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STABILITY AND BIFURCATION - MEMORIAL ISSUE DEDICATED TO JORGE SOTOMAYOR

An algebraic characterization of entire polynomial diffeomorphisms

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Abstract
It is shown that a polynomial map $F : \mathbb{R}^n \to \mathbb{R}^n$ with nowhere zero Jacobian determinant is invertible if and only if an explicit auxiliary polynomial system admits only the trivial solution. The main corollary is a concrete invertibility criterion in the Jacobian conjecture. The proof, conceptually related to differential geometry, represents a simple but infrequent application of differential equations to algebra.

1 Introduction

The primary goal of this note is the algebraic characterization of the systems $\mathbf{F}(x) = y$ of n polynomial equations in n real variables for which the solution exists, is unique, and varies differentiably with y.

Equivalently, we formulate algebraic necessary and sufficient conditions in order for the polynomial map $F : \mathbb{R}^n \to \mathbb{R}^n$ underlying the system to admit a differentiable inverse. The last condition requires $J_F(x) = \det DF(x)$ to be nowhere zero, and so we are really searching for an algebraic characterization of invertible polynomial local diffeomorphisms.

A. Bialynicki-Birula and M. Rosenlicht proved in [1] the intriguing result that, just as in the trivial case of linear operators in finite dimensions (i.e. polynomial maps $\mathbb{R}^n \to \mathbb{R}^n$ of degree one), all injective polynomial self-maps of $\mathbb{R}^n$ are surjective. The complex analogue of this result is the celebrated Ax-Grothendieck theorem [2].

To Jorge Sotomayor, in memoriam

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These developments raise the possibility that perhaps other elementary features of linear systems also carry over to systems of polynomials of arbitrary degrees.

With this in mind, and since a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if $\det L \neq 0$, it is natural to ask if, more generally, polynomial maps on $\mathbb{R}^n$ with non-vanishing Jacobian determinants (i.e. polynomial local diffeomorphisms) are invertible. This was answered in the negative by Pinchuk in [3] (see also [4, 5]).

An equivalent statement for the invertibility of L is the finite-dimensional version of the classical "Fredholm alternative": Either $Lx = y$ has a unique solution for every $y \in \mathbb{R}^n$, or the auxiliary $n \times n$ system $Lx = 0$ has a non-trivial solution.

It turns out that, unlike the condition $\det L \neq 0$ for invertibility, the Fredholm alternative does admit a version for *non-singular* arbitrary polynomial maps. The caveat is that the auxiliary system that controls invertibility is now of type $(2n) \times (2n)$:

Theorem 1.1 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map with $|J_F| > 0$. Then either the system $F(x) = y$ has a unique solution for every $y \in \mathbb{R}^n$, or the auxiliary system*

$$\begin{aligned} F(x) - F(y) &= 0 \\ [DF(x)^{-1}]^* x + [DF(y)^{-1}]^* y &= 0 \end{aligned} \tag{1.1}$$

has a non-trivial solution. Otherwise said, the local diffeomorphism F is invertible if and only if the only solution of the above system in $\mathbb{R}^n \times \mathbb{R}^n$ is $x = y = 0$.

Matrix inversion can be computationally onerous. In this regard, the version of Theorem 1.1 given below, obtained by setting $[DF(x)^{-1}]^* x = z = -[DF(y)^{-1} DF(x)]^* y$, may be more useful:

Corollary 1.2 *A polynomial local diffeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if $x = y = z = 0$ is the only solution of the homogeneous polynomial system*

$$\begin{aligned} F(x) - F(y) &= 0 \\ DF(x)^* z - x &= 0 \\ DF(x)^* z + y &= 0. \end{aligned} \tag{1.2}$$

Remark 1.3

(a) Note that (1.1) is equivalent to

$$\begin{aligned} & F(x) - F(y) = 0 \\ & x + [DF(y)^{-1} DF(x)]^* y = 0. \end{aligned} \tag{1.3}$$

It follows from the theorem that this system has a non-trivial solution in all counterexamples to the strong real Jacobian conjecture (for instance, the ones in [3–5]).

(b) Why systems come into play? Naively, the injectivity question can be understood conceptually as a matter of uniqueness in infinitely many problems: one is tasked with showing that for y fixed—and yet arbitrary, $F(x) = F(y)$ has precisely one solution.

