A non-Laplace harmonic structure in infinite networks

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Abstract By using potential-theoretic methods on locally compact spaces, two harmonic sheaves (resembling the solutions of the Laplace and the Schrödinger operators) on an infinite network are compared.

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1 Introduction

Let X be an infinite network provided with a set of (not necessarily symmetric) conductance on the edges, like an infinite electrical network with a symmetric conductance which is the reciprocal of the resistance or a random walk with its transition probabilities as non-symmetric conductance. By using the Laplace operator on X , harmonic functions and potentials are then defined on X , rendering it a harmonic space behaving,

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but for topology, like the Euclidean spaces, Riemannian manifolds or locally compact spaces endowed with harmonic sheaves in the Brelot axiomatic potential theory.

There are other interesting harmonic structures on X , for example the one defined by using the Schrödinger operator on X . These are different from the Laplace harmonic structure, yet in many ways associated to it. In this note a particular class of non-Laplace harmonic structures is considered, calling them subordinate harmonic structures on X . We develop the potential theory associated with them and, in particular, discuss the relations between bounded and positive harmonic functions in each harmonic structure.

Laplace potential theory on infinite networks has been greatly developed by using the methods of electrical networks and random walks, see for example Cartier [4], Woess [10] and Zemanian [13]. When the conductance is symmetric or can be made symmetric as in the case of electrical networks or reversible Markov chains, one could introduce the Dirichlet norm on X . By using this norm and the analogues of the potential-theoretic methods on Riemannian manifolds, Lyons [5], Soardi [9], Yamasaki [11, 12] and many others have developed a discrete potential theory on X . To do this, the symmetry of the conductance or at least the reversibility condition is necessary. Here we dispense with this requirement and use the methods of the Brelot axiomatic potential theory on locally compact spaces [3].

The questions posed and the methods used to answer them are different from those encountered in electrical networks or random walks; however, the methods of proofs given here can be used in the context of non-reversible Markov chains.

2 Preliminaries

Let X be an infinite graph with countably infinite vertices and countably infinite edges. We say that x and y are neighbours, and write $x \sim y$, if and only if there is an edge joining x and y . We do not assume that each vertex has a finite number of neighbours. When it is necessary to suppose that every vertex in X has only a finite number of neighbours, then we say that X is locally finite. Assume X is connected, that is any two vertices x and y can be joined by a path $\{x = x_0, x_1, \dots, x_n = y\}$. We refer to such a graph X as an infinite network if there is a *transition index* $k(x, y)$ associated with each pair of vertices x and y in X satisfying the conditions: $k(x, y)$ is a non-negative real number, $k(x, y) > 0$ if and only if $x \sim y$, $k(x, y)$ and $k(y, x)$ need not be same and for any vertex x in X , $k(x) = \sum_y k(x, y)$ is always a positive real number.

Henceforth X will refer to just the set of vertices in the infinite network X . If E is a subset of X , $x \in E$ is said to be an *interior vertex* of E if all the neighbours of x in X are also in E . Let $\mathring{E}$ denote the set of all interior vertices of E and $\partial E = E \setminus \mathring{E}$ the boundary of E .

We consider only real-valued functions u defined on E such that

$$\sum_y k(x, y)|u(y)| < \infty, \quad x \in \mathring{E}.$$

If u is a real-valued function on E , then u is said to be *A-superharmonic* on E if $u(x) \geq Au(x)$ for every $x \in \mathring{E}$, where $Au(x) = \sum_y k(x, y)u(y)$. *A-harmonic*

and A -subharmonic functions are defined accordingly. An A -superharmonic function $p \geq 0$ on E is said to be an A -potential on E if for any A -subharmonic function u on E such that $u \leq p$ it follows that $u \leq 0$. If there exists a positive A -potential on X , then X is called an A -hyperbolic network, otherwise X is an A -parabolic network. If $\sum_y k(z, y) = 1$ for every $z \in X$, we use the symbol

$$\Delta u(x) = -u(x) + \sum_y k(x, y)u(y) = \sum_y k(x, y)[u(y) - u(x)].$$

In the case of random walks with $p(x, y)$ as the *transition probability*, if we take $k(x, y) = p(x, y)$ then $\sum_y k(x, y) = 1$ for every x . In the case of infinite electrical networks with $t(x, y)$ as *conductance*, if we write $k(x, y) = t(x, y)/t(x)$, where $t(x) = \sum_y t(x, y)$, then $\sum_y k(x, y) = 1$ for every x . There are other interesting infinite networks for which the condition $\sum_y k(x, y) = 1$ is not satisfied for every x . For example, in the case of a *Schrödinger operator* $\Delta_q u(x) = \Delta u(x) - q(x)u(x)$, where $\Delta u(x) = \sum_y p(x, y)[u(y) - u(x)]$, $\sum_y p(x, y) = 1$, $q(x) \geq 0$ for all x and $q(x_0) > 0$ for at least one x_0 in X , it is convenient to consider $k(x, y) = p(x, y)/(1 + q(x))$ as the transition index on X , in which case $\sum_y k(x_0, y) = 1/(1 + q(x_0)) < 1$.

3 Non-locally finite networks

In non-locally finite networks all the properties of superharmonic functions given in [1] may not be valid. For example, the limit of a sequence of A -superharmonic functions may not be A -superharmonic. However, some properties of superharmonic functions preserve for A -superharmonic functions.

Theorem 3.1 *Let $\mathfrak{F}$ be a family of non-negative A -superharmonic functions defined on a subset E of X . If $h = \inf_{s \in \mathfrak{F}} s$ then h is A -superharmonic on E .*

Proof Let $x \in \mathring{E}$. Since s is A -superharmonic at x , we have $s(x) \geq As(x) \geq Ah(x)$. Taking infimum $h(x) = \inf_{s \in \mathfrak{F}} s(x) \geq Ah(x)$. Hence h is A -superharmonic on E . $\square$

Theorem 3.2 *Let s_n be an increasing sequence of non-negative A -superharmonic (respectively, A -harmonic) functions on a subset E of X . If $s = \sup s_n$ is finite at each vertex then s is A -superharmonic (respectively, A -harmonic) on E .*

Proof Since s_n is A -superharmonic on E we have $s_n(x) \geq As_n(x)$, $x \in \mathring{E}$. Let the neighbours of x be $y_1, y_2, y_3, \dots$ and

$$a_{mn}(x) = \sum_{i=1}^m k(x, y_i)s_n(y_i).$$

Then $a_{mn}(x)$ is increasing in m and $s_n(x) \geq \lim_{m \rightarrow \infty} a_{mn}(x)$. For each fixed m , $a_{mn}(x)$ is also an increasing sequence in n . Hence,

$$\begin{aligned}
s(x) &= \lim_{n \rightarrow \infty} s_n(x) \geq \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} a_{mn}(x) = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} a_{mn}(x) \\
&= \lim_{m \rightarrow \infty} \sum_{i=1}^m k(x, y_i) s(y_i) = \sum_y k(x, y) s(y). \quad \square
\end{aligned}$$

The following theorems can be proved as in the locally finite networks.

