac{\xi^{\mathrm{Re}\,m}(\xi + x)}{\left[(x - \xi)^2 + (y - \eta)^2 \right]^{\mathrm{Re}\,m/2+1}} \right). \tag{12}$$

We can improve this inequality on D_1, by using inequality (10) instead of (9), then there are two constants C_2'' and C_2''' (which do not depend on N) such that for all $(\xi, \eta) \in D_1$,

$$
\begin{aligned}
\left| \frac{\partial F_m}{\partial \xi} \right| &\leq C_2'' \Bigg(\frac{\xi^{\operatorname{Re} m}}{\left[(x-\xi)^2 + (y-\eta)^2 \right]^{1+\operatorname{Re} m/2}} \\
&\qquad + \frac{\xi^{\operatorname{Re} m}(\xi+x)}{\left[(x-\xi)^2 + (y-\eta)^2 \right]^{\operatorname{Re} m/2+1}} \Bigg) \qquad (13) \\
&\leq C_2''' \frac{\xi^{\operatorname{Re} m}}{\left[(x-\xi)^2 + (y-\eta)^2 \right]^{1+\operatorname{Re} m/2}}.
\end{aligned}
$$

Finally,

$$
\frac{\partial F_m}{\partial \eta} = \frac{m(\eta-y)\xi^m}{2\pi} \int_0^\pi \sin^{m-1}\theta \Bigg(\frac{1}{\left[(x-\xi)^2 + 4x\xi \sin^2\theta/2 + (y-\eta)^2 \right]^{m/2+1}} \\
- \frac{1}{\left[(x+\xi)^2 - 4x\xi \sin^2\theta/2 + (y-\eta)^2 \right]^{m/2+1}} \Bigg) d\theta.
$$

Similarly, there is a constant C_3' such that for all N large enough and all $(\xi, \eta) \in \mathbb{H}^+$, we have

$$
\left| \frac{\partial F_m}{\partial \eta} \right| \leq C_3' \frac{|\eta-y|\xi^{\operatorname{Re} m}}{\left[(x-\xi)^2 + (y-\eta)^2 \right]^{\operatorname{Re} m/2+1}}. \qquad (14)
$$

Thanks to these inequalities, we can now estimate the corresponding integrals over domains D_i.

On D_1: Due to (10), we have

$$
\begin{aligned}
\int_{D_1} |F_m| \, d\xi d\eta &= \mathrm{O}(1) \int_{1/2N}^{1/N} d\xi \int_{-N/2}^{N/2} \frac{\xi^{\operatorname{Re} m+1} d\eta}{\left[(x-\xi)^2 + (\eta-y)^2 \right]^{1+\operatorname{Re} m/2}} \\
&= \mathrm{O}(N) \int_{1/2N}^{1/N} \xi^{\operatorname{Re} m+1} d\xi = \mathrm{O}(N) \left[\left(\frac{1}{N} \right)^{\operatorname{Re} m+2} - \left(\frac{1}{2N} \right)^{\operatorname{Re} m+2} \right] \\
&= \mathrm{O}\left(\frac{1}{N^{\operatorname{Re} m+1}} \right).
\end{aligned}
$$

Then thanks to (13),

$$
\begin{aligned}
\int_{D_1} \left| \frac{\partial F_m}{\partial \xi} \right| d\xi d\eta &= \mathrm{O}(1) \int_{1/2N}^{1/N} d\xi \int_{-N/2}^{N/2} \frac{\xi^{\operatorname{Re} m} d\eta}{\left[(x-\xi)^2 + (\eta-y)^2 \right]^{1+\operatorname{Re} m/2}} \\
&= \mathrm{O}(N) \int_{1/2N}^{1/N} \xi^{\operatorname{Re} m} d\xi = \mathrm{O}\left(\frac{1}{N^{\operatorname{Re} m}} \right).
\end{aligned}
$$

On D_2: Due to (9), we have

$$\int_{D_2} |F_m|\,d\xi d\eta = O(1) \int_{1/N}^{N/2} d\xi \int_{N/2}^{N} \frac{\xi^{\mathrm{Re}\,m}\,d\eta}{\left[(x-\xi)^2 + (y-\eta)^2\right]^{\mathrm{Re}\,m/2}}$$

$$= O(1) \int_{1/N}^{N/2} d\xi \int_{N/2}^{N} \frac{\xi^{\mathrm{Re}\,m}\,d\eta}{|y - N/2|^{\mathrm{Re}\,m}} = O(N^2),$$

because we integrate a bounded function (independent of N) on a domain with measure controlled by $O(N^2)$.

Then, inequality (14) implies

$$\int_{D_2} \left|\frac{\partial F_m}{\partial \eta}\right| d\xi d\eta = O(1) \int_{1/N}^{N/2} d\xi \int_{N/2}^{N} \frac{|\eta - y|\xi^{\mathrm{Re}\,m}\,d\eta}{\left[(x-\xi)^2 + (y-\eta)^2\right]^{\mathrm{Re}\,m/2+1}}$$

$$= O(1) \int_{1/N}^{N/2} d\xi \int_{N/2}^{N} \frac{\xi^{\mathrm{Re}\,m}\,d\eta}{|y - \eta|^{\mathrm{Re}\,m+1}}$$

$$= O(1) \int_{1/N}^{N/2} d\xi \int_{N/2}^{N} \frac{N^{\mathrm{Re}\,m}\,d\eta}{|N/2 - y|^{\mathrm{Re}\,m+1}} = O(N).$$

On D_3: Due to (9), we have

$$\int_{D_3} |F_m|\,d\xi d\eta = O(1) \int_{N/2}^{N} d\xi \int_{-N/2}^{N/2} \frac{\xi^{\mathrm{Re}\,m}\,d\eta}{\left[(x-\xi)^2 + (y-\eta)^2\right]^{\mathrm{Re}\,m/2}}$$

$$= O(1) \int_{N/2}^{N} d\xi \int_{-N/2}^{N/2} \frac{\xi^{\mathrm{Re}\,m}\,d\eta}{\left[(x-N/2)^2 + (y-\eta)^2\right]^{\mathrm{Re}\,m/2}}$$

$$= O(N^{\mathrm{Re}\,m+1}) \int_{-N/2}^{N/2} \frac{d\eta}{\left[(x-N/2)^2 + (y-\eta)^2\right]^{\mathrm{Re}\,m/2}} = O(N^2).$$

Then, thanks to (12), we have

$$\int_{D_3} \left|\frac{\partial F_m}{\partial \xi}\right| d\xi d\eta = O(1) \int_{N/2}^{N} d\xi \int_{-N/2}^{N/2} \left(\frac{\xi^{\mathrm{Re}\,m-1}}{\left[(x-\xi)^2 + (y-\eta)^2\right]^{\mathrm{Re}\,m/2}} \right.$$

$$\left. + \frac{\xi^{\mathrm{Re}\,m}(\xi + x)}{\left[(x-\xi)^2 + (y-\eta)^2\right]^{\mathrm{Re}\,m/2+1}} \right) d\eta$$

$$= O(N^{\mathrm{Re}\,m}) \int_{-N/2}^{N/2} \frac{d\eta}{\left[(x-N/2)^2 + (y-\eta)^2\right]^{\mathrm{Re}\,m/2}}$$

$$+ O(N^{\mathrm{Re}\,m+2}) \int_{-N/2}^{N/2} \frac{d\eta}{\left[(x-N/2)^2 + (y-\eta)^2\right]^{\mathrm{Re}\,m/2+1}}$$

$$= O(N) + O(N) = O(N).$$

On D_4: This case is analogous to the case D_2.

On D_5: Due to (9), we have

$$\int_{D_5} |F_m|\, d\xi\, d\eta = \mathrm{O}(1) \int_{1/2N}^{1/N} d\xi \int_{N/2}^{N} \frac{\xi^{\mathrm{Re}\, m}\, d\eta}{\left[(x-\xi)^2 + (y-\eta)^2\right]^{\mathrm{Re}\, m/2}}$$
$$= \mathrm{O}\!\left(\frac{1}{N^{\mathrm{Re}\, m+1}}\right) \int_{N/2}^{N} \frac{d\eta}{|y-N/2|^{\mathrm{Re}\, m}} = \mathrm{O}\!\left(\frac{1}{N^{2\mathrm{Re}\, m}}\right).$$

Then, thanks to (12), we have

$$\int_{D_5} \left|\frac{\partial F_m}{\partial \xi}\right| d\xi\, d\eta = \mathrm{O}(1) \int_{1/2N}^{1/N} d\xi \int_{N/2}^{N} \left(\frac{\xi^{\mathrm{Re}\, m-1}}{\left[(x-\xi)^2 + (y-\eta)^2\right]^{\mathrm{Re}\, m/2}}\right.$$
$$\left. + \frac{\xi^{\mathrm{Re}\, m}(\xi + x)}{\left[(x-\xi)^2 + (y-\eta)^2\right]^{\mathrm{Re}\, m/2+1}}\right) d\eta$$
$$= \mathrm{O}(1) \int_{1/2N}^{1N} d\xi \int_{N/2}^{N} \left(\frac{\xi^{\mathrm{Re}\, m-1}}{|y-\eta|^{\mathrm{Re}\, m}} + \frac{\xi^{\mathrm{Re}\, m}(\xi + x)}{|y-\eta|^{\mathrm{Re}\, m+2}}\right) d\eta$$
$$= \mathrm{O}\!\left(\frac{1}{N^{2\mathrm{Re}\, m-1}}\right).$$

Applying inequality (14), we have

$$\int_{D_5} \left|\frac{\partial F_m}{\partial \eta}\right| d\xi\, d\eta = \mathrm{O}(1) \int_{1/2N}^{1/N} d\xi \int_{N/2}^{N} \frac{|\eta - y|\xi^{\mathrm{Re}\, m}\, d\eta}{\left[(x-\xi)^2 + (y-\eta)^2\right]^{\mathrm{Re}\, m/2+1}}$$
$$= \mathrm{O}(1) \int_{1/2N}^{1/N} d\xi \int_{N/2}^{N} \frac{\xi^{\mathrm{Re}\, m}\, d\eta}{|y-\eta|^{\mathrm{Re}\, m+1}} = \mathrm{O}\!\left(\frac{1}{N^{2\mathrm{Re}\, m+1}}\right).$$