Theorem 1.1 introduces a device that reduces the injectivity issue to the examination of uniqueness in only one problem, rather than infinitely many. The trade-off is that one has to work in a space having twice the dimension of the original one.

(c) Unlike sufficiency, to be established in Sect. 2, the necessity half of the theorem is utterly trivial: injectivity implies that (1.3) reduces to $x-y=0,\ x+y=0$.

The main application of Theorem 1.1 is to the study of the Jacobian conjecture in algebraic geometry [6–8]. The latter can be formulated, equivalently, over $\mathbb{C}$ or $\mathbb{R}$:

(JC) $\forall n \in \mathbb{N}$, if $G : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map with $\det DG = 1$, then G is invertible.

(RJC) $\forall n \in \mathbb{N}$, if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial map with $\det DF = 1$, then F is invertible.

To be clear, the complex Jacobian conjecture in dimension n implies the real version in dimension n (by complexification), whereas the real version in dimension $2n$ implies the complex one in dimension n (by realification).

We also point out that (RJC) fails if $\det DF = 1$ is replaced by $\det DF \geq 1$ [4].

The conjecture below is about Systems:

Conjecture (SJC) $\forall n \in \mathbb{N}$, *if* $F : \mathbb{R}^n \to \mathbb{R}^n$ *is a polynomial map with* $\det DF = 1$, *then* $(x, y) = (0, 0)$ *is the only solution in* $\mathbb{R}^n \times \mathbb{R}^n$ *of the polynomial system* (1.1).

From Theorem 1.1 and (JC) $\Longleftrightarrow$ (RJC), one obtains:

Theorem 1.4 *The Jacobian conjecture holds if and only if* (*SJC*) *is true*.

(SJC) deals with a fairly explicit object, namely a polynomial system. This stands in sharp contrast with the abstract criteria in [7, Thm. (2.1)], stating that any of the assertions below is equivalent to version (JC) of the Jacobian conjecture:

(i) $\mathbb{C}(X)$ is Galois over $\mathbb{C}(F)$.
(ii) $\mathbb{C}[X]$ is a projective $\mathbb{C}[F]$-module.
(iii) The integral closure of $\mathbb{C}[F]$ in $\mathbb{C}[X]$ is unramified over $\mathbb{C}[F]$.

The idea for the proof of Theorem 1.1 comes from differential geometry. It is often the case that the solution of a geometric minimization problem yields a canonical representative of some class. For instance, there is a unique closed geodesic that minimizes length in each non-trivial free homotopy class of loops in a compact Riemannian manifold of negative curvature. An illustrative example is the “waist" of a truncated catenoid.

In our setting, the strategy is a simple one. Given a non-injective local diffeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$, we look for an “optimal" pair (x, y) off the diagonal D of $\mathbb{R}^n \times \mathbb{R}^n$ that realizes lack of injectivity. Specifically, we take the infimum of the quantity $|x|^2 + |y|^2$ over those points $(x, y) \in (\mathbb{R}^n \times \mathbb{R}^n) - D$ for which $F(x) = F(y)$, and then analyze a minimizer whose existence follows from general compactness arguments and the inverse function theorem.

This note is similar in the spirit of [9], a work that is also informed by geometry and ODE's, two subjects to which Jorge Sotomayor made significant contributions. Both papers extend to the non-linear realm particular aspects of linear algebra.

2 A variational approach to injectivity

As injective polynomial maps $\mathbb{R}^n \to \mathbb{R}^n$ are surjective [1], Theorem 1.1 follows from

Theorem 2.1 *A local diffeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$ of class C^1 is injective if and only if $x = y = 0$ is the only solution of*

$$\begin{aligned} F(x) - F(y) &= 0 \\ [DF(x)^{-1}]^* x + [DF(y)^{-1}]^* y &= 0. \end{aligned} \tag{2.1}$$

It was observed already that if F is injective the system Σ_F above reduces to $x - y = 0,\ 2[DF(x)^{-1}]^* x = 0$. For the converse, assume by contradiction that Σ_F has only the null solution but F is not injective. Select $x_0, y_0 \in \mathbb{R}^n$, $x_0 \neq y_0$, such that $F(x_0) = F(y_0)$. Let $J \subset (0, \infty)$ be defined by the condition that a positive number r belongs to J if and only if there exist $x, y \in \mathbb{R}^n$ such that $x \neq y,\ F(x) = F(y),\ r = \sqrt{|x|^2 + |y|^2}$. In particular, $r_0 = \sqrt{|x_0|^2 + |y_0|^2} \in J$.

