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DIFFERENTIAL EQUATIONS AND DYNAMICAL SYSTEMS. DEDICATED TO GIORGIO FUSCO

On the structure of the infinitesimal generators of scalar one-dimensional semigroups with discrete Lyapunov functionals

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Abstract

Dynamical systems generated by scalar reaction-diffusion equations on an interval enjoy special properties that lead to a very simple structure for the semiflow. Among these properties, the monotone behavior of the number of zeros of the solutions plays an essential role. This discrete Lyapunov functional contains important information on the spectral behavior of the linearization and leads to a Morse-Smale description of the dynamical system. Other systems, like the linear scalar delay differential equations under monotone feedback conditions, possess similar kinds of discrete Lyapunov functions. Here we discuss and characterize classes of linear equations that generate semiflows acting on $C^0[0,1]$ or on $C^1[0,1]$ which admit discrete Lyapunov functions related to the zero number. We show that, if the space is $C^1[0,1]$, the corresponding equations are essentially parabolic partial differential equations. In contrast, if the space is $C^0[0,1]$, the corresponding equations are generalizations of monotone feedback delay differential equations.

Keywords Partial differential equations · Delay differential equations · Infinitesimal generators · Discrete Lyapunov functions

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1 Introduction

A large amount of the research conducted on the theory of nonlinear dynamical systems has addressed the study of semiflows possessing certain types of discrete Lyapunov functionals. This particularity is shared by certain systems generated either by scalar parabolic partial differential equations on an interval or by scalar delay differential equations in one space variable. These systems, or better, their linearizations along the flow, possess a certain integer valued functional (often called a zero number or a lap number) which is a nonincreasing function of time. This feature has shown to be essential for many of the special properties that these systems have in common. Such studies have been conducted by many authors and, for references to the decreasing property of the zero number, see [2, 34–37]. See also [32] for a more recent extension of this result. Regarding the far reaching consequences of this property to the study of the corresponding infinite dimensional dynamical systems, see [1, 3, 4, 7, 9, 15, 25, 27]. Here we also mention the equivalent results for ordinary differential equations obtained in [16, 17] and the result for maps [42].

For references on the general theory of dynamical systems generated by parabolic partial differential equations or by delay differential equations we refer to [22, 23, 45, 46].

In these notes, we address the problem of characterizing classes of linear semigroups acting on $C^0[0,1]$ or on $C^1[0,1]$ (with suitable boundary conditions) which admit discrete Lyapunov functionals related to the zero number. We will see that this characterization depends on the particular discrete functional considered and also on the smoothness of the function space.

In the next Section, we recall the decreasing property of the zero number in the setting of linear parabolic differential equations. Then, we recollect the discrete Lyapunov functions, also derived from the zero number, for scalar delay differential equations under monotone feedback conditions.

We present, in the third Section, our Theorem 3.1 as a main result. It consists of a characterization of the infinitesimal generator of a C_0-semigroup of bounded linear operators in $C^1[0,1]$ with the zero number decreasing property. We show that, in the appropriate setting, this infinitesimal generator corresponds necessarily to a second order differential operator.

In Sect. 4 we prepare to also consider the characterization of the infinitesimal generators of semiflows with different discrete Lyapunov functions $V^{\mp}$ derived from the zero number like in the case of delay differential equations. Our main result in this Section, Theorem 4.1, yields a characterization of generators of semigroups in $C^1[0,1]$ with such discrete Lyapunov functionals. We show that the discrete Lyapunov functions $V^{\mp}$ are not specific of delay differential equations. In fact, in Theorem 4.1, we prove that these discrete Lyapunov functions determine, in the appropriate settings, classes of infinitesimal generators corresponding to second order differential operators under non-separated boundary conditions.

In Sect. 5 we illustrate this result with a degenerate example showing that linear scalar delay equations naturally fit this description.

Finally, in Sect. 6 we consider the characterization of generators of semigroups on $C^0[0,1]$ with discrete Lyapunov functions $V^{\mp}$. Our main result, Theorem 6.1, shows that these semigroups include delay differential equations. We complete Theorem 6.1 with a Remark where we discuss how the result in Theorem 3.1 changes when we relax the smoothness of the function space by replacing $C^1[0,1]$ with $C^0[0,1]$.

The notion of zero or lap number has been also extended and used for the description of the dynamics of fully non linear scalar parabolic equations [29, 30] and to the case where the differential operator is the p-Laplacian [20].

The global dynamics of semiflows that preserve an ordering of the function space is known to have various interesting properties [26, 44, 47]. For the case of equation (2.1) where a complete description of the set of equilibria and the proof of the Morse-Smale property are possible the key point is that the zero number induces a structure of total order in the function space.

2 Discrete Lyapunov functions

We next recall the remarkable decreasing property of the zero number in the setting of linear parabolic differential equations. Loosely speaking, this property can be described by saying that if $t \mapsto u(\cdot, t)$ is a solution of a linear parabolic differential equation with the corresponding boundary conditions, then the number $z(u(\cdot,t))$ of zeros of the function $u(\cdot,t)$ is a nonincreasing function of t. A rigorous description is the following Theorem 2.1 (see [2, 32]), where for simplicity we choose Neumann boundary conditions by taking $C^1_n[0,1] = \left\{\varphi \in C^1[0,1] \,:\, \varphi'(0) = \varphi'(1) = 0\right\}$.

Let

$$u_t = a(x)u_{xx} + b(t,x)u_x + c(t,x)u \;,\; x \in (0,1)\;. \tag{2.1}$$

If the coefficients a, b, c are sufficiently smooth and a is positive, problem (2.1) generates an evolution operator $S(t,t_0) \,:\, C^1_n[0,1] \to C^1_n[0,1], t \geq t_0$, corresponding to the solutions $S(t,t_0)u_0(x) = u(t,x)$ with $u(t_0,x) = u_0(x)$. If, in addition, b, c are independent of t then (2.1) generates a linear C_0-semigroup $\{T(t) = S(t,0)\}_{t\geq 0}$. See, for example, [24, 43].

For $\varphi \in C^1_n[0,1]$ let the *zero number* $z(\varphi) \in \mathbb{N}_0 \cup \{\infty\}$ denote the number of strict sign changes of $x \mapsto \varphi(x)$. The precise definition, appearing in [2] for $\varphi \in C^0[0,1]$, see also [35], is the following. Let

$$\begin{aligned} z(\varphi) = \sup\{k \geq 0 \,:\, \exists\ \{x_i\}_{i=0}^k \subset [0,1] \text{ such that } x_{i-1} < x_i \\ \text{and } \varphi(x_{i-1})\varphi(x_i) < 0\;,\; i = 1,\ldots,k\}\;. \end{aligned} \tag{2.2}$$

If $\varphi \in C^0[0,1]$ satisfies $\varphi(x) \geq 0$ or $\varphi(x) \leq 0$ for $x \in [0,1]$ the zero number is set as $z(\varphi) = 0$. Of course this definition also holds for $\varphi \in C^1_n[0,1]$. Notice that, according to this definition, $z(\varphi) = 0$ for $\varphi \equiv 0$.

Then, $z(S(t,t_0)\varphi)$ is a monotone nonincreasing function of $t > t_0$, [2, 32] and see also [7]. More specifically let $\mathcal{N} = \left\{\varphi \in C^1_n[0,1]) \,:\, \varphi(x) = 0 \Rightarrow \varphi'(x) \neq 0\right\}$ be the open dense subset of functions with all zeros nondegenerate. Restricted to $\mathcal{N}$ the zero number z is the map which associates to each $\varphi \in \mathcal{N}$ the number of zeros of φ. Note that $\mathcal{N} = \cup_{k=0}^{\infty} \mathcal{N}_k$, $\mathcal{N}_k = \{\varphi \in \mathcal{N} : z(\varphi) = k\}$ and that $z|_{\mathcal{N}}$ is continuous and locally constant. Then

Theorem 2.1 *For* $\varphi \in C^1_n[0,1]$, $\varphi \not\equiv 0$,

(i) *the set* $\Theta = \left\{ t \in (t_0, +\infty) \, : \, S(t,t_0)\varphi \notin \mathcal{N} \right\}$ *is a finite set;*
(ii) *for* $t \in \Theta$ *there exists a positive* ε_0 *such that for all* $0 < \varepsilon < \varepsilon_0$,

$$z(S(t+\varepsilon, t_0)\varphi) < z(S(t-\varepsilon, t_0)\varphi) \, . \tag{2.3}$$

This nonincreasing character of z along solutions of (2.1) is essential for the description of the global dynamics of semilinear parabolic equations of the form

$$u_t = a(x)u_{xx} + f(x, u, u_x) \, , \; x \in (0,1) \, , \tag{2.4}$$

with the appropriate boundary conditions. See [3, 4, 7, 9, 18] for the pioneering results. See also [30] for the fully nonlinear case. For a smooth nonlinearity $f \in C^2([0,1] \times \mathbb{R}^2, \mathbb{R})$ satisfying suitable dissipative conditions, the semilinear reaction-diffusion equation (2.4) defines a global semiflow in the space $X = H^1(0,1)$. Then, z provides a discrete Lyapunov function for the linearized flow around any given solution of (2.4) or for the evolution of the difference between any two solutions u_1, u_2, of (2.4). In fact,

$$t \mapsto z(u_1(t,\cdot) - u_2(t,\cdot)) \tag{2.5}$$

is nonincreasing for $t > 0$ since $u_1 - u_2$ satisfies a variational equation of the form (2.1). Moreover, considering the difference $u(t+\tau,\cdot) - u(t,\cdot)$ we obtain that

$$t \mapsto z(u_t(t,\cdot)) \tag{2.6}$$

is also nonincreasing for $t > 0$, see [15] and Lemma 4.5 in [39]. The existence of a lap number for degenerate parabolic equations is considered in [20].

Different examples of discrete Lyapunov functions are provided by scalar delay differential equations of the form

$$\dot{x}(t) = h(x(t), x(t-1)) \, , \tag{2.7}$$

with $h \in C^2(\mathbb{R}^2, \mathbb{R})$ and monotone feedback conditions

$$h_v(u,v) < 0 \text{ or } h_v(u,v) > 0 \, . \tag{2.8}$$

These equations define semiflows $x_t = T(t)x_0, t \geq 0$, in $X = C^0[-1,0]$ by $x_t(\theta) = x(t+\theta), \theta \in [-1,0]$.

The following functionals rooted on z are defined on X for each feedback condition (2.8), respectively

$$V^-(\varphi) = 2\left\lfloor \frac{z(\varphi)}{2} \right\rfloor + 1 \quad , \quad V^+(\varphi) = 2\left\lfloor \frac{z(\varphi)+1}{2} \right\rfloor , \tag{2.9}$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Notice that for $\varphi \in X$, $V^-(\varphi)$ is always odd while $V^+(\varphi)$ is always even.

