European Journal of Mathematics (2015) 1:260–278
DOI 10.1007/s40879-014-0015-3

RESEARCH ARTICLE

The ambiguity index of an equipped finite group

Fedor A. Bogomolov · Viktor S. Kulikov

Received: 7 May 2014 / Revised: 23 September 2014 / Accepted: 10 October 2014 /
Published online: 8 April 2015

Abstract In the paper (Kulikov in Sb Math 204(2):237–263, 2013), the ambiguity index $a_{(G,O)}$ was introduced for each equipped finite group (G, O). It is equal to the number of connected components of a Hurwitz space parametrizing coverings of a projective line with Galois group G assuming that all local monodromies belong to conjugacy classes O in G and the number of branch points is greater than some constant. We prove in this article that the ambiguity index can be identified with the size of a generalization of so called Bogomolov multiplier (Kunyavskiĭ in Cohomological and Geometric Approaches to Rationality Problems. Progress in Mathematics, vol 282, pp 209–217, 2010), see also (Bogomolov in Math USSR-Izv 30(3):455–485, 1988) and hence can be easily computed for many pairs (G, O). In particular, the ambiguity indices are completely counted in the cases when G are the symmetric or alternating groups.

Keywords Equipped group · C-group · Bogomolov multiplier · Hurwitz space

The financial support from the Government of the Russian Federation within the framework of the implementation of the 5-100 Programme Roadmap of the National Research University, Higher School of Economics, AG Laboratory is acknowledged by both authors. The first author was also supported by Simons Travel Grant. The second author was partially supported by NSh-2998.2014, RFBR 14-01-00160.

F. A. Bogomolov
Courant Institute of Mathematical Sciences, New York University, 251 Mercer Str., New York, NY 10012, USA
e-mail: bogomolo@cims.nyu.edu

F. A. Bogomolov
Laboratory of Algebraic Geometry, NRU HSE, 7 Vavilova Str., 117312 Moscow, Russia

V. S. Kulikov (✉)
Steklov Mathematical Institute, 8 Gubkina Str., 119991 Moscow, Russia
e-mail: kulikov@mi.ras.ru

Mathematics Subject Classification 20C25

1 Introduction

Let G be a finite group and O be a subset of G consisting of conjugacy classes C_i of G, $O = C_1 \cup \cdots \cup C_m$, which together generate G. The pair (G, O) is called an *equipped group* and O is called an *equipment* of G. We fix the numbering of conjugacy classes contained in O. One can associate a C-group $(\widetilde{G}, \widetilde{O})$ to each equipped group (G, O). The C-group $\widetilde{G}$ is generated by the letters of the alphabet $Y = Y_O = \{y_g : g \in O\}$ subject to relations

$$y_{g_1} y_{g_2} = y_{g_2} y_{g_2^{-1} g_1 g_2} = y_{g_1 g_2 g_1^{-1}} y_{g_1}.$$

We assume $\widetilde{O} = Y_O$ in the definition of $\widetilde{G}$. There is an obvious natural homomorphism $\beta \colon \widetilde{G} \to G$ given by $\beta(y_g) = g$. It was shown in [11], that the commutator subgroup $[\widetilde{G}, \widetilde{G}]$ is finite. The order $a_{(G,O)}$ of the group $\ker \beta \cap [\widetilde{G}, \widetilde{G}]$ was called the *ambiguity index* of the equipped finite group (G, O).

The notion of equipped groups is related to the description of Hurwitz spaces parametrizing maps between projective curves with G as the monodromy group and the ambiguity index $a_{(G,O)}$ is equal to the properly defined "asymptotic" number of connected components of Hurwitz space parametrizing covering of curves with fixed ramification data. More precisely, let $f \colon E \to F$ be a morphism of a non-singular complex irreducible projective curve E onto a non-singular projective curve F. Let us choose a point $z_0 \in F$ such that z_0 is not a branch point of f hence the points $f^{-1}(z_0) = \{w_1, \ldots, w_d\}$, where $d = \deg f$, are simple. If we fix the numbering of points in $f^{-1}(z_0)$ then we call f a *marked covering*.

Let $B = \{z_1, \ldots, z_n\} \subset F$ be the set of branch points of f. The numbering of the points of $f^{-1}(z_0)$ defines a homomorphism $f_* \colon \pi_1(F \setminus B, z_0) \to \Sigma_d$ of the fundamental group $\pi_1 = \pi_1(F \setminus B, z_0)$ to the symmetric group Σ_d. Define $G \subset \Sigma_d$ as $\operatorname{im} f_* = G$. It acts transitively on $f^{-1}(z_0)$. Let $\gamma_1, \ldots, \gamma_n$ be simple loops around, respectively, the points $z_1, \ldots, z_n$ starting at z_0. The image $g_j = f_*(\gamma_j) \in G$ is called a *local monodromy* of f at the point z_j. Each local monodromy g_j depends on the choice of γ_j, therefore, it is defined uniquely up to conjugation in G.

Denote by $O = C_1 \cup \cdots \cup C_m \subset G$ the union of conjugacy classes of all local monodromies and by τ_i the number of local monodromies of f belonging to the conjugacy class C_i. The collection $\tau = (\tau_1 C_1, \ldots, \tau_m C_m)$ is called the *monodromy type* of f. Assume that the elements of O generate the group G. Then the pair (G, O) is an equipped group. Let $\mathrm{HUR}^m_{d,G,O,\tau}(F, z_0)$ be the Hurwitz space (see the definition of Hurwitz spaces in [4] or in [12]) of marked degree d coverings of F with Galois group $G \subset \Sigma_d$, local monodromies in O, and monodromy type τ.

The number of irreducible components of $\mathrm{HUR}^m_{d,G,O,\tau}(F, z_0)$ for fixed d, G, O, F is a function of an integer vector $\tau = (\tau_1, \ldots, \tau_m)$. It was proved in [12] that this number is constant for big τ. More precisely, for each equipped finite group (G, O) there is T such that if for all $i = 1, \ldots, m$ we have $\tau_i \geqslant T$, $i = 1, \ldots, m$, then the number of irreducible components of the Hurwitz space $\mathrm{HUR}^m_{d,G,O,\tau}(F, z_0)$ is equal to $a_{(G,O)}$. The (minimal) number T does not depend on the base curve F and the degree d of the covering.

The subgroup $B_0(G) \subset H^2(G, \mathbb{Q}/\mathbb{Z})$ was defined and studied in [1]. It consists of elements of $H^2(G, \mathbb{Q}/\mathbb{Z})$ which restrict trivially onto abelian subgroups of G. It was conjectured in [2] that $B_0(G)$ is trivial for simple groups. This conjecture was partially solved already in [2] and it was completely solved by Kunyavskiĭ in [13],[1] and by Kunyavskiĭ–Kang in [8] for a wider class of almost simple groups. The latter consists of groups G which contain some simple group L and in turn are contained in the automorphism group Aut L. Kunyavskiĭ in [13] called $B_0(G)$ as *Bogomolov multiplier* and we are going to use his terminology here. Denote by $b_0(G)$ the order of the group $B_0(G)$ and denote by $h_2(G)$ the order of the Schur multiplier of the group G, that is, the order of the group $H_2(G, \mathbb{Z})$.

The aim of this article is to prove

Theorem 1.1 *For an equipped finite group* (G, O) *we have the following inequalities:*

$$b_0(G) \leqslant a_{(G,O)} \leqslant h_2(G).$$

In particular, $a_{(G,G\setminus\{1\})} = b_0(G)$.

Since, by [13], $b_0(G) = 1$ for a finite almost simple group G, we conclude

Corollary 1.2 *Let G be a finite almost simple group. Then there is a constant T such that for any projective irreducible non-singular curve F each non-empty Hurwitz space* $\mathrm{HUR}^m_{d,G,G\setminus\{1\},\tau}(F, z_0)$ *is irreducible if all* $\tau_i \geqslant T$.

It was shown in [11] that if $O_1 \subset O_2$ are two equipments of a finite group G, then $a_{(G,O_2)} \leqslant a_{(G,O_1)}$.

For a symmetric group Σ_d, the famous Clebsch–Hurwitz theorem [3,6] implies that the ambiguity index $a_{(\Sigma_d,T)} = 1$, where T is the set of transpositions in Σ_d, and it was shown in [10] that the ambiguity index $a_{(\Sigma_d,O)} = 1$ if the equipment O contains an odd permutation $\sigma \in \Sigma_d$ such that σ leaves fixed at least two elements. Theorem 4.14 (see Sect. 4.4) gives the complete answer on the value of $a_{(\Sigma_d,O)}$ for each equipment O of Σ_d. Also in Sect. 4.4, we give the complete answer on the value of $a_{(\mathbb{A}_d,O)}$ for each d and for each equipment O of the alternating group $\mathbb{A}_d$.

In Sect. 2, we remind some properties of C-groups and prove one of the inequalities claimed in Theorem 1.1. In Sect. 3, we complete the proof of this theorem.

In Sect. 4, we investigate the properties of ambiguity indices of a quasi-cover of an equipped finite group (G, O), and in Sect. 5, we give a cohomological description of the ambiguity indices.

In Sect. 6, we give examples of finite groups G with Bogomolov multiplier $b_0(G) > 1$. Therefore, for such G each non-empty space $\mathrm{HUR}^m_{d,G,O,\tau}(F, z_0)$ consists of at least $b_0(G) > 1$ irreducible components for any $\tau = (\tau_1, \dots, \tau_m)$ with big enough τ_i.

In this article, if $\mathbb{F}$ is a free group freely generated by an alphabet X, N is a normal subgroup of $\mathbb{F}$, and a group $G = \mathbb{F}/N$, then a word $w = w(x_{i_1}, \dots, x_{i_n})$ in letters

[1] The proof of triviality of $B_0(G)$ for finite almost simple groups G was recently completed in [8] after a minor gap was discovered in the argument in [12]. The gap was due to the reference on a result in [1] which contained an error and was corrected by Jezernik and Moravec [7].

$x_{i_j} \in X$ and their inverses will be considered as an element of G in case if it does not lead to misunderstanding.

2 C-groups and their properties

Let us remind the definition of a C-group (see, for example, [9]).