Theorem 3.3 (minimum principle) *If $s \geq 0$ is an A -superharmonic function on X and $s(x) = 0$ for some $x \in X$ then $s \equiv 0$.*

Theorem 3.4 *Let t and s be A -subharmonic and A -superharmonic functions, respectively, defined on a subset E of X such that $t \leq s$. Then there exists a function h (called A -greatest harmonic minorant of s) which is A -harmonic and $t \leq h \leq s$ on E .*

Theorem 3.5 (A -Dirichlet problem) *Let F be an arbitrary set in X and f a real-valued function on ∂F such that $|f| \leq s$, where s is a positive A -superharmonic function on F . Then there exists a function h on F such that $|h| \leq s$ on F , $Ah(z) = h(z)$ for all $z \in \overset{\circ}{F}$ and $h = f$ on ∂F .*

Let $f \geq 0$ be a non-negative real valued function defined on X . Let

$$Sf = \{u : u \geq 0 \text{ is } A\text{-superharmonic such that } u(x) \geq f(x) \text{ on } X\}$$

and $Rf(x) = \inf \{u(x) : u \in Sf\}$ if Sf is a non-empty set.

Theorem 3.6 *The function Rf is A -superharmonic on X and A -harmonic at x if f is A -subharmonic at x .*

Let E be a subset of X and $u \geq 0$ be a real valued function on X . The function

$$R_u^E = \inf \{v(x) : v \geq 0 \text{ is } A\text{-superharmonic on } X, v(x) \geq u(x) \text{ on } E\}$$

is called the A -reduced function (or A -balayage) of u on E .

Theorem 3.7 *The function R_u^E is A -superharmonic in X and A -harmonic at every vertex in $X \setminus E$.*

Theorem 3.8 *Let X be an A -hyperbolic network. For any $a \in X$ there exists a unique A -potential $G_a(x)$ such that $AG_a(x) - G_a(x) = -\delta_a(x)$.*

Theorem 3.9 (Harnack property) *Let F be a subset of X and E a connected subset of $\overset{\circ}{F}$. Let a and b be two vertices in E . Then there exist two constants $\alpha, \beta > 0$ such that for any non-negative A -superharmonic function s on F , $\alpha s(b) \leq s(a) \leq \beta s(b)$.*

Note 3.10 Since we assumed that X is not locally finite there may not be an infinite path starting at any vertex in X . For example let $X = \{e, x_n\}$, $n = 1, 2, 3, \dots$, where x_n are neighbours of e and no two x_n, x_m are neighbours. In this infinite network X there is no infinite path starting at any vertex.

We assume that there exists at least one e such that there is a path (with distinct vertices) of infinite length starting at e . With this assumption since X is connected, there is an infinite path starting at any vertex in X . Let $E_n = \{x : d(e, x) \leq n\}$. Note that E_n need not be finite. We have $E_n \subseteq E_{n+1} \subseteq \dots$ and $\bigcup_n E_n = X$. Let

$$[e, e_i] = \{x : \text{there exists a path joining } x \text{ to } e \text{ which passes through } e_i\}$$

called a section determined by e and e_i , $e_i \sim e$. It is assumed that e and e_i are in $[e, e_i]$. Then $[e, e_i]$ and $[e, e_i] \setminus \{e\}$ are connected, may be finite or infinite sets. But there exists at least one infinite set because of our assumption that there is a path of infinite length starting at e .

Let $\{X, k(x, y)\}$ be a network. Note that it is possible that there may not be any positive A -superharmonic function on X .

Example Let $X = \{0, 1, 2, \dots\}$, $k(x, y) = 2$ for all $x \sim y$. Let $v \geq 0$ be A -superharmonic on X . Let $v(0) = b$ and $v(i) = b_i$ for every i . Then we have a chain of implications

$$v(0) \geq k(0, 1)v(1) \implies b \geq 2b_1 \implies b \geq \frac{1}{2}b \geq b_1 \implies b \geq b_1.$$

Now,

$$\begin{aligned} v(1) &\geq k(1, 2)v(2) + k(1, 0)v(0) \implies b_1 \geq 2b_2 + 2b \\ \implies b_1 &\geq \frac{1}{2}b_1 \geq b + b_2 \implies b_1 \geq b \implies b = b_1. \end{aligned}$$

Moreover, $b_1 \geq b + b_2 \Rightarrow b \geq b + b_2 \Rightarrow b_2 \leq 0$, hence, since $v \geq 0$, $b_2 = 0$.

$$v(2) \geq 2b_3 + 2b_1 \implies 0 = b_2 \geq 2(b_1 + b_3) \geq b_1 + b_3 \implies b + b_3 \leq 0,$$

which implies $b = 0$ and $b_3 = 0$. Note that for $i \geq 1$, if $b_i = 0$, then since $b_{i-1} + b_{i+1} \leq b_i$, we should have $b_{i-1} = 0$ and $b_{i+1} = 0$. Consequently, $b_i = 0$ for every i which implies $v = 0$.

Let $\{X, k(x, y)\}$ be a network. Assume there exists an A -superharmonic function $\xi > 0$ on X . Let $\mathcal{H}$ be the A -harmonic sheaf on X . Define

$$k'(x, y) = k(x, y) \frac{\xi(y)}{\xi(x)}.$$

Then a function $u(x)$ on X is $A = \{k(x, y)\}$ -superharmonic (respectively A -harmonic) if and only if $u(x)/\xi(x)$ is $A' = \{k'(x, y)\}$ -superharmonic (respectively A' -harmonic). Indeed, we have

$$\begin{aligned}
 u(x) &\geq \sum_y k(x, y) u(y), \\
 u(x) &\geq \sum_y k'(x, y) \frac{\xi(x)}{\xi(y)} u(y), \\
 \frac{u(x)}{\xi(x)} &\geq \sum_y k'(x, y) \frac{u(y)}{\xi(y)}.
 \end{aligned}$$

In particular, taking $u(x) = \xi(x)$ we see that positive constants are A' -superharmonic. Consider the new sheaf

$$\frac{\mathcal{H}}{\xi} = \left\{ \frac{h(x)}{\xi(x)} : h \in \mathcal{H} \right\}.$$

Then with respect to the new sheaf, the constant 1 is A -superharmonic. In case 1 is A -superharmonic on X , we have $1 \geq \sum_y k(x, y)$ for any $x \in X$. That is $k(x) \leq 1$ for all $x \in X$. Thus, the assumption that there exists an A -superharmonic function $\xi > 0$ on X comes down to the assumption that $k(x) \leq 1$ for any $x \in X$.

Definition 3.11 A network X is called a non-degenerate network if $k(x) = \sum_y k(x, y) \leq 1$ for any $x \in X$. A non-degenerate network is called a *Schrödinger network* if $k(x) < 1$ for at least one $x \in X$. If $k(x) = 1$ for all x , then X is called a *Laplace network*.

Proposition 3.12 Let F be a finite set in a non-degenerate network X . If u is A -superharmonic on F and $u \geq 0$ on ∂F , then $u \geq 0$ on F .

Theorem 3.13 Let X be a non-degenerate network. Let $E_i = [e, e_i]$ be an infinite section. Then there exists $h \geq 0$ on E_i such that $h(e) = 0$, $h(x) > 0$ if $x \in E_i \setminus \{e\}$ and $Ah(x) = h(x)$ at every vertex $x \in E_i \setminus \{e\}$.

