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RESEARCH ARTICLE

On α-embedded subsets of products

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Abstract We prove that every continuous function $f: E \to Y$ depends on countably many coordinates if E is an $(\aleph_1, \aleph_0)$-invariant pseudo-$\aleph_1$-compact subspace of a product of topological spaces and Y is a space with a regular G_δ-diagonal. Using this fact for any $\alpha < \omega_1$, we construct an $(\alpha + 1)$-embedded subspace of a completely regular space which is not α-embedded.

Keywords κ-Invariant set · Pseudo-$\aleph_1$-compact set · α-Embedded set

Mathematics Subject Classification 54B10 · 54C45 · 54C20 · 54H05

1 Introduction

If P is a property of functions, then by $P(X)$ ($P^*(X)$) we denote the collection of all real-valued (bounded) functions on a topological space X with the property P. By the symbol C we denote the property of continuity. Let B_α be the property of being a function of α-th Baire class, where $0 \leq \alpha < \omega_1$.

Recall that a subset A of X is called *functionally closed* (open) *in* X if there is $f \in C^*(X)$ with $A = f^{-1}(0)$ ($A = X \setminus f^{-1}(0)$). The system of all functionally open

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(closed) subsets of X we denote by $\mathcal{G}_0^*$ ($\mathcal{F}_0^*$). Assume that the classes $\mathcal{G}_\xi^*$ and $\mathcal{F}_\xi^*$ are defined for all $\xi < \alpha$, where $0 < \alpha < \omega_1$. Then, if α is odd, the class $\mathcal{G}_\alpha^*$ ($\mathcal{F}_\alpha^*$) consists of all countable intersections (unions) of sets of lower classes, and, if α is even, the class $\mathcal{G}_\alpha^*$ ($\mathcal{F}_\alpha^*$) consists of all countable unions (intersections) of sets of lower classes. The classes $\mathcal{F}_\alpha^*$ for odd α and $\mathcal{G}_\alpha^*$ for even α are said to be *functionally additive*, and the classes $\mathcal{F}_\alpha^*$ for even α and $\mathcal{G}_\alpha^*$ for odd α are called *functionally multiplicative*. A set A is called *functionally measurable* if $A \in \bigcup_{0\le\alpha<\omega_1}(\mathcal{F}_\alpha^* \cup \mathcal{G}_\alpha^*)$. Notice that the σ-algebra of functionally measurable subsets of X is also called the *σ-algebra of Baire sets*.

An important role in the extension theory play z-embedded sets (a subset A of a topological space X is called *z-embedded* in X if for any functionally closed set F in A there exists a functionally closed set B in X such that $B \cap A = F$). In [8] for any $\alpha < \omega_1$ the notion of an *α-embedded set* was introduced, i.e. such a set $A \subseteq X$ that every its subset B of functionally multiplicative class α in A is the restriction on A of some set of functionally multiplicative class α in X. Obviously, the class of 0-embedded sets coincides with the class of z-embedded sets. It is not hard to verify that any α-embedded set is β-embedded if $\alpha \le \beta$ [8, Proposition 2.5]. The converse statement is not true as [8, Theorem 2.6] shows: there is a 1-embedded subset E of the product $X = [0,1] \times \prod_{t\in[0,1]} X_t$, $X_t = \mathbb{N}$ for all $t \in [0,1]$, which is not 0-embedded in X. Here we generalize this result and show that for any $\alpha < \omega_1$ there exists a set $E \subseteq X$ which is $(\alpha+1)$-embedded but not α-embedded in X.

A convenient tool in the investigation of properties of α-embedded subsets E of $\prod_{t\in T} X_t$ is the fact that under some conditions on E every continuous function $f\colon E \to \mathbb{R}$ depends on countably many coordinates (see definitions in Sect. 2). Mazur introduced in [10] sets invariant under projection (see Definition 2.1 (a)) and proved that every continuous function $f\colon E \to Y$ depends on countably many coordinates if $E \subseteq \Sigma(a)$ for some $a \in E$ and E is invariant under projection, X_t is a metrizable separable space for each $t \in T$ and Y is a Hausdorff space with a G_δ-diagonal. Engelking [5] established the same result in the case when E is a set which is invariant under composition (see Definition 2.1 (b)) which is contained in $\Sigma(a)$ for some $a \in E$, X_t is a T_1-space with countable base for each $t \in T$ and Y is a Hausdorff space in which every one-point set is G_δ (see also [7]). Noble and Ulmer [11] obtained the dependence of a continuous function $f\colon E \to Y$ on countably many coordinates if E is a subset of a pseudo-$\aleph_1$-compact space $\prod_{t\in T} X_t$, which contains $\sigma(a)$ for some $a \in E$ and Y is a space with a regular G_δ-diagonal. The result of Noble and Ulmer was generalized by Comfort and Gotchev in [2]. Here we consider the so-called $(\aleph_1, \aleph_0)$-invariant subsets of products and, developing the methods of Mazur and of Noble and Ulmer, we show that every continuous function $f\colon E \to Y$ depends on countably many coordinates if E is an $(\aleph_1, \aleph_0)$-invariant pseudo-$\aleph_1$-compact subspace of $\prod_{t\in T} X_t$ and Y is a space with a regular G_δ-diagonal.