On D_6: Due to (9), we have

$$\int_{D_6} |F_m|\, d\xi\, d\eta = \mathrm{O}(1) \int_{N/2}^{N} d\xi \int_{N/2}^{N} \frac{\xi^{\mathrm{Re}\, m}\, d\eta}{\left[(x-\xi)^2 + (y-\eta)^2\right]^{\mathrm{Re}\, m/2}}$$
$$= \mathrm{O}\!\left(N^{\mathrm{Re}\, m+1}\right) \int_{N/2}^{N} \frac{d\eta}{(N/2-y)^{\mathrm{Re}\, m}} = \mathrm{O}(N^2).$$

Then, thanks to (12), we obtain

$$
\int_{D_6} \left| \frac{\partial F_m}{\partial \xi} \right| d\xi d\eta = \mathrm{O}(1) \int_{N/2}^{N} d\xi \int_{N/2}^{N} \left(\frac{\xi^{\mathrm{Re}\,m-1}}{[(x-\xi)^2 + (y-\eta)^2]^{\mathrm{Re}\,m/2}} \right.
$$
$$
\left. + \frac{\xi^{\mathrm{Re}\,m}(\xi+x)}{[(x-\xi)^2 + (y-\eta)^2]^{\mathrm{Re}\,m/2+1}} \right) d\eta
$$
$$
= \mathrm{O}(1) \int_{N/2}^{N} d\xi \int_{N/2}^{N} \left(\frac{\xi^{\mathrm{Re}\,m-1}}{|y-\eta|^{\mathrm{Re}\,m}} + \frac{\xi^{\mathrm{Re}\,m}(\xi+x)}{|y-\eta|^{\mathrm{Re}\,m+2}} \right) d\eta
$$
$$
= \mathrm{O}(N) + \mathrm{O}(N^{\mathrm{Re}\,m+2}) \int_{N/2}^{N} \frac{d\eta}{|y-\eta|^{\mathrm{Re}\,m+2}} = \mathrm{O}(N).
$$

Finally, inequality (14) implies

$$
\int_{D_6} \left| \frac{\partial F_m}{\partial \eta} \right| d\xi d\eta = \mathrm{O}(1) \int_{N/2}^{N} d\xi \int_{N/2}^{N} \frac{|\eta - y| \xi^{\mathrm{Re}\,m} d\eta}{[(x-\xi)^2 + (y-\eta)^2]^{\mathrm{Re}\,m/2+1}}
$$
$$
= \mathrm{O}(N^{\mathrm{Re}\,m+1}) \int_{N/2}^{N} \frac{d\eta}{|y-\eta|^{\mathrm{Re}\,m+1}} = \mathrm{O}(N).
$$

On D_7, D_8: These cases are analogous to the cases D_6 and D_8, respectively. ∎

In the following table, we summarize results obtained on the previous lemmas:

i	$\sup_{D_i} |L_{-m}\phi_N|$	$\int_{D_i} |F_m|$	$(|\partial_\xi \phi_N|, |\partial_\eta \phi_N|)$	$\int_{D_i} |\partial_\xi F_m|$	$\int_{D_i} |\partial_\eta F_m|$
1	$\mathrm{O}(N^2)$	$\mathrm{O}(1/N^2)$	$(\mathrm{O}(N), 0)$	$\mathrm{O}(1/N)$	×
2	$\mathrm{O}(1/N^2)$	$\mathrm{O}(N^2)$	$(0, \mathrm{O}(1/N))$	×	$\mathrm{O}(N)$
3	$\mathrm{O}(1/N^2)$	$\mathrm{O}(N^2)$	$(\mathrm{O}(1/N), 0)$	$\mathrm{O}(N)$	×
4	$\mathrm{O}(1/N^2)$	$\mathrm{O}(N^2)$	$(0, \mathrm{O}(1/N))$	×	$\mathrm{O}(N)$
5	$\mathrm{O}(N^2)$	$\mathrm{O}(1/N^2)$	$(\mathrm{O}(N), \mathrm{O}(1/N))$	$\mathrm{O}(1/N)$	$\mathrm{O}(1/N^2)$
6	$\mathrm{O}(1/N^2)$	$\mathrm{O}(N^2)$	$(\mathrm{O}(1/N), \mathrm{O}(1/N))$	$\mathrm{O}(N)$	$\mathrm{O}(N)$
7	$\mathrm{O}(1/N^2)$	$\mathrm{O}(N^2)$	$(\mathrm{O}(1/N), \mathrm{O}(1/N))$	$\mathrm{O}(N)$	$\mathrm{O}(N)$
8	$\mathrm{O}(N^2)$	$\mathrm{O}(1/N^2)$	$(\mathrm{O}(N), \mathrm{O}(1/N))$	$\mathrm{O}(1/N)$	$\mathrm{O}(1/N^2)$

We can easily check that for each $i \in \{1, \ldots, 8\}$, the quantities

$$
\sup_{D_i} |L_{-m}\phi_N| \int_{D_i} |F_m|, \quad \sup_{D_i} |\partial_\xi \phi_N| \int_{D_i} |\partial_\xi F_m|, \quad \sup_{D_i} |\partial_\eta \phi_N| \int_{D_i} |\partial_\eta F_m|
$$

are bounded. Therefore,

$$
u(x, y) = \mathrm{o}(1) \quad \text{as} \quad N \to +\infty.
$$

Thus $u \equiv 0$ and this completes the proof of Proposition 5.3. □

Proposition 5.7 *Let $u \in \mathcal{D}(\mathbb{H}^+)$ and let $(x, y) \in \mathbb{H}^+$, define*

$$U(x, y) = \int_{\mathbb{H}^+} u(\xi, \eta) F_m(x, y, \xi, \eta) \, d\xi \, d\eta,$$

then $\lim_{\|(x,y)\| \to +\infty} U = 0$, and for all $y \in \mathbb{R}$, $\lim_{(0,y)} U = 0$. Moreover, $U \in C^\infty(\mathbb{H}^+ \setminus \operatorname{supp} u)$ and for all $(x, y) \notin \operatorname{supp} u$ we have $L_{m,x,y} U(x, y) = 0$.

Proof Fix (ξ, η). For Re $m < 1$,

$$F_m(x, y, \xi, \eta) = -\frac{\xi x^{1-m}}{2\pi} \int_0^\pi \frac{\sin^{1-m} \theta \, d\theta}{\left[(x - \xi)^2 + 4x\xi \sin^2 \theta/2 + (y - \eta)^2\right]^{1-m/2}},$$

hence $F_m(x, y, \xi, \eta) \to 0$ as $\|(x, y)\| \to +\infty$. For Re $m \geq 1$,

$$F_m(x, y, \xi, \eta) = -\frac{\xi^m}{2\pi} \int_0^\pi \sin^{m-1} \theta \left[\frac{1}{\left[(x - \xi)^2 + 4x\xi \sin^2 \theta/2 + (y - \eta)^2\right]^{m/2}} \right. $$
$$\left. - \frac{1}{\left[(x + \xi)^2 - 4x\xi \sin^2 \theta/2 + (y - \eta)^2\right]^{m/2}} \right] d\theta,$$

hence $F_m(x, y, \xi, \eta) \to 0$ as $\|(x, y)\| \to +\infty$. So the first statement of the proposition is shown.

For the second statement, for Re $m < 1$, we have

$$F_m(x, y, \xi, \eta) \sim -\frac{\xi x^{1-m}}{2\pi \left[\xi^2 + (y' - \eta)^2\right]^{1-m/2}} \int_0^\pi \sin^{1-m} \theta \, d\theta$$

as $(x, y) \to (0, y')$, which implies the desired result.

Now, assume that Re $m \geq 1$. Let (ξ, η) be fixed in the support of u, which is a compact set in $\mathbb{H}^+$. In particular, there exist $M > 0$ and $\alpha > 0$ which do not depend on u such that $\|(\xi, \eta)\| \leq M$ and $\xi \geq 2\alpha$. Let y be in $\mathbb{R}$. Denote

$$f_m(x) = \frac{1}{\left[(x - \xi)^2 + 4x\xi \sin^2 \theta/2 + (y - \eta)^2\right]^{m/2}}, \qquad x \in [-\alpha, \alpha].$$

By the mean value inequality, for $x > 0$ near 0, we have

$$|f_m(x) - f_m(0)| \leq x \sup_{[0,\alpha]} |f_m'|$$

and

$$|f_m(-x) - f_m(0)| \leq x \sup_{[-\alpha,0]} |f_m'|,$$

then

$$|f_m(x) - f_m(-x)| \le 2x \sup_{[-\alpha,\alpha]} |f'_m| \le 2x|m| \frac{3M + \alpha}{\alpha^{\operatorname{Re} m + 2}}.$$

In particular,

$$\sup_{\substack{(\xi,\eta)\in \operatorname{supp} u \\ y\in\mathbb{R}}} |F_m(x, y)| = \mathrm{O}(x)$$

as $x \to 0+$. The second statement is proved.

The last statement can be deduced from the fact that if $(x, y) \ne (\xi, \eta)$ are both in $\mathbb{H}^+$, then

$$L_{m,x,y} F_m(x, y, \xi, \eta) = 0. \qquad \square$$

Remark 5.8 If $U \in \mathcal{D}(\mathbb{H}^+)$, then $L_{m,x,y}U = u$, but this identity is not necessarily true if $U \notin \mathcal{D}(\mathbb{H}^+)$. In particular, we cannot say that in Proposition 5.7 we have $L_m U = u$.

Now, we will prove a decomposition theorem for axisymmetric potentials, it is interesting to compare it with the known result in [6, Section 4, Theorem 2]. The fundamental difference is that in this paper, the conductivity is not extended by reflection through the boundary $\partial\Omega$ to the whole domain.

Note that, due to our construction of fundamental solutions, the proof of this theorem is more or less the same as the proof of the decomposition theorem in [4, Chapter 9]. Note also that in our situation, the domain of our functions is $\mathbb{H}^+$ not $\mathbb{C}$.