Proof Let $B_n = \{x : |x| \leq n\}$ and $B'_n = B_n \cap [e, e_i]$. Then, by using an A -Dirichlet solution, there is a function u_n on B'_n such that $0 \leq u_n \leq 1$, $u_n(e) = 0$, and $u_n(y) = 1$ if $y \in \partial B'_n \setminus \{e\}$, $u_n(x) > 0$ and $Au_n(x) = u_n(x)$ if x is an interior vertex of $B'_n \setminus \{e\}$. Assume u_n is extended to the rest of E_i by giving the value $u_n(a) = 1$ for each $a \in E_i \setminus B'_n$. Clearly, u_n is an A -superharmonic function on E_i . Fix a vertex, $a \neq e$ in E_i , and define $v_n(x) = u_n(x)/u_n(a)$ on E_i . Then $v_n \geq 0$ is an A -superharmonic function on E_i , $v_n(a) = 1$, $v_n(e) = 0$ and $v_n > 0$ on $E_i \setminus \{e\}$. Take a vertex b in $E_i \setminus \{e\}$. Then by the Harnack property, there is $\alpha > 0$ such that $v_n(b) \leq \alpha v_n(a) = \alpha$ for all n . Hence, we can extract a subsequence $\{v'_n\}$ from $\{v_n\}$ such that $\lim_{n \rightarrow \infty} v'_n(b)$ exists. Take another vertex k in $E_i \setminus \{e\}$. Then, by using the same argument, we can extract a subsequence $\{v''_n\}$ from $\{v'_n\}$ such that $\lim_{n \rightarrow \infty} v''_n(k)$ exists. Note that $\lim_{n \rightarrow \infty} v''_n(b)$ also exists. Since E_i is countably infinite, this method shows that there exists a subsequence $\{v^*_n\}$ of $\{v_n\}$ such that $h(x) = \lim_{n \rightarrow \infty} v^*_n(x)$ exists for each $x \in E_i \setminus \{e\}$. Since $v^*_n(e) = 0$ and $v^*_n(a) = 1$, we find $h(e) = 0$ and $h(a) = 1$. Finally, at any vertex z in $E_i \setminus \{e\}$, $Av^*_n(z) = v^*_n(z)$ for all n sufficiently large, so that $h(x)$ is A -harmonic at z . $\square$

Note that if we extend the above function h on X by giving 0 outside E_i , then $h(e) = 0$ and $Ah(x) = h(x)$ if $x \neq e$.

4 A non-Laplace harmonic structure

Let $\{X, k(x, y)\}$ be a Laplace network, that is the conductance $k(x, y)$ satisfies the condition $\sum_y k(x, y) = 1$ for every x . The Laplace operator

$$\Delta u(x) = -u(x) + \sum_y k(x, y)u(y)$$

defines a harmonic structure on X . An important feature of the Laplace operator is its usefulness in expressing the mean-value property of certain functions. There are also other linear operators on X which possess many properties similar to the Laplace operator properties, modulo the mean-value property. Specifically, similar to the Laplace operator Δ , we shall introduce another operator

$$\Delta' u(x) = -u(x) + \sum_y k'(x, y)u(y),$$

where $0 \leq k'(x, y) \leq k(x, y)$ for every pair of vertices x and y ; and $k'(x, y) < k(x, y)$ for at least one pair x and y . We say that Δ' is *subordinate* to Δ and note that Δ' also defines a harmonic structure on X in the same way as the Laplace operator Δ does. We refer to Δ' as a *non-Laplace operator* on X . The important features of this subordination are that a non-negative Δ -superharmonic function is a Δ' -superharmonic function and a non-negative Δ' -subharmonic function is a Δ -subharmonic function. Some examples of non-Laplace operators are:

1. The Schrödinger operator is subordinate to the Laplace operator. More generally, let $\{X, k(x, y)\}$ be an infinite network in which $k(x) = \sum_y k(x, y) \leq 1$ for each x and $k(x_0) < 1$ for at least one x_0 in X . Then

$$\Delta u(x) = -u(x) + \sum_y \frac{k(x, y)}{k(x)} u(y)$$

is Laplacian and

$$\Delta' u(x) = -u(x) + \sum_y k(x, y)u(y)$$

is a non-Laplace operator subordinate to Δ .

Another way of presenting the above example is as follows: If $k(x) < 1$, then introduce a new vertex $\hat{x}$ and define $\hat{k}(x, \hat{x}) = 1 - k(x)$, $\hat{k}(\hat{x}, x) = 1$ and $\hat{k}(x, y) = k(x, y)$ if $y \neq \hat{x}$. If $k(x) = 1$, then no additional vertex is introduced as a neighbour to x . Let $\hat{X}$ denote the union of X and all $\hat{x}$ whenever they are defined. Then we call $(\hat{X}, \hat{k})$ as the *completion* of the infinite network (X, k) . If s is a real valued function on X , define $\hat{s}$ on $\hat{X}$ such that $\hat{s}(\hat{x}) = s(x)$ and $\hat{s}(x) = s(x)$ for $x \in X$.

Let u be defined on a subset E of X . For $x \in \hat{E}$, define

$$Au(x) = \sum_{y \in X} k(x, y)u(y).$$

If $u(x) = Au(x)$ for every $x \in \mathring{E}$, then say that u is (X, A) -harmonic on E . These harmonic functions define a harmonic sheaf on X , denoted by (X, A) . Similarly, define

$$\widehat{A}u(x) = \sum_{y \in \widehat{X}} \widehat{k}(x, y) \widehat{u}(y)$$

and introduce another harmonic sheaf on X , denoted by $(X, \widehat{A})$. Note that if $v \geq 0$ is $(X, \widehat{A})$ -superharmonic on E , then v is (X, A) -superharmonic on E ; the constant 1 is $(X, \widehat{A})$ -harmonic on X ; if $f \geq 0$ is (X, A) -subharmonic on E , then f is $(X, \widehat{A})$ -subharmonic on E . We can thus say that the harmonic sheaf (X, A) is subordinate to the harmonic sheaf $(X, \widehat{A})$.

2. Let $\{X, k(x, y)\}$ be an infinite network such that $\sum_y k(x, y) = 1$ for every x in X . Let T be a spanning tree of X , obtained from X leaving out at least one edge; the vertex set of T is the same as that of X . If $x \sim y$ in T , write $k'(x, y) = k(x, y)$. If x and y are not neighbours in T , but neighbours in X , then $k'(x, y) = 0 < k(x, y)$. Then $\Delta'u(x) = -u(x) + \sum_y k'(x, y)u(y)$ for $x \in X$ is subordinate to the Laplace operator $\Delta u(x) = -u(x) + \sum_y k(x, y)u(y)$.

3. Let X be an infinite electrical network with conductance $t(x, y)$. Write $p(x, y) = t(x, y)/t(x)$, where $t(x) = \sum_y t(x, y)$. Suppose some branches of X are removed to obtain another electrical network X' that is connected. Then the vertex set of X' is the same as that of X . If $x \sim y$ in X' , then take $p'(x, y) = p(x, y)$. Consequently, if we write

$$\Delta u(x) = -u(x) + \sum_y p(x, y)u(y)$$

and

$$\Delta'u(x) = -u(x) + \sum_y p'(x, y)u(y),$$

then Δ' is subordinate to Δ in X .