Definition 2.1 A group G is a C-*group* if there is a set of generators $x \in X$ in G such that a basis of relations between $x \in X$ consists of the following relations:

$$x_i^{-1} x_j x_i = x_k, \qquad (x_i, x_j, x_k) \in M, \tag{1}$$

where M is a subset of X^3.

Thus the C-structure of G is defined by $X \subset G$ and $M \subset X^3$.

Let $\mathbb{F}$ be a free group freely generated by an alphabet X. Denote by N the subgroup of $\mathbb{F}$ normally generated by the elements $x_i^{-1} x_j x_i x_k^{-1}$, $(x_i, x_j, x_k) \in M$. The group N is a normal subgroup of $\mathbb{F}$. Let $f\colon \mathbb{F} \to G = \mathbb{F}/N$ be the natural epimorphism given by presentation (1). In the sequel, we consider each C-group G as an equipped group (G, O) with the equipment $O = f(X^{\mathbb{F}})$ (where $X^{\mathbb{F}}$ is the orbit of X under the action of the group of inner automorphisms of $\mathbb{F}$). The elements of O are called C-*generators* of the C-group G. In particular, the equipped group $(\mathbb{F}, X^{\mathbb{F}})$ is a C-group.

A homomorphism $f\colon G_1 \to G_2$ of a C-group (G_1, O_1) to a C-group (G_2, O_2) is called a C-*homomorphism* if it is a homomorphism of equipped groups, that is, $f(O_1) \subset O_2$. In particular, two C-groups (G_1, O_1) and (G_2, O_2) are C-*isomorphic* if they are isomorphic as equipped groups.

Claim 2.2 ([9, Lemma 3.6]) *Let N be a normal subgroup of $\mathbb{F}$ normally generated by a set of elements of the form $w_i^{-1} x_j w_i w_l x_k^{-1} w_l^{-1}$, where w_i and w_l are elements of $\mathbb{F}$ and $x_j, x_k \in X$. Let $f\colon \mathbb{F} \to G \simeq \mathbb{F}/N$ be the natural epimorphism. Then $(G, f(X^{\mathbb{F}}))$ is a C-group and f is a C-homomorphism.*

To each C-group (G, O), one can associate a C-graph. By definition, the C-*graph* $\Gamma = \Gamma_{(G,O)}$ *of* a C-*group* (G, O) is a directed labeled graph whose set of vertices $V = \{v_{g_i} : g_i \in O\}$ is in one-to-one correspondence with the set O. Two vertices v_{g_1} and v_{g_2}, $g_1, g_2 \in O$, are connected by a labeled edge $e_{v_{g_1} v_{g_2} v_g}$ (here v_{g_1} is the tail of $e_{v_{g_1} v_{g_2} v_g}$, v_{g_2} is the head of $e_{v_{g_1} v_{g_2} v_g}$, and v_g is the label of $e_{v_{g_1} v_{g_2} v_g}$) if and only if in G we have the relation $g^{-1} g_1 g = g_2$ with some $g \in O$.

A C-homomorphism $f\colon (G_1, O_1) \to (G_2, O_2)$ of C-groups induces a map $f_*\colon \Gamma_{(G_1,O_1)} \to \Gamma_{(G_2,O_2)}$ from the C-graph $\Gamma_{(G_1,O_1)}$ in the C-graph $\Gamma_{(G_2,O_2)}$, where by definition, $f_*(v_g) = v_{f(g)}$ for each vertex v_g of $\Gamma_{(G_1,O_1)}$ and

$$f_*\bigl(e_{v_{g_1} v_{g_2} v_g}\bigr) = e_{v_{f(g_1)} v_{f(g_2)} v_{f(g)}}$$

for each edge $e_{v_{g_1} v_{g_2} v_g}$ of $\Gamma_{(G_1,O_1)}$. The following claim is obvious.

Claim 2.3 *A C-homomorphism $f\colon (G_1, O_1) \to (G_2, O_2)$ is a C-isomorphism if f_* is one-to-one between the sets of vertices of $\Gamma_{(G_1,O_1)}$ and $\Gamma_{(G_2,O_2)}$.*

In the sequel, we will consider only finitely presented C-groups (as groups without equipment) and C-graphs consisting of finitely many connected components. Denote by m the number of connected components of a C-graph $\Gamma_{(G,O)}$.

Then it is easy to see that $G/[G,G] \simeq \mathbb{Z}^m$ and any two C-generators g_1 and g_2 are conjugated in the C-group G if and only if v_{g_1} and v_{g_2} belong to the same connected component of $\Gamma_{(G,O)}$. Thus the set O of C-generators of the C-group (G, O) is the union of m conjugacy classes of G and there is a one-to-one correspondence between the conjugacy classes of G contained in O and the set of connected components of $\Gamma_{(G,O)}$.

Denote by $\tau\colon G \to H_1(G,\mathbb{Z}) = G/[G,G]$ the natural epimorphism. In the sequel, we fix some numbering of the connected components of $\Gamma_{(G,O)}$. Then the group $H_1(G,\mathbb{Z}) \simeq \mathbb{Z}^m$ obtains a natural base consisting of vectors $\tau(g) = (0,\dots,0,1,0,\dots,0)$, where 1 stands on the ith place if g is a C-generator of G and v_g belongs to the ith connected component of $\Gamma_{(G,O)}$. For $g \in G$ the image $\tau(g)$ is called the *type* of g.

Lemma 2.4 *Let g_1, g_2 be two C-generators of a C-group (G, O), N the normal closure of $g_1g_2^{-1}$ in G, and $f\colon G \to G_1 = G/N$ the natural epimorphism. Then*

(i) *(G_1, O_1) is a C-group, where $O_1 = f(O)$, and f is a C-homomorphism;*

(ii) *the map $f_*\colon \Gamma_{(G,O)} \to \Gamma_{(G_1,O_1)}$ is a surjection;*

(iii) *if $g_1g_2^{-1}$ belongs to the center $Z(G)$ of the group G and v_{g_1} and v_{g_2} belong to different components of $\Gamma_{(G,O)}$, then*

  (iii$_1$) *the number of connected components of the C-graph $\Gamma_{(G_1,O_1)}$ is less than the number of connected components of the C-graph $\Gamma_{(G,O)}$,*

  (iii$_2$) *$f\colon [G,G] \to [G_1,G_1]$ is an isomorphism.*

Proof Claims (i), (ii), and (iii$_1$) are obvious. To prove (iii$_2$), note that N is a cyclic group generated by $g_1g_2^{-1}$, since $g_1g_2^{-1}$ belongs to the center $Z(G)$. The type $\tau\big((g_1g_2^{-1})^n\big)$ is non-zero for $n \neq 0$, since v_{g_1} and v_{g_2} belong to different connected components of $\Gamma_{(G,O)}$. Therefore, to complete the proof, it suffices to note that the groups N and $[G,G]$ have trivial intersection, since $\tau(g) = 0$ for all $g \in [G,G]$. □

A C-group (G, O) is called a *C-finite group* if the set of vertices of C-graph $\Gamma_{(G,O)}$ is finite or, the same, if the equipment O of G is a finite set.

Proposition 2.5 ([11]) *Let (G, O) be a C-finite group. Then the commutator $[G,G]$ is a finite group.*

As it was mentioned in Sect. 1, to each finite equipped group (G, O), one can associate a C-group $(\widetilde{G}, \widetilde{O})$ defined as follows. The group $\widetilde{G}$ is generated by the letters of the alphabet $Y = Y_O = \{y_g : g \in O\}$ subject to relations

$$y_{g_1}y_{g_2} = y_{g_2}y_{g_2^{-1}g_1g_2} = y_{g_1g_2g_1^{-1}}y_{g_1}.$$

Here $\widetilde{O} = Y_O$ and there is a natural epimorphism $\beta_O\colon \widetilde{G} \to G$ given by $\beta_O(y_g) = g$.

Note also that a homomorphism of equipped groups $f\colon (G_1, O_1) \to (G, O)$ induces a C-homomorphism $\widetilde{f}\colon (\widetilde{G}_1, \widetilde{O}_1) \to (\widetilde{G}, \widetilde{O})$ such that $f \circ \beta_{O_1} = \beta_O \circ \widetilde{f}$.

Let the elements of a subset S of an equipment O of a group G generate the group G and $O = S^G$, where S^G is the orbit of S under the action $\mathrm{Inn}(G)$. Denote by $\mathbb{F}_S$ a free group freely generated by the alphabet $Y_S = \{y_g : g \in S\}$ and by R_S the normal subgroup of $\mathbb{F}_S$ such that the natural epimorphism $h_S : \mathbb{F}_S \to \mathbb{F}_S/R_S \simeq G$ gives a presentation of the group G.

Claim 2.6 *Let $\widetilde{R}_S \subset R_S$ be the normal subgroup normally generated by the elements of R_S of the form $w_{ij}^{-1} y_{g_i} w_{ij} y_{g_j}^{-1}$, where $w_{ij} \in \mathbb{F}_S$ and $y_{g_i}, y_{g_j} \in Y_S$. Then the C-group $(\widetilde{G}, \widetilde{O})$ has the presentation $\widetilde{G} \simeq \mathbb{F}_S/\widetilde{R}_S$ and the images of the elements of Y_S are C-generators of $\widetilde{G}$.*

Proof Denote by $G_1 = \mathbb{F}_S/\widetilde{R}_S$. By Claim 2.2, G_1 is a C-group with C-equipment $O_1 = Y_S^{G_1}$ and there is a natural epimorphism $\beta_S : (G_1, O_1) \to (G, O)$ given by $\beta_S(y_g) = g$ for $g \in S$.