Theorem 5.9 *Let $m \in \mathbb{C}$. Let Ω be an open set in $\mathbb{H}^+$ and let K be a compact set in Ω. If $u \in C^2(\Omega \setminus K)$ satisfies $L_m u = 0$ in $\Omega \setminus K$, then u has a unique decomposition as*

$$u = v + w,$$

where $v \in C^2(\Omega)$ satisfies $L_m v = 0$ in Ω and $w \in C^2(\mathbb{H}^+ \setminus K)$ satisfies $L_m w = 0$ in $\mathbb{H}^+ \setminus K$ with $\lim_{\partial\mathbb{H}^+} w = 0$.

Proof For $E \subset \mathbb{C}$ and $\rho > 0$, define $E_\rho = \{x \in \mathbb{C} : d(x, E) < \rho\}$, i.e. E_ρ is a neighborhood of E.

First, assume that Ω is a relatively compact open set in $\mathbb{H}^+$. Choose ρ small enough so that K_ρ and $(\partial\Omega)_\rho$ are disjoint. There is a function $\varphi_\rho \in \mathcal{D}(\mathbb{H}^+)$ compactly supported on $\Omega \setminus K$ such that $\varphi_\rho \equiv 1$ in a neighborhood of $\Omega \setminus (K_\rho \cup (\partial\Omega)_\rho)$ (Fig. 4).

For $z = x + iy \in \Omega \setminus (K_\rho \cup (\partial\Omega)_\rho)$, denote

$$F_z(\zeta) = F_m(x, y, \xi, \eta), \qquad L_\zeta = L_{m,\xi,\eta} \qquad \text{for} \quad \zeta = \xi + i\eta.$$
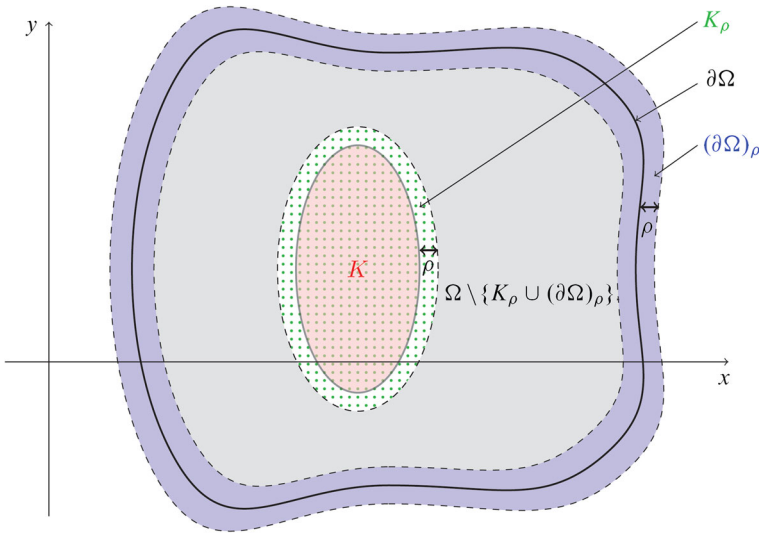
Fig. 4 $\varphi_\rho \equiv 1$ on the grey domain

Thanks to Proposition 4.7, we have

$$
u(z) = u\varphi_\rho(z) = \int_{\Omega_\rho} F_z(\zeta) L_\zeta(u\varphi_\rho)(\zeta)\, d\xi\, d\eta
$$

$$
= \int_{(\partial\Omega)_\rho} F_z(\zeta) L_\zeta(u\varphi_\rho)(\zeta)\, d\xi\, d\eta + \int_{K_\rho} F_z(\zeta) L_\zeta(u\varphi_\rho)(\zeta)\, d\xi\, d\eta
$$

$$
= v_\rho(z) + w_\rho(z).
$$

Then, the last result of Proposition 5.7 shows us that v_ρ satisfies $L_m v_\rho = 0$ on $\Omega \setminus (\partial\Omega)_\rho$ and w_ρ satisfies $L_m w_\rho = 0$ on $\mathbb{H}^+ \setminus K_\rho$. We also have $\lim_{\partial\mathbb{H}^+} w_\rho = 0$.

Now, assume that $\sigma < \rho$. As previously, we obtain the decomposition $u = v_\sigma + w_\sigma$ on $\Omega \setminus (K_\rho \cup (\partial\Omega)_\sigma)$. We claim that $v_\rho = v_\sigma$ on $\Omega \setminus (\partial\Omega)_\rho$ and $w_\rho = w_\sigma$ on $\mathbb{H}^+ \setminus K_\rho$. To see this, note that if $z \in \Omega \setminus (K_\rho \cup (\partial\Omega)_\rho)$, then $v_\rho(z) + w_\rho(z) = v_\sigma(z) + w_\sigma(z)$.

The function $w_\rho - w_\sigma$ satisfies (1) on $\mathbb{H}^+ \setminus K_\rho$, which is equal to $v_\sigma - v_\rho$ on $\Omega \setminus (K_\rho \cup (\partial\Omega)_\rho)$, therefore $v_\sigma - v_\rho$ extends to a solution of (1) on $\Omega \setminus (\partial\Omega)_\rho$. Finally, $w_\rho - w_\sigma$ extends to a solution of (1) on $\mathbb{H}^+$, and $\lim_{\partial\mathbb{H}^+} (w_\rho - w_\sigma) = 0$. Due to Proposition 5.3, we have $w_\rho = w_\sigma$, and hence $v_\rho = v_\sigma$.

For $z \in \Omega$, we can define $v(z) = v_\rho(z)$ for ρ small enough so that $z \in \Omega \setminus (\partial\Omega)_\rho$. Similarly, for $z \in \mathbb{H}^+ \setminus K$, we put $w(z) = w_\rho(z)$ for small ρ. Thus we have established the desired decomposition $u = v + w$.

Now, assume that Ω is an arbitrary domain of $\mathbb{H}^+$ and let u be a solution of $L_m u = 0$ on $\Omega \setminus K$. Choose $a \in \mathbb{H}^+$ and R large enough so that $K \subset D(a, R)$ and $D(a, R)$ is a relatively compact set in $\mathbb{H}^+$. Let $\omega = \Omega \cap D(a, R)$. Note that K is a compact set in ω which is a relatively compact open set in $\mathbb{H}^+$ and u satisfies (1) on $\omega \setminus K$. Applying

the results demonstrated for relatively compact open sets, we obtain

$$u(z) = \widetilde{v}(z) + \widetilde{w}(z)$$

for $z \in \omega \setminus K$, where $\widetilde{v}$ satisfies (1) on the set ω and $\widetilde{w}$ satisfies (1) on $\mathbb{H}^+ \setminus K$ with $\lim_{\partial \mathbb{H}^+} \widetilde{w} = 0$. Note that $V = u - \widetilde{w}$ satisfies (1) on $\Omega \setminus K$ and V can be extended to a solution of (1) in a neighborhood of K because $V = \widetilde{v}$ on ω. The sum $u = V + \widetilde{w}$ provides with the desired decomposition of u.

If we have another decomposition $u = v + w$ with $v \in C^2(\Omega)$, $L_m v = 0$ and with $w \in C^2(\mathbb{H}^+ \setminus K)$, $L_m w = 0$ and $\lim_{\partial \mathbb{H}^+} w = 0$, then we have $V - v = w - \widetilde{w}$ on $\Omega \setminus K$. The function $w - \widetilde{w}$ can be extended on $\mathbb{H}^+$ to a solution of $L_m(w - \widetilde{w}) = 0$ on $\mathbb{H}^+$ with $\lim_{\partial \mathbb{H}^+}(w - \widetilde{w}) = 0$. Thanks to Proposition 5.3, we obtain $w = \widetilde{w}$, then $V = v$, which completes the proof of the decomposition theorem. $\square$

The following proposition is a Poisson formula for axisymmetric potentials in $\mathbb{H}^+$.

Proposition 5.10 *Let $m \in \mathbb{C}$ be such that $\operatorname{Re} m < 1$ and $u : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function. Then there is a unique axisymmetric potential $U \in C^2(\mathbb{H}^+)$ such that $\lim_{\|(x,y)\| \to +\infty} U(x, y) = 0$ and for all $y \in \mathbb{R}$,*

$$\lim_{(0,y)} U = u(y).$$

Moreover, we have for all $(x, y) \in \mathbb{H}^+$,

$$U(x, y) = C_m x^{1-m} \int_{-\infty}^{\infty} \frac{u(\eta)\, d\eta}{(x^2 + (y - \eta)^2)^{1-m/2}}, \tag{15}$$

where

$$C_m = \frac{1 - m}{2\pi} \int_0^\pi \sin^{1-m} \theta\, d\theta = \frac{1}{2^m \pi} \frac{\Gamma^2(1 - m/2)}{\Gamma(1 - m)}.$$

Proof Let us first show that (15) is a solution of $L_m U = 0$. Let $f(x, y) = x^{1-m}/(x^2 + (y - \eta)^2)^{1-m/2}$. Interchanging differentiation and integration, it suffices to prove that $L_m f = 0$. We have

$$\partial_x f = \frac{(1 - m)x^{-m}}{(x^2 + (y - \eta)^2)^{1-m/2}} - \frac{(2 - m)x^{2-m}}{(x^2 + (y - \eta)^2)^{2-m/2}},$$

$$\partial_{xx} f = -\frac{m(1 - m)x^{-m-1}}{(x^2 + (y - \eta)^2)^{1-m/2}} - \frac{(2 - m)(3 - 2m)x^{1-m}}{(x^2 + (y - \eta)^2)^{2-m/2}}$$
$$+ \frac{(2 - m)(4 - m)x^{3-m}}{(x^2 + (y - \eta)^2)^{3-m/2}},$$

$$\partial_{yy} f = -\frac{(2 - m)x^{1-m}}{(x^2 + (y - \eta)^2)^{2-m/2}} + \frac{(2 - m)(4 - m)(y - \eta)^2 x^{1-m}}{(x^2 + (y - \eta)^2)^{3-m/2}}.$$

Then,

$$\Delta f = \frac{m(2-m)x^{1-m}}{(x^2+(y-\eta)^2)^{2-m/2}} - \frac{m(1-m)x^{-m-1}}{(x^2+(y-\eta)^2)^{1-m/2}}$$

and we deduce that $L_m f(x,y) = 0$.