Some of the following relations between Δ -superharmonic functions and Δ' -superharmonic functions are easy to check [2]:

- (i) If $u \geq 0$ is Δ -superharmonic on a subset E of X , then u is Δ' -superharmonic on E .
- (ii) If $v \geq 0$ is Δ' -subharmonic on E , then v is Δ -subharmonic on E .
- (iii) If $p > 0$ is a Δ -potential on X , then p is a Δ' -potential on X .
- (iv) Constants are Δ -harmonic.
- (v) The constant 1 is Δ' -superharmonic but not Δ' -harmonic, so that X is always a Δ' -hyperbolic network; that is there exist positive Δ' -potentials on X .
- (vi) The constant 1 is a Δ' -potential on X if and only if there is no nonzero bounded Δ' -harmonic function on X . If 1 is a Δ' -potential on X , we refer to X as Δ' -parahyperbolic network. For, in this case, though X is Δ' -hyperbolic, yet many of the properties of Δ' -superharmonic functions are similar to those defined on a parabolic Riemannian manifold.

(vii) If X is Δ -parabolic, then X is Δ' -parahyperbolic. For, if X is not Δ' -parahyperbolic, then 1 is not a Δ' -potential. Hence there exists a bounded Δ' -harmonic function h on X , $0 < h < 1$. Now h is Δ -subharmonic. Since h is bounded subharmonic and X is Δ -parabolic, h should be a constant. This is a contradiction since constants are not Δ' -harmonic.

Example of a Δ -hyperbolic network X that is Δ' -parahyperbolic. Let $X = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. Define the conductance as $p(n, n+1) = 3/4$ and $p(n, n-1) = 1/4$. Then $h(n) = 3^{-n}$ is a positive Δ -harmonic function. Hence X is Δ -hyperbolic. Now on the network X , define a new set of conductance $p'(n, n+1) = 1/4 = p'(n, n-1)$ for all n . This introduces a subordinate structure Δ' on X . Let u be a Δ' -harmonic function on X . Then solving the equation $u(n+2) - 4u(n+1) + u(n) = 0$, we find that $u(n) = \lambda(2 + \sqrt{3})^n + \mu(2 - \sqrt{3})^n$, consequently, u is bounded if and only if $\lambda = \mu = 0$, that is 0 is the only bounded Δ' -harmonic function on X . Hence X is Δ' -parahyperbolic.

(viii) Let $G'_e(x)$ be the Δ' -Green function with pole at e . If $\lim_{x \rightarrow \infty} G'_e(x) > 0$, then X is Δ -parabolic and hence X is Δ' -parahyperbolic.

For, suppose on the contrary X is Δ -hyperbolic. Let $q(x)$ be a Δ -potential on X such that $q(e) = G'_e(e)$. Note $\lim_{x \rightarrow \infty} q(x) = 0$. Since q is a Δ -potential, it is a Δ' -potential. Hence $G'_e(x) \leq q(x)$ for all $x \in X$, by the Domination Principle [1] which implies that $\lim_{x \rightarrow \infty} G'_e(x) \leq \lim_{x \rightarrow \infty} q(x) = 0$, a contradiction. Hence X is Δ -parabolic.

(ix) Let X be a Δ -hyperbolic network. If Δ' is subordinate to Δ , then every Δ' -potential with finite Δ' -harmonic support is dominated by a Δ -potential.

For, it is known [1, Theorem 4.3.2] that for a fixed vertex a , if $G_a(x)$ is the Δ -Green function with Δ -harmonic support at a and if $G'_a(x)$ is the Δ' -Green function with Δ' -harmonic support at a , then $G'_a(x) \leq G_a(x)$ for every $x \in X$. Consequently, if u is a Δ' -potential with finite Δ' -harmonic support at $a_1, a_2, \dots, a_m$, then for $\lambda_i > 0$

$$u(x) = \sum_{i=1}^m \lambda_i G'_{a_i}(x) \leq \sum_{i=1}^m \lambda_i G_{a_i}(x) = v(x),$$

where $v(x)$ is a Δ -potential on X .

5 Δ - and Δ' -positive harmonic functions outside a vertex in X

Let e be a fixed vertex in X . Let $'\mathcal{H}_e^+$ denote the set of non-negative functions on X such that $h(e) = 0$ and $\Delta'h(x) = 0$ if $x \neq e$. If $'\mathcal{H}^+$ denotes the set of non-negative Δ' -harmonic functions on X , then there is a one-one relation between $'\mathcal{H}_e^+$ and $'\mathcal{H}^+$ [1, Theorem 3.2.10]. Similarly $\mathcal{H}_e^+$ and $\mathcal{H}^+$ are defined with respect to the Δ -harmonic structure. It is possible in some cases [1, p.9] that $\mathcal{H}^+$ consists only of constants, whereas $\mathcal{H}_e^+$ may contain many non-proportional elements; consequently, there cannot be one-one relation between $\mathcal{H}_e^+$ and $\mathcal{H}^+$. So it is of interest to establish a correspondence between $'\mathcal{H}_e^+$ and $\mathcal{H}_e^+$ rather than between $'\mathcal{H}^+$ and $\mathcal{H}^+$.

Operators S and T . We introduce now two positive linear operators S and T defined on $'\mathcal{H}_e^+$ and $\mathcal{H}_e^+$. Such operators are considered by Ozawa [6] in a Riemann surface of infinite genus having precisely one ideal boundary element.

Let $u \in '\mathcal{H}_e^+$. Then u is Δ -subharmonic on X . If u has a Δ -harmonic majorant in $\mathcal{H}_e^+$, let Su denote the least Δ -harmonic majorant of u . Let Π' denote the subset of $'\mathcal{H}_e^+$ such that $u \in \Pi'$ if and only if Su exists.

Let $u \in \mathcal{H}_e^+$. Then $u(x)$ with the value $u(e) = 0$ is Δ -subharmonic on X . Measuring distances from e , write $E_n = \{x : |x| \leq n\}$; let $D_n u(x)$ be the Dirichlet solution in E_n with boundary values u on ∂E_n , $u(e) = 0$ and $\Delta D_n u(x) = 0$ if $x \in \dot{E}_n \setminus e$. Then $D_n u(x)$ is an increasing function of n for each x . Clearly, $u \in \Pi'$ if and only if $\lim_{n \rightarrow \infty} D_n u(x)$ is finite for each x ; in particular, all bounded functions in $'\mathcal{H}_e^+$ are in Π' . We write $\Pi = S(\Pi')$.

The following lemma is a discrete analogue of a result proved by Ozawa in a Riemann surface F of infinite genus which has precisely one ideal boundary element and there is no non-zero bounded positive solution in F for the equation $\Delta u(x) = P(x)u(x)$, where $P \geq 0$ is continuous and P is different from 0. Let $\{F_n\}_{n \geq 0}$ be an exhaustion of F such that each F_n has a compact analytic curve Γ_n as its relative boundary. Let v be a positive solution of $\Delta u(x) = P(x)u(x)$ on $F \setminus \bar{F}_0$ such that

$$0 < \iint_{F \setminus \bar{F}_0} v(z) P(z) d\sigma < \infty,$$

$v = 0$ on Γ_0 . Then Ozawa [7, p. 15] proves that $\lim_{n \rightarrow \infty} v_n$ exists and is not ∞ , where v_n is a harmonic function on $F_n \setminus \bar{F}_0$ such that $v_n = v$ on $\Gamma_n \cup \Gamma_0$.