Assume that S consists of elements $g_1, \dots, g_n \in O$. If $S \neq O$ then choose an element $g_{n+1} \in O \setminus S$. It is conjugated to some $g_i \in S$. Denote by $R_{g_{n+1}}$ the set of all presentations of g_{n+1} in the form

$$g_{n+1} = w(g_1, \dots, g_n)^{-1} g\, w(g_1, \dots, g_n), \qquad g \in S. \tag{2}$$

Note that if

$$\begin{aligned} g_{n+1} &= w_i(g_1, \dots, g_n)^{-1} g_i\, w_i(g_1, \dots, g_n), \\ g_{n+1} &= w_j(g_1, \dots, g_n)^{-1} g_j\, w_j(g_1, \dots, g_n), \end{aligned}$$

then $w_j w_i^{-1} g_i w_i w_j^{-1} = g_j$, that is,

$$\begin{aligned} &w_j(y_{g_1}, \dots, y_{g_n}) w_i(y_{g_1}, \dots, y_{g_n})^{-1} y_{g_i} \\ &\qquad w_i(y_{g_1}, \dots, y_{g_n}) w_j(y_{g_1}, \dots, y_{g_n})^{-1} y_{g_j}^{-1} \in R_S. \end{aligned} \tag{3}$$

Similarly, if $g_{n+1} = w_i(g_1, \dots, g_n)$ and $g_{n+1}^{-1} g_i g_{n+1} = g_j$ for some $g_i, g_j \in S$, then

$$w(y_{g_1}, \dots, y_{g_n})^{-1} y_{g_i} w(y_{g_1}, \dots, y_{g_n}) y_{g_j}^{-1} \in R_S. \tag{4}$$

Therefore, if $S_1 = S \cup \{g_{n+1}\}$, $\mathbb{F}_{S_1}$ is a free group freely generated by the alphabet $Y_{S_1} = \{y_g : g \in S_1\}$, $R_{g_{n+1}}$ is the set of words of the form

$$w(y_{g_1}, \dots, y_{g_n})^{-1} y_g\, w(y_{g_1}, \dots, y_{g_n}) y_{g_{n+1}}^{-1}$$

defined by all relations (2), and $\widetilde{R}_{S_1}$ is the normal closure in $\mathbb{F}_{S_1}$ of the set $\widetilde{R}_S \cup R g_{n+1}$, then $G_1 \simeq \mathbb{F}_{S_1}/\widetilde{R}_{S_1}$ in view of relations (3) and (4).

Note that if we have a relation $g_i^{-1} g_j g_i = g_k$ for some $g_i, g_j, g_k \in S_1$ then

$$y_{g_i}^{-1} y_{g_j} y_{g_i} y_{g_k}^{-1} \in \widetilde{R}_{S_1}.$$

If $S_1 \neq O$, then we can repeat the construction described above and obtain a presentation $G_1 \simeq \mathbb{F}_{S_2}/\widetilde{R}_{S_2}$, and so on. After several steps we obtain a presentation $G_1 \simeq \mathbb{F}_O/\widetilde{R}_O$. Note that, by induction, we deduce that for any relation in G of the form $g_i^{-1}g_jg_i = g_k$ for some $g_i, g_j, g_k \in O$ we have $y_{g_i}^{-1}y_{g_j}y_{g_i}y_{g_k}^{-1} \in \widetilde{R}_O$. Therefore, there is a natural C-homomorphism $f\colon (\widetilde{G}, \widetilde{O}) \to (G_1, O_1)$. By Claim 2.3, f is a C-isomorphism. □

For an equipped finite group (G, O), consider a presentation of G of the following form. Let us take a free group $\mathbb{F} = \mathbb{F}_O$ freely generated by the alphabet $X_O = \{x_g : g \in O\}$. Consider a normal subgroup $R_O \subset \mathbb{F}$ such that $\mathbb{F}/R_O \simeq G$. Let $h_O\colon \mathbb{F} \to \mathbb{F}/R_O \simeq G$ be the natural epimorphism.

We can associate to (G, O) a group $\overline{G} = \mathbb{F}/[\mathbb{F}, R_O]$. Denote by $\alpha_O\colon \overline{G} \to G$ the natural epimorphism. By Claim 2.2, $(\overline{G}, \overline{O})$ is a C-group, where $\overline{O} = h_O(X_O^{\mathbb{F}})$. Evidently, there is a natural epimorphism of C-groups $\kappa_O\colon (\overline{G}, \overline{O}) \to (\widetilde{G}, \widetilde{O})$ sending $\kappa_O(x_g) = y_g$ for all $g \in O$ and such that $\alpha_O = \beta_O \circ \kappa_O$. The C-group $(\overline{G}, \overline{O})$ is called the *universal central C-extension* of the equipped finite group (G, O). It is easy to see that $\alpha_O\colon \overline{G} \to G$ is a central extension of groups, that is, $\ker \alpha_O$ is a subgroup of the center $Z(\overline{G})$.

We have

$$\ker \alpha_O \cap [\overline{G}, \overline{G}] = (R_O \cap [\mathbb{F}, \mathbb{F}])/[\mathbb{F}, R_O].$$

By Hopf's integral homology formula, we have

$$H_2(G, \mathbb{Z}) \simeq (R_O \cap [\mathbb{F}, \mathbb{F}])/[\mathbb{F}, R_O].$$

Denote by $h_2(G)$ the order of the group $H_2(G, \mathbb{Z})$ and denote by $K_{(G,O)}$ the subgroup of $(R_O \cap [\mathbb{F}, \mathbb{F}])/[\mathbb{F}, R_O]$ generated by the elements of R_O of the form $[w, x_g]$, where $g \in O$ and $w \in \mathbb{F}$, and let $k_{(G,O)}$ be its order.

Theorem 2.7 *For an equipped finite group (G, O) we have*

$$h_2(G) = k_{(G,O)}a_{(G,O)}.$$

Proof We have $\ker \kappa_O \subset \ker \alpha_O$. Therefore, $\ker \kappa_O \subset Z(\overline{G})$. Let us show that for some $n \geqslant 0$ there exist a sequence of C-groups $\overline{G}_0 = \mathbb{F}/R_0, \dots, \overline{G}_n = \mathbb{F}/R_n$, a sequence of C-homomorphisms

$$\varphi_i\colon (\overline{G}_i, \overline{O}_i) \to (\overline{G}_{i+1}, \overline{O}_{i+1}), \qquad 0 \leqslant i \leqslant n-1,$$

where $(\overline{G}_0, \overline{O}_0) = (\overline{G}, \overline{O})$, and a C-homomorphism $\overline{\kappa}\colon (\overline{G}_n, \overline{O}_n) \to (\widetilde{G}, \widetilde{O})$ such that

(i) $\kappa = \overline{\kappa} \circ \varphi$, where $\varphi = \varphi_{n-1} \circ \dots \circ \varphi_0$;
(ii) for each i the homomorphism $\varphi_i\colon [\overline{G}_i, \overline{G}_i] \to [\overline{G}_{i+1}, \overline{G}_{i+1}]$ is an isomorphism;
(iii) $\overline{\kappa}_*$ induces a one-to-one correspondence between the connected components of the C-graphs $\Gamma_{(\overline{G}_n, \overline{O}_n)}$ and $\Gamma_{(\widetilde{G}, \widetilde{O})}$.

Indeed, let us put $R_0 = R_O$ and consider the map κ_*. If it induces a one-to-one correspondence between the connected components of the C-graphs $\Gamma_{(\overline{G},\overline{O})}$ and $\Gamma_{(\widetilde{G},\widetilde{O})}$, then $n = 0$ and it is nothing to prove.

Otherwise, for some $g \in O$ there is a vertex v_{y_g} of $\Gamma_{(\widetilde{G},\widetilde{O})}$ whose preimage $\kappa_*^{-1}(v_{y_g})$ contains at least two vertices, say v_{x_g} and $v_{\overline{g}}$ (here $\overline{g}$ is an element of $X^{\mathbb{F}}$), of $\Gamma_{(\overline{G},\overline{O})}$ belonging to different connected components of $\Gamma_{(\overline{G},\overline{O})}$.

Denote by R_1 the normal closure of $R_O \cup \{x_g\overline{g}^{-1}\}$ in $\mathbb{F}$ and consider the natural homomorphism $\varphi_0\colon \overline{G} \to \overline{G}_1 = \mathbb{F}/R_1$. The element $x_g\overline{g}^{-1}$, considered as an element of $\overline{G}$, belongs to $\ker\kappa$. Therefore, $x_g\overline{g}^{-1} \in Z(\overline{G})$.

Denote by $\kappa_1\colon \overline{G}_1 \to \widetilde{G}$ the homomorphism induced by κ. By Lemma 2.4, the homomorphism φ_1 is a C-homomorphism of C-groups. It is easy to see that $\varphi_0\colon [\overline{G}_0, \overline{G}_0] \to [\overline{G}_1, \overline{G}_1]$ is an isomorphism and the number of connected components of the C-graph $\Gamma_{(\overline{G}_1,\overline{O}_1)}$ is less than the number of connected components of the C-graph $\Gamma_{(\overline{G},\overline{O})}$.

Assume now that κ_{1*} is not a one-to-one correspondence between the connected components of the C-graphs $\Gamma_{(\overline{G}_1,\overline{O}_1)}$ and $\Gamma_{(\widetilde{G},\widetilde{O})}$. Then for some $g_1 \in O$ there is a vertex $v_{y_{g_1}}$ of $\Gamma_{(\widetilde{G},\widetilde{O})}$ which preimage $\kappa_{1*}^{-1}(v_{y_{g_1}})$ contains at least two vertices $v_{x_{g_1}}$ and $v_{\overline{g}_1}$ of $\Gamma_{(\overline{G}_1,\overline{O}_1)}$ belonging to different connected components of $\Gamma_{(\overline{G}_1,\overline{O}_1)}$.

Hence we can repeat the construction described above and obtain a C-group $(\overline{G}_2, \overline{O}_2)$ and C-homomorphisms $\varphi_1\colon \overline{G}_1 \to \overline{G}_2 = \mathbb{F}/R_2$, $\kappa_2\colon \overline{G}_2 \to \widetilde{G}$ such that $\varphi_1\colon [\overline{G}_1, \overline{G}_1] \to [\overline{G}_2, \overline{G}_2]$ is an isomorphism and the number of connected components of the C-graph $\Gamma_{(\overline{G}_2,\overline{O}_2)}$ is less than the number of connected components of the C-graph $\Gamma_{(\overline{G}_1,\overline{O}_1)}$. Since the number of connected components of the C-graph $\Gamma_{(\overline{G},\overline{O})}$ is finite, after several (n) steps of our construction we obtain the desired sequences of C-groups and C-homomorphisms.