We have

$$U(x,y) = C_m x^{1-m} \int_{-\infty}^{\infty} \frac{u(\eta)\,d\eta}{(x^2+(y-\eta)^2)^{1-m/2}}$$

$$= \frac{C_m}{x} \int_{-\infty}^{\infty} \frac{u(\eta)\,d\eta}{(1+((y-\eta)/x)^2)^{1-m/2}}.$$

By a change of variable $t = (y-\eta)/x$, we obtain

$$U(x,y) = C_m \int_{-\infty}^{\infty} \frac{u(y-tx)\,dt}{(1+t^2)^{1-m/2}}.$$

Thanks to the dominated convergence theorem, it suffices to show that

$$C_m \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{1-m/2}} = \frac{1-m}{2\pi} \int_0^{\pi} \sin^{1-m}\theta\,d\theta \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{1-m/2}} = 1.$$

To see this, according [1, p. 258], note that

$$\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{1-m/2}} = B\left(\frac{1}{2}, \frac{1-m}{2}\right) = \frac{\Gamma(1/2)\Gamma((1-m)/2)}{\Gamma(1-m/2)},$$

where B is the Euler beta function and

$$\frac{1-m}{2\pi} \int_0^{\pi} \sin^{1-m}\theta\,d\theta = \frac{1-m}{2\pi} 2^{1-m} B\left(1-\frac{m}{2}, 1-\frac{m}{2}\right)$$

$$= \frac{1-m}{2\pi} 2^{1-m} \frac{\Gamma^2(1-m/2)}{\Gamma(2-m)}.$$

By the duplication formula for the Γ function,

$$\Gamma(2z) = \pi^{-1/2} 2^{2z-1} \Gamma(z) \Gamma\left(z+\frac{1}{2}\right),$$

and by the recurrence formula $\Gamma(z+1) = z\Gamma(z)$, we obtain the desired result:

$$\frac{\Gamma(1/2)\Gamma((1-m)/2)}{\Gamma(1-m/2)} \frac{1-m}{2\pi} 2^{1-m} \frac{\Gamma^2(1-m/2)}{\Gamma(2-m)} = 1.$$

The uniqueness follows from Proposition 5.3. The proof is complete. □

Remark 5.11 One may ask if there is a reproducing formula for the case $\operatorname{Re} m \geq 1$. Let $m \in \mathbb{N}^*$ and let $u \in C^2(\overline{\mathbb{H}^+})$ be such that $L_m u = 0$ on $\mathbb{H}^+$, then the function v defined on $\mathbb{R}^{m+2}$ by

$$v(x_1, \ldots, x_{m+2}) = u\left(\sqrt{x_1^2 + \cdots + x_{m+1}^2}, x_{m+2}\right)$$

is harmonic on $(\mathbb{R}^{m+1})_* \times \mathbb{R}$. In particular, if $m \geq 2$, by [19, Proposition 18, p. 310], v can be extended to a harmonic function on $\mathbb{R}^{m+2}$, which tends to 0 at infinity. We then deduce that $v \equiv 0$ hence $u \equiv 0$. This shows that solving $L_m u = 0$ with u tending to 0 at infinity and with prescribed values of u on the y-axis is a problem which does not make sense. In this case, the fact that there is no solution to this Dirichlet problem is a consequence of the loss of ellipticity of L_m on the boundary of $\mathbb{H}^+$. Therefore, we do not deal with the case $\operatorname{Re} m \geq 1$.

6 Fourier–Legendre decomposition

First, we will introduce a specific system of coordinates (τ, θ) called bipolar coordinates, see [39]. The numerical applications on extremal bounded problems using this system of coordinates can be found in [25–27].

Let $\alpha > 0$. Suppose that there is a positive charge at $A = (-\alpha, 0)$ and a negative charge at $B = (\alpha, 0)$ (the absolute values of the two charges are identical). The potential generated by these charges at a point M is $\ln(MA/MB)$ (modulo a multiplicative constant) (Fig. 5).
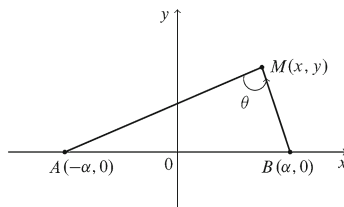
Definition 6.1 The coordinates

$$\tau = \ln \frac{MA}{MB}, \qquad \theta = \widehat{AMB}$$

are called *bipolar coordinates*.

The bipolar coordinates are related to the Cartesian coordinates by the following formulas:

$$x = \frac{\alpha \operatorname{sh} \tau}{\operatorname{ch} \tau - \cos \theta}, \qquad y = \frac{\alpha \sin \theta}{\operatorname{ch} \tau - \cos \theta}.$$

Fig. 5 Bipolar coordinates

Let $R > 0$ and $a = \sqrt{R^2 + \alpha^2}$, the disk with center $(a, 0)$ and radius R is defined in terms of bipolar coordinates as

$$\tau \geq \tau_0 = \ln\left(\frac{a}{R} + \sqrt{\frac{a^2}{R^2} - 1}\right) = \text{argch}\,\frac{a}{R}.$$

The right half-plane is defined as

$$\mathbb{H}^+ = \big\{(\tau, \theta) : \tau \in (0 + \infty], \theta \in [0, 2\pi)\big\}.$$

The level lines $\tau = \tau_0$ are circles with center $(\alpha \coth \tau_0, 0)$ and radii $\alpha/\text{sh}\,\tau_0$. This implies that for all τ_0, τ_1 such that $0 < \tau_0 < \tau_1$, the set $\{(\tau, \theta) : \tau \geq \tau_0\}$ is a closed disk and the set $\{(\tau, \theta) : 0 < \tau < \tau_1\}$ is the complement in $\mathbb{H}^+$ of the closed disk $\{\tau \geq \tau_1\}$ (Fig. 6).

The following theorem is well known in physics for $m = -1$ [3,15,42,45,47,48]. We extend it to $m \in \mathbb{C}$.

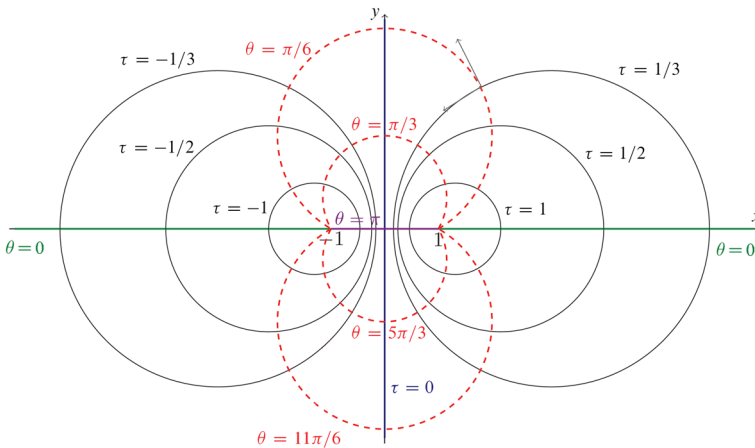**Theorem 6.2** *Let u be a solution to $L_m u = 0$ in an open set in $\mathbb{H}^+$. Let*

$$v_m(\tau, \theta) = \text{sh}^{(m-1)/2}\tau\,(\text{ch}\,\tau - \cos\theta)^{-m/2}u(\tau, \theta),$$

where, by definition,

$$\text{sh}^{(m-1)/2}\tau\,(\text{ch}\tau - \cos\theta)^{-m/2} = \exp\left(\frac{m-1}{2}\ln\text{sh}\,\tau - \frac{m}{2}\ln(\text{ch}\,\tau - \cos\theta)\right).$$

Then

$$\frac{\partial^2 v_m}{\partial \tau^2} + \frac{\partial^2 v_m}{\partial \theta^2} + \coth\tau\,\frac{\partial v_m}{\partial \tau} + \left(\frac{1}{4} - \frac{(m-1)^2}{4\,\text{sh}^2\tau}\right)v_m = 0. \tag{16}$$

Fig. 6 Level lines (with $\alpha = 1$)

Proof We have

$$\frac{\partial u}{\partial \tau} = \alpha \left[\frac{1 - \mathrm{ch}\,\tau \cos \theta}{(\mathrm{ch}\tau - \cos \theta)^2} \frac{\partial u}{\partial x} - \frac{\mathrm{sh}\,\tau \sin \theta}{(\mathrm{ch}\,\tau - \cos \theta)^2} \frac{\partial u}{\partial y} \right],$$

$$\frac{\partial u}{\partial \theta} = \alpha \left[\frac{-\mathrm{sh}\,\tau \sin \theta}{(\mathrm{ch}\,\tau - \cos \theta)^2} \frac{\partial u}{\partial x} + \frac{\mathrm{ch}\,\tau \cos \theta - 1}{(\mathrm{ch}\,\tau - \cos \theta)^2} \frac{\partial u}{\partial y} \right].$$

Thus, we obtain

$$\frac{\partial u}{\partial x} = \frac{1}{\alpha} \left((1 - \mathrm{ch}\,\tau \cos \theta) \frac{\partial u}{\partial \tau} - \mathrm{sh}\,\tau \sin \theta \frac{\partial u}{\partial \theta} \right),$$

and

$$\begin{aligned}
\frac{\partial^2 u}{\partial \tau^2} = {} & \frac{\alpha^2}{(\mathrm{ch}\tau - \cos \theta)^4} \left[(1 - \mathrm{ch}\,\tau \cos \theta)^2 \frac{\partial^2 u}{\partial x^2} + \mathrm{sh}^2\tau \sin^2 \theta \frac{\partial^2 u}{\partial y^2} \right. \\
& \left. - 2(1 - \mathrm{ch}\,\tau \cos \theta)\,\mathrm{sh}\,\tau \sin \theta \frac{\partial^2 u}{\partial x \partial y} \right] \\
& + \frac{\alpha}{(\mathrm{ch}\,\tau - \cos \theta)^3} \left[\mathrm{sh}\,\tau \left(\cos^2 \theta + \mathrm{ch}\,\tau \cos \theta - 2 \right) \frac{\partial u}{\partial x} \right. \\
& \left. + \sin \theta \left(\mathrm{ch}^2\tau - 2 + \cos \theta\,\mathrm{ch}\,\tau \right) \frac{\partial u}{\partial y} \right]
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 u}{\partial \theta^2} = {} & \frac{\alpha^2}{(\mathrm{ch}\,\tau - \cos \theta)^4} \left[\mathrm{sh}^2\tau \sin^2 \theta \frac{\partial^2 u}{\partial x^2} + (\mathrm{ch}\,\tau \cos \theta - 1) \frac{\partial^2 u}{\partial y^2} \right. \\
& \left. + 2(1 - \mathrm{ch}\,\tau \cos \theta)\,\mathrm{sh}\tau \sin \theta \frac{\partial^2 u}{\partial x \partial y} \right] \\
& + \frac{\alpha}{(\mathrm{ch}\,\tau - \cos \theta)^3} \left[\mathrm{sh}\,\tau \left(2 - \cos^2 \theta - \cos \theta\,\mathrm{ch}\,\tau \right) \frac{\partial u}{\partial x} \right. \\
& \left. + \sin \theta \left(2 - \mathrm{ch}^2\tau - \mathrm{ch}\,\tau \cos \theta \right) \frac{\partial u}{\partial y} \right].
\end{aligned}$$