2 Some properties of pseudo-$\aleph_1$-compact invariant sets

Let $(X_t : t \in T)$ be a family of non-empty topological spaces, $X = \prod_{t\in T} X_t$ and let $a = (a_t)_{t\in T}$ be a fixed point of X. For $S \subseteq T$ we denote by p_S the projection $p_S\colon X \to \prod_{t\in S} X_t$, where $p_S(x) = (x_t)_{t\in S}$ for each $x = (x_t)_{t\in T} \in X$; by x_S^a we denote the point with coordinates $(y_t)_{t\in T}$, where $y_t = x_t$ if $t \in S$ and $y_t = a_t$ if $t \in T \setminus S$. For a basic open set $U = \prod_{t\in T} U_t \subseteq X$ let $N(U) = \{t \in T : U_t \neq X_t\}$.

Definition 2.1 A set $E \subseteq X$ is called

(a) *invariant under projection* [10] if $x_S^a \in E$ for any $x \in E$ and $S \subseteq T$;
(b) *invariant under composition* [5] if for any $x, y \in E$ and $S \subseteq T$ we have $z = (z_t)_{t\in T} \in E$, where $z_t = x_t$ for every $t \in S$ and $z_t = y_t$ for every $t \in T \setminus S$.

Clearly, every set E invariant under composition is invariant under projection for any $a \in E$.

Following Engelking [5], Hušek in [7, p. 132] introduced a notion of κ-invariant set for $\kappa \geq \aleph_0$.

Definition 2.2 A set E is called *κ-invariant* if for any $x, y \in E$ and $S \subseteq T$ with $|S| < \kappa$ there is a point $z \in E$ such that $z_t = x_t$ for every $t \in S$ and $z_t = y_t$ for every $t \in T \setminus S$.

Developing the above-mentioned concepts of Mazur and Hušek, we introduce the following notions.

Definition 2.3 Let $\aleph_i$ and $\aleph_j$ be infinite cardinals, $E \subseteq X$ and $a \in E$. Then E is called

- *$\aleph_i$-invariant with respect to a* if $x_S^a \in E$ for every $x \in E$ and $S \subseteq T$ with $|S| < \aleph_i$;
- *$(\aleph_i, \aleph_j)$-invariant with respect to a* if $x_{S_1}^a \in E$ and $x_{T\setminus S_2}^a \in E$ for any point $x \in E$ and for any sets $S_1, S_2 \subseteq T$ with $|S_1| < \aleph_i$ and $|T \setminus S_2| < \aleph_j$.

Obviously, every set $(\aleph_i, \aleph_j)$-invariant with respect to a is $\aleph_i$-invariant with respect to a.

Definition 2.4 A topological space X is said to be

- *pseudo-$\aleph_1$-compact* if any locally finite family of open subsets of X is at most countable;
- *hereditarily pseudo-$\aleph_1$-compact* if each subspace of X is pseudo-$\aleph_1$-compact.

It is easy to check that continuous mappings preserve pseudo-$\aleph_1$-compactness.

The following theorem gives a characterization of pseudo-$\aleph_1$-compactness of $\aleph_0$-invariant sets and is an analogue of the similar result of Noble and Ulmer [11, Corollary 1.5] for products.