Now, consider the C-homomorphism $\overline{\kappa}\colon \overline{G}_n \to \widetilde{G}$. The C-graph $\Gamma_{(\widetilde{G},\widetilde{O})}$ consists of connected components $\Gamma_1, \dots, \Gamma_m$. Let $\{v_{g_{i1}}, \dots, v_{g_{il_i}}\}$ be the set of the vertices of Γ_i. We have $O = \{g_{ij}\}_{1\leqslant i\leqslant m, 1\leqslant j\leqslant l_i}$. Then $\overline{\Gamma}_i = \overline{\kappa}_*^{-1}(\Gamma_i)$ are the connected components of $\Gamma_{(\overline{G}_n,\overline{O}_n)}$. Let

$$\overline{\kappa}_n^{-1}(v_{y_{g_{ij}}}) = \{v_{x_{g_{ij}}}, v_{\overline{g}_{ij1}}, \dots, v_{\overline{g}_{ijr_{ij}}}\}, \qquad \overline{g}_{ijk} \in \overline{O}_n, \quad 1 \leqslant k \leqslant r_{ij}.$$

Since the graph $\overline{\Gamma}_i$ is connected, there are words w_{ijk} in letters of X_O and their inverses such that

$$\overline{g}_{ijk} = w_{ijk} x_{g_{ij}} w_{ijk}^{-1}, \qquad 1 \leqslant k \leqslant r_{ij}.$$

Obviously, the elements $u_{ijk} = [w_{ijk}, x_{g_{ij}}] = \overline{g}_{ijk} x_{g_{ij}}^{-1}$ belong to $[\overline{G}_n, \overline{G}_n] \cap \ker\overline{\kappa}$. Therefore, u_{ijk} as elements of $\mathbb{F}$ belong to $R_O \cap [\mathbb{F}, \mathbb{F}]$.

Consider the group $\overline{G}_{n+1} = \mathbb{F}/R_{n+1}$, where the group R_{n+1} is the normal closure of $R_n \cup \{u_{ijk}\}_{1\leqslant i\leqslant m, 1\leqslant j\leqslant l_i, 1\leqslant k\leqslant r_{ij}}$ in $\mathbb{F}$. Then, by Claim 2.2, $\overline{G}_{n+1} = \mathbb{F}/R_{n+1}$ is a C-group and the natural map $\overline{\kappa}_1\colon \overline{G}_{n+1} \to \widetilde{G}$, induced by $\overline{\kappa}$, is a C-homomorphism. Moreover, $\ker\varphi_n$ of the natural epimorphism $\varphi_n\colon \overline{G}_n \to \overline{G}_{n+1}$ is a subgroup of

$[\overline{G}_n, \overline{G}_n] \simeq [\overline{G}, \overline{G}] = [\mathbb{F}, \mathbb{F}]/[\mathbb{F}, R_O]$ generated by the elements $u_{ijk} = \big[w_{ijk}, x_{g_{ij}}\big]$, where $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant l_i$, and $1 \leqslant k \leqslant r_{ij}$.

To complete the proof, it suffices to note that $\overline{\kappa}_{1*}$ induces a one-to-one correspondence between the sets of vertices of the C-graphs $\Gamma_{(\overline{G}_{n+1}, \overline{O}_{n+1})}$ and $\Gamma_{(\widetilde{G}, \widetilde{O})}$, since all $u_{ijk} = \overline{g}_{ijk} x_{g_{ij}}^{-1}$ belong to $\ker \varphi_n$. Therefore, $\overline{\kappa}_1$ is an isomorphism. □

Lemma 2.8 *Let the order of $g \in O$ be n and let $[x_g, w] \in ([\mathbb{F}, \mathbb{F}] \cap R_O)/[\mathbb{F}, R_O] \subset \mathbb{F}/[\mathbb{F}, R_O]$. Then the order of the element $[x_g, w]$ is a divisor of n.*

Proof The elements x_g^n and $[x_g, w]$ belong to the center of the group $\mathbb{F}/[\mathbb{F}, R_O]$. Therefore,

$$\big[x_g^n, w\big] = x_g^{n-1}[x_g, w] x_g^{1-n}\big[x_g^{n-1}, w\big] = [x_g, w]\big[x_g^{n-1}, w\big] = \cdots = [x_g, w]^n$$

is the unity of $\mathbb{F}/[\mathbb{F}, R_O]$. □

From Lemma 2.8 and Theorem 2.7 we have

Proposition 2.9 *Let the equipment O of an equipped finite group (G, O) consist of conjugacy classes of elements of orders coprime with $h_2(G)$. Then $a_{(G,O)} = h_2(G)$.*

3 Proof of Theorem 1.1

By definition, the *Bogomolov multiplier* $b_0(G)$ of a finite group G is the order of the group

$$B_0(G) = \ker \left[H^2(G, \mathbb{Q}/\mathbb{Z}) \to \bigoplus_{A \subset G} H^2(A, \mathbb{Q}/\mathbb{Z}) \right],$$

where A runs over all abelian subgroups of G.

Remark 3.1 Note that it suffices to consider only restrictions to abelian groups with two generators in order to determine that the element $w \in H^2(G, \mathbb{Q}/\mathbb{Z})$ is contained in $B_0(G)$.

There is a natural duality between $H^2(G, \mathbb{Q}/\mathbb{Z})$ and $H_2(G, \mathbb{Z})$ since the groups $Q/\mathbb{Z}$ and Z are Pontryagin dual (see, for example, [15]). Both groups are finite for finite groups G and hence the duality implies an isomorphism of $H^2(G, \mathbb{Q}/\mathbb{Z})$ and $\mathrm{Hom}(H_2(G, \mathbb{Z}), \mathbb{Q}/\mathbb{Z})$ as abstract groups.

By Theorem 2.7, we have the inequality $h_2(G) \geqslant a_{(G,O)}$ for any equipped finite group (G, O). By [11, Corollary 2], we have $a_{(G,O)} \geqslant a_{(G,G\setminus\{1\})}$ for each equipment O of G. Therefore, to prove Theorem 1.1 it suffices to show that for the equipped finite group $(G, G\setminus\{1\})$ its ambiguity index $a_{(G,G\setminus\{1\})}$ is equal to $b_0(G)$.

In notation used in Sect. 2 and by Theorem 2.7, we have

$$a_{(G,G\setminus\{1\})} = \frac{h_2(G)}{k_{(G,G\setminus\{1\})}},$$

where $k_{(G,G\setminus\{1\})}$ is the order of the subgroup $K_{G\setminus\{1\}}$ of the group

$$\big(R_{G\setminus\{1\}}\cap\big[\mathbb{F}_{G\setminus\{1\}},\mathbb{F}_{G\setminus\{1\}}\big]\big)/\big[\mathbb{F}_{G\setminus\{1\}},R_{G\setminus\{1\}}\big]\simeq H_2(G,\mathbb{Z})$$

generated by the elements of $R_{G\setminus\{1\}}$ of the form $[w,x_g]$, where $g\in G\setminus\{1\}$ and $w\in\mathbb{F}_{G\setminus\{1\}}$.

Lemma 3.2 *Let for some $w_1,w_2\in\mathbb{F}_{G\setminus\{1\}}$ the commutator $[w_1,w_2]$ belong to $R_{G\setminus\{1\}}$. Then $[w_1,w_2]$, considered as an element of $\mathbb{F}_{G\setminus\{1\}}/\big[\mathbb{F}_{G\setminus\{1\}},R_{G\setminus\{1\}}\big]$, belongs to $K_{G\setminus\{1\}}$.*

Proof First of all, note that if $[x_g,w]\in K_{G\setminus\{1\}}$, then $[x_g,w]=\big[w,x_g^{-1}\big]=\big[x_g^{-1},w^{-1}\big]=\big[x_g^{-1},w\big]$ in $K_{G\setminus\{1\}}$, since $K_{G\setminus\{1\}}$ is a subgroup of the center of the C-group $\overline{G}_{G\setminus\{1\}}=\mathbb{F}_{G\setminus\{1\}}/\big[\mathbb{F}_{G\setminus\{1\}},R_{G\setminus\{1\}}\big]$ and these four commutators are conjugated to each other in $\mathbb{F}_{G\setminus\{1\}}$. Similarly, $[w,x_g]=\big[x_g,w^{-1}\big]=\big[w^{-1},x_g^{-1}\big]=\big[x_g^{-1},w^{-1}\big]\in K_{G\setminus\{1\}}$, since $[w,x_g]$ is the inverse element to the element $[x_g,w]$. Note also that for any w_1 the element $w_1[w,x_g]w_1^{-1}$ belongs to $K_{G\setminus\{1\}}$ if $[w,x_g]\in K_{G\setminus\{1\}}$.

Next, the elements w_1^{-1} and w_2^{-1}, considered as elements of G, are equal to some elements g_1 and g_2 of G. Therefore, if $[w_1,w_2]\in R_{G\setminus\{1\}}$ then

$$w_1x_{g_1},\,w_2x_{g_2},\,[x_{g_1},x_{g_2}],\,[w_2,x_{g_1}],\,[w_1,x_{g_2}]\in R_{G\setminus\{1\}}.$$

In addition, we have $[w_1,w_2x_{g_2}]\in\big[\mathbb{F}_{G\setminus\{1\}},R_{G\setminus\{1\}}\big]$ and

$$[w_1,w_2x_{g_2}]=[w_1,w_2]\big(w_2[w_1,x_{g_2}]w_2^{-1}\big).$$

Therefore, $[w_1,w_2]\in R_{G\setminus\{1\}}\cap\big[\mathbb{F}_{G\setminus\{1\}},\mathbb{F}_{G\setminus\{1\}}\big]$ (as an element of $K_{G\setminus\{1\}}$) is the inverse element to the element $[w_1,x_{g_2}]\in K_{G\setminus\{1\}}$ and hence $[w_1,w_2]\in K_{G\setminus\{1\}}$. □

To complete the proof of Theorem 1.1, note that, by Lemma 3.2, for each imbedding $i\colon H\to G$ of an abelian group H generated by two elements the image of $i_*\colon H_2(H,\mathbb{Z})\to H_2(G,\mathbb{Z})$ is a subgroup of $K_{G\setminus\{1\}}$ and the group $K_{G\setminus\{1\}}$ is generated by the images of such elements. Therefore, the group

$$K^{\perp}_{G\setminus\{1\}}=\big\{\varphi\in\mathrm{Hom}(H_2(G,\mathbb{Z}),\mathbb{Q}/\mathbb{Z}):\varphi(w)=0\text{ for all }w\in K_{G\setminus\{1\}}\big\}$$

coincides with the group $B_0(G)$ and $a_{(G,G\setminus\{1\})}=h_2(G)/k_{(G,G\setminus\{1\})}$. □

4 Quasi-covers of equipped finite groups

In this section we use notation introduced in Sect. 2.