Then, for each feedback case, $V^{\mp}$ provides a discrete Lyapunov function for the difference of any two solutions x_t^1, x_t^2, since

$$t \mapsto V^{\mp}(x_t^1 - x_t^2) \tag{2.10}$$

is nonincreasing for $t > 0$. For references, see [35] and [40]. These examples will be relevant for the characterization of infinitesimal generators in our final three Sections.

3 Generators of semiflows with zero number decay

It is natural to ask if the property of possessing a discrete Lyapunov functional z in the sense of Theorem 2.1 is characteristic of differential second order operators. To answer this question, our main result is the following theorem.

Theorem 3.1 *Let A be the infinitesimal generator of a C_0-semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$, $T(t) : C_n^1[0,1] \to C_n^1[0,1]$, with $D(A) = C^2[0,1] \cap C_n^1[0,1]$ and such that*

(i) *the set $\Theta = \{t \in (0,+\infty) : T(t)\varphi \notin \mathcal{N}\}$ is a finite set for every $\varphi \in C_n^1[0,1]$;*
(ii) *for all $0 < t_1 \leq t_2$, with $t_1, t_2 \notin \Theta$ the following holds*

$$z(T(t_1)\varphi) \geq z(T(t_2)\varphi) . \tag{3.1}$$

Then there exist $\alpha, \gamma \in C^0[0,1]$ with α nonnegative and bounded $\beta \in C^0(0,1)$, such that for all $\varphi \in D(A)$ we have

$$(A\varphi)(x) = \alpha(x)\varphi_{xx}(x) + \beta(x)\varphi_x(x) + \gamma(x)\varphi(x) \quad , \quad 0 < x < 1 . \tag{3.2}$$

Here we assume Neumann boundary conditions, but similar results can be obtained for the cases of Dirichlet, Robin or even periodic boundary conditions. In Section 4 we show that this characterization extends also to other types of non-separated boundary conditions.

The main point in the proof of Theorem 3.1, see Lemma 3.3 below, concerns the analysis of the behavior of A on functions with degenerate zeros. We prepare with an auxiliary lemma.

Lemma 3.2 *Given $\xi \in [0,1]$, assume $\varphi : [0,1] \to \mathbb{R}$, $\psi : [0,1] \to \mathbb{R}$, are C^2 functions such that*

(i) $\varphi(\xi) = \varphi'(\xi) = \psi(\xi) = \psi'(\xi) = \psi''(\xi) = 0$;
(ii) $\varphi''(\xi) > 0$;
(iii) $\varphi'(x) \neq 0$ *for every* $x \in (0,1) \setminus \{\xi\}$.

For each $\lambda \in \mathbb{R}$ define $\varphi^\lambda : [0,1] \to \mathbb{R}$ by

$$\varphi^\lambda(x) = \varphi(x) + \lambda\psi(x)\,,\ x \in [0,1]\,. \tag{3.3}$$

Then the set of $\lambda \in \mathbb{R}$ *such that* φ^λ *has a degenerate zero in* $[0,1] \setminus \{\xi\}$ *has Lebesgue measure zero.*

Proof By (ii) and (iii) it follows that $\varphi(x) > 0$ for every $x \neq \xi$. Assume first that $x \in \{0,1\}, x \neq \xi$. Then there is at most one value of λ such that $\varphi^\lambda(x) = 0$. It remains to show that the set of λ such that φ^λ has a degenerate zero in $(0,1) \setminus \{\xi\}$ has Lebesgue measure zero. To show this we note that $x \in (0,1)\setminus\{\xi\}$ is a degenerate zero of φ^λ if and only if

$$\begin{cases} \varphi(x) + \lambda\psi(x) = 0, \\ \varphi'(x) + \lambda\psi'(x) = 0 \end{cases} \Leftrightarrow \quad \frac{\psi(x)}{\varphi(x)} = \frac{\psi'(x)}{\varphi'(x)} = -\frac{1}{\lambda}\,, \tag{3.4}$$

where we have used assumption (iii) and the positivity of $\varphi(x)$ for $x \neq \xi$ which, in particular, imply $\lambda \neq 0$.

Consider now the function $\Psi : (0,1) \setminus \{\xi\} \to \mathbb{R}$ defined by $\Psi(x) = \frac{\psi(x)}{\varphi(x)}$. Since φ and φ' do not vanish for $x \neq \xi$, Ψ is well defined, of class C^1 and

$$\Psi'(x) = \frac{\varphi'(x)}{\varphi(x)}\left(\frac{\psi'(x)}{\varphi'(x)} - \Psi(x)\right)\,, \tag{3.5}$$

vanishes if and only if $\Psi(x) = \frac{\psi'(x)}{\varphi'(x)}$. From this and (3.4) we conclude that $x \in (0,1) \setminus \{\xi\}$ is a degenerate zero of φ^λ if and only if $-\lambda^{-1}$ is a critical value of Ψ. Since Ψ is a C^1 function defined in an open set an application of Sard's theorem concludes the proof, see ex.2 pag.170 in [6]. □

Next we associate to each $\xi \in [0,1]$ three functions $a^\xi, b^\xi, c^\xi \in D(A)$ such that (here we use the partial derivative notation $v' = v_x$, see Fig. 1)

$$\begin{cases} a^\xi(\xi) = a^\xi_x(\xi) = b^\xi(\xi) = 0\,,\ \text{for every } \xi \in [0,1]\,, \\ a^\xi_{xx}(\xi) = c^\xi(\xi) = 1\,,\ \text{for every } \xi \in [0,1]\,, \\ b^\xi_x(\xi) = 1\,,\ \text{for every } \xi \in (0,1)\,, \\ a^\xi_x(x)(x-\xi) > 0\,,\ \text{for every } x \in (0,1) \setminus \{\xi\}\,. \end{cases} \tag{3.6}$$

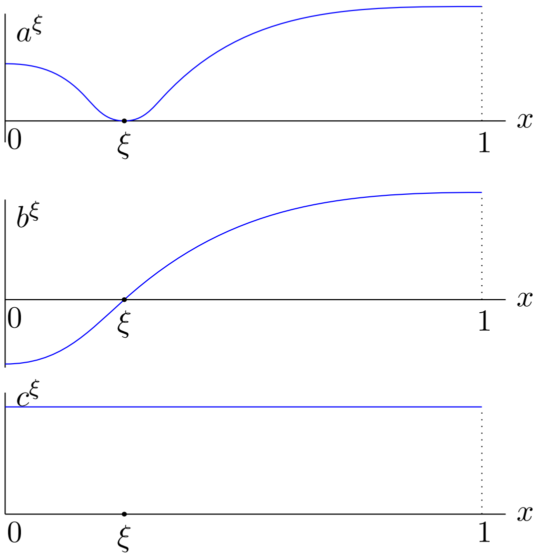

Fig. 1 The functions a^ξ, b^ξ and c^ξ

Lemma 3.3 *Let $\xi \in [0, 1]$ and $\psi^\xi \in D(A)$ be such that*

$$\psi^\xi(\xi) = \psi^\xi_x(\xi) = \psi^\xi_{xx}(\xi) = 0 \,. \tag{3.7}$$

Then, if A is the generator of a semigroup that satisfies the assumptions in Theorem 3.1, it results that

$$\left(A\psi^\xi\right)(\xi) = 0 \,. \tag{3.8}$$

Proof Let $\varphi^{\xi,\lambda} = a^\xi + \lambda\psi^\xi$. Then, on the basis of Lemma 3.2, there is a sequence $\{\lambda_k\}_{k=1}^\infty$ such that

$$\lambda_k\lambda_{k+1} < 0 \quad , \quad \lim_{k\to+\infty} |\lambda_k| = +\infty \tag{3.9}$$

and such that all zeros of φ^{ξ,λ_k} are nondegenerate except the one at ξ which is the unique degenerate zero of φ^{ξ,λ_k}. From the condition on the second derivative of a^ξ in (3.6) it also follows that, for each $k = 1, 2, \ldots$, there is a $\delta_k > 0$ such that for every $x \in (\xi - \delta_k, \xi + \delta_k)$ we have

$$\varphi^{\xi,\lambda_k}(x) \geq \frac{1}{4}(x - \xi)^2 \,. \tag{3.10}$$

This and the fact that the zeros of φ^{ξ,λ_k} outside $(\xi - \delta_k, \xi + \delta_k)$ are nondegenerate imply that the number of zeros of φ^{ξ,λ_k} is finite and equal to $N + 1$, where N is the number of zeros of φ^{ξ,λ_k} in $[0, 1]\backslash(\xi - \delta_k, \xi + \delta_k)$. Therefore

$$\varphi^{\xi,\lambda_k} \in \partial\mathcal{N}_N \,. \tag{3.11}$$

Let Θ_k be the set defined in (i) in Theorem 3.1 with $\psi = \varphi^{\xi,\lambda_k}$. Since Θ_k is a finite set, there exists $t_k > 0$ such that $(0, t_k] \cap \Theta_k = \emptyset$. It follows that, for $t \in (0, t_k]$, $T(t)\varphi^{\xi,\lambda_k}$ has only nondegenerate zeros and, since $\lim_{t\to 0^+} T(t)\varphi^{\xi,\lambda_k} = \varphi^{\xi,\lambda_k}$, has exactly N zeros in $[0,1] \setminus (\xi - \delta_k, \xi + \delta_k)$. Hence

$$
\begin{aligned}
&T(t_k)\varphi^{\xi,\lambda_k} \in \mathcal{N}_{\bar N}, \quad \text{for some } \; \bar N \geq N \\
&\text{and} \quad T(t_k)\varphi^{\xi,\lambda_k}(\xi) < 0, \quad \Rightarrow \quad \bar N \geq N + 2\,.
\end{aligned} \tag{3.12}
$$

Claim: Assume $(A\psi^{\xi})(\xi) \neq 0$. Then k large and $t_k > 0$ sufficiently small imply

$$
\lambda_k (A\psi^{\xi})(\xi)(T(t_k)\varphi^{\xi,\lambda_k})(\xi) > 0\,. \tag{3.13}
$$

From the assumption that $a^{\xi}, \psi^{\xi} \in D(A)$ it follows

$$
\begin{cases}
T(t)a^{\xi} - a^{\xi} - tAa^{\xi} = o_1(t) \\
T(t)\psi^{\xi} - \psi^{\xi} - tA\psi^{\xi} = o_2(t)
\end{cases}, \tag{3.14}
$$

where $o_i(t) \in C^1(0,1), i = 1, 2$, are small order terms of $t > 0$, that is, $\| \frac{o_i(t)}{t} \| \to 0$ as $t \searrow 0$. Hence $t_k > 0$ sufficiently small and $\big(A\psi^{\xi}\big)(\xi) \neq 0$ yield

$$
\left| \frac{o_1(t_k)}{t_k}(\xi) \right| < 1, \qquad \left| \frac{o_2(t_k)}{t_k}(\xi) \right| < \frac{1}{2} |(A\psi^{\xi})(\xi)|\,. \tag{3.15}
$$

This, $\varphi^{\xi,\lambda_k}(\xi) = 0$ and

$$
\begin{aligned}
&\big(T(t_k)\varphi^{\xi,\lambda_k}\big)(\xi) = \big(T(t_k)\varphi^{\xi,\lambda_k}\big)(\xi) - \varphi^{\xi,\lambda_k}(\xi) \\
&= t_k \big((Aa^{\xi})(\xi) + \lambda_k (A\psi^{\xi})(\xi) + (o_1(t_k)/t_k)(\xi) + \lambda_k (o_2(t_k)/t_k)(\xi) \big)\,,
\end{aligned} \tag{3.16}
$$

imply that, for k sufficiently large, $\big(T(t_k)\varphi^{\xi,\lambda_k}\big)(\xi)$ has the same sign as $\lambda_k \big(A\psi^{\xi}\big)(\xi)$ and the claim is established.