Theorem 2.5 *Let $(X_t : t \in T)$ be a family of topological spaces, $X = \prod_{t\in T} X_t$, $a \in X$ and let $E \subseteq X$ be an $\aleph_0$-invariant set with respect to a. Then the following conditions are equivalent:*

(i) *E is pseudo-$\aleph_1$-compact;*
(ii) *for any finite non-empty set $S \subseteq T$ and for any uncountable family $(U_i : i \in I)$ of open sets U_i in X with $U_i \cap E \neq \emptyset$ the family $\big(p_S(U_i \cap E) : i \in I\big)$ is not locally finite in $p_S(E)$.*

Proof (i)$\Rightarrow$(ii) Let $S \subseteq T$ be a finite non-empty set, $(U_i : i \in I)$ be an uncountable family of basic open sets U_i in X with $U_i \cap E \neq \varnothing$ and let $V_i = p_S(U_i \cap E)$ for each $i \in I$. If the family $(V_i : i \in I)$ is locally finite in $p_S(E)$, then the family $\big(p_S^{-1}(V_i) \cap E : i \in I\big)$ is locally finite in E and $U_i \cap E \subseteq p_S^{-1}(V_i) \cap E$ for each $i \in I$, which contradicts pseudo-$\aleph_1$-compactness of E.

(ii)$\Rightarrow$(i) Consider an uncountable family $\big(U_i = \prod_{t \in T} U_i^t : i \in I\big)$ of basic open sets in X such that $U_i \cap E \neq \varnothing$ for all $i \in I$. By Šanin's lemma [12] we choose a finite set Z and uncountable set $J \subseteq I$ such that $N(U_i) \cap N(U_j) = Z$ for all distinct $i, j \in J$.

Let $V_i = p_Z(U_i \cap E)$ for all $i \in J$. It follows from (ii) that the family $(V_i : i \in J)$ has a cluster point $v \in p_Z(E)$. Take $y \in E$ such that $v = p_Z(y)$ and put $x = y_Z^a$. We shall show that x is a cluster point of $(U_i \cap E : i \in J)$. Indeed, let $W = \prod_{t \in T} W_t$ be a basic open neighborhood of x in X and $V = \prod_{t \in Z} W_t \cap p_Z(E)$. Choose such infinite set $K \subseteq J$ that $V \cap V_i \neq \varnothing$ and $N(W) \cap N(U_i) \subseteq Z$ for all $i \in K$. Take an arbitrary $i \in K$ and a point $b \in V \cap V_i$. Consider a point $c \in U_i \cap E$ with $b = p_Z(c)$ and put $d = c^a_{Z \cup N(U_i)}$. Clearly, $d \in U_i$. Since E is $\aleph_0$-invariant with respect to a and $c \in E$, $d \in E$. Moreover, $p_Z(d) = p_Z(c) = b \in V$ and $d_t = a_t \in W_t$ for every $t \in N(W) \setminus Z$. Therefore, $d \in W$. Hence, $d \in W \cap E \cap U_i$. □

The example below shows that condition (ii) in the previous theorem cannot be weakened to the following: *the set $p_S(E)$ is pseudo-$\aleph_1$-compact for any non-empty finite set $S \subseteq T$.*

Example There exists a set $E \subseteq \prod_{t \in T} X_t$, $(\aleph_1, \aleph_1)$-invariant with respect to a point $a \in E$ such that $p_S(E)$ is pseudo-$\aleph_1$-compact for any non-empty finite set $S \subseteq T$, but E is not pseudo-$\aleph_1$-compact.

Proof Let $T = [0, 1]$, $X_0 = \mathbb{P} = \mathbb{R} \times [0, +\infty)$ be the Niemytzki plane [6, p. 21], $X_t = \{0, 1\}$ for each $t \in (0, 1]$, $X = \prod_{t \in T} X_t$ and let $a = (a_t)_{t \in T} \in X$, where $a_t = 0$ for each $t \in (0, 1]$ and $a_0 = (0, 0)$. For each $t \in (0, 1]$ define $y^{(t)} = \big(y_s^{(t)}\big)_{s \in T}$ and $z^{(t)} = \big(z_s^{(t)}\big)_{s \in T} \in X$ as follows:

$$y_s^{(t)} = \begin{cases} 0, & s \in (0, 1] \setminus \{t\}, \\ 1, & s = t, \\ (t, 0), & s = 0, \end{cases} \qquad z_s^{(t)} = \begin{cases} 0, & s \in (0, 1] \setminus \{t\}, \\ 1, & s = t, \\ (0, 0), & s = 0. \end{cases}$$

