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ORIGINAL ARTICLE

On $SL(2, \mathbb{C})$-representations of torus knot groups

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Abstract
The aim of this paper is to study the algebraic structure of the space $R(\Gamma_{n,m})$ of representations of the torus knot groups, $\Gamma_{n,m} = \langle x, y : x^n = y^m \rangle$, into the linear special group $SL(2, \mathbb{C})$.

Keywords Knot groups presentation · Representation · Knot groups · Parabolic representation

Mathematics Subject Classification 57M25 · 57M05 · 57Q45 · 14A10

1 Introduction

Knot theory is understood as the study of the equivalence classes of embeddings of the circle S^1 or the disjoint union of m copies of S^1 into S^3, considered up to ambient isotopy of S^3. A *torus knot* is a knot isotopic to one that lies on the boundary $\partial(V)$ of an unknoted solid torus $V \subset S^3$. A knot $K \subset S^3$ is said to be *hyperbolic* if $S^3 \setminus K = \mathbb{H}^3/\Gamma$, where $\mathbb{H}^3$ is the hyperbolic 3–space and Γ is a discrete, torsion-free subgroup of $\mathrm{Iso}^+(\mathbb{H}^3)$, isomorphic to the fundamental group $\pi_1(S^3 \setminus K)$ of the knot complement $S^3 \setminus K$.

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R. Riley studied the geometry of knot complements throughout representation theory in which he considered $SL(2,\mathbb{C})$-representations and used the theory of Haken manifolds to prove that the complement of the figure-eight knot is homeomorphic to $\mathbb{H}^3/\Gamma$, where Γ is a Kleinian subgroup of $PSL(2,\mathbb{C})$, see [36] and [35]. Following this line, Riley also proved that many other knots are hyperbolic and he conjectured that the knots that are neither torus knots nor satellite knots are hyperbolic knots, see [37]. Riley's work prepared the setting for Thurston's geometrization conjecture. In the 80's W. Thurston in [42] showed that every knot in S^3 is either a torus knot, a satellite knot or a hyperbolic knot.