Since $\mathcal{N}_{\bar N}$ is open we can choose an open neighborhood $V_k \subset \mathcal{N}_{\bar N}$ of $T(t_k)\varphi^{\xi,\lambda_k}$ so small that

$$
\operatorname{sign}(v(\xi)) = \operatorname{sign}((T(t_k)\varphi^{\xi,\lambda_k})(\xi)) = \operatorname{sign}(\lambda_k (A\psi^{\xi})(\xi))\,, \; v \in V_k\,, \tag{3.17}
$$

where we have also used (3.13). The continuity of $T(t_k)$ implies $T(t_k)U_k \subset V_k$ for some open nonempty neighborhood U_k of φ^{ξ,λ_k} and (3.11) implies in particular $T(t_k)O_k \subset V_k$ where we have set $O_k = U_k \cap \mathcal{N}_N$. Then (3.9) and (3.17) imply that k can be chosen so that

$$
\big(T(t_k)\varphi^{\xi,\lambda_k}\big)(\xi) < 0\,. \tag{3.18}
$$

Then (3.13) yields

$$N = z(\varphi) < z(T(t_k)\varphi) = \bar{N}\ ,\ \varphi \in O_k\ , \tag{3.19}$$

in contradiction with assumption (ii) of Theorem 3.1. This contradiction proves Lemma 3.3. □

Now, based on Lemma 3.3, we complete the proof of Theorem 3.1. Given $x \in [0,1]$ and $\varphi \in D(A)$ we write

$$\varphi = \varphi(x)c^x + \varphi_x(x)b^x + \varphi_{xx}(x)a^x + \psi^{x,\varphi} \tag{3.20}$$

where a^x, b^x and c^x are the functions defined in (3.6). Since φ and a^x, b^x, c^x belong to $D(A)$ also $\psi^{x,\varphi} \in D(A)$ and (3.6) implies

$$\psi^{x,\varphi}(x) = \psi^{x,\varphi}_x(x) = \psi^{x,\varphi}_{xx}(x) = 0\ . \tag{3.21}$$

Therefore, Lemma 3.3 implies $(A\psi^{x,\varphi})(x) = 0$ and it follows that

$$(A\varphi)(x) = \varphi(x)(Ac^x)(x) + \varphi_x(x)(Ab^x)(x) + \varphi_{xx}(x)(Aa^x)(x)\ ,\ x \in [0,1]\ , \tag{3.22}$$

which is (3.2) with α, β, γ defined by

$$\gamma(x) = (Ac^x)(x)\ ,\ \beta(x) = (Ab^x)(x)\ ,\ \alpha(x) = (Aa^x)(x)\ ,\ x \in [0,1]\ . \tag{3.23}$$

On the basis of Lemma 3.3, $(Aa^\xi)(\xi) = \alpha(\xi) < 0$ will lead again to (3.18). This proves that α is a nonnegative function. It remains to prove the smoothness of α, β, γ. Consider the constant map $\varphi \equiv 1$ in $D(A)$. Then, from the abstract theory of C_0-semigroups on $C^1_n[0,1]$ we have $A1 \in C^1_n[0,1]$. Hence, by (3.22) and (3.23)

$$\gamma = A1 \in C^1_n[0,1] \subset C^0[0,1]\ . \tag{3.24}$$

Given $x_0 \in (0,1)$ let $\varphi, \psi \in D(A)$ be such that:

$$\begin{aligned} \varphi(x_0) = 0,\ \varphi'(x_0) = 1,\ \varphi''(x_0) = 0\ ,\\ \psi(x_0) = 0,\ \psi'(x_0) = 0,\ \psi''(x_0) = 1\ . \end{aligned} \tag{3.25}$$

Let $d(x) = \varphi'(x)\psi''(x) - \psi'(x)\varphi''(x)$ and observe that $d(x_0) = 1$. This and $\varphi, \psi \in D(A)$ imply that there is $\delta_0 > 0$ such that

$$\left.\begin{aligned} & d(x) \geq \tfrac{1}{2}\ ,\\ & |\varphi' - 1| \leq \tfrac{1}{2}, \quad |\psi'' - 1| \leq \tfrac{1}{2}\ , \\ & |\psi'| \leq \tfrac{1}{2}, \quad |\varphi''| \leq \tfrac{1}{2}\ , \end{aligned}\right\} \quad \text{for } x \in (x_0 - \delta_0, x_0 + \delta_0)\ . \tag{3.26}$$

With this choice of φ and ψ, from (3.22) and (3.23) we get a system of two equations in $\beta(x)$ and for $\alpha(x)$ that can be solved for $x \in (x_0 - \delta_0, x_0 + \delta_0)$ yielding:

$$\beta(x) = \frac{((A\varphi)(x) - \varphi(x)\gamma(x))\psi''(x) - ((A\psi)(x) - \psi(x)\gamma(x))\varphi''(x)}{\varphi'(x)\psi''(x) - \psi'(x)\varphi''(x)},$$

$$\alpha(x) = \frac{((A\psi)(x) - \psi(x)\gamma(x))\varphi'(x) - ((A\varphi)(x) - \varphi(x)\gamma(x))\psi'(x)}{\varphi'(x)\psi''(x) - \psi'(x)\varphi''(x)}. \tag{3.27}$$

From this and the continuity of $A\varphi$ and $A\psi$ it follows that β and α are continuous in $(x_0 - \delta_0, x_0 + \delta_0)$ and therefore in $(0, 1)$ since $x_0 \in (0, 1)$ is arbitrary. From (3.22) we also have that β and α are bounded in $(0, 1)$. It remains to discuss the behavior of α for $x \to 0, 1$. To this end choose $\varphi \in D(A)$ such that $\varphi(0) = \varphi'(0) = 0$ and $\varphi''(0) = 1$. Then, for x in a neighborhood of 0 it results

$$A\varphi(x) = \varphi(x)\gamma(x) + \varphi'(x)\beta(x) + \varphi''(x)\alpha(x). \tag{3.28}$$

This and the fact that α and β are bounded implies the existence of the

$$\lim_{x\to 0^+} \alpha(x) = (A\varphi)(0) \tag{3.29}$$

and we can continuously extend α to the interval $[0, 1)$ by setting $\alpha(0) = (A\varphi)(0)$. Similarly we obtain an extension of α to $[0, 1]$ by setting $\alpha(1) = (A\varphi)(1)$. This concludes the proof of Theorem 3.1. □

We can ask if $\alpha \geq 0$ in Theorem 3.1 can be upgraded to α strictly positive. Our guess is that this is not possible under only the assumptions of Theorem 3.1. See Sect. 5 and Remark 6.3 at the end of Sect. 6.

As we have seen, the zero number is well defined in $C^0[0, 1]$. An interesting issue is to understand the role of the regularity of the phase space on the structure of semigroups and of the corresponding generators that enjoy the decay property of a discrete Lyapunov functional. Since, by lowering the regularity, the class of motions on which the decay property must hold is enlarged, we expect that changing $C^1_n[0, 1]$ to $C^0[0, 1]$ restricts the class of semigroups that admit z as a discrete Lyapunov functional. For a discussion of this question we refer to Remark 6.2.

4 Generators of semigroups in $C^1[0, 1]$ with discrete Lyapunov functions $V^{\mp}$

In this Section we consider the problem of characterizing the generators of C_0 -semigroups $T(t)$, $t \geq 0$, which admit the discrete Lyapunov functional V^- or V^+ defined in (2.9). We assume that the function space is $C^1[0, 1]$ endowed with suitable boundary conditions. We observe that the arguments developed in the proofs of Lemma 3.2 and Lemma 3.3 apply also to the case where the discrete Lyapunov functional is V^- or V^+. This is a consequence of the fact that, if (3.8) is violated, as shown in Lemma 3.3, there are t_k and λ_k such that (3.12) holds and the number of zeros increases at least by two contradicting the decay property of z and V^- and V^+. From (3.8) it follows that again the generator is a second order operator and the problem is to understand what restrictions to the boundary conditions derive from the assumption that the semigroup admits V^- or V^+ as a discrete Lyapunov functional, see Theorem 4.1 below.

In Sect. 2 we have introduced the functionals $V^\pm$ in connection with the scalar delay equation (2.7) in the phase space $C^0[-1,0]$ which is the standard choice for equation (2.7). Obviously, with the appropriate definition of the map $\theta \to x_t(\theta)$ we can replace $C^0[-1,0]$ with the space $C^0[0,1]$. The fact is that the right space for the delay equation is a space of continuous functions. In contrast, as stated in Theorem 4.1, the generator of a semigroup with the same discrete Lyapunov functional, acting on a function space of C^1 functions, is a second order operator. This suggests that the smoothness of the function space plays an important role on the characterization of generators with discrete Lyapunov functionals. We confirm this in Sect. 6 where we show that changing C^1 to C^0 implies that the generator has $\alpha \equiv 0$ and corresponds to a generalized kind of delay equation.

We consider the space

$$C^1_{ns}[0,1] = \left\{ \varphi \in C^1[0,1] \, : \, B_0(\varphi) = B_1(\varphi) = 0 \right\} , \tag{4.1}$$

where $B_0, B_1 \, : \, C^1[0,1] \to \mathbb{R}$ are the boundary operators

$$\begin{cases} B_0(\varphi) = \varphi'(0) + \delta_{00}\varphi(0) + \delta_{01}\varphi(1) , \\ \\ B_1(\varphi) = \varphi'(1) + \delta_{10}\varphi(0) + \delta_{11}\varphi(1) . \end{cases} \tag{4.2}$$

For this type of boundary value problems we refer to ch.11,12 of [5]. See also [32] for results regarding the zeros at the boundary. We remark that $\delta_{01} = \delta_{10} = 0$ correspond to the separated Robin boundary conditions not under consideration here.