Consider the $(\aleph_1, \aleph_1)$-invariant set

$$E = \big\{y^{(t)} : t \in (0, 1]\big\} \cup \big\{z^{(t)} : t \in (0, 1]\big\} \cup \left(X_0 \times \prod_{t \in (0,1]} \{0\}\right)$$

with respect to the point a. Observe that for any finite set $S \subseteq [0, 1]$ the sets $p_S\big(\{y^{(t)} : t \in (0, 1]\}\big)$ and $p_S\big(\{z^{(t)} : t \in (0, 1]\}\big)$ are finite and the set $p_S\big(X_0 \times \prod_{t \in (0,1]} \{0\}\big)$ is separable. Hence, E satisfies the condition mentioned above. But $\big(\{y^{(t)}\} : t \in (0, 1]\big)$ is a locally finite family of open sets in E. Therefore, E is not pseudo-$\aleph_1$-compact. □

3 Dependence of continuous mappings on countably many coordinates

Definition 3.1 Let $E \subseteq \prod_{t\in T} X_t$. We say that a function $f\colon E \to Y$ *depends on a set* $S \subseteq T$ [3, p. 231] if for all $x, y \in E$ the equality $p_S(x) = p_S(y)$ implies $f(x) = f(y)$. If $|S| \le \aleph_0$ then we say that f *depends on countably many coordinates*. Similarly, E *depends on* S if for all $x \in E$ and $y \in X$ with $p_S(x) = p_S(y)$ we have $y \in E$.

Definition 3.2 We say that a space Y has a *regular* G_δ*-diagonal* [14] if there exists a sequence $(G_n)_{n=1}^\infty$ of open subsets of Y^2 such that

$$\{(y, y) : y \in Y\} = \bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \overline{G_n}. \tag{1}$$

We denote $\sigma(a) = \big\{x \in X : |t \in T : x_t \ne a_t| < \aleph_0\big\}$ as in [4].

Theorem 3.3 *Let Y be a space with a regular G_δ-diagonal, $(X_t : t \in T)$ be a family of topological spaces, $X = \prod_{t\in T} X_t$, $a \in X$ and let $E \subseteq X$ be a pseudo-$\aleph_1$-compact subspace which is $(\aleph_1, \aleph_0)$-invariant with respect to a. Then for any continuous mapping $f\colon E \to Y$ there exist a countable set $T_0 \subseteq T$ and a continuous mapping $f_0\colon p_{T_0}(E) \to Y$ such that $f = f_0 \circ (p_{T_0}|_E)$. In particular, f depends on countably many coordinates.*

Proof Let $(G_n)_{n=1}^\infty$ be a sequence of open sets in Y^2 which satisfies (1) and let $f\colon E \to Y$ be a continuous function. Denote by T_0 the set of all $t \in T$ for which there exist points $x^t, y^t \in E \cap \sigma(a)$ such that

$$\begin{aligned} x^t_s &= y^t_s \qquad \text{for all} \quad s \ne t,\\ x^t_t &= a_t,\\ f(x^t) &\ne f(y^t). \end{aligned} \tag{2}$$

Assume that T_0 is uncountable and choose an uncountable subset $B \subseteq T_0$ and a number $n_0 \in \mathbb{N}$ such that

$$\big(f(x^t), f(y^t)\big) \in Y^2 \setminus \overline{G_{n_0}} \qquad \text{for all} \quad t \in B.$$

Using the continuity of f at x^t and y^t for every $t \in B$, we find open basic neighborhoods U^t and V^t of x^t and y^t, respectively, such that

$$p_s(U^t) = p_s(V^t) \qquad \text{for} \quad s \ne t, \tag{3}$$

$$f(U^t \cap E) \times f(V^t \cap E) \subseteq Y^2 \setminus \overline{G_{n_0}}. \tag{4}$$

Since E is pseudo-$\aleph_1$-compact and the family $(V^t \cap E : t \in B)$ is uncountable, there exists a point $x^* \in E$ such that for any basic open neighborhood W of x^* the set $C_W = \{t \in B : V^t \cap E \cap W \ne \varnothing\}$ is infinite. The continuity of f at x^* implies that there is an basic open neighborhood W of x^* such that $f(W \cap E) \times f(W \cap E) \subseteq G_{n_0}$.

Notice that $C = C_W \setminus N(W) \neq \varnothing$. Fix $t \in C$ and $y \in V^t \cap E \cap W$. Let $x = y^a_{T\setminus\{t\}}$. Then (2) and (3) imply that $x \in U^t$. Since E is $(\aleph_1, \aleph_0)$-invariant with respect to a, $x \in E$. Moreover, $x \in W$, since $t \notin N(W)$. Then $(f(x), f(y)) \in G_{n_0}$, which contradicts (4). Hence, the set T_0 is countable.