4.1 Definitions

Let $f\colon (G_1, O_1) \to (G, O)$ be a homomorphism of equipped groups. We say that f is a *cover of equipped groups* (or, equivalently, (G_1, O_1) is a *cover* of (G, O)) if
(i) f is an epimorphism such that $f(O_1) = O$;
(ii) $\ker f$ is a subgroup of the center ZG_1 of G_1;
(iii) $f_*\colon H_1(G_1, \mathbb{Z}) \to H_1(G, \mathbb{Z})$ is an isomorphism.
Let $f\colon (G_1, O_1) \to (G, O)$ be a homomorphism of equipped finite groups. We say that $S \subset O_1$ is a *section* of f if $f_{|S}\colon S \to O$ is a one-to-one correspondence. Denote by $O_S \subset O_1$ the orbit of S under the action of the group of the inner automorphisms of G_1.

Let $f\colon (G_1, O_1) \to (G, O)$ be an epimorphism of equipped groups such that $\ker f \subset ZG_1$. We say that f is a *quasi-cover of equipped groups* (or, equivalently, (G_1, O_1) is a *quasi-cover* of (G, O)) if there is a section S of f such that $O_S = O_1$.

Below, we will assume that for a quasi-cover f of equipped groups a section S is chosen and fixed.

4.2 Properties of quasi-covers

Lemma 4.1 *Let $f\colon (G_1, O_1) \to (G, O)$ be a cover of equipped finite groups and $S \subset O_1$ a section. Then G_1 is generated by the elements of S.*

Proof Denote by G_S the subgroup of G_1 generated by the elements of S. Obviously, $\varphi = f_{|G_S}\colon G_S \to G$ is an epimorphism and $\ker\varphi \subset \ker f \subset ZG_1$. Therefore, to prove lemma it suffices to show that $\ker f \subset G_S$. To show this, let us consider the natural epimorphism $f_1\colon G_1 \to G_2 = G_1/\ker\varphi$ and the natural epimorphism $\psi\colon G_2 \to G$ induced by the cover f. Obviously, $\psi\colon (G_2, f_1(O_1)) \to (G, O)$ is a cover of equipped finite groups and $\psi_{|H}\colon H \to G$ is an isomorphism, where $H = f_1(G_S)$. Therefore, $G_2 \simeq \ker\psi \times G$. Consequently, $\ker\psi = 0$, since $\psi_*\colon H_1(G_2, \mathbb{Z}) \to H_1(G, \mathbb{Z})$ is an isomorphism and $\ker\psi$ is an abelian group. □

If S is a section of a cover $f\colon (G_1, O_1) \to (G, O)$, then Lemma 4.1 implies that $O_S = S^{G_1}$ is an equipment of G_1 and $f\colon (G_1, O_S) \to (G, O)$ is also a cover of equipped groups.

Below, we fix a section S of a cover $f\colon (G_1, O_1) \to (G, O)$. Then the cover f can be considered as a quasi-cover.

In notation used in Sect. 2, consider the universal central C-extension $\alpha_O\colon (\overline{G}, \overline{O}) \to (G, O)$ of an equipped finite group (G, O). We have two natural epimorphisms $h_O\colon \mathbb{F}_O \to G = \mathbb{F}_O/R_O$ and $\beta_O\colon \mathbb{F}_O \to \overline{G} = \mathbb{F}_O/[\mathbb{F}_O, R_O]$ such that $h_O = \alpha_O \circ \beta_O$.

Lemma 4.2 *Let $f\colon (G_1, O_1) \to (G, O)$ be a quasi-cover of equipped finite groups. Then there is an epimorphism $\alpha_S\colon (\overline{G}, \overline{O}) \to (G_1, O_S)$ of equipped groups such that $\alpha_O = f \circ \alpha_S$.*

Proof By Lemma 4.1, there is an epimorphism $h_S\colon \mathbb{F}_O \to G_1$ defined by $h_S(x_g) = \widehat{g} \in S$ for all $g \in G$, where $\widehat{g} = f_{|S}^{-1}(g)$. Denote by $R_S = \ker h_S$. Obviously, we have $f \circ h_S = h_O$. Therefore, $R_S \subset R_O$.

Let us show that the group $[\mathbb{F}_O, R_O]$ is a subgroup of R_S. Indeed, consider any $w \in R_O$. Then, as an element of G_1, the element $w \in \ker f$ and, consequently, w belongs to the center of G_1. In particular, it commutes with any generator $\widehat{g} \in S$ of G_1 and hence $[w, x_g] \in R_S$, that is, $[\mathbb{F}_O, R_O] \subset R_S$. The inclusion $[\mathbb{F}_O, R_O] \subset R_S$ implies the desired epimorphism α_S. □

We say that a cover (respectively, a quasi-cover) of equipped finite groups $f\colon (G_1, O_1) \to (G, O)$ is *maximal* if for any cover of equipped finite groups $f_1\colon (G_2, O_2) \to (G_1, O_1)$ such that $f_2 = f \circ f_1$ is also a cover (respectively, quasi-cover) of equipped finite groups, the epimorphism f_1 is an isomorphism.

Theorem 4.3 *For any cover* (*respectively, quasi-cover*) *of equipped finite groups* $f\colon (G_1, O_1) \to (G, O)$, *there is a maximal cover* (*respectively, quasi-cover*) $f_2\colon (G_2, O_2) \to (G, O)$ *for which there is a cover* $f_1\colon (G_2, O_2) \to (G_1, O_S)$ *such that*

(i) $f_2 = f \circ f_1$;
(ii) $\ker f_2 \simeq H_2(G, \mathbb{Z})$ (*respectively,* $[\overline{G}, \overline{G}] \cap \ker f_2 \simeq H_2(G, \mathbb{Z})$).

Proof Consider the epimorphism $\alpha_S\colon (\overline{G}, \overline{O}) \to (G_1, O_S)$ defined in the proof of Lemma 4.2. The group $\ker \alpha_S$ is a subgroup of the center of $\overline{G}$.

Since $(\overline{G}, \overline{O})$ is a C-group and $\overline{O}$ consists of M conjugacy classes, where $M \leqslant |O| = \operatorname{rk} \mathbb{F}_O$, then $H_1(\overline{G}, \mathbb{Z}) = \overline{G}/[\overline{G}, \overline{G}] = \mathbb{Z}^M$. Let $\tau\colon \overline{G} \to \mathbb{Z}^M$ be the natural homomorphism (that is, τ is the type homomorphism $\overline{G} \to H_1(\overline{G}, \mathbb{Z})$, see Sect. 1). The image $\tau(\ker \alpha_S)$ is a sublattice of maximal rank in $\mathbb{Z}^M$. Let us choose a $\mathbb{Z}$-free basis $a_1, \dots, a_M$ in $\tau(\ker \alpha_S)$ and choose elements $\overline{g}_i \in \ker \alpha_S$, $1 \leqslant i \leqslant M$, such that $\tau(\overline{g}_i) = a_i$.

Denote by H_S a group generated by the elements $\overline{g}_i$, $1 \leqslant i \leqslant M$, and denote by $K_S = [\overline{G}, \overline{G}] \cap \ker \alpha_S$. Then it is easy to see that $H_S \simeq \mathbb{Z}^M$ is a subgroup of the center of $\overline{G}$, the intersection $H_S \cap [\overline{G}, \overline{G}]$ is trivial, and $\ker \alpha_S \simeq K_S \times H_S$.

Denote by $G_2 = \overline{G}/H_S$ the quotient group and by $\alpha_{H_S}\colon \overline{G} \to G_2$ and $f_1\colon G_2 \to G_1$ the natural epimorphisms. We have $\alpha_S = f_1 \circ \alpha_{H_S}$. Denote also by $O_2 = \alpha_{H_S}(\overline{O})$. Then it is easy to see that $\alpha_{H_S}\colon (\overline{G}, \overline{O}) \to (G_2, O_2)$ and $f_1\colon (G_2, O_2) \to (G_1, O_S)$ are central extensions of equipped groups.

By construction, it is easy to see that $[\overline{G}, \overline{G}] \cap \ker \alpha_{H_S}$ is trivial and $\ker f_1 \subset [G_1, G_1]$ is a subgroup of the center of G_1. Therefore, the epimorphism f_1 is a cover of equipped groups. In addition, it is easy to see that $\alpha_O = f_1 \circ \alpha_{H_S}$ and $f_2 = f \circ f_1\colon (G_2, O_2) \to (G, O)$ is a cover (respectively, quasi-cover) of equipped groups. We have

$$K_S \simeq \ker f_1 \subset \alpha_{H_O}\big([\overline{G}, \overline{G}] \cap \ker \alpha_0\big) = \alpha_{H_O}(H_2(G, \mathbb{Z})) \subset [G_2, G_2].$$

Therefore, if $k_{f_i} = |\ker f_i|$, $i = 1, 2$, is the order of the group $\ker f_i$ and k_f is the order of $\ker f$, then

$$h_2(G) = k_{f_2} = k_{f_1} k_f. \tag{5}$$

Since we can repeat the construction described above to the cover (respectively, quasi-cover) f_2 and applying again equality (5), where the new f is our f_2 and the new f_1

is a cover existence of which follows from assumption that the old f_2 is not maximal, we obtain that the new f_1 is an isomorphism, that is, the covering f_2 is maximal. □

In the case when $f_1\colon (G, G\setminus\{1\}) \to (G, G\setminus\{1\})$ is an isomorphism of equipped finite groups, the maximal cover $f_2\colon (G_2, O_2) \to (G, G\setminus\{1\})$, constructed in the proof of Theorem 4.3, will be called a *universal maximal cover*.