In particular, we have

$$\frac{\partial^2 u}{\partial \tau^2} + \frac{\partial^2 u}{\partial \theta^2} = \frac{\alpha^2}{(\mathrm{ch}\,\tau - \cos \theta)^2} \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right].$$

Therefore, we obtain

$$L_{m,x,y} u = \left(\frac{\operatorname{ch}\tau - \cos\theta}{\alpha} \right)^2 \left(\frac{\partial^2 u}{\partial \tau^2} + \frac{\partial^2 u}{\partial \theta^2} + \frac{m(1 - \operatorname{ch}\tau \cos\theta)}{\operatorname{sh}\tau (\operatorname{ch}\tau - \cos\theta)} \frac{\partial u}{\partial \tau} \right. $$
$$\left. - \frac{m \sin\theta}{\operatorname{ch}\tau - \cos\theta} \frac{\partial u}{\partial \theta} \right).$$

According to definition of v_m,

$$u(\tau, \theta) = \frac{(\operatorname{ch}\tau - \cos\theta)^{m/2}}{\operatorname{sh}^{(m-1)/2}\tau} v_m(\tau, \theta).$$

Denote

$$r_m(\tau, \theta) = \frac{(\operatorname{ch}\tau - \cos\theta)^{m/2}}{\operatorname{sh}^{(m-1)/2}\tau},$$

then

$$\frac{\partial r_m}{\partial \theta} = \frac{m}{2} \frac{\sin\theta}{\operatorname{ch}\tau - \cos\theta} r_m,$$
$$\frac{\partial^2 r_m}{\partial \theta^2} = \frac{m}{4(\operatorname{ch}\tau - \cos\theta)^2} \left(2\cos\theta \operatorname{ch}\tau + m\sin^2\theta - 2 \right) r_m$$

and

$$\frac{\partial r_m}{\partial \tau} = \frac{1}{(\operatorname{ch}\tau - \cos\theta) \operatorname{sh}\tau} \left(\operatorname{ch}^2\tau + (m-1)\operatorname{ch}\tau \cos\theta - m \right) r_m,$$
$$\frac{\partial^2 r_m}{\partial \tau^2} = \frac{1}{4(\operatorname{ch}\tau - \cos\theta)^2 \operatorname{sh}^2\tau} \left[\operatorname{ch}^4\tau - 2\operatorname{ch}^3\tau \cos\theta + (m-1)^2\operatorname{ch}^2\tau \cos^2\theta \right.$$
$$+ 2(m-1)\operatorname{ch}^2\tau + (4 - 2m^2)\operatorname{ch}\tau \cos\theta$$
$$\left. + 2(m-1)\cos^2\theta + m(m-2) \right] r_m.$$

Hence the equation $L_{m,x,y} u = 0$ can be rewritten as

$$r_m \left(\frac{\partial^2 v_m}{\partial \tau^2} + \frac{\partial^2 v_m}{\partial \theta^2} \right) + \frac{\partial v_m}{\partial \tau} \left(2\frac{\partial r_m}{\partial \tau} + \frac{m}{\operatorname{sh}\tau} \frac{1 - \operatorname{ch}\tau \cos\theta}{\operatorname{ch}\tau - \cos\theta} r_m \right)$$
$$+ \frac{\partial v_m}{\partial \theta} \left(2\frac{\partial r_m}{\partial \theta} - \frac{m \sin\theta}{\operatorname{ch}\tau - \cos\theta} r_m \right)$$
$$+ v_m \left(\frac{\partial^2 r_m}{\partial \tau^2} + \frac{\partial^2 r_m}{\partial \theta^2} + \frac{m(1 - \operatorname{ch}\tau \cos\theta)}{\operatorname{sh}\tau (\operatorname{ch}\tau - \cos\theta)} \frac{\partial r_m}{\partial \tau} \right.$$
$$\left. - \frac{m \sin\theta}{\operatorname{ch}\tau - \cos\theta} \frac{\partial r_m}{\partial \theta} \right) = 0$$

with

$$2\frac{\partial r_m}{\partial \tau} + \frac{m}{\operatorname{sh}\tau}\frac{1 - \operatorname{ch}\tau \cos\theta}{\operatorname{ch}\tau - \cos\theta} r_m = r_m \coth\tau,$$

$$2\frac{\partial r_m}{\partial \theta} - \frac{m \sin\theta}{\operatorname{ch}\tau - \cos\theta} r_m = 0$$

and

$$\frac{\partial^2 r_m}{\partial \tau^2} + \frac{\partial^2 r_m}{\partial \theta^2} + \frac{m(1 - \operatorname{ch}\tau \cos\theta)}{\operatorname{sh}\tau\,(\operatorname{ch}\tau - \cos\theta)}\frac{\partial r_m}{\partial \tau}$$

$$- \frac{m \sin\theta}{\operatorname{ch}\tau - \cos\theta}\frac{\partial r_m}{\partial \theta} = \left(\frac{1}{4} - \frac{(m-1)^2}{4\operatorname{sh}^2\tau}\right) r_m.$$

This completes the proof.   □

Let us apply the method of separation of variables, i.e. assume v_m has the form $v_m(\tau,\theta) = A_m(\tau)B_m(\theta)$. Then (16) becomes

$$\frac{A_m''}{A_m} + \coth\tau\,\frac{A_m'}{A_m} + \frac{1}{4} - \frac{(m-1)^2}{4\operatorname{sh}^2\tau} = -\frac{B_m''}{B_m}.$$

The term on the right depends only on θ and the left-hand side depends only on τ, thus we deduce that both are constant. Let $n \in \mathbb{C}$ be such that this constant is equal to n^2. Then we have

$$\begin{cases} A_m'' + \coth\tau\, A_m' + \left(\dfrac{1}{4} - \dfrac{(m-1)^2}{4\operatorname{sh}^2\tau} - n^2\right) A_m = 0, \\ B_m'' + n^2 B_m = 0. \end{cases}$$

The function B_m is a 2π-periodic function as θ represents an angle, therefore n should necessarily be an integer.

To examine the equation satisfied by A_m, we carry out the following change of function

$$A_m(\tau) = C_m(\operatorname{ch}\tau).$$

Then, C_m satisfies

$$\operatorname{sh}^2\tau\, C_m''(\operatorname{ch}\tau) + 2\operatorname{ch}\tau\, C_m'(\operatorname{ch}\tau) + \left(\frac{1}{4} - n^2 - \frac{(m-1)^2}{4\operatorname{sh}^2\tau}\right) C_m(\operatorname{ch}\tau) = 0$$

which can be rewritten as

$$(1 - \operatorname{ch}^2\tau)\, C_m''(\operatorname{ch}\tau) - 2\operatorname{ch}\tau\, C_m'(\operatorname{ch}\tau)$$

$$+ \left(n^2 - \frac{1}{4} - \frac{((m-1)/2)^2}{1 - \operatorname{ch}^2\tau}\right) C_m(\operatorname{ch}\tau) = 0. \tag{LAH}$$

This equation is called the *hyperbolic associated Legendre equation*. Note that if we put $z = \operatorname{ch}\tau$ and $u(z) = C_m(\operatorname{ch}\tau)$, then

$$(1 - z^2)u'' - 2zu' + \left[\nu(\nu+1) - \frac{\mu^2}{1 - z^2} \right] u = 0, \qquad \text{(LA)}$$

where

$$\nu = n - \frac{1}{2}, \qquad \mu = \frac{m-1}{2}.$$

This equation is called the *associated Legendre equation*, and it can be reduced to the Legendre equation if $\mu = 0$:

$$(1 - z^2)u'' - 2zu' + \nu(\nu+1)u = 0. \qquad \text{(L)}$$

Two independent solutions of (LA) are given in Appendix, where they are denoted by $P_\nu^\mu(\operatorname{ch}\tau)$ and $Q_\nu^\mu(\operatorname{ch}\tau)$.

Theorem 6.3 *Let $m \in \mathbb{C}$ and $0 < \tau_0$. Let u be a smooth solution to $L_m u = 0$ on the disk $\tau \geq \tau_0$ and let v be a smooth solution to $L_m v = 0$ on $\mathbb{H}^+ \setminus \{\tau > \tau_0\}$, which is the complement in $\mathbb{H}^+$ of the disk $\{\tau > \tau_0\}$, and assume that $\lim_{\partial\mathbb{H}^+} v = 0$. Then there are two sequences $(a_n)_{n\in\mathbb{Z}}$ and $(b_n)_{n\in\mathbb{Z}}$ of $\ell^2(\mathbb{Z})$ (rapidly decreasing) such that*

$$u = \sum_{n=-\infty}^{+\infty} a_n Q_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau)\,\operatorname{sh}^{(1-m)/2}\tau\,(\operatorname{ch}\tau - \cos\theta)^{m/2} e^{in\theta},$$

$$v = \sum_{n=-\infty}^{+\infty} b_n P_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau)\,\operatorname{sh}^{(1-m)/2}\tau\,(\operatorname{ch}\tau - \cos\theta)^{m/2} e^{in\theta}.$$

The sequence (a_n) is unique. In addition, the convergence of the first series is uniform on every compact set $[\tau_1, \tau_2]$ of the disk $\tau > \tau_0$ with $\tau_0 \leq \tau_1 < \tau_2$. And the convergence of the second one is uniform on every compact set $[\tau_3, \tau_4]$ of the complement of the disk $\tau > \tau_0$ in $\mathbb{H}^+$ with $0 < \tau_3 < \tau_4 \leq \tau_0$.