Our result regarding the characterization of the infinitesimal generators is the following theorem.

Theorem 4.1 *Let A be the infinitesimal generator of a C_0-semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}, T(t) \, : \, C^1_{ns}[0,1] \to C^1_{ns}[0,1]$ with $D(A) = C^2[0,1] \cap C^1_{ns}[0,1]$ and such that $T(t)$ admits the discrete Lyapunov functional defined by V^- (or V^+, alternatively) in (2.8).*

Then, there exist $\alpha, \gamma \in C^0[0,1]$, with α nonnegative and $\beta \in C^0(0,1)$ bounded, such that for all $\varphi \in D(A)$ we have

$$(A\varphi)(x) = \alpha(x)\varphi_{xx}(x) + \beta(x)\varphi_x(x) + \gamma(x)\varphi(x) \, , \;\; 0 < x < 1 \, . \tag{4.3}$$

Furthermore, the cross-boundary constants in (4.2) satisfy

$$\delta_{01} < 0 \, , \; \delta_{10} > 0 \qquad (\text{ or } \delta_{01} > 0 \, , \; \delta_{10} < 0, \text{ respectively}) \, . \tag{4.4}$$

Proof We focus on the case of V^-. The discussion of the case V^+ is similar. We let $\mathcal{N} \subset C^1_{ns}[0,1]$ denote the open dense subset of functions with no zeros on the boundary of [0, 1] and all (interior) zeros nondegenerate. We remark that in this case we have $\mathcal{N} = \cup_{k=0}^{\infty} \mathcal{N}_{2k+1}$, with $\mathcal{N}_{2k+1} = \{\varphi \in \mathcal{N} : \; V^-(z(\varphi)) = 2k+1\}$.

We first consider the case of degenerate interior zeros and address later the zeros on the boundary.

As we have already observed Lemma 3.2 holds as stated and we can consider the family $\varphi^{\xi,\lambda} = \varphi^{\xi} + \lambda\psi^{\xi}$ with φ^{ξ} and ψ^{ξ} as in Lemma 3.3. It results

$$\varphi^{\xi,\lambda} \in \partial\mathcal{N}_{2k+1}\,, \tag{4.5}$$

for some k and, proceeding as in the proof of Lemma 3.3, we have that $(A\psi^{\xi})(\xi) \neq 0$ implies the existence of $t > 0$ and $\lambda \in \mathbb{R}$ such that $(T(t)\varphi^{\xi,\lambda})(\xi) < 0$. It follows that $z((T(t)\varphi^{\xi,\lambda})(\xi)) = 2k' + 1 \geq 2k + 1 + 2$ and, as in the proof of Lemma 3.3, this implies the existence of open sets $O \in \mathcal{N}_{2k+1}$ and $V \in \mathcal{N}_{2k'+1}$ such that, for all $\varphi \in O$

$$2k + 1 = V^{-}(z(\varphi)) < V^{-}(z(T(t)\varphi)) = 2k' + 1\,, \tag{4.6}$$

with $T(t)\varphi \in V$, contradicting the decreasing property of the Lyapunov function of the semigroup $T(t)$. This shows that $(A\psi^{\xi})(\xi) = 0$ and the form (4.3) of the infinitesimal generator A follows as in the proof of Theorem 3.1.

To prove (4.4) we now deal with the zeros on the boundary. Then, let $\varphi \in C^1_{ns}[0,1]$ denote a function with only nondegenerate zeros and which is zero at $x = 0$ or $x = 1$. Note that, due to the boundary conditions (4.1),(4.2), nondegeneracy implies that these zeros cannot occur simultaneously on both boundaries.

The cross-boundary conditions (4.4) essentially prevent zeros to occur on the boundary when $z(\varphi)$ is even. In fact, in this case, the cross-boundary values of φ' and φ have the wrong sign, since (4.1),(4.2), imply

$$\varphi(0) = 0 \;\Rightarrow\; \varphi'(0) = -\delta_{01}\varphi(1), \tag{4.7}$$

$$\varphi(1) = 0 \;\Rightarrow\; \varphi'(1) = -\delta_{10}\varphi(0). \tag{4.8}$$

Then, if $z(\varphi)$ is even, $\varphi(0) = 0$ forces $\varphi(1)$ and $\varphi'(0)$ to have opposite signs, contrary to (4.7), (4.4). Likewise, $\varphi(1) = 0$ forces $\varphi(0)$ and $\varphi'(1)$ to have the same sign, contrary to (4.8), (4.4). Hence, zeros on the boundaries can only occur when $z(\varphi)$ is odd, in which case $V^{-}(T(t)\varphi)$ is constant in some small interval $t \in [0, \varepsilon)$ and $T(t)\varphi \in \mathcal{N}$ for small $t > 0$.

The proof of smoothness of α, β, γ follows the same lines as in the proof of Theorem 3.1. This completes the proof. □

5 Parabolic and delay operators as members of the same class determined by the decay of V^{-}.

Theorem 4.1 describes a quite large class of operators parameterized by the functions $\alpha \geq 0$, $\beta, \gamma \in \mathbb{R}$, and for the discrete Lyapunov functional V^{-}, with the constants $\delta_{01} < 0$ and $\delta_{10} > 0$.

It is interesting to observe that this class includes second order operators as well as linear delay equations. Indeed the operator A corresponding to $\alpha, \gamma \equiv 0$, $\beta \equiv 1$ and $\delta_{10} = p > 0$, $\delta_{11} = q \in \mathbb{R}$ is given by

$$(A\phi)(x) = \phi'(x)\ ,\ x \in [0,1)\ ,$$
$$(A\phi)(1) = -p\phi(0) + q\phi(1)\ , \tag{5.1}$$

which is the infinitesimal generator of the C_0-semigroup defined by the linear delay equation

$$\dot{y}(t) = -py(t-1) + qy(t)\ , \tag{5.2}$$

see Sect. 7.1 in [21]. As shown in [35], (5.2) admits the discrete Lyapunov functional V^-.

We also observe that on the basis of these considerations we cannot hope to deduce the strict inequality $\alpha > 0$ simply from the decay property of V^-. It is appropriate here to question the role of the cross-boundary conditions (4.4) and compare with the motivating examples provided by the scalar delay differential equations under monotone feedback conditions. The corresponding semiflows also preserve the decreasing properties for the Lyapunov functions defined by $V^{\mp}$. Our illustration example indicates that the cross-boundary conditions in these equations play the role of the monotone feedback conditions, e.g. $p > 0$, in delay equations.

6 Generators of semigroups on $C^0[0,1]$ with discrete Lyapunov functionals $V^{\mp}$

Let $T(t) : C^0[0,1] \to C^0[0,1]$, $t \geq 0$, be a C_0-semigroup and let A be the infinitesimal generator of the semigroup with $D(A) \subset C^0[0,1]$. We present some results on the structure imposed on A by the assumption that the semigroup admits the discrete Lyapunov functional V^- or V^+.

For $k \in \{0,1\}$ we set $C^k_{00} = \left\{\varphi \in C^k[0,1] : \mathrm{supp}(\varphi) \subset (0,1)\right\}$, and prove the following

Theorem 6.1 *Assume that*

(i) $T(t), t \geq 0$ *is a* C_0*-semigroup acting on* $C^0[0,1]$.
(ii) $V^{\mp}(T(t)\varphi) \leq V^{\mp}(T(\tau)\varphi),\ t \geq \tau,\ \varphi \in C^0[0,1]$.
(iii) *The domain* $D(A) \subset C^0[0,1]$ *of the infinitesimal generator* A *of* $T(t)$ *is a subset of* $C^1[0,1]$ *and contains* C^1_{00}.
(iv) *If* $\{\varphi_k\}_{k=1}^{\infty} \subset D(A)$ *is a sequence that converges to* φ *in* $C^1[0,1]$, *then* $\varphi \in D(A)$ *and*

$$\lim_{k\to+\infty} \|A\varphi_k - A\varphi\|_{C^0[0,1]} = 0\ . \tag{6.1}$$

Then, there exist functions $b, c \in C^0[0,1]$ such that

$$(A\varphi)(x) = b(x)\varphi'(x) + c(x)\varphi(x)\ ,\ x \in [0,1]\ ,\ \varphi \in D(A). \tag{6.2}$$

Moreover, if $b(x) \neq 0$ *for* $x \in [0,1]$, *then there is* $\widehat{\alpha} \in \mathbb{R}$ *and* $\mp a > 0$, *for* $V^{\mp}$ *respectively, such that*

$$b > 0 \Rightarrow D(A) = \left\{ \varphi \in C^1[0,1] \,:\, a\varphi(0) + \widehat{\alpha}\varphi(1) = b(1)\varphi'(1) + c(1)\varphi(1) \right\},$$

$$b < 0 \Rightarrow D(A) = \left\{ \varphi \in C^1[0,1] \,:\, a\varphi(1) + \widehat{\alpha}\varphi(0) = b(0)\varphi'(0) + c(0)\varphi(0) \right\}. \tag{6.3}$$

Proof In the proof we consider the case where (ii) holds with the negative sign. The analysis of the other case is similar. We divide the proof in various lemmas.

Lemma 6.2 *The assumptions in Theorem 6.1 imply*:

$$\mathrm{supp}(A\varphi) \subset \mathrm{supp}(\varphi),\ \ \varphi \in D(A)\,. \tag{6.4}$$

Proof We first prove (6.4) under the assumption $z(\varphi) < +\infty$. Assume instead that there exists $\varphi \in D(A)$ and an open interval $I = (\alpha, \beta)$ such that

$$\bar{I} \cap \mathrm{supp}(\varphi) = \emptyset \text{ and } (A\varphi)(x) \neq 0,\ \ x \in \bar{I}\,. \tag{6.5}$$

Let $I_i = (\alpha_i, \beta_i) \subset I$, $i = 1, 2, 3$ open intervals such that $\beta_1 < \alpha_2$ and $\beta_2 < \alpha_3$.