Corollary 4.4 *For any equipped finite group (G, O) there is a maximal cover of equipped groups. For any cover (respectively, quasi-cover) $f\colon (G_1, O_1) \to (G, O)$ of equipped finite groups, $k_f = |\ker f| \leqslant h_2(G)$ (respectively, $k_f = |\ker f \cap [G_1, G_1]| \leqslant h_2(G)$) and f is maximal if and only if $k_f = h_2(G)$.*

4.3 The ambiguity index of a quasi-cover of an equipped group

Let $(\widetilde{G}, \widetilde{O})$ be the C-group associated with an equipped group (G, O) and $\beta_O\colon (\widetilde{G}, \widetilde{O}) \to (G, O)$ the natural epimorphism of equipped groups (see definitions in Sect. 2).

Theorem 4.5 *Let $f\colon (G_1, O_1) \to (G, O)$ be a quasi-cover of equipped finite groups. Then there is a natural C-epimorphism $\kappa_S\colon (\overline{G}, \overline{O}) \to (\widetilde{G}_1, \widetilde{O}_S)$ such that $\kappa_O = \widetilde{f}\circ\kappa_S$ and $\alpha_O = \beta_O\circ\widetilde{f}\circ\kappa_S = f\circ\beta_{O_S}\circ\kappa_S$, where the C-epimorphism $\kappa_O\colon (\overline{G}, \overline{O}) \to (\widetilde{G}, \widetilde{O})$ is defined in Sect. 2 and the C-epimorphism $\widetilde{f}\colon (\widetilde{G}_1, \widetilde{O}_S) \to (\widetilde{G}, \widetilde{O})$ is associated with f.*

Proof In notation used in the proof of Lemma 4.2, we have an inclusion $R_S \subset R_O$ of normal subgroups of $\mathbb{F}_O$ which induces $f\colon G_1 = \mathbb{F}_O/R_S \to G = \mathbb{F}_O/R_O$.

Let $\widetilde{R}_S \subset R_S$ be the normal subgroup normally generated by the elements of R_S of the form $w_{ij}^{-1}x_{g_i}w_{ij}x_{g_j}^{-1}$, where $w_{ij} \in \mathbb{F}_O$ and $x_{g_i}, x_{g_j} \in X_O$. For any $w \in R_O$ and any generator x_g, $g \in O$, the commutator $[x_g, w] \in R_S$, since f is a central extension of groups. Therefore,

$$[\mathbb{F}_O, R_O] \subset \widetilde{R}_S. \tag{6}$$

By Claim 2.6, $\widetilde{G}_1 \simeq \mathbb{F}_S/\widetilde{R}_S$. Therefore, (6) induces an epimorphism $\kappa_S\colon \overline{G} = \mathbb{F}_O/[\mathbb{F}_O, R_O] \to \mathbb{F}/\widetilde{R}_S \simeq \widetilde{G}_1$. Obviously, the C-epimorphism $\kappa_S\colon (\overline{G}, \overline{O}) \to (\widetilde{G}_1, \widetilde{O}_S)$ satisfies all properties claimed in Theorem 4.5. □

Let $f\colon (G_1, O_1) \to (G, O)$ be a cover (respectively, quasi-cover) of equipped finite groups and $\widetilde{f}_S\colon (\widetilde{G}_1, \widetilde{O}_S) \to (\widetilde{G}, \widetilde{O})$ a C-epimorphism associated with $f\colon (G_1, O_S) \to (G, O)$. Denote by k_f the order of the group $\ker f \cap [G_1, G_1]$ and by $k_{\widetilde{f}_S}$ the order of the group $\ker \widetilde{f}_S \cap [\widetilde{G}_1, \widetilde{G}_1]$.

Corollary 4.6 *Let $f\colon (G_1, O_1) \to (G, O)$ be a quasi-cover of equipped finite groups, S a section of f. Then*

$$h_2(G) = a_{(G,O)}k_{\widetilde{f}_S}k_S = k_f a_{(G_1,O_S)}k_S,$$

where k_S is the order of the group $\ker\kappa_S \cap [\overline{G}, \overline{G}]$.

Corollary 4.7 *Let $f\colon (G_1, O_1) \to (G, O)$ be a cover (respectively, quasi-cover) of equipped finite groups, S a section of f. Then for any equipment $\widehat{O}$ of G_1 (respectively, such that $O_1 \subset \widehat{O}$) we have an inequality $a_{(G_1,\widehat{O})} \leqslant h_2(G)$. If f is maximal, then $a_{(G_1,\widehat{O})} = 1$.*

Proof If f is a cover, then $f\colon (G_1, \widehat{O}) \to \big(G, f(\widehat{O})\big)$ is also a cover of equipped groups and $a_{(G_1,\widehat{O})} \leqslant h_2(G)$ by Corollary 4.6.

As it was mentioned in Sect. 1, we have $a_{(G_1,\widehat{O})} \leqslant a_{(G_1,O_1)}$ if $O_1 \subset \widehat{O}$ and if f is a quasi-cover, then $a_{(G_1,O_1)} \leqslant h_2(G)$ by Corollary 4.6.

If f is maximal, then $k_f = h_2(G)$ by Corollary 4.4 and, therefore, if f is a cover then $f\colon (G_1, \widehat{O}) \to \big(G, f(\widehat{O})\big)$ is also maximal. It follows from Corollary 4.6 that $a_{(G_1,\widehat{O})} = 1$ in the case of maximal covers, and $a_{(G_1,\widehat{O})} \leqslant a_{(G_1,O_1)} = 1$ in the case of maximal quasi-covers f. □

Let $f\colon (G_1, O_1) \to (G, O)$ be a cover of equipped finite groups such that $f^{-1}(O) = O_1$. We say that f *splits* over a conjugacy class $C \subset O$ if $f^{-1}(C)$ consists of at least two conjugacy classes of G_1. The number $s_f(C)$ of the conjugacy classes contained in $f^{-1}(C)$ is called the *splitting number* of the conjugacy class C for f. We say that f *splits completely* over C if $s_f(C) = k_f$, where $k_f = |\ker f|$.

Let C be a conjugacy class in G. Consider the subgroups $K_C \subset K_{G\setminus\{1\}}$ of the group

$$\big(R_{G\setminus\{1\}} \cap \big[\mathbb{F}_{G\setminus\{1\}}, \mathbb{F}_{G\setminus\{1\}}\big]\big)/\big[\mathbb{F}_{G\setminus\{1\}}, R_{G\setminus\{1\}}\big] \simeq H_2(G, \mathbb{Z}),$$

where K_C is generated by the elements of $R_{G\setminus\{1\}}$ of the form $[x_h, x_g]$, $h \in G\setminus\{1\}$. Let k_C be the order of the group K_C.

Proposition 4.8 *Let $f\colon (G_1, O_1) \to (G, G\setminus\{1\})$ be a universal maximal cover of equipped finite groups and let C be a conjugacy class in G. Then $h_2(G) = s_f(C)k_C$.*

Proof For $g \in C$ the preimage $f^{-1}(C)$ consists of the conjugacy classes of the elements zx_g, where

$$z \in \ker f = \big(R_{G\setminus\{1\}} \cap \big[\mathbb{F}_{G\setminus\{1\}}, \mathbb{F}_{G\setminus\{1\}}\big]\big)/\big[\mathbb{F}_{G\setminus\{1\}}, R_{G\setminus\{1\}}\big] \simeq H_2(G, \mathbb{Z}).$$

Note that $\ker f \subset ZG_1$ and $\ker f$ acts transitively on the set of the conjugacy classes $C_1, \dots, C_{k_f(C)}$ involving in $f^{-1}(C)$, $z(C_i) = C_j$ if $z\overline{g} \in C_j$ for $\overline{g} \in C_i$.

Let $x_g \in C_1$, where $g \in C$. Then $z(C_1) = C_1$ if and only if for some $w \in G_1$ we have $wx_gw^{-1} = zx_g$, that is, $z = [w, x_g]$.

If $f(w) = h$ then $w = z_1x_h$ for some $z_1 \in \ker f$ and, therefore, $z = [x_h, x_g]$, that is, $z \in K_C$. The converse statement that each element $z \in K_C$ leaves fixed the conjugacy class C_1 is obvious. □

Proposition 4.9 *Let $f\colon (G_1, O_1) \to (G, G\setminus\{1\})$ be a universal maximal cover of equipped finite groups. Then $a_{(G,O)} = h_2(G)$ if and only if f splits completely over each conjugacy class $C \subset O$. If $s_f(C) = 1$ for some conjugacy class $C \subset O$ then $a_{(G,O)} = 1$.*

Proof We have $k_f = h_2(G)$. The map $g \mapsto x_g$ is a section in O_1. Denote by $\overline{O}$ the equipment of G_1 consisting of the elements conjugated to x_g, $g \in O$. Therefore, $f: (G_1, \overline{O}) \to (G, O)$ is a maximal cover of equipped groups and Proposition 4.9 follows from Corollary 4.6. □

Proposition 4.10 *Let $f: (G_1, O_1) \to (G, G \setminus \{1\})$ be a universal maximal cover of equipped finite groups and let $C_1 \subset O$ and $C_2 \subset O$ be two conjugacy classes contained in an equipment of G. Then $a_{(G,O)} = 1$ if $s_f(C_1)$ and $s_f(C_2)$ are coprime.*

Proof The group $\ker \widetilde{f}_S \cap [\widetilde{G}_1, \widetilde{G}_1] \subset H_2(G, \mathbb{Z})$ contains two subgroups K_{C_1} and K_{C_2} whose indices in $H_2(G, \mathbb{Z})$ are coprime. This fact and Corollary 4.6 imply the statement. □

Proposition 4.11 *Let $f: (G_1, O_1) \to (G, G \setminus \{1\})$ be a universal maximal cover of equipped finite groups and let $h_2(G) = pq$, where p and q are coprime integers. Let $C_1 \subset O$ be a conjugacy class such that $s_f(C_1) = q$ and let $s_f(C)$ be coprime with p for each conjugacy class $C \subset O$. Then the ambiguity index $a_{(G,O)} = p$.*

Proof Similarly, the statement follows from Corollary 4.6, since the group $\ker \widetilde{f}_S \cap [\widetilde{G}_1, \widetilde{G}_1] \subset H_2(G, \mathbb{Z})$ is generated by subgroups K_{C_1} of index p in $\ker f$ and subgroups of indices coprime to p. □

4.4 The ambiguity indices of symmetric groups and alternating groups

In [5], the following theorems were proved.