We show that f depends on T_0. To do this it is sufficient to check the equality $f(x) = f(x^a_{T_0})$ for every $x \in E$. Consider the case $x \in E \cap \sigma(a)$. Let $\{t \in T \setminus T_0 : x_t \neq a_t\} = \{t_1, \ldots, t_m\}$. Then

$$f(x) = f\big(x^a_{T\setminus\{t_1\}}\big) = f\big(\big(x^a_{T\setminus\{t_1\}}\big)^a_{T\setminus\{t_2\}}\big) = \cdots$$
$$= f\big(\big(\big(x^a_{T\setminus\{t_1\}}\big)\cdots\big)^a_{T\setminus\{t_m\}}\big) = f\big(x^a_{T_0}\big).$$

Now let $x \in E$. Notice that $E \cap \sigma(a)$ is a dense set in E. Indeed, if $b = (b_t)_{t\in T} \in E$ and W is a basic open neighborhood of b in X, then $b^a_{N(W)} \in W \cap E \cap \sigma(a)$. Hence, there exists a net (x_i) of points $x_i \in E \cap \sigma(a)$ such that $\lim_i x_i = x$. Then $\lim_i (x_i)^a_{T_0} = x^a_{T_0}$. It follows from continuity of f that

$$f(x) = f\big(\lim_i x_i\big) = \lim_i f(x_i) = \lim_i f\big((x_i)^a_{T_0}\big) = f\big(\lim_i (x_i)^a_{T_0}\big) = f\big(x^a_{T_0}\big).$$

Consider the function $f_0 \colon p_{T_0}(E) \to Y$ defined by $f_0(z) = f(x)$ if $z = p_{T_0}(x)$ for $x \in E$. Observe that f_0 is defined correctly, because f depends on T_0. It remains to prove that f_0 is continuous on $p_{T_0}(E)$. Fix $z \in p_{T_0}(E)$ and a net (z_i) of points $z_i \in p_{T_0}(E)$ such that $\lim_i z_i = z$. Take $x \in E$ and $x_i \in E$ with $z = p_{T_0}(x)$ and $z_i = p_{T_0}(x_i)$. Let $y_i = (x_i)^a_{T_0}$ and $y = x^a_{T_0}$. Then $y_i, y \in E$ and $\lim_i y_i = y$. Moreover, since f is continuous at y, we have

$$\lim_i f_0(z_i) = \lim_i f(x_i) = \lim_i f(y_i) = f(y) = f(x) = f_0(z).$$

Hence, f_0 is continuous at z. □

Notice that the proof of dependence of f on T_0 in Theorem 3.3 is similar to the proof of [1, Lemmas 2.27 (a) and 2.32].

Theorem 3.4 *Let $(X_t : t \in T)$ be an uncountable family of topological spaces, $X = \prod_{t\in T} X_t$, $a \in X$ and let $E \subseteq X$ be an $(\aleph_1, \aleph_0)$-invariant set with respect to a. Consider the following conditions:*

(i) *E is pseudo-$\aleph_1$-compact;*
(ii) *for any space Y with a regular G_δ-diagonal and for any continuous mapping $f \colon E \to Y$ there exist a countable set $T_0 \subseteq T$ and a continuous mapping $f_0 \colon p_{T_0}(E) \to Y$ such that $f = f_0 \circ (p_{T_0}|_E)$;*
(iii) *for any continuous function $f \colon E \to \mathbb{R}$ there exist a countable set $T_0 \subseteq T$ and a continuous mapping $f_0 \colon p_{T_0}(E) \to \mathbb{R}$ such that $f = f_0 \circ (p_{T_0}|_E)$.*

Then (i) ⇒ (ii) ⇒ (iii).
If E is completely regular and

(iv) *for any non-empty open set U in E there exists an uncountable set $T_U \subseteq T$ such that for every $t \in T_U$ there are $y^{(t)} = \big(y^{(t)}_s\big)_{s\in T}$ and $z^{(t)} = \big(z^{(t)}_s\big)_{s\in T} \in U$ with $y^{(t)}_t \neq z^{(t)}_t$ and $y^{(t)}_s = z^{(t)}_s$ for every $s \in T\setminus\{t\}$,*

then (iii) ⇒ (i).