Choose a map $\psi \in C^1_{00}$ with $\mathrm{supp}(\psi) = \bar{I}_1 \cup \bar{I}_3$ and such that (see Fig. 2)

$$\psi(x)(A\varphi)(x) < 0,\ \ x \in I_1 \cup I_3\,. \tag{6.6}$$

For $\lambda \neq 0$ define $\varphi_\lambda = \lambda\varphi + \psi$ and observe that

$$\begin{aligned} z(\varphi_1) &\leq z(\varphi) + 2 < +\infty\,,\\ z(\varphi_\lambda) &= z(\varphi_1)\,,\ \ \lambda \neq 0\,. \end{aligned} \tag{6.7}$$

Fix $\bar{\lambda} > 0$ so that

$$\bar{\lambda} \min_{x \in \bar{I}_2} |(A\varphi)(x)| > \max_{x \in \bar{I}_2} |(A\psi)(x)|\,. \tag{6.8}$$

Then we have

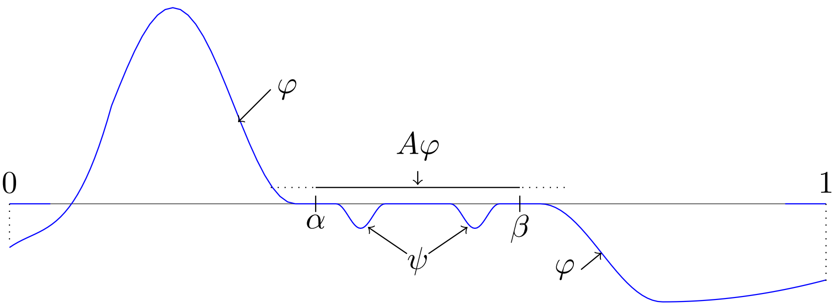

Fig. 2 The maps φ, ψ and $A\varphi$

$$\lim_{t\to 0^+}(T(t)\varphi_{\bar{\lambda}})(x) = \psi(x)\ ,\ x \in \bar{I}_1 \cup \bar{I}_3\ , \tag{6.9}$$

and, for small $t > 0$,

$$\begin{aligned}(T(t)\varphi_{\bar{\lambda}})(x)(T(t)\varphi)(x) &= t^2 A\varphi_{\bar{\lambda}}A\varphi + o(t^2)\\ &= t^2(\bar{\lambda}(A\varphi)^2 + A\psi A\varphi) + o(t^2) > 0\ ,\ x \in \bar{I}_2\ .\end{aligned} \tag{6.10}$$

For $0 < t << 1$ this implies that, when x describes the interval I, $(T(t)\varphi_{\bar{\lambda}})(x)$ changes sign twice and we have

$$z(T(t)\varphi_{\bar{\lambda}}) = z(\varphi_{\bar{\lambda}}) + 2\ , \tag{6.11}$$

which, in contradiction with (ii), yields $V^-(T(t)\varphi_{\bar{\lambda}}) > V^-(\varphi_{\bar{\lambda}})$. This contradiction concludes the proof for the case $z(\varphi) < +\infty$.

For the general case we construct a sequence $\{\varphi_k\}_{k=1}^{\infty} \subset D(A)$ which converges to φ in $C^1[0,1]$ and satisfies

$$\begin{aligned}&\mathrm{supp}\varphi_k \subset \mathrm{supp}\varphi\ ,\\ &z(\varphi_k) < +\infty\ .\end{aligned} \tag{6.12}$$

Assume that there exists a non empty open interval I such that $\bar{I} \cap \mathrm{supp}\varphi = \emptyset$ and $|(A\varphi)(x)| > 0$, for $x \in \bar{I}$. From assumption (iv) in Theorem 6.1, we have that for k sufficiently large $|(A\varphi_k)(x)| > 0$, for $x \in \bar{I}$. This is in contradiction with the first part of the proof which on the basis of (6.12) implies $\mathrm{supp}(A\varphi_k) \subset \mathrm{supp}(\varphi)$. This contradiction shows that (6.4) holds true also if $z(\varphi) = +\infty$. It remains to construct the sequence $\{\varphi_k\}_{k=1}^{\infty}$.

Let $E = \{x \in [0,1]\ :\ \varphi(x) = 0\}$ be the set of the zeros of φ and $\widetilde{E}$ the set of the accumulation points of E. For $x \in \widetilde{E}$ we have $\varphi(x) = \varphi'(x) = 0$ and therefore, for $\delta > 0$ sufficiently small, we obtain

$$\begin{aligned}&|\varphi(y)| = o(\delta)\ ,\\ &|\varphi'(y)| \le f(\delta)\ ,\end{aligned} \quad 0 \le |y - x| \le 2\delta\ ,\ y \in [0,1]\ , \tag{6.13}$$

where $s \to f(s)$ is a positive function that converges to zero as $s \to 0^+$. From (6.13) it follows that, given $\epsilon > 0$, for each $x \in \widetilde{E}$ there exists $\delta_{x,\epsilon} > 0$ such that

$$\begin{aligned}&|\varphi(y)| \le o(\delta_{x,\epsilon}) \le \epsilon\delta_{x,\epsilon}\ ,\\ &|\varphi'(y)| \le \epsilon\ ,\end{aligned} \quad 0 \le |y - x| \le 2\delta_{x,\epsilon}\ ,\ y \in [0,1]\ . \tag{6.14}$$

Since $\widetilde{E}$ is compact and $\{(x - \delta_{x,\epsilon}, x + \delta_{x,\epsilon}) \cap [0,1]\}_{x\in\widetilde{E}}$ is a open covering of $\widetilde{E}$, there exist an integer N and $x_j \in \widetilde{E}$, $j = 1, \ldots, N$ such that $\widetilde{E} \subset \cup_{j=1}^{N}[0,1] \cap (x_j - \delta_{x_j,\epsilon}, x_j + \delta_{x_j,\epsilon})$. Set $I_{i,j} = [0,1] \cap (x_j - i\delta_{x_j,\epsilon}, x_j + i\delta_{x_j,\epsilon})$, $i = 1,2$, $j = 1, \ldots, N$ and let $a_{i,j}, b_{i,j}$ be the extremes of $I_{i,j}$.

To conclude the proof of the existence of the sequence $\{\varphi_k\}_{k=1}^{\infty}$ we define a map $\varphi_\epsilon \in D(A) \subset C^1[0,1]$ with only a finite number of zeros and such that

$\|\varphi_\epsilon - \varphi\|_{C^1[0,1]} \le C\epsilon$ with $C > 0$ independent of ϵ. We define φ_ϵ in each connected component of the set $\cup_{j=1}^N I_{2,j}$. Let I be one of these connected components and let $p, q \in \{1, \dots, N\}$ be defined by $a_{1,p} = \min_{x_j \in I} a_{1,j}$ and $b_{1,q} = \max_{x_j \in I} b_{1,j}$. Define $\bar{\alpha}, \bar{\beta} \in I$ by setting

$$\bar{\alpha} = \begin{cases} a_{1,p} \,, & \text{if } a_{2,p} > 0 \,, \\ \min_{x \in \widetilde{E} \cup [a_{1,p}, x_p]} \,, & \text{if } a_{2,p} = 0 \,. \end{cases}$$
$$\bar{\beta} = \begin{cases} b_{1,q} \,, & \text{if } b_{2,q} < 1 \,, \\ \max_{x \in \widetilde{E} \cup [x_q, b_{1,q}]} \,, & \text{if } b_{2,q} = 1 \,. \end{cases} \tag{6.15}$$

Set

$$\varphi_\epsilon(x) = 0 \,, \; x \in [\alpha, \beta] \,. \tag{6.16}$$

To complete the definition of φ_ϵ in I we focus on the cases $a_{2,p} > 0$ and $a_{2,p} = 0$, the discussion of the cases $b_{2,q} < 1$ and $b_{2,q} = 1$ is analogous. We observe that $a_{2,p} > 0$ implies $a_{2,p} = x_p - 2\delta_{x_p,\epsilon}$, $\bar{\alpha} = x_p - \delta_{x_p,\epsilon}$. Then, define φ_ϵ in the interval $[x_p - 2\delta_{x_p,\epsilon}, x_p - \delta_{x_p,\epsilon})$ by setting

$$\begin{aligned} &\varphi_\epsilon(x) = \varphi(x) \,, && x \in [0, x_p - 2\delta_{x_p,\epsilon}) \,, \\ &\varphi_\epsilon(x) = \chi\Big((\delta_{x_p,\epsilon})^{-1}(x - x_p + 2\delta_{x_p,\epsilon})\Big)\varphi(x) \,, && x \in [x_p - 2\delta_{x_p,\epsilon}, x_p - \delta_{x_p,\epsilon}) \,, \end{aligned} \tag{6.17}$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is a C^∞ nonnegative map that satisfies $\chi(s) = 1$, $s \in (-\infty, 0)$ and $\chi(s) = 0$, $s \in [1, +\infty]$. From (6.16) and (6.17) it follows that to estimate the C^1 norm of $\varphi_\epsilon - \varphi$ in $I \setminus (\bar{\beta}, 1]$ it suffices to look at the interval $[x_\eta - 2\delta_{x_\eta,\epsilon}, x_\eta - \delta_{x_\eta,\epsilon})$ where

$$\varphi_\epsilon(x) - \varphi(x) = \Big(\chi(\delta_{x_\eta,\epsilon}{}^{-1}(x - x_\eta + 2\delta_{x_\eta,\epsilon})) - 1\Big)\varphi(x) \,. \tag{6.18}$$

From this expression and (6.13) we obtain

$$\begin{aligned} &|\varphi_\epsilon(x) - \varphi(x)| \le |\varphi(x)| \le \epsilon \,, \\ &|\varphi'_\epsilon(x) - \varphi'(x)| \le |\varphi'(x)| + (\delta_{x_\eta,\epsilon})^{-1}\chi'\Big((\delta_{x_\eta,\epsilon})^{-1}(x - x_\eta + 2\delta_{x_\eta,\epsilon})\Big)|\varphi(x)| \le C\epsilon \,, \end{aligned} \tag{6.19}$$

where $C = 1 + \max_s \chi'(s)$. Consider now the case $a_{2,p} = 0$. In this case we have $\bar{\alpha} = \min_{x \in \widetilde{E} \cup [a_{1,p}, x_p]}$ and therefore $\varphi(\bar{\alpha}) = \varphi'(\bar{\alpha}) = 0$. Hence

$$\varphi_\epsilon(x) = \varphi(x) \,, \; x \in [0, \bar{\alpha}) \,, \tag{6.20}$$

yields a well defined C^1 map, and again from (6.16) and (6.13), we obtain

$$\|(\varphi_\epsilon - \varphi)|_{[0,\bar\beta]}\|_{C^1[0,\bar\beta]} \leq C\epsilon \ . \tag{6.21}$$

By repeating the above construction for each connected component of $\cup_{j=1}^{N} I_{2,j}$ we end up with a map $\varphi_\epsilon \in C^1[0,1]$ that satisfies $\|\varphi_\epsilon - \varphi\|_{C^1[0,1]} \leq C\epsilon$. Since by construction $\varphi_\epsilon - \varphi \in C^1_{00}$, we have that $\varphi_\epsilon \in D(A)$. Moreover, φ_ϵ has only a finite number of zeros and

$$\varphi_\epsilon(x) \neq 0 \ \Rightarrow \ \varphi(x) \neq 0 \quad \text{and} \quad \operatorname{sign}(\varphi_\epsilon(x)) = \operatorname{sign}(\varphi(x)) \ . \tag{6.22}$$