It is well known that torus knots complements do not support an unique hyperbolic geometry, because it is not possible to get faithful representations of those groups that admit a presentation of the form $\langle x, y : x^n = y^m \rangle$ into $PSL(2,\mathbb{C})$ with Kleinian groups as their image. Despite that, in this paper we study representations of torus knot groups into $SL(2,\mathbb{C})$, and among other things, we give a simple proof of the fact that the image of any $SL(2,\mathbb{C})$-representation φ of $\pi_1(S^3 \backslash K_{n,m})$ has torsion elements. Thus, a natural question arose, what type of topological space $\mathbb{H}^3/\Gamma$ is, where $\Gamma = \varphi(\pi_1(S^3 \setminus K_{n,m}))$. On the other hand, the complement of a torus knot $S^3 \setminus K_{n,m}$ can be decomposed into pieces $M_1, \cdots, M_k$ such that each of them have one of the eight types of geometric structure, this decomposition is studied from an irreducible decomposition of the $SL(2,\mathbb{C})$-representation affine space $R(\Gamma_{n,m})$ of $\Gamma_{n,m} := \pi_1(S^3 \setminus K_{n,m})$, in this way it is of great interest to get a complete characterization of these irreducible components of $R(\Gamma_{n,m})$. We know that the structure of the representation variety $R(\Gamma_{n,m})$ and the character variety $X(\Gamma_{n,m})$ have been widely studied and determined in many references, for example, in [28], Muñoz and Porti give a geometric description of the character variety $X(\Gamma_{n,m})$ of $\Gamma_{n,m}$ into $SL(2,\mathbb{C})$, $GL(2,\mathbb{C})$ and $PGL(2,\mathbb{C})$, in [16], Liriano computes the dimension of $R(\Gamma)$, where Γ is an one-relator group with presentation $\left\langle x_1, \cdots, x_n, y \mid w(x_1, \cdots, x_n) = y^k \right\rangle$, $w(x_1, \cdots, x_n)$ is a word in the free group $F(x_1, \cdots, x_n)$ and $k \geq 2$, thereby, he proves that the dimension of $R(\Gamma_{n,m})$ is 4, see [16, Theorem 0.4], also in [17], he provides a formula to compute the number of four-dimensional irreducible components of $R(\Gamma_{n,m})$; following this line J. Martín-Morales and A. M. Oller-Marcen, in [22], give a complete description of the character variety $X(\Gamma_{n,m})$ and they prove that it is possible, in most cases, to recover n, m from $X(\Gamma_{n,m})$. Moreover, in [23], they compute the total number of irreducible components of $R(\Gamma_{n,m})$ and their corresponding dimension, extending, in this way, the work of S. Liriano. It is appropriate to emphasize here that they do not get such number from an explicit decomposition, but instead, by using a topological result and a well known irreducible decomposition of the character variety $X(\Gamma_{n,m})$. Thus, as we note, there are a lot of important results in this line. In addition, in this paper, we give a description of the matrices and the explicit changes of basis of the representation variety $R(\Gamma_{n,m})$. We also yield a complete characterization of reducible and irreducible $SL(2,\mathbb{C})$-representations of torus knot groups; see also [29] and [30], and several results in order to compute, explicitly, non-abelian representations of torus knot groups. We also prove that the abelian representations in $R(\Gamma_{n,m})$ define an affine algebraic set that is a closed subset of the variety representation $V(J_{n,m})$ of $\Gamma_{n,m}$, endowed with the Zariski topology. Thus, the non-abelian representations in $R(\Gamma_{n,m})$ correspond to open subsets of $V(J_{n,m})$. We denote by $A(\Gamma_{n,m})$ the abelian representations and by $N(\Gamma_{n,m})$ the non-abelian representations, then, $R(\Gamma_{n,m}) = A(\Gamma_{n,m}) \cup N(\Gamma_{n,m})$. We decompose these two subsets as a disjoint union of open sets. On the other hand, we obtain a decomposition of $R(\Gamma_{n,m})$ into closed subsets just by taking closures. Since, $A(\Gamma_{n,m})$ is a closed set, we have that $R(\Gamma_{n,m}) = A(\Gamma_{n,m}) \cup \overline{N(\Gamma_{n,m})}$. It is well known that every abelian representation of $\Gamma_{n,m}$ in $SL(2,\mathbb{C})$ is reducible. Because $\overline{N(\Gamma_{n,m})} = \overline{N_I(\Gamma_{n,m})} \cup \overline{N_R(\Gamma_{n,m})}$, where $N_I(\Gamma_{n,m})$ and $N_R(\Gamma_{n,m})$ denote the set of non-abelian irreducible and reducible representations, respectively, of $\Gamma_{n,m}$, and $\overline{N_R(\Gamma_{n,m})} \subset \overline{N_I(\Gamma_{n,m})}$, we conclude that $R(\Gamma_{n,m}) = A(\Gamma_{n,m}) \cup \overline{N_I(\Gamma_{n,m})}$,, see [24] for more details. We also find the irreducible components of $R(\Gamma_{n,m})$ and some of their ideals. We prove that the closed subset $A(\Gamma_{n,m})$ is irreducible. Since the set of non-abelian representations in $R(\Gamma_{n,m})$ can be decomposed as a disjoint union of the set of reducible non-abelian representations and the set of irreducible non-abelian representations, we also study the irreducible components of each of these subsets. In the last section we use that decomposition in order to present a complete description of the character variety $X(\Gamma_{n,m})$ of the group of torus knots.