Lemma 5.1 *Let X be a locally finite infinite network with a symmetric Laplace operator Δ . Let $v \geq 0$ be defined on X such that $v(e) = 0$, $\Delta v(x) \geq 0$ if $x \neq e$, and $\sum_x \Delta v(x) < \infty$. Then there exists a function h on X such that $h \geq v$, $h(e) = 0$ and $\Delta h(x) = 0$ if $x \neq e$.*

Proof We consider the two cases of X being a parabolic network or hyperbolic separately.

Case 1. X parabolic: Let $s = -v$. Then s is Δ -superharmonic on X (note $\Delta v(e) \geq 0$). Since $\sum_x \Delta s(x)$ is finite, then s has a Δ -harmonic minorant outside a finite set [1, Theorem 3.5.2]. Hence $v(x)$ has a harmonic majorant $\xi(x)$ outside a finite set. By [1, Lemma 3.4.2], there exists a function $\eta(x)$ on X such that $\Delta \eta(x) = 0$ if $x \neq e$ and $\xi(x) - \eta(x)$ is bounded outside a finite set in X . Hence, for some α , $\alpha + \eta(x) \geq v(x)$ on X and $\Delta[\alpha + \eta(x)] = 0$ if $x \neq e$. Let $h(x) = \eta(x) - \eta(e)$. Then $h(x) - v(x)$ is lower bounded on $X \setminus e$ and $h(e) - v(e) = 0$. Hence by the minimum principle [1, Theorem 3.4.1], the lower bounded function $h - v$ on X which is Δ -superharmonic at each vertex $x \neq e$ satisfies the condition $h - v \geq 0$ on X . Thus $h \geq v$ on X , $\Delta h(x) = 0$ if $x \neq e$ and $h(e) = 0$.

Case 2. X hyperbolic: As in the case of a parabolic network, the fact that s is Δ -superharmonic and $\sum_x \Delta s(x)$ is finite implies that s has a Δ -harmonic minorant outside a finite set in X . Then by [1, Corollary 3.2.7], we conclude that s has a Δ -harmonic minorant in X . That is, v has a Δ -harmonic majorant H on X . Choose now

an exhaustion of X by finite sets E_m such that $X = \bigcup_m E_m$, $e \in \dot{E}_1$ and $E_m \subset \dot{E}_{m+1}$ for $m \geq 1$. Let v_m be the Dirichlet solution in E_m with boundary values v on ∂E_m and 0 at e . Then v_m is increasing in m and $v_m \leq H$. Hence $h = \lim_{m \rightarrow \infty} v_m$ exists, $\Delta h(x) = 0$ if $x \neq e$, $h \geq v \geq 0$ and $h(e) = 0$. $\square$

Proposition 5.2 *Let X be a locally finite infinite network with a symmetric Laplace operator Δ . Let $v \in {}'\mathcal{H}_e^+$ be a function such that $\sum_x \Delta v(x)$ is finite. Then $v \in \Pi'$.*

Proof For, $\Delta v(x) \geq 0$ for each $x \neq e$ since $v \in {}'\mathcal{H}_e^+$. Since $v(e) = 0$ and $\sum_x \Delta v(x)$ is finite by assumption, by the above lemma, there exists h on X such that $h \geq v$, $h(e) = 0$ and $\Delta h(x) = 0$ if $x \neq e$. Hence Sv exists, that is $v \in \Pi'$. $\square$

Let $v \in \mathcal{H}_e^+$. Then v is Δ' -superharmonic at each vertex $x \neq e$. Let Tv denote the greatest Δ' -harmonic minorant of v in $X \setminus e$. Take $Tv(e) = 0$. Thus, for every $v \in \mathcal{H}_e^+$, $Tv \in {}'\mathcal{H}_e^+$. Just as we have defined $\mathcal{H}_e^+$, we can define $\mathcal{S}_e^+$ as the class of functions $s \geq 0$ on X such that $s(e) = 0$ and $\Delta s(x) \leq 0$ for every $x \neq e$. Let $\mathcal{G}_e$ be the subclass of $\mathcal{S}_e^+$ consisting of $s \in \mathcal{S}_e^+$ such that if $h \in \mathcal{H}_e^+$ and $h \leq s$ on X , then $h = 0$. Suppose p is a Δ -potential on X and let R_p^e stand for the balayage of p on e , that is $R_p^e = \inf_v v$, where $v > 0$ is Δ -superharmonic and $v \geq p$ on e . Then $p - R_p^e \in \mathcal{G}_e$. For if $h \in \mathcal{H}_e^+$ and $h \leq p - R_p^e$, then $h(x)$ with the value $h(e) = 0$ is a Δ -subharmonic function on X majorized by the Δ -potential p on X and hence $h = 0$. We define similarly the classes ${}'\mathcal{S}_e^+$ and ${}'\mathcal{G}_e$. Note that $\mathcal{G}_e \subset {}'\mathcal{G}_e$. For, if $p \in \mathcal{G}_e$, then $p \in {}'\mathcal{S}_e^+$. Moreover, if $h \in {}'\mathcal{H}_e^+$ and $h \leq p$, then $\Delta h(x) \geq 0$ if $x \neq e$. Hence by the Perron method, there exists $H \in \mathcal{H}_e^+$ such that $h \leq H \leq p$ on X . Since $p \in \mathcal{G}_e$, $H = 0$ so that $h = 0$. Hence $p \in {}'\mathcal{G}_e$. Thus $\mathcal{G}_e \subset {}'\mathcal{G}_e$.

Proposition 5.3 *If $h \in \Pi'$, then $TSh = h$. Hence S is one-one on Π' .*

Proof Note $p = Sh - h \in \mathcal{G}_e$ and hence $Sh - h = p'_1$ for some $p'_1 \in {}'\mathcal{G}_e$. Since $Sh \geq h$ and $h \in {}'\mathcal{H}_e^+$, $T(Sh) \geq h$ and $Sh - TSh = p'_2$ for some $p'_2 \in {}'\mathcal{G}_e$. Let $u = h - T(Sh)$ in X with $u(e) = 0$. Then $\Delta'|u|(x) \geq 0$ for every $x \neq e$; also since $|u| \leq |h - Sh| + |Sh - T(Sh)| = p'_1 + p'_2 \in {}'\mathcal{G}_e$, then $|u| = 0$ and hence $h = T(Sh)$. If $h_1, h_2 \in \Pi'$ and $Sh_1 = Sh_2$, then $h_1 = T(Sh_1) = T(Sh_2) = h_2$. $\square$

Corollary 5.4 *T is one-one on Π .*

Proof Let $\varphi_1, \varphi_2 \in \Pi$ such that $T\varphi_1 = T\varphi_2$. Since $\Pi = S(\Pi')$, there exist $\psi_1, \psi_2 \in \Pi'$ such that $S\psi_1 = \varphi_1$ and $S\psi_2 = \varphi_2$. Consequently, $\psi_1 = T(S\psi_1) = T\varphi_1 = T\varphi_2 = T(S\psi_2) = \psi_2$. This implies that $\varphi_1 = S\psi_1 = S\psi_2 = \varphi_2$, that is T is one-one on Π . $\square$

Note 5.5 T need not be one-one on $\mathcal{H}_e^+$. For, suppose X is Δ' -parahyperbolic. Let R_1^e stand as usual for the Δ -balayage of 1 on e . Then R_1^e is a Δ -potential on X such that $\Delta R_1^e(x) = 0$ if $x \neq e$ and $R_1^e(e) = 1$. Hence $h = 1 - R_1^e \in \mathcal{H}_e^+$ and $h \neq 0$. Now, $Th(x)$ extended by 0 at e is a bounded Δ' -subharmonic function on X . Then $Th = 0$ since 1 is a Δ' -potential on X . Hence T is not one-one.