Theorem 4.12 ([5, Theorem 3.8]) *Let $\widetilde{\Sigma}_d$ be a maximal cover of the symmetric group Σ_d. The conjugacy classes of Σ_d which split in $\widetilde{\Sigma}_d$ are: (a) the classes of even permutations which can be written as a product of disjoint cycles with no cycles of even length; and (b) the classes of odd permutations which can be written as a product of disjoint cycles with no two cycles of the same length (including 1).*

Theorem 4.13 ([5, Theorem 3.9]) *Let $\widetilde{\mathbb{A}}_d$ be the maximal cover of the alternating group $\mathbb{A}_d$. The conjugacy classes of $\mathbb{A}_d$ which split in $\widetilde{\mathcal{A}_d}$ are: (a) the classes of permutations whose decompositions into disjoint cycles have no cycles of even length; and (b) the classes of permutations which can be expressed as a product of disjoint cycles with at least one cycle of even length and with no two cycles of the same length (including 1).*

Remind that, by definition, an equipment O of Σ_d must contain a conjugacy class of odd permutation since the elements of the equipment must generate the group.

It is well known that for the symmetric group Σ_d, $d \geqslant 4$, and for the alternating group $\mathbb{A}_d$, $d \neq 6, 7$, $d \geqslant 4$, the order of the Schur multiplier $h_2(\Sigma_d) = h_2(\mathbb{A}_d) = 2$. The following theorems are straightforward consequences of Proposition 4.8 and Theorems 4.9–4.13.

Theorem 4.14 *Let O be an equipment of a symmetric group Σ_d. Then $a_{(\Sigma_d,O)} = 2$ if and only if O consists of conjugacy classes of odd permutations such that they can be* *written as a product of disjoint cycles with no two cycles of the same length* (*including* 1) *and conjugacy classes of even permutations such that they can be written as a product of disjoint cycles with no cycles of even length. Otherwise,* $a_{(\Sigma_d,O)} = 1$.

Theorem 4.15 *Let O be an equipment of an alternating group $\mathbb{A}_d$, $d \neq 6, 7$. Then $a_{(\mathbb{A}_d,O)} = 2$ if and only if O consists of conjugacy classes of permutations whose decompositions into disjoint cycles have no cycles of even length and the classes of permutations which can be expressed as a product of disjoint cycles with at least one cycle of even length and with no two cycles of the same length* (*including* 1). *Otherwise,* $a_{(\mathbb{A}_d,O)} = 1$.

It is well known that in the case when $d = 6, 7$, the order of the Schur multiplier $h_2(\mathbb{A}_d) = 6$.

For $\sigma \in \mathbb{A}_d$ denote by $c(\sigma) = (l_1, \dots, l_m)$ the cycle type of permutation σ, that is, the collection of lengths l_i of non-trivial (that is $l_i \geqslant 2$) cycles entering into the factorization of σ as a product of disjoint cycles. For a conjugacy class C in $\mathbb{A}_d$ the collection $c(C) = c(\sigma)$ is called the *cycle type* of C if $\sigma \in C$. It is well known that the cycle type $c(C)$ does not depend on the choice of $\sigma \in C$ and there are at most two conjugacy classes in $\mathbb{A}_d$ of a given cycle type c.

The group $\mathbb{A}_d$, $d = 6, 7$, has the following non-trivial conjugacy classes:

(I) two conjugacy classes of each cycle type (5), (2, 4), and (if $d = 7$) (7);

(II) two conjugacy classes of cycle type (3) and one conjugacy class of cycle type (3, 3);

(III) one conjugacy class of cycle type (2, 2) and one conjugacy class of cycle type (2, 2, 3) if $d = 7$.

Proposition 4.16 *The ambiguity index $a_{(\mathbb{A}_d,O)}$, $d = 6, 7$, takes the following values:*

(I) $a_{(\mathbb{A}_d,O)} = 6$ *if O contains only the elements of conjugacy classes of type* (I)*;*

(II) $a_{(\mathbb{A}_d,O)} = 2$ *if O contains only the elements of conjugacy classes of type* (I) *and the elements of at least one conjugacy class of type* (II)*;*

(III) $a_{(\mathbb{A}_d,O)} = 3$ *if O contains only the elements of conjugacy classes of type* (I) *and the elements of at least one conjugacy class of type* (III)*;*

(II+III) $a_{(\mathbb{A}_d,O)} = 1$ *if O contains the elements of at least one conjugacy class of type* (II) *and the elements of at least one conjugacy class of type* (III).

Proof Let $f\colon (G_1, O_1) \to (\mathbb{A}_d, \mathbb{A}_d \setminus \{1\})$ be the universal maximal cover. Note that, by [13], $a_{(\mathbb{A}_d,\mathbb{A}_d\setminus\{1\})} = 1$. Therefore, there exist elements $\sigma_1, \dots, \sigma_4$ in $\mathbb{A}_d$ such that $[x_{\sigma_1}, x_{\sigma_2}]$ and $[x_{\sigma_3}, x_{\sigma_4}]$ in $\big([\mathbb{F}_{\mathbb{A}_d\setminus\{1\}}, \mathbb{F}_{\mathbb{A}_d\setminus\{1\}}] \cap R_{\mathbb{A}_d}\big)/[\mathbb{F}_{\mathbb{A}_d\setminus\{1\}}, R_{\mathbb{A}_d}]$ have, respectively, order two and three.

It is easy to see that for an element σ belonging to a conjugacy class C of type (I) the centralizer $Z(\sigma) \subset \mathbb{A}_d$ of the element σ is a cyclic group generated by σ. Therefore, K_C is the trivial group and hence $s_f(C) = h_2(\mathbb{A}_d)$. Therefore, by Proposition 4.9, $a_{(\mathbb{A}_d,O)} = 6$ if O contains only the elements of conjugacy classes of type (I).

Let σ be of cycle type (2, 2, 3). Without loss of generality, we can assume that $\sigma = \sigma_1\sigma_2$, where $\sigma_1 = (1, 2)(3, 4)$ and $\sigma_2 = (5, 6, 7)$. Then the centralizer $Z(\sigma) \subset \mathbb{A}_d$ of σ is $\mathrm{Kl}_4 \times \langle\sigma_2\rangle$, where $\mathrm{Kl}_4 = \langle\sigma_1\rangle \times \langle\sigma_3\rangle$ and $\sigma_3 = (1, 3)(2, 4)$. We have $\big[x_\sigma, x_{\sigma_3,\sigma_2^{\pm 1}}\big] = [x_{\sigma_1}, x_{\sigma_3}]$ in the group $\mathbb{F}_{\mathbb{A}_d\setminus\{1\}}/\big[\mathbb{F}_{\mathbb{A}_d\setminus\{1\}}, R_{\mathbb{A}_d}\big]$. Therefore K_C, where

C has type $(2, 2, 3)$, is a group of order at most two since the order of σ_1 is two (see Lemma 2.8) and it is of order two if and only if $[x_{\sigma_1}, x_{\sigma_3}]$ is not the unity in $\mathbb{F}_{\mathbb{A}_d\setminus\{1\}}/\big[\mathbb{F}_{\mathbb{A}_d\setminus\{1\}}, R_{\mathbb{A}_d}\big]$. But, the embeddings $\langle\sigma_1, \sigma_3\rangle \subset \mathbb{A}_d \subset \Sigma_d$ define a sequence of homomorphisms

$$H_2(\langle\sigma_1, \sigma_3\rangle, \mathbb{Z}) \to H_2(\mathbb{A}_d, \mathbb{Z}) \to H_2(\Sigma_d, \mathbb{Z})$$

such that the image of the non-trivial element $[x_{\sigma_1}, x_{\sigma_3}]$ in $H_2(\langle\sigma_1, \sigma_3\rangle, \mathbb{Z})$ is non-trivial in $H_2(\Sigma_d, \mathbb{Z})$. Therefore, $s_f(C) = 3$ for the conjugacy class C of cyclic type $(2, 2, 3)$ and, similarly, $s_f(C) = 3$ for the conjugacy class C of cyclic type $(2, 2)$, since K_C is a subgroup of $H_2(\mathbb{A}_d, \mathbb{Z}) \simeq \mathbb{Z}/6\mathbb{Z}$ generated by the elements of the second order (see Proposition 4.8) and only the elements of K_{C_1} and K_{C_2} can generate the subgroup of order two in $H_2(\mathbb{A}_d, \mathbb{Z})$.