We let α be the infimum of the (supposedly non-empty) set J and proceed to show that neither of the alternatives $\alpha = 0,\ \alpha > 0$, can occur. Choose a sequence $r_j \in J$, $r_j \le r_0$, with $\lim_{j\to\infty} r_j = \alpha$, with corresponding $x_j, y_j \in \mathbb{R}^n$ such that

$$x_j \neq y_j, \quad F(x_j) = F(y_j), \quad r_j = \sqrt{|x_j|^2 + |y_j|^2} \le r_0. \tag{2.2}$$

Passing to subsequences, if necessary, we may assume that (x_j) and (y_j), which are bounded in view of (2.2), converge to points $\underline{x}, \underline{y}$ in $\mathbb{R}^n$. By continuity, (2.2) implies

$$F(\underline{x}) = F(\underline{y}),\ \alpha = \sqrt{|\underline{x}|^2 + |\underline{y}|^2}. \tag{2.3}$$

Next, we will see that the remaining piece of information from (2.2) is also preserved in the limit, namely that $\underline{x} \neq \underline{y}$. If not, since F is non-singular, by the inverse function theorem we can choose a neighborhood U of the common value $\underline{x} = \underline{y}$ for which $F|U$ is injective. As $\lim x_j = \underline{x} = \underline{y} = \lim y_j$, we can pick j sufficiently large so that the distinct points $x_j,\ y_j$ belong to $\bar{U}$. But since $F|U$ is injective one must have $F(x_j) \neq F(x_j)$, a contradiction to (2.2).

Thus, one obtains an enhanced version of (2.3):

$$\underline{x} \neq \underline{y},\ F(\underline{x}) = F(\underline{y}), \quad \alpha = \sqrt{|\underline{x}|^2 + |\underline{y}|^2}. \tag{2.4}$$

An immediate consequence of (2.4) is that the alternative $\alpha = 0$ cannot occur. Assume therefore that $\alpha > 0$ and consider

$$\begin{aligned}&\xi = (\xi_1, \xi_2) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n,\\ &\xi_1(x, y) = -\big(x + [DF(y)^{-1}DF(x)]^* y\big),\\ &\xi_2(x, y) = DF(y)^{-1}DF(x)\xi_1(x, y).\end{aligned} \tag{2.5}$$

As the map F is of class C^1, ξ can be regarded as a continuous vector field on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, and as such it has local trajectories by Peano's theorem on existence of solutions of systems of ordinary differential equations with continuous coefficients. Let therefore $\phi : [0, \epsilon_1) \to \mathbb{R}^n \times \mathbb{R}^n$, $\phi(t) = (x(t), y(t))$, be a solution, for some $\epsilon_1 > 0$, of the problem

$$\begin{aligned}&\frac{dx}{dt} = \xi_1(x, y),\\ &x(0) = \underline{x}\\ &\frac{dy}{dt} = \xi_2(x, y)\\ &y(0) = \underline{y}.\end{aligned} \tag{2.6}$$

One has $\frac{d}{dt}\big(F(x(t)) - F(y(t))\big) = DF(x)\frac{dx}{dt} - DF(y)\frac{dy}{dt} = DF(x)\xi_1 - DF(y)\xi_2 = 0$, and so, for $t \in (0, \epsilon_1)$,

$$F(x(t)) - F(y(t)) = F(x(0)) - F(y(0)) = F(\underline{x}) - F(\underline{y}) = 0. \tag{2.7}$$

From (2.5) and (2.6),

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}(|x|^2 + |y|^2) = \langle x, \frac{dx}{dt}\rangle + \langle y, \frac{dy}{dt}\rangle =\\ &\langle x, \frac{dx}{dt}\rangle + \langle y, DF(y)^{-1}DF(x)\frac{dx}{dt}\rangle =\\ &\langle x + [DF(y)^{-1}DF(x)]^* y, \frac{dx}{dt}\rangle =\\ &-|\xi_1(x, y)|^2 \le 0.\end{aligned} \tag{2.8}$$

We claim that the previous inequality is strict at $(\underline{x}, \underline{y})$, i.e.

$$\frac{d}{dt}(|x|^2 + |y|^2)\big|_{t=0} < 0. \tag{2.9}$$

Indeed, by (2.8) if (2.9) fails one must have $\xi_1(\underline{x}, \underline{y}) = 0$ and, by (2.5),

$$\underline{x} + [DF(\underline{y})^{-1}DF(\underline{x})]^* \underline{y} = 0. \tag{2.10}$$