If T is one-one on $\mathcal{H}_e^+$, then $ST\varphi = \varphi$ for every $\varphi \in \mathcal{H}_e^+$. For, $T\varphi \leq \varphi$ so that $S(T\varphi) \leq \varphi$. Suppose $ST\varphi = \psi$. Then $T\psi = T(ST\varphi) = TS(T\varphi) = T\varphi$. Hence the assumption that T is one-one implies that $\psi = \varphi$. That is, $ST\varphi = \varphi$. Consequently, in this case, $\mathcal{H}_e^+ = \Pi$. For, if $\varphi \in \mathcal{H}_e^+$, then $\varphi = S(T\varphi) \in \Pi$.

Proposition 5.6 *X is Δ' -parahyperbolic if and only if $Tu = 0$ for every bounded $u \in \mathcal{H}_e^+$.*

Proof Let X be Δ' -parahyperbolic. Suppose $u \in \mathcal{H}_e^+$ is bounded. Then $Tu \geq 0$ is bounded and in Π' ; with the value $Tu(e) = 0$, Tu becomes a bounded Δ' -subharmonic function on X . Since X is Δ' -parahyperbolic by assumption, 1 is a Δ' -potential on X and hence $Tu = 0$. Conversely, suppose $Tu = 0$ for every bounded $u \in \mathcal{H}_e^+$. Then X is Δ' -parahyperbolic. For, otherwise there exists a bounded Δ' -harmonic function $v > 0$ on X . Since R_v^e is a Δ' -potential on X , $b = v - R_v^e$ is bounded and hence in Π' . Then by Proposition 5.3, $TS(b) = b$ which is non-zero, a contradiction to the assumption that $Tu = 0$ for every bounded u in $\mathcal{H}_e^+$. $\square$

Corollary 5.7 *X is not Δ' -parahyperbolic if and only if there is a bounded non-zero element in Π .*

Proof Let $u \in \Pi$ be bounded and non-zero. That is, for some $\varphi \in \Pi'$ we have $S\varphi = u$. Then $Tu = TS\varphi = \varphi \neq 0$. Hence by the above proposition, X is not Δ' -parahyperbolic.

Suppose now X is not Δ' -parahyperbolic. Then there exists a Δ' -harmonic function b on X , $0 < b < 1$. Let $u = b - R_b^e$. Then u is bounded non-zero in $\mathcal{H}_e^+$ and hence in Π' . Since S is one-one Su exists and is bounded non-zero, $Su \in \Pi$. $\square$

Let $\Pi_0 \subset \mathcal{H}_e^+$ be such that $h \in \Pi_0$ if and only if $Th = 0$. Remark that the only common element between Π and Π_0 is 0. For, if $h \in \Pi \cap \Pi_0$, then $h = S\varphi$ for some $\varphi \in \Pi'$. Then $Th = T(S\varphi) = \varphi$. But $h \in \Pi_0$ means that $Th = 0$. Hence $\varphi = 0$ so that $h = 0$.

Proposition 5.8 *Any element $u \in \mathcal{H}_e^+$ is of the form $u = v + h$ where $v \in \Pi$ and $h \in \Pi_0$. This decomposition is unique.*

Proof Since u is in $\mathcal{H}_e^+$, $\Delta'u(x) \leq 0$ if $x \neq e$ and $Tu \leq u$. Since $\Delta Tu(x) \geq 0$ if $x \neq e$, and $u \in \mathcal{H}_e^+$, $Tu \leq S(Tu) \leq u$. Let $h = u - STu$. Then $Tu = Th + T(STu) = Th + Tu$, so that $Th = 0$; that is $h \in \Pi_0$. Write $v = STu$. Then $v \in \Pi$ since $Tu \in \Pi'$. Thus, $u = v + h$ where $v \in \Pi$ and $h \in \Pi_0$. Suppose $u = v_1 + h_1$ is another such decomposition. Then $Tv + Th = Tv_1 + Th_1$, so that $Tv = Tv_1$. Since the operator T is one-one on Π , Corollary 5.4, $v = v_1$ and the decomposition is unique. $\square$

Corollary 5.9 $T(\Pi) = T(\mathcal{H}_e^+) = \Pi'$.

Proof From the above proposition, it is clear that $T(\Pi) = T(\mathcal{H}_e^+)$. Let $\varphi \in \mathcal{H}_e^+$. Then $T\varphi \leq \varphi$. Since $T\varphi(x)$ is Δ -subharmonic and $\varphi(x)$ is Δ -harmonic at every vertex $x \neq e$, we have $T\varphi \leq S(T\varphi) \leq \varphi$. This means that $T\varphi \in \Pi'$. Hence $T(\mathcal{H}_e^+) \subset \Pi'$. Let $\psi \in \Pi'$. Then $S\psi \in \Pi$ and $T(S\psi) = \psi$. Hence $\Pi' \subset T(\Pi)$. Thus $T(\mathcal{H}_e^+) \subset \Pi' \subset T(\Pi)$ which proves the corollary. $\square$

Note 5.10 $T: \Pi \rightarrow \Pi'$ is a one-one and onto map with its inverse $S: \Pi' \rightarrow \Pi$. For, if $\varphi \in \Pi$, then $\varphi = Su$ for some $u \in \Pi'$. Hence $ST\varphi = S(TSu) = Su = \varphi$. The other side $TSh = h$ for every $h \in \Pi'$ is already proved, see Proposition 5.3.

Let $\# \Pi'$ denote the cardinality of the normalized minimal harmonic functions in Π' , that is $u \in \Pi'$ is minimal if $\varphi \in \Pi'$ and $\varphi \leq u$ then $\varphi = \lambda u$. Similarly, $\# \Pi$ is defined.

Proposition 5.11 $\# \Pi' = \# \Pi$.