This completes the proof. □

Lemma 6.3 *Assume the same as in Theorem 6.1. Then*

(i) $\varphi(x) = \varphi'(x) = 0, \ \Rightarrow \ (A\varphi)(x) = 0, \ \varphi \in D(A)$.
(ii) *For $\varphi \in D(A)$, $x \in (0,1)$, $(A\varphi)(x)$ depends only on the 1-jet of φ at x.*
(iii) *There exist $b, c \in C^0(0,1)$ such that*

$$(A\varphi)(x) = b(x)\varphi'(x) + c(x)\varphi(x) \ , \ x \in (0,1) \ , \ \varphi \in D(A) \ . \tag{6.23}$$

Proof To prove (i) we use an argument similar to the one used in the definition of the function φ_ϵ in the proof of Lemma 6.2. For each $\delta > 0$ sufficiently small we construct a map $\varphi_\delta \in D(A)$ which vanishes in $(x-\delta, x+\delta)$ and satisfies

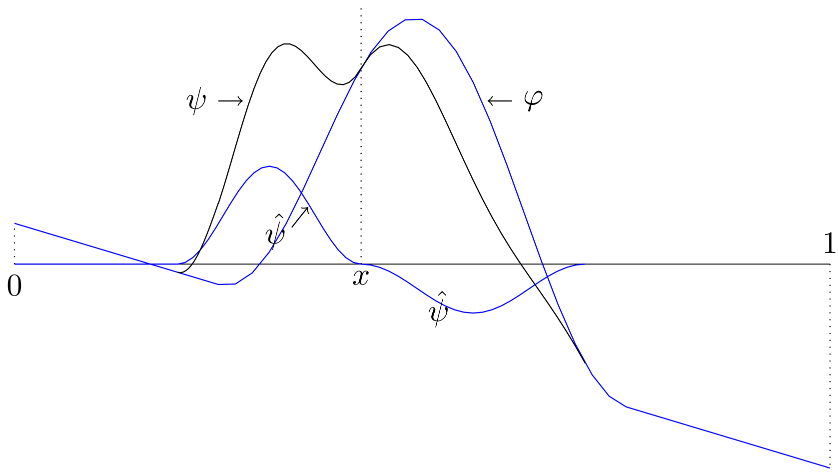

Fig. 3 The maps φ, $\hat\psi$ and ψ

$$|(\varphi - \varphi_\delta)(y)| \le |\varphi(y)| \le o(\delta)\,,$$

$$|(\varphi' - \varphi'_\delta)(y)| \le |\varphi'(y)| + \delta^{-1}\chi'\big(\delta^{-1}(y - x + 2\delta)\big)|\varphi(y)| \le f(\delta) + \delta^{-1}o(\delta)\,. \tag{6.24}$$

These estimates and the analogous in the intervals $[x-\delta, x+\delta]$ and $[x+\delta, x+2\delta]$ show that φ_δ converges in $C^1[0,1]$ to φ as $\delta \to 0$. This and assumption (iv) in Theorem 6.1 imply $\lim_{\delta\to 0} \|A\varphi_\delta - A\varphi\|_{C^0[0,1]} = 0$ and, in particular, using also Lemma 6.2 which yields $(A\varphi_\delta)(x) = 0$, we conclude $(A\varphi)(x) = 0$.

To show (ii), let $x \in \text{supp}(\varphi)$ be a point where $\varphi(x)$ and $\varphi'(x)$ are not both zero. Let $J^1_x\varphi \subset D(A)$ be the 1-jet of $\varphi \in D(A)$ at $x \in (0,1)$. From $\psi \in J^1_x\varphi$ it follows that $\hat{\psi} = \psi - \varphi$ satisfies $\hat{\psi}(x) = \hat{\psi}'(x) = 0$ (see Fig. 3). This and (i) imply $(A\hat{\psi})(x) = 0$ and therefore

$$(A\psi)(x) = (A\varphi)(x) + (A\hat{\psi})(x) = (A\varphi)(x)\,. \tag{6.25}$$

This proves (ii).

From (ii) and the linearity of A it follows that there is a continuous vector $(0,1) \ni x \to ((c(x), b(x))$ such that

$$\begin{aligned}(A\varphi)(x) &= ((c(x), b(x)) \cdot \begin{pmatrix}\varphi(x)\\ \varphi'(x)\end{pmatrix}\\ &= c(x)\varphi(x) + b(x)\varphi'(x) \quad , \quad x \in (0,1)\,,\ \varphi \in D(A)\,.\end{aligned} \tag{6.26}$$

The proof is complete. □

Remark 6.1 The functions b and c are defined and continuous in (0, 1). Since $A\varphi \in C^0[0,1]$ and $(A\varphi)(x) = b(x)\varphi'(x) + c(x)\varphi(x)$, $x \in (0,1)$, for all $\varphi \in D(A)$ it follows that there exist

$$\begin{aligned}L_0\varphi &= \lim_{x\to 0^+}(A\varphi)(x) = b(0)\varphi'(0) + c(0)\varphi(0)\,,\\ L_1\varphi &= \lim_{x\to 1^-}(A\varphi)(x) = b(1)\varphi'(1) + c(1)\varphi(1)\,,\end{aligned} \tag{6.27}$$

and actually $b, c \in C^0[0,1]$. For $\varphi \in D(A)$ and $0 \le t \ll 1$, L_0 and L_1 satisfy

$$\begin{aligned}(T(t)\varphi - \varphi)(0) &= tL_0\varphi + o(t)\,,\\ (T(t)\varphi - \varphi)(1) &= tL_1\varphi + o(t)\,.\end{aligned} \tag{6.28}$$

The operators L_0 and L_1 may be defined for all $\varphi \in C^1[0,1]$ or impose some conditions on $\varphi \in D(A)$ that restrict $D(A)$ to a proper subspace of $C^1[0,1]$.

From the previous analysis it follows that, if $T(t)$, $t \geq 0$, is a semigroup on $C^0[0,1]$ and the domain $D(A)$ of the infinitesimal generator A coincides with $C^1[0,1]$ (possibly restricted by the conditions imposed by the operators L_0 and L_1), then there are functions $b, c \in C^0[0,1]$ such that

$$(A\varphi)(x) = b(x)\varphi'(x) + c(x)\varphi(x)\ ,\ x \in [0,1]\ ,\ \varphi \in C^1[0,1]\ . \tag{6.29}$$

The definition of infinitesimal generator implies that the function u defined by $u(x,t) = (T(t)\varphi)(x)$ satisfies the scalar hyperbolic equation

$$\begin{cases} b(x)\frac{\partial}{\partial x}u(x,t) - \frac{\partial}{\partial t}u(x,t) = -c(x)u(x,t)\ , \\ u(x,0) = \varphi(x)\ . \end{cases} \tag{6.30}$$

We assume $b(x) \neq 0$, $x \in [0,1]$ and solve this equation in case $b > 0$. The case $b < 0$ is similar. The characteristic equations are

$$\frac{dx}{d\tau} = b(x)\ ,\ \frac{dt}{d\tau} = -1\ ,\ \frac{d\zeta}{d\tau} = -c(x)\zeta\ , \tag{6.31}$$

and it follows that

$$\int_{x_0}^{x} \frac{ds}{b(s)} + t = 0\ . \tag{6.32}$$

Since $b > 0$ by assumption, the integral in this equation is monotone increasing in x and decreasing in x_0. Furthermore, (6.32) can be solved for x and x_0 defining functions $x = g(x_0,t)$ and $x_0 = g^0(x,t)$ such that

$$\int_{x_0}^{g(x_0,t)} \frac{ds}{b(s)} + t = \int_{g^0(x,t)}^{x} \frac{ds}{b(s)} + t = 0\ . \tag{6.33}$$

We note the identity

$$g(g^0(x,t),s) = g^0(x,t-s) = g(x_0,s)\ . \tag{6.34}$$

This follows from (6.33) which implies

$$\begin{aligned} &\int_{g^0(x,t-s)}^{x} \frac{d\tau}{b(\tau)} + t - s = 0\ , \\ &\int_{g(g^0(x,t),s)}^{x} \frac{d\tau}{b(\tau)} + t - s = \int_{g^0(x,t)}^{x} \frac{d\tau}{b(\tau)} + t - \Big(\int_{g^0(x,t)}^{g(g^0(x,t),s)} \frac{d\tau}{b(\tau)} + s\Big) = 0\ . \end{aligned} \tag{6.35}$$

The equation $x = g(x_0,t)$ is the equation of the characteristic or better of the projection of the characteristic curve on the x, t plane. By differentiating (6.33) with respect to t one obtains

$$\frac{\partial}{\partial t} g(x_0,t) = -b(g(x_0,t))\ . \tag{6.36}$$

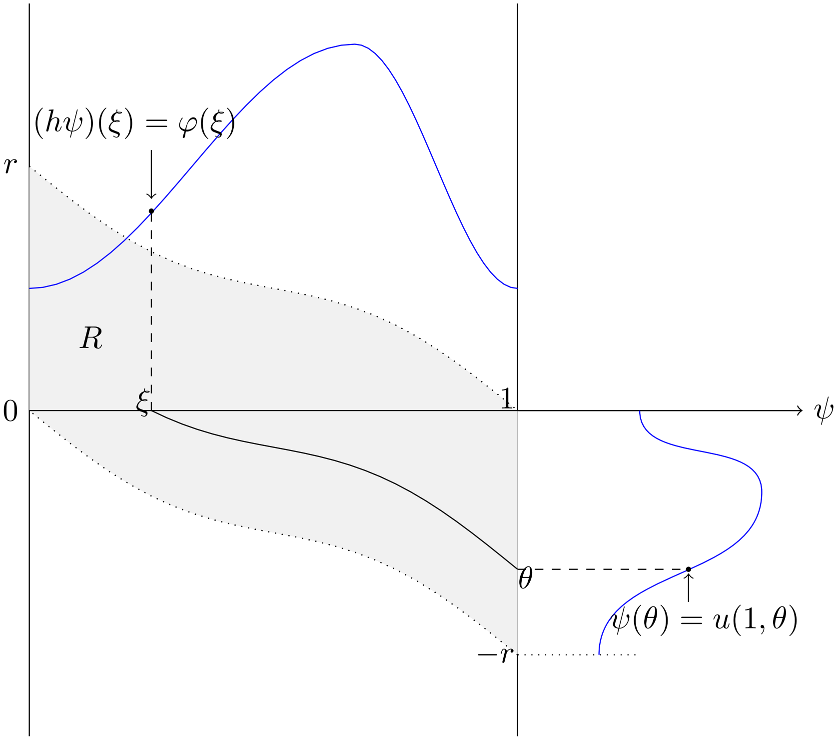

Fig. 4 Illustration, for $b > 0$, of $\psi(\theta) = u(1, \theta)$ and $\varphi(\xi) = h\psi(\xi)$

