Sampling the Lindelöf hypothesis for Dirichlet L -functions by the Cauchy random walk

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Abstract We study the average value of Dirichlet L -function along vertical lines $\sigma + i\mathbb{R}$ inside the critical strip

$$\frac{1}{N} \sum_{1 \leq n \leq N} L(\sigma + iS_n, \chi),$$

where S_n is the sequence of partial sums of the infinite series of independent Cauchy-distributed random variables and χ is a primitive character modulo q . By using Atkinson's formula and a proposition of Lifshits and Weber (Proc Lond Math Soc 98(1):241–270, 2009), we show that the expectation value of $L(s, \chi)$ on the Cauchy random walk $s = \sigma + iS_n$ equals one, which implies that the values of Dirichlet L -function are small on average.

Keywords Atkinson's formula · Cauchy random walk · Dirichlet L -function · Lindelöf hypothesis

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1 Introduction and statement of results

Let $s = \sigma + it$ be a complex variable, χ be a Dirichlet character modulo q and let $L(s, \chi)$ denote the corresponding Dirichlet L -function defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}.$$

This Dirichlet series converges for $\sigma > 1$, and can be extended analytically to all $s \in \mathbb{C}$, except for a simple pole with residue 1 if χ is the principal character. An important open problem in analytic number theory is the Lindelöf hypothesis, which is a consequence of the Riemann hypothesis. The generalized Lindelöf hypothesis for Dirichlet L -functions $L(s, \chi)$ with a character modulo q states that, for all $\epsilon > 0$,

$$L\left(\frac{1}{2} + it, \chi\right) \ll_{\epsilon} ((1 + |t|)q)^{\epsilon}.$$

Let $X_1, X_2, \dots$ denote an infinite sequence of independent Cauchy-distributed random variables and consider the sequence of partial sums $S_n = X_1 + \dots + X_n$, $n = 1, 2, \dots$. In order to understand the behaviour of $L(\sigma + it, \chi)$ when t tends to infinity, we investigate the almost sure asymptotic behaviour of the system

$$L_n(\sigma, \chi) = L(\sigma + iS_n, \chi), \quad n = 1, 2, \dots,$$

for $\sigma \geq 1/2$, where χ is a primitive character modulo q . For any positive integer n , let

$$WL_n(\sigma, \chi) = L(\sigma + iS_n, \chi) - \mathbb{E}L(\sigma + iS_n, \chi) = L_n(\sigma, \chi) - \mathbb{E}L_n(\sigma, \chi),$$

where $\mathbb{E}X$ is the expectation of the random variable X . In this paper we study the average value of Dirichlet L -function along vertical lines $\sigma + i\mathbb{R}$ inside the critical strip

$$\frac{1}{N} \sum_{1 \leq n \leq N} L(\sigma + iS_n, \chi),$$

where S_n is the sequence of partial sums of the infinite series of independent Cauchy-distributed random variables and χ is a primitive character modulo q . We shall prove the following theorem.

Theorem 1.1 *For any real $b > 2$, $\sigma \geq 1/2$,*

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n L(\sigma + iS_k, \chi) - n}{\sqrt{n} \log^b(1 + n)} = 0 \quad \text{almost surely,}$$

and

$$\left\| \sup_{n \geq 1} \frac{\left| \sum_{k=1}^n L(\sigma + iS_k, \chi) - n \right|}{\sqrt{n} \log^b(1+n)} \right\|_2 < \infty.$$

We note that our results are similar to [4, Theorem 1.2]. It seems that statements of this type mainly make use of the following Dirichlet series representation for $\sigma > 1$ and its analytic continuation to the left.

For $\sigma > 1$,

$$L(s, \chi) = q^{-s} \sum_{b=1}^{q-1} \chi(b) \zeta\left(s, \frac{b}{q}\right),$$

where

$$\zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s}, \quad \sigma > 1, \quad 0 < a \leq 1,$$

is the Hurwitz zeta-function. The Hurwitz zeta-function has an analytic continuation to the whole complex plane with a simple pole at $s = 1$ with residue 1. For the Hurwitz zeta-function we obtain a similar result.

Theorem 1.2 *For any real $b > 2$, $0 < a \leq 1$ and $\sigma \geq 1/2$,*

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n a^{\sigma+iS_k} \zeta(\sigma+iS_k, a) - n}{\sqrt{n} \log^b(1+n)} = 0 \quad \text{almost surely,}$$

and

$$\left\| \sup_{n \geq 1} \frac{\left| \sum_{k=1}^n a^{\sigma+iS_k} \zeta(\sigma+iS_k, a) - n \right|}{\sqrt{n} \log^b(1+n)} \right\|_2 < \infty.$$

In 2009, Lifshits and Weber [4] studied the average value of the Riemann zeta-function $\zeta(s)$ at randomly distributed points along the critical line $\sigma = 1/2$. They showed that almost surely

$$\frac{1}{N} \sum_{1 \leq n \leq N} \zeta\left(\frac{1}{2} + iS_n\right) = 1 + o(N^{-1/2} \log^b N)$$

for any $b > 2$; here S_n , as in our case, is a Cauchy random walk. Hence, Lifshits and Weber showed that the average value of $\zeta(s)$ on the Cauchy random walk $s = 1/2 + iS_n$ equals one, which implies that the values of the zeta-function are small on average. Their proof is based on the following proposition of Weber [7].