Let σ be of cycle type $(3, 3)$. Without loss of generality, we can assume that $\sigma = \sigma_1\sigma_2$, where $\sigma_1 = (1, 2, 3)$ and $\sigma_2 = (4, 5, 6)$. Then the centralizer $Z(\sigma) \subset \mathbb{A}_d$ of σ is $\langle\sigma_1\rangle \times \langle\sigma_2\rangle$. Therefore, $[x_{\overline{\sigma}}, x_\sigma]$ is not the unity in $\mathbb{F}_{\mathbb{A}_d\setminus\{1\}}/\big[\mathbb{F}_{\mathbb{A}_d\setminus\{1\}}, R_{\mathbb{A}_d}\big]$ only if $\overline{\sigma} = \sigma_1^{\pm 1}$, either $\overline{\sigma} = \sigma_2^{\pm 1}$, or $\overline{\sigma} = \sigma_1\sigma_2^{-1}$, or $\overline{\sigma} = \sigma_1^{-1}\sigma_2$. We have

$$\big[x_{\sigma_1\sigma_2^{-1}}, x_\sigma\big] = [x_{\sigma_1}, x_{\sigma_2}]\big[x_{\sigma_2^{-1}}, x_{\sigma_1}\big] = [x_{\sigma_1}, x_{\sigma_2}]^2$$

in $\mathbb{F}_{\mathbb{A}_d\setminus\{1\}}/\big[\mathbb{F}_{\mathbb{A}_d\setminus\{1\}}, R_{\mathbb{A}_d}\big]$ and, similarly, $\big[x_{\sigma^{-1}\sigma_2}, x_\sigma\big] = [x_{\sigma_1}, x_{\sigma_2}]$, since the elements $x_\sigma x_{\sigma_2}^{-1} x_{\sigma_1}^{-1}$ and $x_{\sigma_1\sigma_2^{-1}} x_{\sigma_2} x_{\sigma_1}^{-1}$ belong to the center of the group $\mathbb{F}_{\mathbb{A}_d\setminus\{1\}}/\big[\mathbb{F}_{\mathbb{A}_d\setminus\{1\}}, R_{\mathbb{A}_d}\big]$. Therefore, the group K_{C_1} is a non-trivial group of order three if and only if K_{C_2} is a non-trivial group of order three, where C_1 is a conjugacy class of the cycle type (3) and C_2 is the conjugacy class of the cycle type $(3, 3)$, and hence $s_f(C_1) = s_f(C_2) = 2$. Now Proposition 4.16 follows from Propositions 4.9–4.11. □

5 Cohomological description of the ambiguity indices

In notation used in Sect. 2, for an equipped finite group (G, O) a subgroup $K_{(G,O)}$ of $H_2(G, \mathbb{Z})$ was defined as follows: $K_{(G,O)}$ is the subgroup of $\big(R_O \cap [\mathbb{F}_O, \mathbb{F}_O]\big)/[\mathbb{F}_O, R_O]$ generated by the elements of R_O of the form $[w, x_g]$, where $g \in O$ and $w \in \mathbb{F}_O$, and $k_{(G,O)}$ is its order.

Denote

$$\begin{aligned} B_{(G,O)} &= K_{(G,O)}^{\perp} \\ &= \big\{\varphi \in \mathrm{Hom}(H_2(G, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) : \varphi(w) = 0 \text{ for all } w \in K_{(G,O)}\big\} \end{aligned}$$

a subgroup of $H^2(G, \mathbb{Q}/\mathbb{Z})$ dual to $K_{(G,O)}$. As in the proof of Theorem 1.1, it is easy to show that

$$B_{(G,O)} = \ker\left[H^2(G, \mathbb{Q}/\mathbb{Z}) \to \bigoplus_{A\subset G} H^2(A, \mathbb{Q}/\mathbb{Z})\right],$$

where A runs over all abelian subgroups of G generated by two elements $g \in O$ and $h \in G$. Let $b_{(G,O)}$ be the order of the group $B_{(G,O)}$. In particular, $b_{(G,G\setminus\{1\})} = b_0(G)$.

The next theorem immediately follows from Theorem 2.7.

Theorem 5.1 *For an equipped finite group* (G, O) *we have* $a_{(G,O)} = b_{(G,O)}$.

The group $H^2(G, \mathbb{Q}/\mathbb{Z})$ is a direct sum of primary components, $H^2(G, \mathbb{Q}/\mathbb{Z}) = \Sigma_p H^2(G, \mathbb{Q}/\mathbb{Z})_p$, where primes p run through a subset of primes dividing the order of of $H^2(G, \mathbb{Q}/\mathbb{Z})$ and hence G. Therefore, we have the following

Proposition 5.2 *If the set of conjugacy classes O consists of all classes of the elements of prime orders then* $a_{(G,O)} = b_0(G)$. *Moreover, it is sufficient to consider such classes only for primes dividing* $h_2(G)$.

Note that $H^2(G, \mathbb{Q}/\mathbb{Z})_p$ embeds into $H^2(\mathrm{Syl}_p(G), \mathbb{Q}/\mathbb{Z})_p$ where $\mathrm{Syl}_p(G)$ is a Sylow p-subgroup of G. Similarly, the p-primary component $B_0(G)_p$ is a subgroup of $B_0(\mathrm{Syl}_p(G))$.

More explicit versions of Proposition 5.2 for different groups provide with simple methods to compute $B_0(G)$.

6 An example of a finite group G with $b_0(G) > 1$

The following groups were constructed in the article of Saltman [14]. Consider a finite p-group G_p of order p^9 which is a central extension of $A_p = \mathbb{Z}_p^4$, where $\mathbb{Z}_p$ is a cyclic group of order p. Denote the generators of A_p by $x_i, i = 1, \ldots, 4$. The center of G_p is generated by pairwise commutators $x_i x_j x_i^{-1} x_j^{-1} = [x_i, x_j]$ with one relation $[x_1, x_2][x_3, x_4] = 1$. Thus there is a natural exact sequence

$$1 \to \mathbb{Z}_p^5 \to G_p \to A_p \to 1.$$

The following lemma first appeared in different notation in [14] and then in [1] in the current form.

Lemma 6.1 $B_0(G_p) = \mathbb{Z}_p$.

Proof It is shown in [1] that for a central extension G of an abelian group A the group $B_0(G)$ is contained in the image of $H^2(A, \mathbb{Q}/\mathbb{Z})$ in $H^2(G, \mathbb{Q}/\mathbb{Z})$ under the cohomology map induced by projection $\pi_A : G \to A$.

The proof is based on analysis of the standard spectral sequence with $E_2^{pq} = H^p(A, H^q(K, \mathbb{Q}/\mathbb{Z}))$ converging to $H^{p+q}(G, \mathbb{Q}/\mathbb{Z})$ for $p + q = 2$, where K is a kernel of π_A.

The group $H^2(A_p, \mathbb{Q}/\mathbb{Z}) = \mathbb{Z}_p^6$ and it is generated by elements $[x_i, x_j]^*$. The kernel of the map $H^2(A_p, \mathbb{Q}/\mathbb{Z}) \to H^2(G_p, \mathbb{Q}/\mathbb{Z})$ is Pontryagin dual to the center $\mathbb{Z}_p^5$ of G_p. Thus the image of $H^2(A_p, \mathbb{Q}/\mathbb{Z})$ in $H^2(G_p, \mathbb{Q}/\mathbb{Z})$ is a cyclic p-group generated by one element w.

Let us show that w is in $B_0(G_p)$. The element w defines an element in $H^2(B, \mathbb{Q}/\mathbb{Z}) = \mathbb{Z}_p$ for any abelian subgroup $B \subset G_p$ of rank 2. The fact that the element w is in $B_0(G_p)$ is equivalent to the triviality of the restriction of w on any abelian subgroup $B \subset G_p$ of rank 2 [1].

Since w is induced from A_p its restriction is automatically trivial on any such B with a projection $\pi_A(B)$ contained in cyclic subgroup of A_p. Thus it is enough to check that w is trivial on any abelian subgroup in G_p which surjects onto rank 2 subgroup $\mathbb{Z}_p^2 \subset \mathbb{Z}_p^4 = A_p$.

However, G_p does not contain such subgroups. Indeed, assume $x_1, y_1 \in G_p$ generate an abelian subgroup B of rank 2 in G_p which projects into the abelian rank 2 subgroup of A_p with generators x, y. Then the commutator $[x, y]$ is contained in the space of non-trivial relations for G_p.

We know, however, that the only non-trivial relation in G_p between commutators of elements in A_p is $[x_1, x_2][x_3, x_4] = 1$. The element $[x_1, x_2][x_3, x_4]$ in $\mathbb{Z}_p^6$ cannot be represented as $[x, y]$ for a pair $x, y \in A_p$. Hence such a group B cannot exist and any abelian subgroup of G_p projects into a cyclic subgroup of A_p. Therefore, w restricts trivially onto any abelian subgroup with two generators in G_p and w is contained $B_0(G_p)$. Since w is non-trivial the group $B_0(G_p)$ is non-trivial and equal to $\mathbb{Z}_p$. □

References

1. Bogomolov, F.A.: The Brauer group of quotient spaces of linear representations. Math. USSR-Izv. **30**(3), 455–485 (1988)
2. Bogomolov, F., Maciel, J., Petrov, T.: Unramified Brauer groups of finite simple groups of Lie type A_ℓ. Am. J. Math. **126**(4), 935–949 (2004)
3. Clebsch, A.: Zur Theorie der Riemann'schen Fläche. Math. Ann. **6**(2), 216–230 (1873)
4. Fulton, W.: Hurwitz schemes and irreducibility of moduli of algebraic curves. Ann. Math. **90**(3), 542–575 (1969)
5. Hoffman, P.N., Humphreys, J.F.: Projective Representations of the Symmetric Groups. In: Oxford Mathematical Monographs. Clarendon Press/Oxford University Press, New York (1992)
6. Hurwitz, A.: Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten. Math. Ann. **39**(1), 1–60 (1891)
7. Jezernik, U., Moravec, P.: Universal commutator relations, Bogomolov multipliers, and commuting probability. arXiv:1307.6533 (2013)
8. Kang, M., Kunyavskiĭ, B.: The Bogomolov multiplier of rigid finite groups. Arch. Math. (Basel) **102**(3), 209–218 (2014)
9. Kulikov, Vik.S.: Hurwitz curves. Rus. Math. Surv. **62**(6), 1043–1119 (2007)
10. Kulikov, Vik.S.: Factorization semigroups and irreducible components of the Hurwitz space. II. Izv. Math. **76**(2), 356–364 (2012)
11. Kulikov, Vik.S.: Factorizations in finite groups. Sb. Math. **204**(2), 237–263 (2013)
12. Kulikov, Vik.S., Kharlamov, V.M.: Covering semigroups. Izv. Math. **77**(3), 594–626 (2013)
13. Kunyavskiĭ, B.: The Bogomolov multiplier of finite simple groups. In: Cohomological and Geometric Approaches to Rationality Problems. Progress in Mathematics, vol. 282, pp. 209–217. Birkhäuser, Boston (2010)
14. Saltman, D.J.: Noether's problem over an algebraically closed field. Invent. Math. **77**(1), 71–84 (1984)
15. Spanier, E.H.: Algebraic Topology. Springer, New York (1981)