If $\operatorname{Re} m < 1$, then the sequence (b_n) is unique.

Proof Indeed, decomposing the function

$$\theta \mapsto u(\tau_0, \theta)(\operatorname{ch}\tau_0 - \cos\theta)^{-m/2}\operatorname{sh}^{(m-1)/2}\tau_0$$

as Fourier series with respect to the variable θ, yields the following Fourier expansion for $u(\tau_0, \cdot)$:

$$u(\tau_0, \theta) = \operatorname{sh}^{(1-m)/2}\tau_0\,(\operatorname{ch}\tau_0 - \cos\theta)^{m/2} \sum_{n=-\infty}^{+\infty} a_n e^{in\theta},$$

where $a_n \in \ell^2(\mathbb{Z})$ satisfies

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} (\mathrm{ch}\,\tau_0 - \cos\theta)^{-m/2} \mathrm{sh}^{(m-1)/2}\tau_0\, u(\tau_0, s) e^{-ins}\, ds.$$

The function $u(\tau_0, \cdot)$ is a smooth function of the variable θ, therefore the sequence $(a_n)_n$ is rapidly decreasing as $|n| \to +\infty$.

The function

$$\tilde{u}(\tau, \theta) = \mathrm{sh}^{(1-m)/2}\tau\,(\mathrm{ch}\,\tau - \cos\theta)^{m/2} \sum_{n=-\infty}^{+\infty} a_n \frac{Q_{n-1/2}^{(m-1)/2}(\mathrm{ch}\,\tau)}{Q_{n-1/2}^{(m-1)/2}(\mathrm{ch}\,\tau_0)} e^{in\theta}$$

coincides with u on the circle $\tau = \tau_0$. Let us see that $\tilde{u}$ is well defined on the disk $\tau \geq \tau_0$. Indeed, thanks to Proposition 8.1, as $|n| \to +\infty$,

$$\frac{Q_{n-1/2}^{(m-1)/2}(\mathrm{ch}\,\tau)}{Q_{n-1/2}^{(m-1)/2}(\mathrm{ch}\,\tau_0)} \sim \sqrt{\frac{\mathrm{sh}\,\tau_0}{\mathrm{sh}\,\tau}}\, e^{|n|(\tau_0 - \tau)}$$

and this equivalence is uniform on all compact sets $[\tau_1, \tau_2]$ with $0 < \tau_0 \leq \tau_1 < \tau_2$. It follows that the series which defines $\tilde{u}$ is norm convergent on any compact set $[\tau_1, \tau_2]$ of the disk $\tau \geq \tau_0$. The same is true for derivatives with respect to τ and θ (which are also expressed through the associated Legendre functions, see Appendix).

Due to the fact that the solution of an elliptic equation is uniquely determined by its boundary values (this follows from the maximum principle), we deduce that $\tilde{u}$ is the unique axisymmetric potential on the disk $\tau \geq \tau_0$ which coincides with u on the circle $\tau = \tau_0$.

For v, the proof is similar. Indeed, decomposing the function

$$\theta \mapsto v(\tau_0, \theta)(\mathrm{ch}\,\tau_0 - \cos\theta)^{-m/2} \mathrm{sh}^{(m-1)/2}\tau_0$$

as Fourier series with respect to the variable θ, yields the following Fourier expansion for $v(\tau_0, \cdot)$:

$$v(\tau_0, \theta) = \mathrm{sh}^{(1-m)/2}\tau_0\,(\mathrm{ch}\,\tau_0 - \cos\theta)^{m/2} \sum_{n=-\infty}^{+\infty} b_n e^{in\theta},$$

where $b_n \in \ell^2(\mathbb{Z})$ satisfies

$$b_n = \frac{1}{2\pi} \int_0^{2\pi} (\mathrm{ch}\,\tau_0 - \cos\theta)^{-m/2} \mathrm{sh}^{(m-1)/2}\tau_0\, v(\tau_0, s) e^{-ins}\, ds.$$

The function $v(\tau_0, \cdot)$ is a smooth function of the variable θ, therefore the sequence $(b_n)_n$ is rapidly decreasing as $|n| \to +\infty$.

The function

$$\widetilde{v}(\tau, \theta) = \mathrm{sh}^{(1-m)/2}\tau \, (\mathrm{ch}\,\tau - \cos\theta)^{m/2} \sum_{n=-\infty}^{+\infty} b_n \frac{P_{n-1/2}^{(m-1)/2}(\mathrm{ch}\,\tau)}{P_{n-1/2}^{(m-1)/2}(\mathrm{ch}\,\tau_0)} e^{in\theta}$$

coincides with v on the circle $\tau = \tau_0$. Let us see that $\widetilde{v}$ is well defined on the complement of the disk $\tau > \tau_0$. Indeed, thanks to Proposition 8.1, as $|n| \to +\infty$,

$$\frac{P_{n-1/2}^{(m-1)/2}(\mathrm{ch}\,\tau)}{P_{n-1/2}^{(m-1)/2}(\mathrm{ch}\,\tau_0)} \sim \sqrt{\frac{\mathrm{sh}\,\tau_0}{\mathrm{sh}\,\tau}} \, e^{|n|(\tau-\tau_0)}$$

and this equivalence is uniform on all compact sets $[\tau_1, \tau_2]$ with $0 < \tau_1 < \tau_2 \leq \tau_0$. It follows that the series which defines $\widetilde{v}$ is norm convergent on any compact set $[\tau_1, \tau_2]$ of the complement of the disc $\tau > \tau_0$. The same is true for derivatives with respect to τ and θ.

We will show that

$$\lim_{\tau\to 0+} \widetilde{v} = 0.$$

If $\mathrm{Re}\, m < 1$, by formula (18), for $n \in \mathbb{N}$ we have

$$P_{n-1/2}^{(m-1)/2}(\mathrm{ch}\,\tau) = \frac{2^{(m-1)/2}}{\sqrt{\pi}\,\Gamma(1-m/2)} \, \mathrm{sh}^{(1-m)/2}\tau \int_0^\pi (\mathrm{ch}\,\tau + \mathrm{sh}\,\tau\cos\theta)^{n+m/2-1} \sin^{1-m}\theta \, d\theta,$$

hence

$$\lim_{\tau\to 0+} P_{n-1/2}^{(m-1)/2}(\mathrm{ch}\,\tau) = 0.$$

In addition, for $n > 1 - \mathrm{Re}\, m/2$, we have

$$\begin{aligned}
\left|P_{n-1/2}^{(m-1)/2}(\mathrm{ch}\,\tau)\right| &\leq \frac{2^{(\mathrm{Re}\, m-1)/2}\,\mathrm{sh}^{(1-\mathrm{Re}\, m)/2}\tau}{\sqrt{\pi}\,|\Gamma(1-m/2)|} \\
&\qquad \int_0^\pi (\mathrm{ch}\,\tau + \mathrm{sh}\,\tau\cos\theta)^{n+\mathrm{Re}\, m/2-1} \sin^{1-\mathrm{Re}\, m}\theta \, d\theta \\
&\leq \frac{2^{(\mathrm{Re}\, m-1)/2}\,\mathrm{sh}^{(1-\mathrm{Re}\, m)/2}\tau}{\sqrt{\pi}\,|\Gamma(1-m/2)|} \\
&\qquad \int_0^\pi (\mathrm{ch}\,\tau + \mathrm{sh}\,\tau)^{n+\mathrm{Re}\, m/2-1} \sin^{1-\mathrm{Re}\, m}\theta \, d\theta \\
&\leq C_m \, \mathrm{sh}^{(1-\mathrm{Re}\, m)/2}\tau \, e^{(n+\mathrm{Re}\, m/2)\tau}
\end{aligned}$$

thus

$$\sum_{n>1-\operatorname{Re} m/2} \sup_{\tau \in [0, \tau_0/2]} \left| b_n \frac{P_{n-1/2}^{(m-1)/2}(\operatorname{ch} \tau)}{P_{n-1/2}^{(m-1)/2}(\operatorname{ch} \tau_0)} e^{in\theta} \right| < +\infty.$$

By Proposition 8.1, we obtain

$$P_{n-1/2}^{(m-1)/2}(\operatorname{ch} \tau_0) \sim \frac{n^{m/2-1}}{\sqrt{2\pi \operatorname{sh} \tau_0}} e^{n\tau_0} \quad \text{as} \quad n \to +\infty.$$

So, $\lim_{\tau \to 0+} \widetilde{v} = 0$.

The uniqueness of (b_n) for $\operatorname{Re} m < 1$ follows from the following fact established in the next section: the families

$$\mathcal{A} = \left(\frac{Q_{n-1/2}^{(m-1)/2}(\operatorname{ch} \tau)}{Q_{n-1/2}^{(m-1)/2}(\operatorname{ch} \tau_0)} \frac{(\operatorname{ch} \tau - \cos \theta)^{m/2}}{\operatorname{sh}^{(m-1)/2} \tau} e^{in\theta} \right)_{n \in \mathbb{Z}} = (a_n)_{n \in \mathbb{Z}},$$

$$\mathcal{B} = \left(\frac{P_{n-1/2}^{(m-1)/2}(\operatorname{ch} \tau)}{P_{n-1/2}^{(m-1)/2}(\operatorname{ch} \tau_1)} \frac{(\operatorname{ch} \tau - \cos \theta)^{m/2}}{\operatorname{sh}^{(m-1)/2} \tau} e^{in\theta} \right)_{n \in \mathbb{Z}} = (b_n)_{n \in \mathbb{Z}}$$

form a Riesz basis.   □

Corollary 6.4 *The solution of the Dirichlet problem for $L_m u = 0$ on $D((a, 0), R)$, with $u = \varphi$ on $\partial D((a, 0), R)$, is given by*

$$u(\tau, \theta) = \operatorname{sh}^{(1-m)/2} \tau \, (\operatorname{ch} \tau - \cos \theta)^{m/2} \sum_{n=-\infty}^{+\infty} c_n \frac{Q_{n-1/2}^{(m-1)/2}(\operatorname{ch} \tau)}{Q_{n-1/2}^{(m-1)/2}(\operatorname{ch} \tau_0)} e^{in\theta},$$

where $\{\tau = \tau_0\}$ corresponds to the circle with center $(a, 0)$ and radius R and

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} (\operatorname{ch} \tau_0 - \cos \theta)^{-m/2} \operatorname{sh}^{(m-1)/2} \tau_0 \, \varphi(a + R \cos s, R \sin s) e^{-ins} \, ds.$$

Similarly, the function

$$v(\tau, \theta) = \operatorname{sh}^{(1-m)/2} \tau \, (\operatorname{ch} \tau - \cos \theta)^{m/2} \sum_{n=-\infty}^{+\infty} c_n \frac{P_{n-1/2}^{(m-1)/2}(\operatorname{ch} \tau)}{P_{n-1/2}^{(m-1)/2}(\operatorname{ch} \tau_0)} e^{in\theta}, \qquad (17)$$

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} (\operatorname{ch} \tau_0 - \cos \theta)^{-m/2} \operatorname{sh}^{(m-1)/2} \tau_0 \, \varphi(a + R \cos s, R \sin s) e^{-ins} \, ds,$$

is a solution to $L_m v = 0$ on $\mathbb{H}^+ \setminus D((a, 0), R)$, which is equal to φ on $\partial D((a, 0), R)$.