Multiplying (2.10) by $[DF(x)^{-1}]^*$ one obtains $[DF(x)^{-1}]^* x + [DF(y)^{-1}]^* y = 0$. From the second relation in (2.4) one then sees that $(\underline{x}, \underline{y})$ is a solution of (1.1) and so, by the main hypothesis of the theorem, $\underline{x} = \underline{y} = 0$. But then (2.4) implies $\alpha = 0$, a case that had been discarded already, and therefore (2.9) must hold.

Next, using (2.4) and (2.9) one can find $\epsilon_2 \in (0, \epsilon_1)$ such that, for all $t \in (0, \epsilon_2)$,

$$|x(t)|^2 + |y(t)|^2 < |x(0)|^2 + |y(0)|^2 = |\underline{x}|^2 + |\underline{y}|^2 = \alpha^2. \tag{2.11}$$

It follows from (2.7), the first relation in (2.4), and the continuity of ϕ, that if $\epsilon_3 \in (0, \epsilon_2)$ is sufficiently small then $F(x(t)) = F(y(t))$ and $x(t) \neq y(t)$ for all $t \in (0, \epsilon_3)$.

The last relations imply, for $t \in (0, \epsilon_3)$, and after taking square roots in (2.11), that

$$\sqrt{|x(t)|^2 + |y(t)|^2} \in J$$
$$\sqrt{|x(t)|^2 + |y(t)|^2} < \alpha,$$

contradicting the definition of α. It follows that $J = \emptyset$, $x_0 = y_0$, and so F is injective.

3 An explicit algebraic criterion in (JC)

In this section we show that Theorem 1.1 has a counterpart for local biholomorphisms (see Remark 1.3 a) and Theorems 3.1, 3.2 below), leading to a direct criterion for (JC) to hold, without having to go through its real version (RJC). Here, A^* stands for the adjoint of the complex matrix A, i.e. its conjugate-transpose.

Theorem 3.1 *A local biholomorphism $F : \mathbb{C}^n \to \mathbb{C}^n$ is injective if and only if $x = y = 0$ is the only solution in $\mathbb{C}^n$ of the system*

$$\begin{aligned} &F(x) - F(y) = 0 \\ &x + [DF(y)^{-1}DF(x)]^* y = 0. \end{aligned} \tag{3.1}$$

From [1] one obtains

Theorem 3.2 *A polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$, $\det DF = 1$, is invertible if and only if $x = y = 0$ is the only solution in $\mathbb{C}^n$ of system* (3.1).

Remark 3.3 Unlike the real case, in Theorem 3.2 the system is no longer polynomial in the (complex) coordinates of x and y, since it clearly involves their conjugates as well. Notice that Corollary 1.2 also admits a complex version.

The proof of Theorem 3.1 follows closely that of Theorem 1.1, but for the benefit of the reader primarily interested in the version (JC) of the Jacobian conjecture - and not on the invertibility of local diffeomorphisms - we go over the arguments again, in a slightly modified way.

If F is injective, (3.1) reduces to $x - y = 0, x + y = 0$ and the conclusion follows. Assume now, by contradiction, that the only solution of (3.1) is the null one but F is not injective, say $F(x_0) = F(y_0)$, $x_0 \neq y_0$.

Choose $R > \sqrt{|x_0|^2 + |y_0|^2}$, let B_R be the closed ball in $\mathbb{C}^n \times \mathbb{C}^n$ with center (0, 0) and radius R, and take D to be the diagonal of $\mathbb{C}^n \times \mathbb{C}^n$. Consider

$$\mathcal{C} = \{(x, y) \in B_R - D \mid F(x) = F(y)\},$$

observing that this set is non-empty since it contains (x_0, y_0).

Next, we show that $\mathcal{C}$ is closed, hence compact. To this end, take a sequence (x_n, y_n) in $\mathcal{C}$ that converges in $\mathbb{C}^n \times \mathbb{C}^n$ to (a, b). Evidently, $(a, b) \in B_R$, $F(a) = F(b)$, and so it remains to check that $a \neq b$, If not, the inverse function theorem applied at $a = b$ implies $x_n = y_n$ for sufficiently large n, contradicting $\mathcal{C} \cap D = \emptyset$.