Proposition 1.3 *Let $\{m_l : l \geq 1\}$ be a sequence of positive reals with partial sums $M_n = \sum_{l=1}^n m_l$ tending to infinity with n . Assume that*

$$\log \frac{M_n}{m_n} \sim \log M_n.$$

Let $\Phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a concave non-decreasing function. Then any sequence $\{\xi_l : l \geq 1\}$ of random variables satisfying the increment condition

$$\mathbb{E} \left| \sum_{l=i}^j \xi_l \right|^2 \leq \Phi \left(\sum_{l=1}^j m_l \right) \sum_{l=i}^j m_l, \quad i \leq j,$$

for any $\tau > 3/2$ verifies

$$\frac{\sum_{l=1}^n \xi_l}{\sqrt{\Phi(M_n)} \log^\tau(1+M_n) \sqrt{M_n}} \xrightarrow{a.s.} 0$$

and

$$\left\| \sup_{n \geq 1} \frac{|\sum_{l=1}^n \xi_l|}{\sqrt{\Phi(M_n)} \log^\tau(1+M_n) \sqrt{M_n}} \right\|_2 < \infty.$$

In order to investigate the almost sure asymptotic behaviour of the system $\zeta(1/2+iS_n)$, $n = 1, 2, \dots$, the increment condition in Proposition 1.3 is necessary. Lifshits and Weber replaced the sequence $\{\xi_n : n \geq 1\}$ by $\{\zeta(1/2+iS_n) - \mathbb{E}\zeta(1/2+iS_n) : n \geq 1\}$ and developed a complete second-order theory for the system $\{\zeta(1/2+iS_n) - \mathbb{E}\zeta(1/2+iS_n)\}$. Their approach is based on a suitable approximation of the Riemann zeta-function. By this the complete second-order theory for the system $\{\zeta(1/2+iS_n) - \mathbb{E}\zeta(1/2+iS_n)\}$ follows from a study of the approximation by a finite sum. Then Lifshits and Weber showed that the approximation by a finite sum approximates the zeta-function well enough and leads to their results.

Our paper is organized as follows. In Sect. 2 we prove Theorem 1.1 by using Atkinson's formula [1] in order to derive the result for the expectation of the system $L(\sigma+iS_n, \chi)$, $\sigma \geq 1/2$, $n = 1, 2, \dots$. Here we use an alternative proof in the case of Dirichlet L -functions, when the character χ is primitive. Instead of an approximation by a finite sum we work directly with the Dirichlet L -function involving Atkinson's formula. In our situation we show that, for $\sigma > 0$ and a primitive character χ modulo q ,

$$\mathbb{E} \sum_{k=1}^{\infty} \chi(k) k^{-(\sigma+iS_n)} = \sum_{k=1}^{\infty} \mathbb{E} \left\{ \chi(k) k^{-(\sigma+iS_n)} \right\} \quad (1)$$

and

$$\mathbb{E} \{ L_n(\sigma, \chi) \overline{L_m(\sigma, \chi)} \} = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \mathbb{E} \left\{ \chi(k) \overline{\chi(l)} k^{-\sigma-iS_n} l^{-\sigma+iS_m} \right\}. \quad (2)$$

Notice that interchanging summation and expectation as in (1) and (2) is not possible for the Riemann zeta-function when $0 < \sigma < 1$ (the case considered by Lifshits and Weber). We will establish (1) and (2) as part of the proof of Lemma 2.1

in the next section. Here, Atkinson's formula is used to calculate the correlation $\mathbb{E} L(\sigma + iS_n, \chi) \overline{L(\sigma + iS_m, \chi)}$, when $m > n + 1$; the idea of Atkinson's formula is to consider the product $\zeta(s_1)\zeta(s_2)$ and represent it as

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1 + s_2) + \zeta_2(s_1, s_2) + \zeta_2(s_2, s_1),$$

where

$$\zeta_2(s_1, s_2) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} m_1^{-s_1} (m_1 + m_2)^{-s_2}.$$

Hence we can avoid the lengthy proof using approximation of the Dirichlet L -function as in [4, Section 2]. In Sect. 3 we prove Theorem 1.2 by using the same method as in the proof of Theorem 1.1 and also we give a remark for the proof by using the technique of Lifshits and Weber [4].

As usual, we write μ for the Möbius function, ϕ for the Euler function, and denote by $d(n)$ the number of divisors of n .

2 Average value of Dirichlet L -function with Cauchy random walk

In this section we prove Theorem 1.1. We develop a complete second-order theory of the system $\{WL_n(\sigma, \chi) : n \geq 1\}$. For this aim we first establish the following result.

Lemma 2.1 *For $\sigma > 1/2$, there exists a constant C_1 such that*

$$\begin{aligned} \mathbb{E} \sum_{\chi} |WL_n(\sigma, \chi)|^2 &= \frac{2\phi^2(q)}{q^{2\sigma}} \zeta(2\sigma - 1) \Gamma(2\sigma - 1) \frac{\Gamma(n + 1 - \sigma)}{\Gamma(n + \sigma)} \\ &\quad + \phi(q) L(2\sigma, \chi_0) + C_1 + o(1), \end{aligned}$$

where

$$C_1 = \lim_{n \rightarrow \infty} \frac{4\phi(q)}{q^{2\sigma-1}} \sum_{e|q} \frac{\mu(e)}{e} \sum_{h=1}^{\infty} h^{1-2\sigma} \sum_{u=1}^{\infty} \int_0^{\infty} \frac{y^{n-\sigma}}{(1+y)^{n+\sigma}} \cos \frac{2\pi h u q y}{e} dy.$$

For $\sigma = 1/2$, there exists a constant C_2 such that

$$\mathbb{E} \sum_{\chi} \left| WL_n \left(\frac{1}{2}, \chi \right) \right|^2 = \frac{\phi^2(q)}{q} \log \left(n + \frac{1}{2} \right) + C_2 + o(1),$$

where

$$\begin{aligned} C_2 &= \lim_{n \rightarrow \infty} 4\phi(q) \sum_{e|q} \frac{\mu(e)}{e} \sum_{v=1}^{\infty} d(v) \int_0^{\infty} \frac{y^{n-1/2}}{(1+y)^{n+1/2}} \cos \frac{2\pi q y v}{e} dy \\ &\quad + \gamma + \gamma_q - \log 2\pi + \log q, \end{aligned}$$

with the Euler constant γ and $\gamma_q = \gamma + \sum_{p|q} \log p/(p-1)$, where the summation is over all prime divisors p of q .