Proof The implication (i) ⇒ (ii) follows from Theorem 3.3, whereas the implication (ii) ⇒ (iii) is obvious.
(iii) ⇒ (i) Suppose that E is not pseudo-$\aleph_1$-compact and choose a locally finite in E family $(U_\alpha : \alpha < \omega_1)$ of non-empty open sets U_α. Note that U_α may be taken to be disjoint. Indeed, let $(V_i : i \in I)$ be a locally finite family of non-empty open subsets of E with $|I| > \aleph_0$. For every $i \in I$ we choose a non-empty open set $W_i \subseteq V_i$ and a finite set $J_i \subseteq I$ such that $W_i \subseteq \bigcap_{j\in J_i} V_j$ and $W_i \cap V_j = \emptyset$ for all $j \in I \setminus J_i$. Since $i \in J_i$ for every $i \in I$, $\bigcup_{i\in I} J_i = I$. Now we take a uncountable set $I_0 \subseteq I$ such that all sets J_i from the family $(J_i : i \in I_0)$ are different. Then the uncountable family $(W_i : i \in I_0)$ consists of mutually disjoint elements.

Since E is completely regular, we may assume that all sets U_α are functionally open. For every $\alpha < \omega_1$ take a continuous function $f_\alpha : E \to [0, 1]$ such that $U_\alpha = f_\alpha^{-1}((0, 1])$. Since T_{U_α} is uncountable, we may construct a family $(t_\alpha : \alpha < \omega_1)$ of distinct points $t_\alpha \in T_{U_\alpha}$. According to (iv) we choose for every $\alpha < \omega_1$ points $y^{(\alpha)} = \big(y^{(\alpha)}_s\big)_{s\in T}$, $z^{(\alpha)} = \big(z^{(\alpha)}_s\big)_{s\in T} \in U_\alpha$ such that $y^{(\alpha)}_{t_\alpha} \neq z^{(\alpha)}_{t_\alpha}$ and $y^{(\alpha)}_s = z^{(\alpha)}_s$ for every $s \in T\setminus\{t_\alpha\}$. Now for every $\alpha < \omega_1$ we choose a continuous function $g_\alpha : E \to [0, 1]$ such that $g_\alpha\big(y^{(\alpha)}\big) = 1$ and $g_\alpha\big(z^{(\alpha)}\big) = 0$.

Consider the continuous function $f : E \to [0, 1]$, $f(x) = \sum_{\alpha<\omega_1} f_\alpha(x) g_\alpha(x)$. Since sets U_α are mutually disjoint,

$$f\big(y^{(\alpha)}\big) - f\big(z^{(\alpha)}\big) = f_\alpha\big(y^{(\alpha)}\big) g_\alpha\big(y^{(\alpha)}\big) - f_\alpha\big(z^{(\alpha)}\big) g_\alpha\big(z^{(\alpha)}\big) = f_\alpha\big(y^{(\alpha)}\big) > 0.$$

Hence, $f\big(y^{(\alpha)}\big) \neq f\big(z^{(\alpha)}\big)$ for every $\alpha < \omega_1$. Since the set $\{t_\alpha : \alpha < \omega_1\}$ is uncountable, the function f does not satisfy (iii). □

4 Functionally measurable sets

Proposition 4.1 *Let E be a subset of $X = \prod_{t\in T} X_t$ such that for any continuous function $f : E \to \mathbb{R}$ there exist a countable set $T_0 \subseteq T$ and a continuous mapping $f_0 : p_{T_0}(E) \to \mathbb{R}$ with $f = f_0 \circ (p_{T_0}|_E)$ and let $0 \le \alpha < \omega_1$. Then for any set A of functionally additive (multiplicative) class α in E there exists a countable set $T_0 \subseteq T$ such that A depends on T_0 and $p_{T_0}(A)$ is of functionally additive (multiplicative) class α in $p_{T_0}(E)$.*

Proof Let $\alpha = 0$. We consider the case when a set A is functionally open in E. Then $A = f^{-1}((0, +\infty))$ for some continuous function $f : E \to \mathbb{R}$. Take a countable set $T_0 \subseteq T$ and a continuous mapping $f_0 : p_{T_0}(E) \to \mathbb{R}$ with $f = f_0 \circ (p_{T_0}|_E)$. Then the set $p_{T_0}(A) = f_0^{-1}((0, +\infty))$ is functionally open in $p_{T_0}(E)$. Moreover, if $x \in A$ and $y \in E$ with $p_{T_0}(x) = p_{T_0}(y)$, then $f(y) = f(x) > 0$. Therefore, $y \in A$ which implies that A depends on T_0.