This and $b > 0$ show that the characteristic line oriented for increasing t crosses the lines $x = Const$ from the right to the left. Moreover the characteristic line through (x_0, t_0) is obtained from the characteristic through $(x_0, 0)$ by a translation of size t_0 in the time direction. After setting $x = g(x_0, t)$ into the third equation (6.29) we obtain

$$\zeta(x_0, t) = \zeta(x_0, 0)e^{\int_0^t c(g(x_0,s))ds} . \tag{6.37}$$

It follows that

$$u(x, t) = \zeta(g^0(x, t), t) = \varphi(g^0(x, t))e^{\int_0^t c(g(g^0(x,t),s))ds} = \varphi(g^0(x, t))e^{\int_0^t c(g^0(x,t-s))ds} , \tag{6.38}$$

where we have also used (6.34). This equation defines the solution of (6.30) in the region $R \subset [0, 1] \times \mathbb{R}$ contained between the characteristic line through (0, 0) and the characteristic line through (1, 0). From (6.38) it follows that the sign of $u(x, t)$ along a characteristic curve $x = g(x_0, t)$ is constant and coincides with the sign of $\varphi(x_0)$ where x_0 is the abscissa of the intersection of the considered characteristic curve with the x axes at time $t = 0$. Indeed we have

$$u(g(x_0,t),t) = \varphi(x_0)e^{\int_0^t c(g(x_0,s))ds} . \tag{6.39}$$

Set $r = \int_0^1 \frac{ds}{b(s)}$. Then, by means of (6.39), we can define a linear homeomorphism $h : C^0[-r,0] \to C^0[0,1]$ by setting (see Fig. 4)

$$\begin{aligned} \psi &:= u(1,\cdot)\,, \\ h\psi &\to \varphi\,. \end{aligned} \tag{6.40}$$

To see this, set $\xi(\theta) = g^0(1,\theta)$ and $\theta(\xi) = -\int_\xi^1 \frac{ds}{b(s)}$. The functions $\xi : [-r,0] \to [0,1]$ and $\theta : [0,1] \to [-r,0]$ are the inverse of each other:

$$\xi(\theta(\xi)) = \xi \in [0,1]\,; \quad \theta(\xi(\theta)) = \theta \in [-r,0]\,. \tag{6.41}$$

Note also that (6.34) implies

$$g(\xi(\theta),s) = g(g^0(1,\theta),s) = g^0(1,\theta - s) = \xi(\theta - s)\,. \tag{6.42}$$

From (6.39) we have

$$\begin{aligned} &\psi(\theta(\xi)) = u(1,\theta(\xi)) = (h\psi)(\xi)e^{\int_0^{\theta(\xi)} c(g(\xi,s))ds} \\ &\Rightarrow\ (h\psi)(\xi) = \psi(\theta(\xi))e^{\int_{\theta(\xi)}^0 c(g(\xi,s))ds}\,, \end{aligned} \tag{6.43}$$

and

$$(h^{-1}\varphi)(\theta) = u(1,\theta) = \varphi(\xi(\theta))e^{\int_0^\theta c(\xi(\theta - s))ds}\,, \tag{6.44}$$

where we have also used (6.42).

Equation (6.38) determines $(T(t)\varphi)(x) = u(x,t)$, $t \geq 0$ only in the set $R \cap [0,1] \times [0,+\infty)$. To extend the solution to the whole of $[0,1] \times [0,+\infty)$ we need to analyze the role and the structure of the boundary operators L_0 and L_1. The fact that the solution is already defined on $R \cap [0,1] \times [0,+\infty)$ implies that L_0 is defined for all $\varphi \in C^1[0,1]$. Instead $L_1 : D(A) \to C^0[0,1]$ may impose effective conditions on φ that restrict $D(A)$ to a proper subspace of $C^1[0,1]$. In any case, if we set $y(t) = (T(t)\varphi)(1)$, then $y : [0,+\infty) \to \mathbb{R}$ is the solution of the problem

$$\begin{cases} y' = L_1 h y_t\,, \\ y_0 = h^{-1}\varphi\,, \end{cases} \tag{6.45}$$

where

$$y_t(\theta) = y(t+\theta)\,,\ \theta \in [-r,0]\,. \tag{6.46}$$

Once y is known, the semigroup is determined by

$$T(t)\varphi = h y_t\,. \tag{6.47}$$

Remark 6.2 We are under the assumption that $b > 0$. If instead we assume $b < 0$, then the operator L_1 does not impose a restriction and can be defined for all $\varphi \in C^1[0,1]$ while the operator L_0 may restrict $D(A)$ to a proper subset of $C^1[0,1]$. Once the operator L_0 is specified, in case $b < 0$, we can define the analogous of the homeomorphism h and show that the semigroup $T(t)$, $t \geq 0$ is determined by equations similar to (6.45) and (6.47).

We also observe that (6.45) includes delay equations as a special case. Indeed if we think of $T(t)$ as a C_0-semigroup on $C^0[-1,0]$ rather than on $C^0[0,1]$, then the delay equation corresponds to the case where h is the identity.

One can also consider the extension of (6.38) to negative time. For $b > 0$ the extension to $t \leq 0$ depends on the choice of L_0 while, for $b < 0$, depends on the choice of L_1.

If $b(x_0) = 0$ for some $x_0 \in (0,1)$, then $x = x_0$ is an asymptote for the characteristics. Moreover, $b(x)(x - x_0) > 0$ for $x \neq x_0$ implies that the characteristics approach the asymptote for $t \to +\infty$. To extend the solution for $t \geq 0$ one needs to specify both operators L_0 and L_1 while the solution is automatically defined for $t \leq 0$. The opposite situation arises when $b(x)(x - x_0) < 0$, for $x \neq x_0$. More complex situations where b has several zeros can be analyzed along the lines developed before. We discussed the case $b > 0$ as the simplest significant case which, for $b \equiv 1$, includes the delay equation.

Next we determine what properties are required for L_1 in order that $V^{\pm}$ be a Lyapunov functional for the semigroup defined by (6.47).

Lemma 6.4 *Let $L_1 : D(A) \to \mathbb{R}$ be a linear functional and assume that the semigroup $T(t), t \geq 0$, defined by (6.45) and (6.47) satisfies*

$$V^{\mp}(T(t)\varphi) \leq V^{\mp}(T(\tau)\varphi)\,, \quad t \geq \tau \geq 0\,. \tag{6.48}$$

Then

$$L_1\varphi = a\varphi(0) + \alpha\varphi(1)\,, \tag{6.49}$$

with $\alpha \in \mathbb{R}$ and $\mp a \geq 0$, respectively for $V^{\mp}$.

Proof We only consider the case of the negative sign in (6.48). The discussion of the other case is similar. Observe that from (6.39) and the definition of h it follows that

$$z(y_t) = z(hy_t)\,. \tag{6.50}$$

Therefore it suffices to show that V^- is a Lyapunov functional for the solution $t \to y_t$ of the generalized delay equation (6.45). We divide the proof in tree steps:

Step 1 Let $\varphi \in C^0[0,1]$ a map that satisfies $\varphi(1) = 0$ and such that $z(\varphi) < +\infty$. For small $t > 0$, the solution y_t of (6.45) satisfies

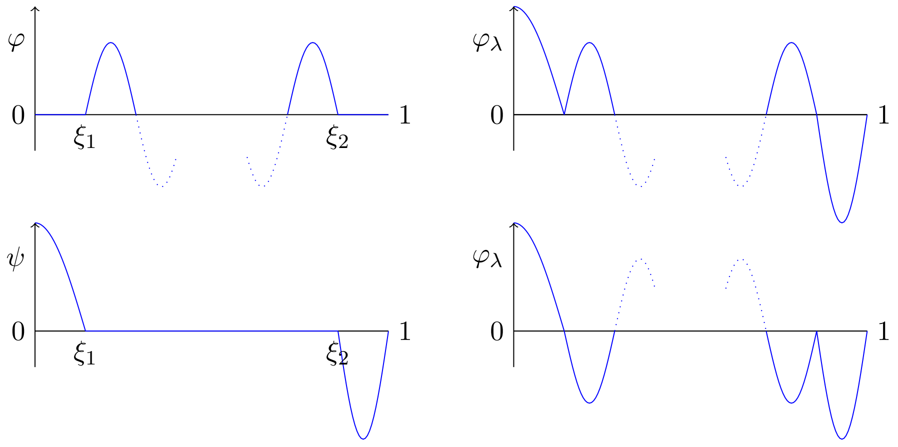

Fig. 5 φ, ψ and φ_λ, for $\lambda > 0$ and $\lambda < 0$

$$y_t(\theta) = \begin{cases} (h^{-1}\varphi)(t+\theta), \; \theta \in [-r+t, 0]\,, \\ \theta L_1\varphi + o(\theta), \; \theta \in (0, t]\,, \end{cases} \tag{6.51}$$

and, if $t > 0$ is sufficiently small

$$z(y_t|_{[-r+t,0]}) = z(y_0) = z(\varphi)\,. \tag{6.52}$$

The continuity of φ, (6.50) and the assumption $z(\varphi) < +\infty$ implies the existence of $\delta \in (0, r)$ and $\theta_0 \in [-\delta, 0]$ such that

$$\mp y_0(\theta) \geq 0\,, \;\; \theta \in [-\delta, 0] \text{ and } \mp y_0(\theta_0) > 0\,, \tag{6.53}$$

for $V^{\mp}$ respectively. This and (6.52) imply that, if $\pm L_1(\varphi) > 0$, then the function y_t has a sign change in the interval $[-\delta, t]$. Together with (4.8) this yields $z(y_t) = z(\varphi) + 1$. If $z(\varphi)$ is odd this implies $V^-(y_t) = V^-(y_0) + 2$ in contradiction with (6.48). This contradiction proves that

$$\varphi \in C^0[0,1]\,, \;\; \varphi(1) = 0 \text{ and } z(\varphi) \text{ odd} \quad \Rightarrow \quad \varphi(\xi(\theta_0))L_1\varphi \geq 0\,. \tag{6.54}$$