For $m > n + 1$, $\sigma \in [1/2, 1)$, there exists a constant C_3 (dependent on q) such that

$$\left| \mathbb{E} \sum_{\chi} W L_n(\sigma, \chi) \overline{W L_m(\sigma, \chi)} \right| \leq C_3 \max \left\{ \frac{1}{2^{m-n}}, \frac{1}{n} \right\}.$$

Proof In order to investigate the covariance structure, we study the behavior of the moments of first and second order of $L_n(\sigma, \chi)$, and the correlation $\mathbb{E} \sum_{\chi} L_n(\sigma, \chi) \overline{L_m(\sigma, \chi)}$, from which the second order distances $\mathbb{E} \sum_{\chi} |L_n(\sigma, \chi) - L_m(\sigma, \chi)|^2$, $m > n$, can be derived easily. The first moments are given by

$$\mathbb{E} L_n(\sigma, \chi) = \mathbb{E} L(\sigma + i S_n, \chi) = \frac{n}{\pi} \int_{\mathbb{R}} L(\sigma + i \tau, \chi) \frac{d\tau}{n^2 + \tau^2}.$$

The integrand on the right-hand side is a regular function of τ except at $\tau = \pm ni$ in the τ -plane, since for primitive χ the Dirichlet L -function is a regular function for $\sigma > 0$. In order to calculate the expectation of $L(\sigma + i S_n, \chi)$ we apply calculus of residues. For a sufficiently large parameter $R > 1 + \sigma$, we denote the counterclockwise oriented semicircle of radius R centered at the origin located in the lower half of the τ -plane by Γ_R . Then

$$\begin{aligned} \int_{-R}^R L(\sigma + i \tau, \chi) \frac{d\tau}{n^2 + \tau^2} \\ = \int_{\Gamma_R} L(\sigma + i \tau, \chi) \frac{d\tau}{n^2 + \tau^2} - 2\pi i \operatorname{Res}_{\tau=-ni} \frac{L(\sigma + i \tau, \chi)}{n^2 + \tau^2}. \end{aligned}$$

By the functional equation for the Dirichlet L -function and Stirling's formula, we have for $1/2 \leq \alpha \leq 3/4$,

$$L(1 - \alpha + it, \chi) \ll_q |L(\alpha + it, \chi)| (1 + |t|)^{\alpha-1/2},$$

see [2]. From this, we obtain

$$\int_{\Gamma_R} L(\sigma + i \tau, \chi) \frac{d\tau}{n^2 + \tau^2} \ll_q \frac{R}{n^2 + R^2} \max_{\tau \in \Gamma_R} |L(\sigma + i \tau, \chi)| \ll_q R^{-1/2}.$$

Hence with $R \rightarrow \infty$, the integral tends to zero. Now we compute the residue

$$\operatorname{Res}_{\tau=-ni} \frac{L(\sigma + i \tau, \chi)}{n^2 + \tau^2} = \lim_{\tau \rightarrow -ni} (\tau + ni) \frac{L(\sigma + i \tau, \chi)}{n^2 + \tau^2} = \frac{L(n + \sigma, \chi)}{-2ni}.$$

Therefore, we have

$$\mathbb{E} L(\sigma + i S_n, \chi) = L(n + \sigma, \chi) \quad (3)$$

for any integer n and $\sigma \geq 1/2$. Moreover, (1) follows from

$$\begin{aligned} \sum_{k=1}^{\infty} \mathbb{E} \chi(k) k^{-(\sigma+iS_n)} &= \sum_{k=1}^{\infty} \chi(k) k^{-\sigma} \mathbb{E} \exp(-iS_n \log k) = \sum_{k=1}^{\infty} \chi(k) k^{-\sigma-n} \\ &= L(n+\sigma, \chi) = \mathbb{E} \sum_{k=1}^{\infty} \chi(k) k^{-(\sigma+iS_n)}. \end{aligned}$$

With the next calculations we find an asymptotic formula for $\mathbb{E} \sum_{\chi} L_n(\sigma, \chi) \overline{L_m(\sigma, \chi)}$, where $m > n + 1$. Let us fix $\sigma \in [1/2, 1)$ and $m > n + 1$. Here, since we shall prove (2) by using the method of Lifshits and Weber, we shall make use of the following approximation of the Dirichlet L -function, see [6]. For $0 < \sigma < 1$ and $t > 0$, let $x > Cqt/(2\pi)$, C is a positive constant, then

$$L(s, \chi) = \sum_{n \leq x} \chi(n) n^{-s} + O(q^{1/2} x^{-\sigma} \log(q+2)).$$

We can consider the second-order theory for the system $WL_n(\sigma, \chi)$ from a study of the same kind for the system

$$Z_n(\sigma, \chi) = \sum_{n \leq x} \chi(n) n^{-\sigma-iS_n}, \quad n = 1, 2, \dots, \quad x > 0.$$