Moreover, if $\operatorname{Re} m < 1$, then v satisfies $\lim_{\partial \mathbb{H}^+} v = 0$, and (17) is the unique solution of the Dirichlet problem $L_m v = 0$ on $\mathbb{H}^+ \setminus D((a, 0), R)$ which vanishes on $\partial \mathbb{H}^+$.

7 Riesz basis

We will prove that the first group of functions of the following family:

$$
\left(\frac{(\operatorname{ch}\tau - \cos\theta)^{m/2}}{\operatorname{sh}^{(m-1)/2}\tau}\left\{\begin{array}{l}\cos(n\theta)\\ \sin(n\theta)\end{array}\right\}\left\{\begin{array}{l}P_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau)\\ Q_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau)\end{array}\right\}\right)_{n\in\mathbb{Z}}
$$

is a basis of solutions on the disk $\tau \geq \tau_1$ and the second one is a basis of solutions on $\tau \leq \tau_0$, which is the complement on $\mathbb{H}^+$ of some disk, with $0 < \tau_0 < \tau_1$. This fact is known for $m = -1$, namely, for $\mu = 1$. We extend this result for complex m.

Let us recall the definition of a Riesz basis. The sequence $(x_n)_{n\in\mathbb{N}}$ is called a *quasi-orthogonal* or *Riesz sequence* of a Hilbert space X if there are two constants $c, C > 0$ such that for all sequences $(a_n)_{n\in\mathbb{Z}}$ in ℓ^2, we have

$$
c^2\sum_n |a_n|^2 \leq \left\|\sum_n a_n x_n\right\|^2 \leq C^2\sum_n |a_n|^2.
$$

If the family $(x_n)_{n\in\mathbb{Z}}$ is complete, it is called a *Riesz basis*. The matrix of scalar products $\{\langle x_i, x_j\rangle\}_{i,j}$ is called the *Gram matrix associated to* $\{x_i\}_i$. Let us recall the following characterization of a Riesz basis by the Gram matrix.

Property ([43, p. 170]) *A family $\{x_i\}_i$ is a Riesz basis for some Hilbert space X if $\{x_i\}_i$ is complete in X and the Gram matrix associated to $\{x_i\}_i$ defines a bounded and invertible operator on $\ell^2(\mathbb{N})$.*

Let $\mathcal{A}$ and $\mathcal{B}$ be the families of solutions to $L_m u = 0$, respectively, inside the disk $\tau > \tau_0$ and outside the disk $\tau > \tau_1$, with $0 < \tau_0 < \tau_1$:

$$
\mathcal{A} = \left(\frac{Q_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau)}{Q_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_0)}\frac{(\operatorname{ch}\tau - \cos\theta)^{m/2}}{\operatorname{sh}^{(m-1)/2}\tau}e^{in\theta}\right)_{n\in\mathbb{Z}} = (a_n)_{n\in\mathbb{Z}},
$$

$$
\mathcal{B} = \left(\frac{P_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau)}{P_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_1)}\frac{(\operatorname{ch}\tau - \cos\theta)^{m/2}}{\operatorname{sh}^{(m-1)/2}\tau}e^{in\theta}\right)_{n\in\mathbb{Z}} = (b_n)_{n\in\mathbb{Z}}.
$$

Let $\mathcal{C}$ be the union of these two families,

$$
\mathcal{C} = (c_n)_{n\in\mathbb{Z}} = \left(c_{2n} = a_n \text{ and } c_{2n+1} = b_n\right)_{n\in\mathbb{Z}}.
$$

Denote the annulus defined in terms of bipolar coordinates $\{0 < \tau_0 < \tau < \tau_1\}$ by $\mathbb{A}$. The space $L^2(\partial\mathbb{A})$ is equipped with the following inner product: for $f, g \in L^2(\partial\mathbb{A})$,

$$
\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\tau_0, \theta)\overline{g(\tau_0, \theta)}\frac{\operatorname{sh}^{\operatorname{Re}m-1}\tau_0}{(\operatorname{ch}\tau_0 - \cos\theta)^{\operatorname{Re}m}}\,d\theta
$$
$$
+ \frac{1}{2\pi}\int_0^{2\pi} f(\tau_1, \theta)\overline{g(\tau_1, \theta)}\frac{\operatorname{sh}^{\operatorname{Re}m-1}\tau_1}{(\operatorname{ch}\tau_1 - \cos\theta)^{\operatorname{Re}m}}\,d\theta.
$$

Proposition 7.1 $\mathcal{C}$ *is a Riesz basis in the Hilbert space* $L^2(\partial\mathbb{A})$.

Proof To find the Gram matrix for $\mathcal{C}$, first calculate all its elements. For $n \in \mathbb{Z}$,

$$\langle c_{2n}, c_{2n} \rangle = 1 + \left| \frac{Q_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_1)}{Q_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_0)} \right|^2,$$

$$\langle c_{2n+1}, c_{2n+1} \rangle = 1 + \left| \frac{P_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_0)}{P_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_1)} \right|^2,$$

$$\langle c_{2n}, c_{2n+1} \rangle = \overline{\frac{P_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_0)}{P_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_1)}} + \frac{Q_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_1)}{Q_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_0)},$$

$$\langle c_{2n+1}, c_{2n} \rangle = \overline{\langle c_{2n}, c_{2n+1} \rangle}.$$

In all other cases, the inner product is zero, hence the Gram matrix is diagonal by blocks and each block is the following 2×2 matrix:

$$M_n = \begin{pmatrix} 1 + \left| \dfrac{Q_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_1)}{Q_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_0)} \right|^2 & \overline{\dfrac{P_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_0)}{P_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_1)}} + \dfrac{Q_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_1)}{Q_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_0)} \\[2em] \dfrac{P_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_0)}{P_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_1)} + \overline{\dfrac{Q_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_1)}{Q_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_0)}} & 1 + \left| \dfrac{P_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_0)}{P_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_1)} \right|^2 \end{pmatrix}.$$

The Gram matrix G has the form

$$G = \begin{pmatrix} M_0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & M_{-1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & 0 & M_1 & 0 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \ddots & 0 & M_{-2} & 0 & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & 0 & \ddots & \ddots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & M_{-n} & \ddots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & M_n & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

The determinant of M_n is

$$\det M_n = \left| 1 - \frac{Q_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_1)\, P_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_0)}{Q_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_0)\, P_{n-1/2}^{(m-1)/2}(\operatorname{ch}\tau_1)} \right|^2.$$

Let us show that M_n is invertible. Suppose the contrary, then $\det M_n = 0$, which is equivalent to

$$Q_{n-1/2}^{(m-1)/2}(\text{ch } \tau_1) \, P_{n-1/2}^{(m-1)/2}(\text{ch } \tau_0) = Q_{n-1/2}^{(m-1)/2}(\text{ch } \tau_0) \, P_{n-1/2}^{(m-1)/2}(\text{ch } \tau_1).$$

It can be rewritten as

$$\begin{vmatrix} Q_{n-1/2}^{(m-1)/2}(\text{ch } \tau_0) & P_{n-1/2}^{(m-1)/2}(\text{ch } \tau_0) \\ Q_{n-1/2}^{(m-1)/2}(\text{ch } \tau_1) & P_{n-1/2}^{(m-1)/2}(\text{ch } \tau_1) \end{vmatrix} = 0,$$

with $P_{n-1/2}^{(m-1)/2}(\text{ch } \tau_0), Q_{n-1/2}^{(m-1)/2}(\text{ch } \tau_1) \neq 0$. Therefore, there is $\lambda \in \mathbb{C} \setminus \{0\}$ (which depends on m, n, τ_0 and τ_1) such that

$$\begin{cases} Q_{n-1/2}^{(m-1)/2}(\text{ch } \tau_0) = \lambda P_{n-1/2}^{(m-1)/2}(\text{ch } \tau_0), \\ Q_{n-1/2}^{(m-1)/2}(\text{ch } \tau_1) = \lambda P_{n-1/2}^{(m-1)/2}(\text{ch } \tau_1). \end{cases}$$

Then, by the asymptotic behavior of associated Legendre functions (see Proposition 8.1 in Appendix), on the one hand, we have

$$\lambda \sim \pi e^{i\pi(m-1)/2} e^{-2n\tau_0} \qquad \text{as} \quad n \to +\infty,$$

and on the other hand, we have

$$\lambda \sim \pi e^{i\pi(m-1)/2} e^{-2n\tau_1} \qquad \text{as} \quad n \to +\infty,$$

what implies that $\tau_0 = \tau_1$, whereas it is not possible. Hence the matrix M_n is invertible and this completes the proof. $\qquad \square$

8 Appendix: Associated Legendre functions of first and second kind

In this section, we provide formulas of integral representation for the associated Legendre functions of the first and the second kind with $z = \text{ch } \tau > 1$, see [2,39,50].

$$P_\nu^\mu(\text{ch } \tau) = \frac{2^{-\nu} \, \text{sh}^{-\mu} \tau}{\Gamma(-\mu - \nu)\Gamma(\nu + 1)} \int_0^\infty (\text{ch } \tau + \text{ch } \theta)^{\mu-\nu-1} \, \text{sh}^{2\nu+1}\theta \, d\theta$$

with $\text{Re } \nu > -1$ and $\text{Re }(\mu + \nu) < 0$.