Let $(\underline{x}, \underline{y})$ be a point of absolute minimum for the function $h(x, y) = \sqrt{|x|^2 + |y|^2}$ on $\mathcal{C}$. Observe that $(\underline{x}, \underline{y})$ lies in the *interior* of B_R, since $h(\underline{x}, \underline{y}) \leq h(x_0, y_0) < R = h|\partial B_R$.

Consider the local solutions, in the interior of B_R, of the initial value problem corresponding to (2.6):

$$\begin{aligned}
&\frac{dx}{dt} = -\left(x + [DF(y)^{-1}DF(x)]^* y\right) \\
&x(0) = \underline{x} \\
&\frac{dy}{dt} = -DF(y)^{-1}DF(x)\left(x + [DF(y)^{-1}DF(x)]^* y\right) \\
&y(0) = \underline{y}.
\end{aligned}$$

Since $y' = DF(y)^{-1}DF(x)x'$, the derivative of $F(x(t)) - F(y(t))$ is zero and so

$$F(x(t)) - F(y(t)) = F(x(0)) - F(y(0)) = F(\underline{x}) - F(\underline{y}) = 0.$$

The computation analogous to (2.8), of the variation of h along the local solution, now needs to take real parts into account:

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}(|x|^2 + |y|^2) &= \mathrm{Re}\langle x, \frac{dx}{dt}\rangle + \mathrm{Re}\langle y, \frac{dy}{dt}\rangle \\
&= \mathrm{Re}\langle x, \frac{dx}{dt}\rangle + \mathrm{Re}\langle y, DF(y)^{-1}DF(x)\frac{dx}{dt}\rangle \\
&= \mathrm{Re}\langle x + [DF(y)^{-1}DF(x)]^* y, \frac{dx}{dt}\rangle \\
&= -\left|x + [DF(y)^{-1}DF(x)]^* y\right|^2 \leq 0.
\end{aligned}$$

From this point on the argument proceeds as in Sect. 2. If the above inequality were strict at $t = 0$, for t close to zero the quantity $|x(t)|^2 + |y(t)|^2$ would be strictly smaller than $|\underline{x}|^2 + |\underline{y}|^2$, a contradiction to the fact that $(\underline{x}, \underline{y})$ is a point of global minimum for h.

Hence, the derivative in (2.8) is zero at $t = 0$, leading to $\underline{x} + [DF(\underline{y})^{-1}DF(\underline{x})]^* \underline{y} = 0$. By the main hypothesis of the theorem this implies $(\underline{x}, \underline{y}) = (0, 0)$, contradicting $\mathcal{C} \cap D = \emptyset$. Thus, there is no (x_0, y_0) off the diagonal for which $F(x_0) = F(y_0)$.

4 Remarks on polynomial homeomorphisms

Since we dealt with the issue of differentiable dependence on y of the solutions x of a polynomial system $F(x) = y$, it is natural to look also into continuous dependence.

By the invariance of domain theorem (i.e., a continuous injective map $F : \mathbb{R}^n \to \mathbb{R}^n$ is open, hence $\mathbb{R}^n \xrightarrow{F} F(\mathbb{R}^n)$ is a homeomorphism), existence and uniqueness of solutions of $F(x) = y$ already guarantees continuous dependence. Thus, the main problem in this circle of ideas is to characterize algebraically those polynomial maps $F : \mathbb{R}^n \to \mathbb{R}^n$ that are *homeomorphisms* (instead of *diffeomorphisms*, as it was done in the present note).

Simply put, one can leave topology aside and try to *characterize algebraically the polynomial maps on* $\mathbb{R}^n$ *that are invertible*.

An obvious necessary condition, arising from the fact that a homeomorphism either preserves or reverses orientation, is that the Jacobian determinant should be everywhere non-negative or non-positive, i.e. it does not change sign. Likewise, by the aforementioned invariance of domain theorem, another necessary condition is that F be an open map. In the smooth (resp. polynomial) case, openness is equivalent to discreteness (resp. finiteness) of fibers, plus the requirement that the Jacobian determinant does not change sign [10–12].

It is conceivable that besides these two conditions one needs just a single extra one in order to ensure invertibility, involving the solutions of a suitable polynomial system. However, as matters stand, it is not clear how to proceed because the passage from $J_F > 0$ to $J_F \geq 0$ introduces several technical difficulties in our variational approach.

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