Since, for $\sigma > 0$, $\mathbb{E} |Z_n(\sigma, \chi) - L(\sigma+iS_n, \chi)|^2 = O(qx^{-2\sigma} \log^2(q+2)) \rightarrow 0$, when $x \rightarrow \infty$, we can show that $Z_n(\sigma, \chi)$ approximates the Dirichlet L -function well enough. From this it follows that

$$\begin{aligned} \mathbb{E} L_n(\sigma, \chi) \overline{L_m(\sigma, \chi)} &= \lim_{x \rightarrow \infty} \mathbb{E} Z_n(\sigma, \chi) \overline{Z_m(\sigma, \chi)} \\ &= \lim_{x \rightarrow \infty} \mathbb{E} \sum_{k \leq x} \chi(k) k^{-\sigma-iS_n} \sum_{l \leq x} \overline{\chi(l)} l^{-\sigma+iS_m} \\ &= \lim_{x \rightarrow \infty} \sum_{k \leq x} \sum_{l \leq x} \mathbb{E} \chi(k) \overline{\chi(l)} k^{-\sigma-iS_n} l^{-\sigma+iS_m} \\ &= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \mathbb{E} \chi(k) \overline{\chi(l)} k^{-\sigma-iS_n} l^{-\sigma+iS_m}. \end{aligned}$$

Thus, we deduce

$$\begin{aligned} \mathbb{E} \sum_{\chi \bmod q} L_n(\sigma, \chi) \overline{L_m(\sigma, \chi)} &= \mathbb{E} \sum_{\chi \bmod q} \sum_{k, l=1}^{\infty} \chi(k) k^{-(\sigma+iS_n)} \overline{\chi(l)} l^{-(\sigma-iS_m)} \\ &= \sum_{\chi \bmod q} \sum_{k, l=1}^{\infty} \chi(k) \overline{\chi(l)} (kl)^{-\sigma} \mathbb{E} \{ \exp(-iS_n \log k) \exp(iS_m \log l) \} \\ &= \sum_{\chi \bmod q} \sum_{k, l=1}^{\infty} \chi(k) \overline{\chi(l)} (kl)^{-\sigma} \mathbb{E} \exp(i(\log l - \log k) S_n + i(S_m - S_n) \log l). \end{aligned}$$

In order to evaluate this we consider the value of $\mathbb{E} \exp(i(\log l - \log k) S_n)$. If $l > k$, then $\mathbb{E} \exp(i(\log l - \log k) S_n) = (l/k)^{-n}$. If $l < k$, then $\mathbb{E} \exp(i(\log l - \log k) S_n) = (l/k)^n$.

Thus we get

$$\mathbb{E} \sum_{\chi \bmod q} L_n(\sigma, \chi) \overline{L_m(\sigma, \chi)} = \sum_{\chi \bmod q} \sum_{k, l=1}^{\infty} \chi(k) \overline{\chi(l)} (kl)^{-\sigma} \left(\frac{\min(k, l)}{\max(k, l)} \right)^n l^{-m+n}.$$

In [5], the terms of the double series are studied according to whether $k = l$, $k > l$ or $k < l$ involving the orthogonality of the Dirichlet characters. Following similar reasoning, we deduce that the above sum can be written as

$$\phi(q) \left\{ L(m-n+2\sigma, \chi_0) + \sum_{\substack{r=1 \\ (r,q)=1}}^{\infty} \sum_{h=1}^{\infty} \frac{r^{n-\sigma}}{(r+qh)^{m+\sigma}} + \sum_{\substack{r=1 \\ (r,q)=1}}^{\infty} \sum_{h=1}^{\infty} \frac{r^{2n-m-\sigma}}{(r+qh)^{n+\sigma}} \right\} + o(1), \quad (4)$$

where χ_0 is the principal character modulo q . For $\sigma \geq 1/2$, $m > n+1$, (4) holds by analytic continuation. In view of the convergence of the double series, for $\sigma \geq 1/2$, $m > n+1$, we have

$$\sum_{\substack{r=1 \\ (r,q)=1}}^{\infty} \sum_{h=1}^{\infty} \frac{r^{n-\sigma}}{(r+qh)^{m+\sigma}} = \sum_{e|q} \mu(e) \sum_{h=1}^{\infty} \sum_{r=1}^{\infty} \frac{(er)^{n-\sigma}}{(er+qh)^{m+\sigma}}.$$

By Poisson's summation formula,

$$\begin{aligned} & \sum_{r=1}^{\infty} \frac{(er)^{n-\sigma}}{(er+qh)^{m+\sigma}} \\ &= \int_0^{\infty} \frac{(ex)^{n-\sigma}}{(ex+qh)^{m+\sigma}} dx + 2 \sum_{u=1}^{\infty} \int_0^{\infty} \frac{(ex)^{n-\sigma}}{(ex+qh)^{m+\sigma}} \cos(2\pi ux) dx \\ &= \frac{(qh)^{1-2\sigma+n-m}}{e} \left(\int_0^{\infty} \frac{y^{n-\sigma}}{(1+y)^{m+\sigma}} dy \right. \\ & \quad \left. + 2 \sum_{u=1}^{\infty} \int_0^{\infty} \frac{y^{n-\sigma}}{(1+y)^{m+\sigma}} \cos \frac{2\pi u q s y}{t} dy \right). \end{aligned}$$

Now we sum with respect to h and e and use the identities

$$\begin{aligned} \int_0^{\infty} \frac{y^{n-\sigma}}{(1+y)^{m+\sigma}} dy &= \Gamma(m-n+2\sigma-1) \frac{\Gamma(n-\sigma+1)}{\Gamma(m+\sigma)}, \\ \sum_{e|q} \mu(e) e^{-1} &= \phi(q) q^{-1}. \end{aligned}$$