Assume that the assertion is true for all $\alpha < \beta$ and consider a set A of functionally additive class α in E. Then $A = \bigcup_{n=1}^{\infty} A_n$, where A_n is of functionally multiplicative class $\alpha_n < \alpha$ for every n. By the assumption, for every n there exists a countable set $T_n \subseteq T$ such that A_n depends on T_n and $p_{T_n}(A_n)$ belongs to functionally multiplicative class α_n in $p_{T_n}(E)$. Notice that $p_{T_0}(A_n)$ is of functionally multiplicative class α_n in $p_{T_0}(E)$ for every n. Then $p_{T_0}(A) = \bigcup_{n=1}^{\infty} p_{T_0}(A_n)$ is of functionally additive class α in $p_{T_0}(E)$. □

Definition 4.2 Let $0 \le \alpha < \omega_1$. A space X is called α-*universal* if any subset of X is α-embedded in X.

Clearly, every perfectly normal space is α-universal for any $\alpha < \omega_1$.

Proposition 4.3 *Let $0 \le \alpha < \omega_1$, $(X_t)_{t\in T}$ be a family of topological spaces such that every countable subproduct is α-universal, $X = \prod_{t\in T} X_t$ and let $E \subseteq X$ be such a set as in Proposition 4.1. Then E is an α-embedded set in X.*

Proof Let $A \subseteq E$ be a set of functionally multiplicative class α in E. According to Proposition 4.1 there exists a countable set $T_0 \subseteq T$ such that A depends on T_0 and $A_0 = p_{T_0}(A)$ is of functionally multiplicative class α in $E_0 = p_{T_0}(E)$. Since $X_0 = \prod_{t\in T_0} X_t$ is α-universal, the set E_0 is α-embedded in X_0. Hence, there exists a set B_0 of functionally multiplicative class α in X_0 such that $B_0 \cap E_0 = A_0$. Let $B = p_{T_0}^{-1}(B_0)$. Then B is of functionally multiplicative class α in X, because the mapping p_{T_0} is continuous. Moreover, it is easy to see that $B \cap E = A$. □

Proposition 4.4 *Let $0 \le \alpha < \omega_1$, $X = \prod_{t\in T} X_t$ be a pseudo-$\aleph_1$-compact space, where $(X_t)_{t\in T}$ is a family of spaces such that every countable subproduct is α-universal and hereditarily pseudo-$\aleph_1$-compact. Then any functionally measurable set $E \subseteq X$ is α-embedded in X.*

Proof Consider a functionally measurable set $E \subseteq X$. Without loss of generality, we may assume that E belongs to functionally multiplicative class β for some $0 \le \beta < \omega_1$. Take a function $f \in B_\beta(X)$ such that $E = f^{-1}(0)$. Since X is pseudo-$\aleph_1$-compact, [11, Theorem 2.3] implies that there exists a countable set $T_0 \subseteq T$ such that for all $x \in E$ and $y \in X$ the equality $p_{T_0}(x) = p_{T_0}(y)$ implies that $y \in E$. Let $E_0 = p_{T_0}(E)$. Then

$$E = E_0 \times \prod_{t\in T\setminus T_0} X_t.$$

Since $\prod_{t\in T_0\cup S} X_t$ is a hereditarily pseudo-$\aleph_1$-compact space, $E_0 \times \prod_{t\in S} X_t$ is pseudo-$\aleph_1$-compact space for any finite set $S \subseteq T\setminus T_0$. Hence, by [11, Corollary1.5] the set E is pseudo-$\aleph_1$-compact. Therefore, E satisfy the condition of Proposition 4.1 by Theorem 3.3 applied to the whole product $E_0 \times \prod_{t\in T\setminus T_0} X_t$. It remains to use Proposition 4.3. □

The following result implies a positive answer to [8, Question 8.1].