Proof Let u be minimal in Π' . Let $\varphi \in \Pi$ be such that $\varphi \leq Su$. Then $T\varphi \leq T(Su) = u$. Since u is minimal, $T\varphi = \alpha u$. Since $\varphi = Sf$ for some $f \in \Pi'$, $ST\varphi = S(TSf) = Sf = \varphi$. Consequently, $\varphi = S(T\varphi) = S(\alpha u) = \alpha(Su)$. That is Su is minimal in Π . Hence $\# \Pi' \leq \# \Pi$. Similarly, let $v \in \Pi$ be a minimal function. Then $Tv \in \Pi'$. Let $\psi \in \Pi'$ and $\psi \leq Tv$. Hence $S\psi \leq STv = v$. Since v is minimal, $S\psi = \lambda v$ so that $\psi = TS\psi = \lambda Tv$; it shows that Tv is minimal in Π' and $\# \Pi \leq \# \Pi'$. This proves the proposition. $\square$

6 Assumption

At the end of Sect. 4 it was remarked that if X is a Δ -hyperbolic network and if Δ' is subordinate to Δ , then every Δ' -potential with finite Δ' -harmonic support is dominated by a Δ -potential. As a generalization, we make the following assumption (only in this section).

Assumption Any Δ' -potential in $'\mathcal{G}_e$ is dominated by a Δ -potential in $\mathcal{G}_e$.

The results in this section are obtained only by making use of Assumption.

Proposition 6.1 If $\varphi \in \mathcal{H}_e^+$, then $ST\varphi = \varphi$ and consequently T is one-one on $\mathcal{H}_e^+$.

Proof When $\varphi \in \mathcal{H}_e^+$, $\varphi - T\varphi = p' \in '\mathcal{G}_e$. Since $T\varphi \leq \varphi$, $ST\varphi \in \Pi$ and we have $T\varphi \leq ST\varphi \leq \varphi$ and $ST\varphi - T\varphi = p \in \mathcal{G}_e$. Hence $0 \leq \varphi - ST\varphi = p' - p \leq p'$. By Assumption, p' is dominated by some $p_0 \in \mathcal{G}_e$. Hence the non-negative Δ -harmonic function in $\mathcal{H}_e^+$ is majorized by the potential p_0 in $\mathcal{G}_e$. This implies that $\varphi = ST\varphi$. To show T is one-one, suppose $T\varphi_1 = T\varphi_2$ for $\varphi_1, \varphi_2 \in \mathcal{H}_e^+$. Then $\varphi_1 = S(T\varphi_1) = S(T\varphi_2) = \varphi_2$. $\square$

Corollary 6.2 Let $\varphi \in \mathcal{H}_e^+$ be majorized by some $v \in \Pi$. Then $\varphi \in \Pi$.

Proof Since $v = Su$ for some $u \in \Pi'$, $\varphi \leq Su$ so that $T\varphi \leq T(Su) = u$. Hence $T\varphi \in \Pi'$. This implies, by the above proposition, $\varphi = ST\varphi \in \Pi$. $\square$

Corollary 6.3 If $v \in \mathcal{H}_e^+$ is minimal, then $Tv \in \Pi'$ is minimal in $'\mathcal{H}_e^+$. If $u \in \Pi'$ is minimal, then Su is minimal in $\mathcal{H}_e^+$.

Proof Let $v \in \mathcal{H}_e^+$ be minimal. Then Tv is in Π' . Let $h \in '\mathcal{H}_e^+$ and $h \leq Tv$. Then $h \in \Pi'$ and $Sh \leq S(Tv) = v$. Since v is minimal in Π , then $Sh = \lambda v$ so that $h = T(Sh) = \lambda Tv$. That is, Tv is minimal.

On the other side, let $u \in \Pi'$ be minimal. Let $f \in \mathcal{H}_e^+$ and $f \leq Su$. Then $Tf \leq T(Su) = u$ so that $Tf = \alpha u$ which implies that $f = S(Tf) = \alpha Su$. Hence Su is minimal. $\square$

Corollary 6.4 $\Pi = \mathcal{H}_e^+$. Moreover, $\# \Pi' = \# \mathcal{H}_e^+ \leq \# '\mathcal{H}_e^+ = \# \mathcal{H}^+$.

Proof $\mathcal{H}_e^+ = \Pi + \Pi_0$, see Proposition 5.8. Since T is one-one, Π_0 consists of the 0 element only. Hence $\mathcal{H}_e^+ = \Pi$. Since T is one-one on $\mathcal{H}_e^+$, and $T(\mathcal{H}_e^+) = \Pi'$, Corollary 5.9, $\#\Pi' = \#\mathcal{H}_e^+$. Since $\Pi' \subset {}'\mathcal{H}_e^+$, we have $\#\Pi' \leq \#{}'\mathcal{H}_e^+$. For, let $u \in \Pi'$ be minimal. Then to show that u is minimal in ${}'\mathcal{H}_e^+$, let $\varphi \in {}'\mathcal{H}_e^+$ and $\varphi \leq u$. Then $S\varphi$ exists and Su is minimal by Corollary 6.3. Since $S\varphi \leq Su$, $S\varphi = \lambda(Su)$ which implies $\varphi = \lambda u$. That is u is minimal in ${}'\mathcal{H}_e^+$. Hence $\#\Pi' \leq \#{}'\mathcal{H}_e^+$. Since $\#\mathcal{H}_e^+ = \#{}'\mathcal{H}_e^+$ [1, Theorem 3.2.10], the corollary follows. $\square$

Proposition 6.5 X is Δ' -parahyperbolic if and only if X is Δ -parabolic.

Proof ($\Rightarrow$) Let X be Δ' -parahyperbolic. Suppose X is Δ -hyperbolic. Then $\varphi = 1 - R_1^e \neq 0$ and $\varphi \in \mathcal{H}_e^+$. Then $T\varphi \in {}'\mathcal{H}_e^+$ and consequently $T\varphi$ is Δ' -subharmonic on X . Since $T\varphi \leq \varphi$, $T\varphi \geq 0$ is a Δ' -subharmonic function on X , bounded above by the Δ' -potential 1 on X . Hence $T\varphi = 0$ which implies that $\varphi = S(T\varphi) = 0$, a contradiction. So X is Δ -parabolic.

($\Leftarrow$) Conversely, if X is Δ -parabolic, then X is always Δ' -parahyperbolic. $\square$

7 Quasi-bounded Δ - and Δ' -harmonic functions in X

In this section, we go back to considering the entire network X instead of $X \setminus e$. Let $\mathcal{H}^+$ and ${}'\mathcal{H}^+$ stand for the class of non-negative Δ - and Δ' -harmonic functions, respectively. Now $T: \mathcal{H}^+ \rightarrow {}'\mathcal{H}^+$ is extended in an obvious manner: if $v \in \mathcal{H}^+$, then $Tv \in {}'\mathcal{H}^+$ stands for the greatest Δ' -harmonic minorant of the Δ' -superharmonic function v in X . Similarly, $S: {}'\mathcal{H}^+ \rightarrow \mathcal{H}^+$ is defined as: if $u \in {}'\mathcal{H}^+$, then Su is the least Δ -harmonic majorant of the Δ -subharmonic function u on X , if it exists. As usual $v \in \mathcal{H}^+$ is said to be Δ -quasi-bounded if $v = \sup_{\varphi \in \mathfrak{B}} \varphi$, where φ is in the class $\mathfrak{B}$ of all bounded non-negative Δ -harmonic functions on X such that $\varphi \leq v$. Two Δ -harmonic functions $v_1, v_2 \in \mathcal{H}^+$ are said to be *mutually Δ -singular* if the greatest Δ -harmonic minorant of $\inf(v_1, v_2)$ is 0; and a Δ -harmonic function $v \in \mathcal{H}^+$ is *Δ -singular* if $\inf(1, v)$ is a Δ -potential. We use similarly the terms Δ' -quasi-bounded and Δ' -singular.