This gives

$$\sum_{\substack{r=1 \\ (r,q)=1}}^{\infty} \sum_{h=1}^{\infty} \frac{r^{n-\sigma}}{(r+qh)^{m+\sigma}} = \frac{\phi(q)}{q^{m-n+2\sigma}} \zeta(m-n+2\sigma-1) \\ \cdot \Gamma(m-n+2\sigma-1) \frac{\Gamma(n+1-\sigma)}{\Gamma(m+\sigma)} + g_q(m, n),$$

where

$$g_q(m, n) = \frac{2}{q^{m-n+2\sigma-1}} \sum_{e|q} \frac{\mu(e)}{e} \sum_{h=1}^{\infty} h^{-m+n-2\sigma+1} \\ \cdot \sum_{u=1}^{\infty} \int_0^{\infty} \frac{y^{n-\sigma}}{(1+y)^{m+\sigma}} \cos \frac{2\pi huqy}{e} dy. \quad (5)$$

Exactly in the same way we obtain that the last sum in (4) is

$$\sum_{\substack{r=1 \\ (r,q)=1}}^{\infty} \sum_{h=1}^{\infty} \frac{r^{2n-m-\sigma}}{(r+qh)^{n+\sigma}} = \frac{\phi(q)}{q^{m-n+2\sigma}} \zeta(m-n+2\sigma-1) \Gamma(m-n+2\sigma-1) \\ \cdot \frac{\Gamma(-m+2n-\sigma+1)}{\Gamma(n+\sigma)} + f_q(m, n),$$

where

$$f_q(m, n) = \frac{2}{q^{m-n+2\sigma-1}} \sum_{e|q} \frac{\mu(e)}{e} \sum_{h=1}^{\infty} h^{-m+n-2\sigma+1} \\ \cdot \sum_{u=1}^{\infty} \int_0^{\infty} \frac{y^{2n-m-\sigma}}{(1+y)^{n+\sigma}} \cos \frac{2\pi huqy}{e} dy.$$

Thus

$$\mathbb{E} \sum_{\chi \bmod q} L_n(\sigma, \chi) \overline{L_m(\sigma, \chi)} \\ = \frac{\phi^2(q)}{q^{m-n+2\sigma}} \zeta(m-n+2\sigma-1) \Gamma(m-n+2\sigma-1) \\ \cdot \left(\frac{\Gamma(n-\sigma+1)}{\Gamma(m+\sigma)} + \frac{\Gamma(-m+2n-\sigma+1)}{\Gamma(n+\sigma)} \right) \\ + \phi(q) L(m-n+2\sigma, \chi_0) + \phi(q) (g_q(m, n) + f_q(m, n)). \quad (6)$$

Now we return to (5) and consider the convergence of its right-hand side. We have, for $\sigma \geq 1/2$ and $k \geq 1$,

$$\begin{aligned}
& 2 \int_0^\infty \frac{y^{n-\sigma}}{(1+y)^{m+\sigma}} \cos(2\pi ky) dy \\
&= k^{\sigma-n-1} \int_0^\infty \frac{y^{n-\sigma}}{(1+y/k)^{m+\sigma}} (e^{2\pi iy} + e^{-2\pi iy}) dy \\
&= k^{\sigma-n-1} \int_0^{i\infty} \frac{y^{n-\sigma}}{(1+y/k)^{m+\sigma}} e^{2\pi iy} dy + k^{\sigma-n-1} \int_0^{-i\infty} \frac{y^{n-\sigma}}{(1+y/k)^{m+\sigma}} e^{-2\pi iy} dy \\
&\ll \left| \frac{k^{\sigma-n-1}}{\sigma-n-1} \right|
\end{aligned}$$

is uniformly bounded for $m > n + 1$. It follows that the double series (5) is absolutely convergent for $\sigma \geq 1/2$ and $m > n + 1$, by comparison with

$$\sum_{h=1}^{\infty} |h^{-m-\sigma}| \sum_{u=1}^{\infty} |u^{\sigma-n-1}| = o(1) \quad \text{as } m, n \rightarrow \infty.$$

Next we shall find an asymptotic formula for $\mathbb{E} \sum_{\chi} |L_n(\sigma, \chi)|^2$. For this we put $m = n$ in (6), then we have, for $\sigma \in (1/2, 1)$,

$$\begin{aligned}
\mathbb{E} \sum_{\chi \bmod q} |L_n(\sigma, \chi)|^2 &= \frac{2\phi^2(q)}{q^{2\sigma}} \zeta(2\sigma-1) \Gamma(2\sigma-1) \frac{\Gamma(n+1-\sigma)}{\Gamma(n+\sigma)} \\
&\quad + \phi(q) L(2\sigma, \chi_0) + g_q(n, n),
\end{aligned} \tag{7}$$

where

$$g_q(n, n) = \frac{4\phi(q)}{q^{2\sigma-1}} \sum_{e|q} \frac{\mu(e)}{e} \sum_{h=1}^{\infty} h^{1-2\sigma} \sum_{u=1}^{\infty} \int_0^\infty \frac{y^{n-\sigma}}{(1+y)^{n+\sigma}} \cos \frac{2\pi huqy}{e} dy.$$

In view of (7), the case $\sigma = 1/2$ is exceptional. Here we use the fact that $g_q(n, n)$ and $f_q(n, n)$ are continuous in n and write $2\sigma - 1 = \delta$, $|\delta| < 1/2$, with the aim of letting $\delta \rightarrow 0$. Then the first two terms on the right-hand side give us

$$\frac{2\phi^2(q)}{q^{1+\delta}} \zeta(\delta) \Gamma(\delta) \frac{\Gamma(1-\sigma+n)}{\Gamma(1-\sigma+n+\delta)} + \phi(q) L(1+\delta, \chi_0).$$