Step 2 Assume now that $\varphi \in C^0_{00}$ has $z(\varphi) < +\infty$ and even. Let $0 < \xi_1 < \xi_2 < 1$ be such that $\text{supp}(\varphi) \subset [\xi_1, \xi_2]$ and let $\psi \in C^0[0,1]$ be a function with support in $[0, \xi_1] \cup [\xi_2, 1]$ that satisfies $\psi(\xi) > 0$, $\xi \in [0, \xi_1)$, $\psi(\xi) < 0$, $\xi \in (\xi_2, 1)$, and $\psi(1) = 0$ (see Fig. 5). The function ψ has $z(\psi) = 1$. For $\lambda \in \mathbb{R}$ define $\varphi_\lambda = \psi + \lambda\varphi \in C^0[0,1]$. It results that $\varphi_\lambda(1) = 0$ and

$$z(\varphi_\lambda) = z(\psi) + z(\varphi) = z(\varphi) + 1\,, \;\; \lambda \in \mathbb{R} \setminus \{0\}\,. \tag{6.55}$$

Hence $z(\varphi_\lambda)$ is odd for all $\lambda \in \mathbb{R}$ (see Fig. 4). Then (6.54) implies

$$L_1\varphi_\lambda = \lambda L_1\varphi + L_1\psi \leq 0\,,\ \lambda \in \mathbb{R}\,, \tag{6.56}$$

since $\varphi_\lambda(\xi(\theta_0)) < 0$. If $L_1\varphi \neq 0$ then there exists $\lambda \in \mathbb{R}$ such that $L_1(\psi) + \lambda L_1\varphi > 0$, in contradiction with (6.54), and therefore we conclude that

$$\varphi \in C^0_{00} \text{ and } z(\varphi) \text{ even } \Rightarrow\ L_1\varphi = 0\,. \tag{6.57}$$

Since the set of functions with these properties is dense in the set $C^0_0 = \{\varphi \in C^0[0,1] : \varphi(0) = \varphi(1) = 0\}$, we have

$$L_1\varphi = 0\,,\ \varphi \in C^0_0\,, \tag{6.58}$$

whatever the parity of $z(\varphi)$.

Step 3 Each $\varphi \in C^0[0,1]$ has a unique decomposition of the form

$$\varphi = \widetilde{\varphi} + \varphi(0)\varphi^* + \varphi(1)\varphi_*\,,\ \widetilde{\varphi} \in C^0_0\,, \tag{6.59}$$

where $\varphi_*(\xi) = \xi$, $\varphi^*(\xi) = 1 - \xi$, $\xi \in [0,1]$. This and (6.58) imply that

$$L_1\varphi = L_1\widetilde{\varphi} + \varphi(0)L_1\varphi^* + \varphi(1)L_1\varphi_* = a\varphi(0) + \widetilde{\alpha}\varphi(1)\,,\ \varphi \in C^0[0,1]\,, \tag{6.60}$$

where we have set $a = L_1\varphi^*$ and $\widetilde{\alpha} = L_1\varphi_*$. Finally, we observe that if $\varphi(1) = 0$, $\varphi(0) \neq 0$ and $z(\varphi)$ is odd we have $\varphi(0)\varphi(\xi(\theta_0)) < 0$. From this and (6.54) it follows that

$$\varphi(\xi(\theta_0))L_1\varphi \geq 0\ \Rightarrow\ \varphi(0)L_1\varphi = \varphi(0)^2 a \leq 0 \tag{6.61}$$

which implies $a \leq 0$ unless $L_1\varphi \equiv 0$. The proof is complete. □

Remark 6.3 Note that the above analysis confirms the idea that relaxing the smoothness assumptions on the phase space restricts the class of semigroups which admit a given discrete Lyapunov functional. Indeed the discussion above shows that we must have $\alpha \equiv 0$ while α is allowed to be ≥ 0 in Theorem 4.1.

Remark 6.4 We can ask how the statement and the proof of Theorem 6.1 change when we replace (ii) by the assumption

$$z(T(t)\varphi) \leq z(T(\tau)\varphi)\,,\ \forall\, t \geq \tau\,,\ \varphi \in C^0[0,1]\,. \tag{6.62}$$

This replacement does not affect the proofs and the statements of Lemma 6.2 and Lemma 6.3. On the other hand, if (6.48) in Lemma 6.4 is replaced by (6.62), then (6.49) becomes

$$L_1\varphi = \alpha\varphi(1)\,,\ \alpha \in \mathbb{R}\,. \tag{6.63}$$

For the proof we observe that, with the new assumption, the argument in Step 1 yields

$$\varphi \in C^0[0,1]\,,\ \varphi(1)=0\ \Rightarrow\ \varphi(\xi(\theta))L_1\varphi \geq 0\,, \tag{6.64}$$

instead of (6.54). Then, arguing as in Step 2 but without accounting for the parity of $z(\varphi)$ one obtains (6.58) and (6.60). Finally, if $\varphi(1)=0$, equations (6.64) and (6.60) imply that $a\varphi(\xi(\theta))\varphi(0) \geq 0$ independently of φ. Therefore, $a=0$ follows.

7 Conclusion/discussion

The existence of a discrete Lyapunov functional has proved to be an essential tool for the detailed description of the global dynamics of Eqs. (2.1) and (2.7), [7–9, 15]. These results rise the question of existence of other classes of equations which admit some other type of discrete Lyapunov functionals.

A systematic study of this question would require the introduction of an abstract notion of discrete functional, say $\mathscr{V}$ (see [19] for an attempt in this direction), which generalizes the known examples: the zero number z and the related functionals $V^{\pm}$. Then, beginning with the finite dimensional case, one should determine all the pairs $(\mathscr{V}, \mathscr{H})$ where $\mathscr{H}$ is the class of linear operators which generates semigroups that admit $\mathscr{V}$ as a discrete Lyapunov functional. This may be the object of future research but it is outside the scope of the present paper.

For Eqs. (2.1) and (2.7) the phase space is a set of scalar functions defined on an interval: a one dimensional domain. We guess that this fact is essential for the existence of a discrete Lyapunov functional. Therefore we conjecture that, besides the known examples, there is not too much to be discovered. In particular we don't expect the existence of a discrete Lyapunov functional for equations defined on higher dimensional domains or for vector valued equations like, for instance, the damped wave equation or parabolic equations on $n>1$ dimensional domains. See [19] where it is shown that no discrete functional exists for a parabolic equation on a two dimensional domain.

In search of other situations where the existence of a discrete Lyapunov functional can be ascertained we note that, if H is a separable Hilbert space with the Hilbert basis $\{e_j\}_{j=1}^{\infty}$, the operators A which in $\{e_j\}_{j=1}^{\infty}$ are represented by a semi-infinite Jacobi matrix with positive bounded off diagonal elements admit a discrete Lyapunov functional. But again this functional is related to the number of sign changes in the sequence of the coordinates of the generic element $x=\sum_{j=1} x_j e_j$ of H. Note also that from Theorem 7.13 in [48] any self-adjoint transformation with a simple spectrum has, in a suitable Hilbert basis, a matrix representation of semi-infinite Jacobi type with positive off diagonal elements. The study of this class of operators and their nonlinear counterparts may have a mathematical interest but does not seem to be related to some problem of physical relevance.

This seems to substantiate our conjecture that the discrete functionals defined via the zero number and their variations are actually the only possible.

It is well known that the parabolic equation (2.1) has a variational structure given by a *continuous* Lyapunov functional, usually called *energy function* [11, 31, 38, 50]. Therefore, a comment is in order to clarify the role played by continuous and discrete Lyapunov functionals on the detailed description of the global dynamics defined by (2.1). Under general dissipative conditions on the nonlinearity f, the decreasing character of the energy function leads to the existence of a global attractor $\mathscr{A}$: a compact connected maximal set which is invariant under the dynamics defined by (2.1). This is almost all the information that can be deduced from the existence of the energy function. On the other hand all the beautiful surprising results on the dynamics restricted to $\mathscr{A}$, like the Morse–Smale property [1, 22, 25] or the description of the global attractor $\mathscr{A}$ from the *meander permutation* $\sigma \in S(n)$ [8, 18], depend on the decay property of the zero number. This decay property induces, on the function space, a geometric structure consisting in a nested family of cones $K_n = \left\{u \in C^1_n[0,1] : z(u) < n\right\}$, $n = 1, \ldots$. The key point is that the decay of the zero number implies the positively invariance of K_n, $n = 1, \ldots$, both under the linearization of (2.1) around any of its solutions or for the difference of any two such solutions.

In the previous Sections, we have characterized the infinitesimal generators of semiflows with given discrete Lyapunov functions derived from the zero number (2.2). We have shown that, besides the discrete Lyapunov functions associated to the semiflows, the smoothness required for the domain of the infinitesimal generators also determines their characterization. In fact, we have shown that infinitesimal generators acting on C^2 correspond to scalar parabolic equations while infinitesimal generators acting on C^1 correspond to transport equations, both with adequate boundary conditions.

We next discuss some topics related to our results and some lines of research that could follow-up our research.

Here we have discussed only equations with *discrete* Lyapunov functions. However, it is well known that most equations considered here are also gradient-like, exhibiting a variational structure given by a *continuous* Lyapunov function. For example, this holds for scalar one-dimensional semilinear parabolic equations, with C^2-smooth dissipative nonlinearities, under separated boundary conditions, see [10, 38, 50]. We remark that such equations define semiflows in $X = C^r$, $1 < r < 2$, [28, 33, 41]. Note that due to the dissipative conditions on f, each semiflow has a global attractor $\mathscr{A}$. Then, assuming hyperbolicity of all the equilibria, the *Sturm attractor* $\mathscr{A}$ has the Morse-Smale property. See, for example, [1, 22, 25]. This argument was also adapted to the case of periodic boundary conditions, see [11–14]. The existence of a continuous Lyapunov function also holds for quasilinear parabolic equations, see [29]. In addition, we remark that Lappicy and Fiedler extended this result to the case of fully nonlinear scalar parabolic equations, see [31].

The continuous Lyapunov function is essential to establish the existence of the global attractor $\mathscr{A}$, but from the mere knowledge of the continuous Lyapunov function very little information on the structure of $\mathscr{A}$ could be derived. On the other hand, from the discrete Lyapunov function one could deduce which equilibria are connected by heteroclinic orbits, and the transversal intersection of stable and unstable manifolds of equilibria, see [7, 18, 49]. We note that all these examples also exhibit discrete Lyapunov functions (lap numbers derived from the zero number). This also holds for the monotone feedback delay differential equations considered.

Finally, we remark that degenerate parabolic equations like p-Laplacian problems already mentioned in the Introduction, generate semiflows in $W^{1,p}(0,1)$, $p > 2$, which possess a continuous Lyapunov function and a discrete Lyapunov function (i.e., the zero number), [20]. This example raises the following questions: Can we obtain characterizations of infinitesimal generators acting on less regular spaces $C^r[0,1]$ with $r < 1$? Also, can we obtain such a characterization for fractional semilinear parabolic equations? How far can we reduce r? We leave these questions to future research.

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Declarations

Conflict of interest On behalf of the authors the corresponding author declares no conflict of interest.

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