$$P_\nu^\mu(\text{ch } \tau) = \frac{2^\mu \, \text{sh}^{-\mu} \tau}{\sqrt{\pi} \, \Gamma(1/2 - \mu)} \int_0^\pi \frac{(\text{ch } \tau + \text{sh } \tau \cos \theta)^{\mu+\nu}}{\sin^{2\mu} \theta} \, d\theta \qquad (18)$$

with Re $\mu < 1/2$.

$$P_\nu^\mu(\text{ch}\,\tau) = \sqrt{\frac{2}{\pi}}\, \frac{\text{sh}^\mu\tau}{\Gamma(1/2-\mu)} \int_0^\tau \frac{\text{ch}\,[(\nu+1/2)\theta]}{(\text{ch}\,\tau - \text{ch}\,\theta)^{\mu+1/2}}\, d\theta$$

with Re $\mu < 1/2$.

$$Q_\nu^\mu(\text{ch}\,\tau) = \frac{e^{i\pi\mu}\sqrt{\pi}}{2^\mu}\, \frac{\text{sh}^\mu\tau\,\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)\,\Gamma(\mu+1/2)} \int_0^\infty \frac{\text{sh}^{2\mu}\theta}{(\text{ch}\,\tau + \text{sh}\,\tau\,\text{ch}\,\theta)^{\nu+\mu+1}}\, d\theta$$

with Re $\mu > -1/2$, Re $(\nu - \mu + 1) < 0$ and Re $(\nu + \mu + 1) > 0$.

$$Q_\nu^\mu(\text{ch}\,\tau) = \sqrt{\frac{\pi}{2}}\, e^{i\pi\mu}\, \frac{\text{sh}^\mu\tau}{\Gamma(1/2-\mu)} \int_\tau^\infty \frac{e^{-(\nu+1/2)\theta}}{(\text{ch}\,\theta - \text{ch}\,\tau)^{\mu+1/2}}\, d\theta$$

with Re $\mu < 1/2$ and Re $(\mu + \nu + 1) > 0$.

$$Q_\nu^\mu(\text{ch}\,\tau) = e^{i\pi\mu}\, 2^{-\nu-1}\, \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+1)}\, \text{sh}^{-\mu}\tau \int_0^\pi (\text{ch}\tau + \cos\theta)^{\mu-\nu-1} \sin^{2\nu+1}\theta\, d\theta$$

with Re $\nu > -1$ and Re $(\mu + \nu + 1) > 0$, see [50, pp. 4–6].

There are the following relations between the Legendre functions, see [50, p. 6] and [2, Formula 8.2.2]:

$$P_\nu^\mu = P_{-\nu-1}^\mu,$$
$$Q_{-\nu-1}^\mu(z) = \frac{-\pi e^{i\pi\mu}\cos(\pi\nu)P_\nu^\mu + \sin[\pi(\nu+\mu)]Q_\nu^\mu}{\sin[\pi(\nu-\mu)]}$$

for $\nu - \mu \notin \mathbb{Z}$. In particular, for $\nu = n - 1/2$ and $n \in \mathbb{Z}$, we have

$$Q_{-\nu-1}^\mu = Q_\nu^\mu, \qquad \mu \in \mathbb{C},$$
$$e^{i\pi\mu}\Gamma(\nu+\mu+1)Q_\nu^{-\mu} = e^{-i\pi\mu}\Gamma(\nu-\mu+1)Q_\nu^\mu,$$
$$P_\nu^{-\mu} = \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\left[P_\nu^\mu - \frac{2}{\pi}e^{-i\pi\mu}\sin(\pi\mu)Q_\nu^\mu\right].$$

There hold the following Whipple formulas relating the associated Legendre functions of first and second kind, see [50, p. 6]:

$$Q_\nu^\mu(\text{ch}\,\tau) = e^{i\pi\mu}\sqrt{\frac{\pi}{2}}\, \frac{\Gamma(\mu+\nu+1)}{\sqrt{\text{sh}\tau}}\, P_{-\mu-1/2}^{-\nu-1/2}(\coth\tau),$$
$$P_\nu^\mu(\text{ch}\,\tau) = \frac{ie^{i\pi\nu}}{\Gamma(-\nu-\mu)}\sqrt{\frac{2}{\pi}}\, \frac{1}{\sqrt{\text{sh}\tau}}\, Q_{-\mu-1/2}^{-\nu-1/2}(\coth\tau).$$

There hold the following recursion formulas, see [50, pp. 6–7]:

$$P_\nu^{\mu+1}(\operatorname{ch}\tau) = \frac{(\nu-\mu)\operatorname{ch}\tau\, P_\nu^\mu(\operatorname{ch}\tau) - (\nu+\mu)\, P_{\nu-1}^\mu(\operatorname{ch}\tau)}{\operatorname{sh}\tau},$$

$$(\nu-\mu+1)\, P_{\nu+1}^\mu(\operatorname{ch}\tau) = (2\nu+1)\operatorname{ch}\tau\, P_\nu^\mu(\operatorname{ch}\tau) - (\nu+\mu)\, P_{\nu-1}^\mu(\operatorname{ch}\tau),$$

$$(z^2-1)\frac{d\,P_\nu^\mu(z)}{dz} = (\nu+\mu)(\nu-\mu+1)(z^2-1)^{1/2} P_\nu^{\mu-1}(z) - \mu z P_\nu^\mu(z).$$

$$(z^2-1)\frac{d\,P_\nu^\mu(z)}{dz} = \nu z P_\nu^\mu(z) - (\nu+\mu)\, P_{\nu-1}^\mu(z).$$

All of these formulas are used to calculate the values $P_\nu^\mu(\operatorname{ch}\tau)$ and $Q_\nu^\mu(\operatorname{ch}\tau)$ for all $\tau > 0$ and $(\mu, \nu) \in \mathbb{C}^2$.

If μ and τ are fixed, the following proposition describes the behavior of associated Legendre functions of the first and second kind when $\nu = n - 1/2$, $n \in \mathbb{Z}$, as $|n| \to +\infty$.

Proposition 8.1 *Fix $\tau > 0$ and $\mu \in \mathbb{C}$. Then if $\nu = n - 1/2$ and $n \in \mathbb{Z}$, we have:*

$$P_\nu^\mu(\operatorname{ch}\tau) \sim \frac{e^{\tau/2}}{\sqrt{2\pi\operatorname{sh}\tau}}\, \nu^{\mu-1/2} e^{\tau\nu} \qquad\qquad as \ \ \nu \to +\infty,$$

$$P_\nu^\mu(\operatorname{ch}\tau) \sim \frac{e^{-\tau/2}}{\sqrt{2\pi\operatorname{sh}\tau}}\, (-\nu)^{\mu-1/2} e^{-\tau\nu} \qquad\qquad as \ \ \nu \to -\infty,$$

$$Q_\nu^\mu(\operatorname{ch}\tau) \sim e^{i\pi\mu} e^{-\tau/2}\sqrt{\frac{\pi}{2\operatorname{sh}\tau}}\, \nu^{\mu-1/2} e^{-\tau\nu} \qquad\qquad as \ \ \nu \to +\infty,$$

$$Q_\nu^\mu(\operatorname{ch}\tau) \sim e^{i\pi\mu} e^{\tau/2}\sqrt{\frac{\pi}{2\operatorname{sh}\tau}}\, (-\nu)^{\mu-1/2} e^{\tau\nu} \qquad\qquad as \ \ \nu \to -\infty.$$

These equivalences are locally uniform with respect to the variable τ, i.e. uniform on the whole interval $[\tau_0, \tau_1]$ with $0 < \tau_0 < \tau_1$.

Proof If $\nu = n - 1/2$ and $n \in \mathbb{N}$, see [50, p. 48], we have

$$P_\nu^\mu(\operatorname{ch}\tau) = \frac{\Gamma(\nu+1)}{\Gamma(\nu-\mu+1)}\frac{1}{\sqrt{2\pi(\nu+1)\operatorname{sh}\tau}}\left[e^{(\nu+1/2)\tau} + e^{-\pi i(\mu-1/2)-(\nu+1/2)\tau}\right]$$

$$\cdot\left[1+O\!\left(\frac{1}{\nu}\right)\right].$$

By the Stirling formula as $\nu \to +\infty$

$$\frac{\Gamma(\nu+1)}{\Gamma(\nu-\mu+1)} \sim \frac{\sqrt{2\pi}\,\nu^{\nu+1/2}e^{-\nu}}{\sqrt{2\pi}\,(\nu-\mu)^{\nu-\mu+1/2}e^{-\nu+\mu}} = \left(\frac{\nu}{\nu-\mu}\right)^{\nu+1/2}(\nu-\mu)^\mu e^{-\mu}$$

$$= (\nu-\mu)^\mu e^{-\mu}\exp\left(-\left(\nu+\frac{1}{2}\right)\ln\left(1-\frac{\mu}{\nu}\right)\right) \sim \nu^\mu,$$

consequently,

$$P_\nu^\mu(\operatorname{ch} \tau) \sim \nu^\mu \frac{1}{\sqrt{2\pi \nu \operatorname{sh} \tau}} e^{\tau/2} e^{\tau \nu} = \frac{e^{\tau/2}}{\sqrt{2\pi \operatorname{sh} \tau}} \nu^{\mu - 1/2} e^{\tau \nu},$$

which gives us the first estimate.

The second estimate is obtained thanks to the relation $P_\nu^\mu = P_{-\nu-1}^\mu$. The third estimate follows from [50, Formula (8.3)]:

$$Q_\nu^\mu(\operatorname{ch} \tau) \sim \sqrt{\frac{\pi}{2 \operatorname{sh} \tau}} \nu^{\mu - 1/2} e^{i\pi\mu} e^{-\tau(\nu+1/2)},$$

and the last estimate follows from the fact that for $\nu = n - 1/2$ and $n \in \mathbb{Z}$, we have

$$Q_{-\nu-1}^\mu = Q_\nu^\mu.$$

The local uniformity of these equivalences implies from explicit expressions of P_ν^μ and Q_ν^μ in terms of hypergeometric functions available in [21, pp. 124–138] and estimates of these hypergeometric functions which are locally uniform [50, pp. 178–182]. □

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