Using Taylor's formula for the gamma-function terms, the functional equation for $\zeta(s)$ and the Laurent expansion of $\zeta(s)$ at the pole at $s = 1$,

$$\zeta(s) = (s-1) + \gamma + O(|s-1|),$$

and writing

$$\gamma_q = \gamma + \sum_{p|q} \frac{\log p}{p-1},$$

where γ is the Euler constant, we obtain

$$\begin{aligned} \phi(q)L(1+\delta, \chi_0) + \frac{\phi^2(q)}{q} \zeta(1-\delta) \left(\frac{q}{2\pi}\right)^{-\delta} \frac{\Gamma(1-\sigma+n)}{\Gamma(1-\sigma+n+\delta)} \sec \frac{\pi\delta}{2} \\ = \frac{\phi^2(q)}{q} \left\{ \frac{1}{\delta} + \gamma_q - \left(\frac{1}{\delta} - \gamma\right) \left(1 - \delta \log \frac{q}{2\pi}\right) \left(1 - \delta \frac{\Gamma'}{\Gamma}(1-\sigma+n)\right) \right\} + O(|\delta|) \\ = \frac{\phi^2(q)}{q} \left\{ \frac{\Gamma'}{\Gamma}(1-\sigma+n) + \gamma + \gamma_q + \log \frac{q}{2\pi} \right\} + O(|\delta|). \end{aligned}$$

Hence, letting $\delta \rightarrow 0$ and setting $\sigma = 1/2$, we have

$$\begin{aligned} \mathbb{E} \sum_{\chi \bmod q} \left| L_n\left(\frac{1}{2}, \chi\right) \right|^2 &= \frac{\phi^2(q)}{q} \left\{ \frac{\Gamma'}{\Gamma}\left(n + \frac{1}{2}\right) + \gamma + \gamma_q + \log \frac{q}{2\pi} \right\} + G_q(n) + o(1) \\ &= \frac{\phi^2(q)}{q} \left\{ \log\left(n + \frac{1}{2}\right) - \frac{1}{2n-1} + \gamma + \gamma_q + \log \frac{q}{2\pi} \right\} \\ &\quad + G_q(n) + o\left(\frac{1}{n^2}\right), \end{aligned}$$

where

$$G_q(n) = 4\phi(q) \sum_{e|q} \frac{\mu(e)}{e} \sum_{v=1}^{\infty} d(v) \int_0^{\infty} \frac{y^{n-1/2}}{(1+y)^{n+1/2}} \cos \frac{2\pi qyv}{e} dy.$$

In the next step, we estimate the covariance of the system $\{WL_n(\sigma, \chi) : n \geq 1\}$. Recall that $WL_n(\sigma, \chi) = L(\sigma + iS_n, \chi) - \mathbb{E}L(\sigma + iS_n, \chi)$. Since

$$\begin{aligned} \mathbb{E} \sum_{\chi} WL_n(\sigma, \chi) \overline{WL_m(\sigma, \chi)} &= \mathbb{E} \sum_{\chi} L_n(\sigma, \chi) \overline{L_m(\sigma, \chi)} \\ &\quad - \sum_{\chi} \mathbb{E}L_n(\sigma, \chi) \overline{\mathbb{E}L_m(\sigma, \chi)}, \end{aligned}$$

we obtain from (3) and (6), for $\sigma \in [1/2, 1)$,

$$\begin{aligned} \mathbb{E} \sum_{\chi} WL_n(\sigma, \chi) \overline{WL_m(\sigma, \chi)} \\ = \frac{\phi^2(q)}{q^{m-n+2\sigma}} \zeta(m-n+2\sigma-1) \Gamma(m-n+2\sigma-1) \\ \cdot \left(\frac{\Gamma(n-\sigma+1)}{\Gamma(m+\sigma)} + \frac{\Gamma(-m+2n-\sigma+1)}{\Gamma(n+\sigma)} \right) \quad (8) \\ + \phi(q)L(m-n+2\sigma, \chi_0) + \phi(q)(g_q(m, n) + f_q(m, n)) \\ - \sum_{\chi} L(n+\sigma, \chi) \overline{L(m+\sigma, \chi)}. \end{aligned}$$

In view of (8), for $\sigma \in (1/2, 1)$,

$$\begin{aligned} \mathbb{E} \sum_{\chi} |WL_n(\sigma, \chi)|^2 &= \frac{2\phi^2(q)}{q^{2\sigma}} \zeta(2\sigma - 1) \Gamma(2\sigma - 1) \frac{\Gamma(n + 1 - \sigma)}{\Gamma(n + \sigma)} \\ &\quad + \phi(q) L(2\sigma, \chi_0) + g_q(n, n) - \sum_{\chi} |L(n + \sigma, \chi)|^2. \end{aligned}$$

For $\sigma = 1/2$, similarly we get

$$\begin{aligned} \mathbb{E} \sum_{\chi} \left| WL_n\left(\frac{1}{2}, \chi\right) \right|^2 &= \frac{\phi^2(q)}{q} \left\{ \log\left(n + \frac{1}{2}\right) - \frac{1}{2n - 1} + \gamma + \gamma_q + \log \frac{q}{2\pi} \right\} \\ &\quad + G_q(n) - \sum_{\chi} \left| L\left(n + \frac{1}{2}, \chi\right) \right|^2. \end{aligned}$$

Now we estimate (8) for $\sigma \geq 1/2$, $m > n + 1$, by

$$\begin{aligned} &\left| \mathbb{E} \sum_{\chi} WL_n(\sigma, \chi) \overline{WL_m(\sigma, \chi)} \right| \\ &\leq \frac{\phi^2(q)}{q^{m-n+2\sigma}} \zeta(m - n + 2\sigma - 1) \Gamma(m - n + 2\sigma - 1) \\ &\quad \cdot \left(\frac{\Gamma(m - \sigma + 1)}{\Gamma(m + \sigma)} + \frac{\Gamma(-m + 2n - \sigma + 1)}{\Gamma(n + \sigma)} \right) \\ &\quad + \left| \phi(q) L(m - n + 2\sigma, \chi_0) - \sum_{\chi} L(n + \sigma, \chi) \overline{L(m + \sigma, \chi)} \right| \\ &\quad + \left| \phi(q) (g_q(m, n) + f_q(m, n)) \right|. \end{aligned}$$