Then we prove as in the case of a Riemann surface, see Parreau [8], that any $v \in \mathcal{H}^+$ can be uniquely represented as $v = v_B + v_S$, where v_B is Δ -quasi-bounded and v_S is Δ -singular.

If s is a Δ -superharmonic function in X , let us write $\widehat{s}$ to denote the greatest Δ -harmonic minorant of s , if it exists. The following Propositions 7.2, 7.4 and Corollary 7.3 have been used in Riemann surfaces by Parreau.

Note 7.1 If $s_n = \inf(v, n)$ then s_n is a non-negative increasing sequence of bounded superharmonic functions. Hence the greatest harmonic minorants $\widehat{s}_n$ are non-negative increasing harmonic functions such that $\widehat{s}_n \leq v$. Then $\lim_{n \rightarrow \infty} \widehat{s}_n$ is harmonic, by Theorem 3.2.

Proposition 7.2 If $v \in \mathcal{H}^+$ has the decomposition $v = v_B + v_S$, then $v_B = \lim_{n \rightarrow \infty} \inf(v, n)$.

Proof $v_B = \sup_{b \in \mathfrak{S}} b$, where $\mathfrak{S}$ is the family of bounded non-negative Δ -harmonic functions $b \leq v$. Remark that each $b \in \mathfrak{S}$ satisfies the condition $b \leq \widehat{\inf(v, n)}$ for some n . Hence

$$v_B \leq \sup_n \widehat{\inf(v, n)} = \lim_{n \rightarrow \infty} \widehat{\inf(v, n)}.$$

But $\widehat{\inf(v, n)} \in \mathfrak{S}$ for each n and hence $\lim_{n \rightarrow \infty} \widehat{\inf(v, n)} \leq v_B$. Thus, we have the relation $v_B = \lim_{n \rightarrow \infty} \widehat{\inf(v, n)}$. $\square$

Corollary 7.3 *A function v is Δ -quasi-bounded in $\mathcal{H}^+$ if and only if $v = \lim_{n \rightarrow \infty} \widehat{\inf(v, n)}$.*

Proof If v is Δ -quasi-bounded

$$v = v_B = \lim_{n \rightarrow \infty} \widehat{\inf(v, n)} \leq \lim_{n \rightarrow \infty} \inf(v, n) \leq v.$$

Hence $v = \lim_{n \rightarrow \infty} \widehat{\inf(v, n)}$. On the other hand, suppose

$$v = \lim_{n \rightarrow \infty} \widehat{\inf(v, n)}.$$

Let $\mathfrak{S}$ be the family of non-negative bounded Δ -harmonic functions $b \leq v$. Then $\widehat{\inf(v, n)} \in \mathfrak{S}$ for each n and hence we have $\sup_{b \in \mathfrak{S}} b \geq \lim_{n \rightarrow \infty} \widehat{\inf(v, n)} = v$. But $\sup_{b \in \mathfrak{S}} b \leq v$ so that $v = \sup_{b \in \mathfrak{S}} b$ is Δ -quasi-bounded. $\square$

Proposition 7.4 *If $\{b_m\}$ is an increasing sequence of bounded Δ -harmonic functions such that $\sup_m b_m = v \in \mathcal{H}^+$, then v is Δ -quasi-bounded.*

Proof For any m , there is n_0 such that

$$b_m \leq \widehat{\inf(v, n_0)} \leq \lim_{n \rightarrow \infty} \widehat{\inf(v, n)}.$$

Consequently, $v = \sup_m b_m \leq \lim_{n \rightarrow \infty} \widehat{\inf(v, n)} \leq v$ so that $v = \lim_{n \rightarrow \infty} \widehat{\inf(v, n)}$. Then, by Corollary 7.3, v is Δ -quasi-bounded. $\square$

Proposition 7.5 *X is Δ' -parahyperbolic if and only if every Δ' -harmonic function $u \in {}'\mathcal{H}^+$ is Δ' -singular.*

Proof If X is Δ' -parahyperbolic, then every bounded Δ' -harmonic function is 0 and hence $\inf(1, u)$ is a Δ' -potential if $u \in {}'\mathcal{H}^+$, that is u is Δ' -singular.

Conversely, suppose every $u \in {}'\mathcal{H}^+$ is singular. Then the only bounded Δ' -harmonic function on X is 0. Hence X is Δ' -parahyperbolic. $\square$

Proposition 7.6 *If $u \in \Pi'$ is Δ' -quasi-bounded, then Su is Δ -quasi-bounded.*

Proof By Corollary 7.3, $u = \lim_{n \rightarrow \infty} u_n$ where $\{u_n\}$ is an increasing sequence of bounded Δ' -harmonic functions. Note that each $u_n \in \Pi'$ and Su_n is an increasing sequence of bounded Δ -harmonic functions. Also, $\lim_{n \rightarrow \infty} Su_n \leq Su$, hence $\lim_{n \rightarrow \infty} Su_n \in \mathcal{H}^+$. Now, $u_n \leq Su_n \leq \lim_{n \rightarrow \infty} Su_n$ so that $u \leq \lim_{n \rightarrow \infty} Su_n$. Since Su is the least Δ -harmonic majorant of u , we have $u \leq Su \leq \lim_{n \rightarrow \infty} Su_n$. Hence $Su = \lim_{n \rightarrow \infty} Su_n$. Then, see Proposition 7.4, Su is Δ -quasi-bounded. $\square$

Proposition 7.7 *If $u \in \Pi'$ is Δ' -singular then Su is Δ -singular, provided we assume that $Tb = 0$ implies that $b = 0$ for any bounded b in $\mathcal{H}^+$.*

Proof Let $h \in \mathcal{H}^+$ be bounded and $h \leq Su$. Then $Th \leq u$. Since Th is a bounded Δ' -harmonic function majorized by the singular Δ' -harmonic function u , $Th = 0$; since T is one-one on the set of bounded non-negative Δ -harmonic functions by the assumption, $h = 0$. That is Su is Δ -singular. $\square$

Proposition 7.8 *If $u \in \Pi$ is Δ -singular, then Tu is Δ' -singular.*

Proof Let b be a bounded Δ' -harmonic function such that $b \leq Tu$. Then $Sb \leq STu \leq u$. Since Sb is bounded and u is Δ -singular, $Sb = 0$. Since S is one-one, $b = 0$. That is, Tu is Δ' -singular. $\square$

Proposition 7.9 *If u and v are mutually Δ -singular, then Tu and Tv are mutually Δ' -singular.*

Proof Since $Tu \leq u$ and $Tv \leq v$, $\inf(Tu, Tv) \leq \inf(u, v)$. By hypothesis, $\inf(u, v)$ is a Δ -potential and hence a Δ' -potential. Consequently $\inf(Tu, Tv)$ is a Δ' -potential; that is, Tu and Tv are mutually Δ' -singular. $\square$

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