Corollary 4.5 *Let $(X_t)_{t\in T}$ be a family of separable metrizable spaces. Then every functionally measurable subset of $X=\prod_{t\in T}X_t$ is α-embedded in X for any $0\le\alpha<\omega_1$.*

Proof The statement follows from Proposition 4.4 and the fact that any countable product of separable metrizable spaces is separable and metrizable, consequently, α-universal and hereditarily pseudo-$\aleph_1$-compact. □

5 The construction of α-embedded sets

Theorem 5.1 *For every $0\le\alpha<\omega_1$ there exists a completely regular space X with an $(\alpha+1)$-embedded subspace $E\subseteq X$ which is not α-embedded.*

Proof Fix $\alpha<\omega_1$. Let $X_0=[0,1]$, $X_t=\mathbb{N}$ for every $t\in(0,1]$, $Y=\prod_{t\in(0,1]}X_t$ and $X=[0,1]\times Y=\prod_{t\in[0,1]}X_t$.

According to [9, p. 371] there exists a set $A_1\subseteq[0,1]$ of additive class α which does not belong to multiplicative class α. Let $A_2=[0,1]\setminus A_1$. For $i=1,2$ put

$$F_i=\bigcap_{n\ne i}\big\{y=(y_t)_{t\in(0,1]}\in Y:|\{t\in(0,1]:y_t=n\}|\le 1\big\}.$$

It is easy to see that F_1 and F_2 are closed disjoint subsets of Y. Let $B_i=A_i\times F_i$ for $i=1,2$ and $E=B_1\cup B_2$. Then B_1 and B_2 are disjoint closed subsets of E.

Claim 5.2 *The set B_i is α-embedded in X for every $i=1,2$.*

Proof We show that B_1 is pseudo-$\aleph_1$-compact (for the set B_2 we argue verbatim). Since A_1 is separable, it is enough to check that F_1 is pseudo-$\aleph_1$-compact. Notice that the set F_1 is $(\aleph_1,\aleph_1)$-invariant with respect to the point $a=(a_t)_{t\in(0,1]}$, where $a_t=1$ for every $t\in(0,1]$. Since for any finite set $S\subseteq(0,1]$ the space $\prod_{t\in S}X_t$ is countable, the set F_1 satisfies condition (ii) of Theorem 2.5. Then by Theorem 2.5 the set F_1 is pseudo-$\aleph_1$-compact.

Now observe that each set B_i is $(\aleph_1,\aleph_1)$-invariant with respect to the point $a^i=(a^i_t)_{t\in[0,1]}$, where $a^i_t=i$ for all $t\in(0,1]$ and $a^i_0\in A_i$. It remains to apply Theorem 3.3 and Proposition 4.3. ■

Claim 5.3 *The set E is not α-embedded in X.*

Proof Assume the contrary and choose a set H of functionally multiplicative class α in X such that $H\cap E=B_1$. It follows from Proposition 4.1 that there is a countable set $S=\{0\}\cup T$, where $T\subseteq(0,1]$, such that H depends on S. Let $y_0\in Y$ be such that $p_T(y_0)$ is a sequence of distinct natural numbers which are not equal to 1 or 2. Take $y_1\in F_1$ and $y_2\in F_2$ with $p_T(y_0)=p_T(y_1)=p_T(y_2)$. Then for all $x\in A_1$ we have $(x,y_1)\in H$ and, consequently, $(x,y_0)\in H$. Moreover, for all $x\in A_2$ we have $(x,y_2)\notin H$ and, consequently, $(x,y_0)\notin H$. Hence, $A_1\times\{y_0\}=\big([0,1]\times\{y_0\}\big)\cap H$. Therefore, $A_1\times\{y_0\}$ is of functionally multiplicative class α in X, which implies that the set A_1 belongs to functionally multiplicative class α in $[0,1]$, a contradiction. ■

Claim 5.4 *The set E is $(\alpha+1)$-embedded in X.*

Proof Let C be a set of functionally multiplicative class $(\alpha+1)$ in E. Denote $E_i = A_i \times Y$ for $i = 1, 2$. Then E_1 is of functionally additive class α and E_2 is of functionally multiplicative class α in X. For $i = 1, 2$ put $C_i = C \cap B_i$. Since each of the sets C_i is of functionally multiplicative class $(\alpha+1)$ in the α-embedded set B_i in X, there exists a set D_i of functionally multiplicative class $(\alpha+1)$ in X such that $D_i \cap B_i = C_i$. Let $D = (D_1 \cap E_1) \cup (D_2 \cap E_2)$. Then D is a set of functionally multiplicative class $(\alpha+1)$ in X and $D \cap E = C$. ■

This completes the proof. □

Notice that the sets F_i were first defined by Stone [13] in his proof of non-normality of the uncountable power $\mathbb{N}^\tau$ of the space $\mathbb{N}$ of natural numbers.

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