We observe

$$\begin{aligned} &\phi(q) L(m - n + 2\sigma, \chi_0) - \sum_{\chi} L(n + \sigma, \chi) \overline{L(m + \sigma, \chi)} \\ &= \phi(q) \sum_{k=1}^{\infty} \chi_0(k) k^{-m+n-2\sigma} - \phi(q) \sum_{k=1}^{\infty} \chi_0(k) k^{-m-\sigma} - \phi(q) \sum_{l=1}^{\infty} \chi_0(l) l^{-n-\sigma} \\ &\quad - \sum_{\chi} \sum_{k=2}^{\infty} \overline{\chi(k)} k^{-m-\sigma} \sum_{l=2}^{\infty} \chi(l) l^{-n-\sigma}. \end{aligned}$$

It follows that

$$\left| \phi(q) L(m - n + 2\sigma, \chi_0) - \sum_{\chi} L(n + \sigma, \chi) \overline{L(m + \sigma, \chi)} \right| \leq C_q \left(\frac{1}{2^{m-n}}, \frac{1}{2^n} \right).$$

For the other terms, in view of the absolute convergence of the double series (5), we have, for $m > n + 1$,

$$|\phi(q)(g_q(m, n) + f_q(m, n))| = O\left(\frac{1}{q^{m-n}}\right).$$

Finally considering the last term, we have

$$\frac{\Gamma(m - n + 2\sigma - 1)\Gamma(n + 1 - \sigma)}{\Gamma(m + \sigma)} = O\left(\frac{1}{n}\right),$$

hence

$$\frac{\phi^2(q)\zeta(m - n + 2\sigma - 1)}{q^{m-n+2\sigma-1}} \frac{\Gamma(m - n + 2\sigma - 1)\Gamma(n + 1 - \sigma)}{\Gamma(m + \sigma)} = O_q\left(\frac{1}{n}\right).$$

Thus, for $q \geq 2$, there exists a constant C (depending on q) such that

$$\left| \mathbb{E} \sum_{\chi} W L_n(\sigma, \chi) \overline{W L_m(\sigma, \chi)} \right| \leq C \max\left(\frac{1}{2^{m-n}}, \frac{1}{n}\right).$$

Here and in the sequel C denotes a positive constant, not necessarily the same at each appearance. $\square$

Proof of Theorem 1.1 Now we consider the asymptotic behaviour along the Cauchy random walk. The essential step consists of controlling the increments

$$\mathbb{E} \sum_{\chi} \left| \sum_{\substack{i \leq n \leq j \\ n \text{ even}}} W L_n(\sigma, \chi) \right|^2 \quad \text{and} \quad \mathbb{E} \sum_{\chi} \left| \sum_{\substack{i \leq n \leq j \\ n \text{ odd}}} W L_n(\sigma, \chi) \right|^2.$$

Since the two increments can be treated in exactly the same way, we consider only the first one. We use Lemma 2.1. For $\sigma \geq 1/2$ we have

$$\begin{aligned} \mathbb{E} \sum_{\chi} \left| \sum_{\substack{i \leq n \leq j \\ n \text{ even}}} W L_n(\sigma, \chi) \right|^2 &= \sum_{\substack{i \leq n \leq j \\ n \text{ even}}} \mathbb{E} \sum_{\chi} |W L_n(\sigma, \chi)|^2 \\ &\quad + \sum_{\substack{i \leq n \leq j \\ n \text{ even}}} \sum_{\substack{i \leq m \leq j \\ m \text{ even}}} \left| \mathbb{E} \sum_{\chi} W L_n(\sigma, \chi) \overline{W L_m(\sigma, \chi)} \right| \\ &\leq C_q \sum_{\substack{i \leq n \leq j \\ n \text{ even}}} \log n + C_q \sum_{\substack{i \leq n < m \leq j \\ n, m \text{ even}}} \frac{1}{2^{m-n}}. \end{aligned}$$

However,

$$\sum_{\substack{i \leq n \leq j \\ n \text{ even}}} \sum_{\substack{i \leq m \leq j \\ m \text{ even}}} \frac{1}{2^{m-n}} \leq \sum_{i \leq n \leq j} 1 \cdot \sum_{m > n} \frac{1}{2^{m-n}} \leq (j-i) \sum_{h \geq 1} 2^{-h} \leq C(j-i),$$

with some positive constant C . Therefore,

$$\mathbb{E} \sum_{\chi} \left| \sum_{\substack{i \leq n \leq j \\ n \text{ even}}} W L_n(\sigma, \chi) \right|^2 \leq C(j-i) \log j.$$

A similar estimate holds for the odd part. Now the conclusion of Theorem 1.1 is easily obtained from Proposition 1.3. We apply this result to $W L_n(\sigma, \chi)$ with the choice $m_l \equiv 1$ and $\Phi(x) = \log(n+1)$ and obtain the assertion of Theorem 1.1. $\square$

Remark 2.2 If we put $q = 1$ in Lemma 2.1, for $\sigma = 1/2$, then some of our results are contained in [4, Theorem 1.1], however, our constants take another form.

Remark 2.3 For $\sigma \in [1/2, 1)$, we can deduce

$$\sum_{\substack{a=1 \\ (a,q)=1}}^q \mathbb{E} \left| \zeta \left(\sigma + i S_n, \frac{a}{q} \right) \right|^2 = \frac{q^{2\sigma}}{\phi(q)} \sum_{\chi \bmod q} \mathbb{E} |L(\sigma + i S_n, \chi)|^2.$$

Thus,

$$\sum_{\substack{a=1 \\ (a,q)=1}}^q \mathbb{E} \left| \zeta \left(\sigma + i S_n, \frac{a}{q} \right) - \mathbb{E} \zeta \left(\sigma + i S_n, \frac{a}{q} \right) \right|^2 \leq \frac{1}{q} \log(n+1-\sigma), \quad n \rightarrow \infty.$$

3 Average value of the Hurwitz zeta-function with Cauchy random walk

In this section we study the same question for the Hurwitz zeta-function. Consider the system

$$\zeta_n(\sigma, a) = a^{\sigma + i S_n} \zeta(\sigma + i S_n, a), \quad n = 1, 2, \dots,$$

where a satisfies $0 < a \leq 1$. For any positive integer n we define

$$W_n(\sigma, a) = a^{\sigma + i S_n} \zeta(\sigma + i S_n, a) - \mathbb{E} a^{\sigma + i S_n} \zeta(\sigma + i S_n, a) = \zeta_n(\sigma, a) - \mathbb{E} \zeta_n(\sigma, a).$$

By a similar reasoning as in Sect. 2 we obtain

Lemma 3.1 For $\sigma > 1/2$, there exists a constant D_n such that

$$\mathbb{E} |W_n(\sigma, a)|^2 = a^{2\sigma} \zeta(2\sigma, a) + \frac{2a^{2\sigma} \zeta(2\sigma-1) \Gamma(2\sigma-1) \Gamma(n+1-\sigma)}{\Gamma(n+\sigma)} + D_n + o(1),$$

where

$$D_n = 4a^{2\sigma} \sum_{h=1}^{\infty} h^{1-2\sigma} \sum_{u=1}^{\infty} \int_0^{\infty} \frac{(y+a)^{n-\sigma}}{(h+y+a)^{n+\sigma}} \cos(2\pi h u y) dy - 2,$$

and $\lim_{n \rightarrow \infty} D_n$ exists. For $\sigma = 1/2$, there exists a constant E_n such that

$$\mathbb{E} \left| W_n \left(\frac{1}{2}, a \right) \right|^2 = a \log \left(n + \frac{1}{2} \right) + E_n + o(1),$$

where

$$E_n = 4a \sum_{v=1}^{\infty} d(v) \int_0^{\infty} \frac{(y+a)^{n-1/2}}{(1+y+a)^{n+1/2}} \cos(2\pi v y) dy + a\gamma + a\gamma_a - a \log 2\pi - 2,$$

here γ_a is the generalized Euler constant given by

$$\gamma_a = \lim_{N \rightarrow \infty} \left(\sum_{m=0}^N \frac{1}{m+a} - \log(N+a) \right)$$

and $\lim_{n \rightarrow \infty} E_n$ exists. For $m > n + 1$, $\sigma \leq 1/2$,

$$|\mathbb{E} W_n(\sigma, a) \overline{W_m(\sigma, a)}| \leq C_3 \max \left\{ \frac{1}{(1+a)^{m-n}}, \frac{1}{n} \right\}.$$

In view of Lemma 3.1 and Proposition 1.3 we obtain Theorem 1.2.

We can use the technique of Lifshits and Weber [4] to prove Lemma 3.1 for the case $\sigma = 1/2$. Namely, similarly to [4], one can use the following approximation for the Hurwitz zeta-function:

$$\zeta(s, a) = \sum_{m \leq x} (m+a)^{-s} + \frac{(x+a)^{1-s}}{s-1} + O(x^{-\sigma}),$$

valid uniformly for $\sigma \geq \sigma_0 > 0$ and $|t| \leq x/2$ [3]. Then the second-order theory for the system $W_n(\sigma, a)$ follows from a study of the same type for the auxiliary system

$$Z_n(\sigma, a) = a^{\sigma+iS_n} \sum_{0 \leq k \leq x} (k+a)^{-(\sigma+iS_n)} - a^{\sigma+iS_n} \frac{(x+a)^{1-\sigma-iS_n}}{1-\sigma-iS_n},$$

$n = 1, 2, \dots, x > 1/2$. Firstly, one would have to investigate the second-order theory for $Z_n(\sigma, a)$ following the technique of the proof of Lifshits and Weber [4]. Next one would have to show that $Z_n(\sigma, a)$ approximates $\zeta_n(\sigma, a)$ well enough by using fourth moment estimates in order to obtain $\mathbb{E}|Z_n(\sigma, a) - \zeta_n(\sigma, a)|^2 \rightarrow 0$. From this one could deduce

Lemma 3.2 For $0 < a \leq 1$, there exist constants C, C_0 such that

$$\mathbb{E} \left| W_n \left(\frac{1}{2}, a \right) \right|^2 = a \log(n+a) + aC + o(1), \quad n \rightarrow \infty,$$

and, for $m > n + 1$,

$$\left| \mathbb{E} W_n \left(\frac{1}{2}, a \right) \overline{W_m \left(\frac{1}{2}, a \right)} \right| \leq C_0 \max \left(\frac{1}{(1+a)^{m-n}}, \frac{1}{n} \right).$$

The explicit value of C is

$$C = \gamma_a - \frac{2}{a} + 2 \int_0^1 \phi(\alpha) d\alpha + 2 \int_1^\infty \left(\phi(\alpha) - \frac{e^\alpha - 1}{2\alpha(e^\alpha - 1)} \right) d\alpha,$$

where $\phi(\alpha)$ is defined by $\phi(\alpha) = (\alpha e^\alpha - 2e^\alpha + \alpha + 2) / (2\alpha^2(e^\alpha - 1))$.

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