2 Definitions and motivation

The set of Möbius transformations $f(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$, acting on $\mathbb{C}_\infty = \mathbb{C} \cup \infty$ is denoted by $\mathcal{M}_{\mathbb{C}}(\mathbb{C}_\infty)$. This set is a group under composition of transformations. On the other hand, let the matrix group of 2×2 non singular matrices with determinant one be denoted by

$$SL(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{C}, ad - bc = 1 \right\}.$$

Then, we have the homomorphism $\pi : SL(2,\mathbb{C}) \to \mathcal{M}_{\mathbb{C}}(\mathbb{C}_\infty)$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \pi_A : \pi_A(z) = \frac{az+b}{cz+d},$$

has kernel $\{\pm I\}$, and provides a natural identification between $\mathcal{M}_{\mathbb{C}}(\mathbb{C}_\infty)$ and $PSL(2,\mathbb{C}) := SL(2,\mathbb{C})/\pm I$.

In this paper, the trace and transposed of the matrix A are denoted by $tr(A)$ and A^t, respectively. The following definition comes from the correspondence between $SL(2,\mathbb{C})$ and the group of Möbius transformations $PSL(2,\mathbb{C})$. A complete classification of matrices in $SL(2,\mathbb{C})$ according to the value of the square of their trace can be found in [2, Theorem 4.3.4].

Definition 1 [2] Given a matrix A in $SL(2,\mathbb{C})$ such that $A \neq I$, it is said to be a *parabolic matrix* if $tr^2(A) = 4$, A is said to be an *elliptic matrix* if $tr^2(A) \in [0, 4)$, the matrix is said to be a *hyperbolic matrix* if $tr^2(A) \in (4, \infty)$, and A is a *strictly loxodromic matrix* if $tr^2(A) \notin [0, \infty)$.

Let G be a group. A *representation* of G on $SL(2,\mathbb{C})$ is a group homomorphism $\varphi : G \to SL(2,\mathbb{C})$. Two representations $\varphi : G \to SL(2,\mathbb{C})$ and $\varphi' : G \to SL(2,\mathbb{C})$ are said to be *equivalent* if there exists $P \in SL(2,\mathbb{C})$ such that, for every $g \in G$, $\varphi(g) = P\varphi'(g)P^{-1}$.

Consider a finitely generated group G with finite presentation given by $G = \langle x_1, x_2, \dots, x_n : r_1, r_2, \dots, r_m\rangle$. Then any representation $\varphi : G \to SL(2,\mathbb{C})$ is completely determined by the n-tuple $(\varphi(x_1), \cdots, \varphi(x_n)) \in SL(2,\mathbb{C})^n$ subject to the relations $r_j(\varphi(x_1), \cdots, \varphi(x_n)) = I$, for all $j = 1, \cdots, m$. Thereby, we define the subset $V(G)$ of $SL(2,\mathbb{C})^n$ as

$$V(G) = \left\{(A_1, \cdots A_n) \in SL(2,\mathbb{C})^n \mid r_j(A_1, \cdots A_n) = I, \forall j = 1, \cdots, m\right\}.$$

Thus, using the natural embedding of $SL(2,\mathbb{C})$ into $\mathbb{C}^4$, $V(G)$ can be endowed with the structure of an affine algebraic set (the zero set of polynomials in $\mathbb{C}^{4n}$). That is, $V(G)$ is in bijection to the zero set in $\mathbb{C}^{4n}$ of the polynomials given by the matrix entries of the group relations r_j and by the determinant equal to one. Therefore, there exists a natural 1-1 correspondence between the set $R(G)$, of representations of G on $SL(2,\mathbb{C})$, and the points of $V(G)$ in which each representation $\varphi : G \to SL(2,\mathbb{C})$ is identified with the point $(\varphi(x_1), \cdots, \varphi(x_n))$. From this correspondence, we refer to $R(G)$ as the space of representations of G in $SL(2,\mathbb{C})$ and also the algebraic variety. The space $R(G)$ is well-defined in the following sense, for two finite sets of generators of G, the unique bijection between the corresponding spaces of representations which preserves the above identification is an isomorphism of algebraic sets, see [8], thereby, $R(G)$ is well-defined up to isomorphism.

Definition 2 A representation $\varphi : G \to SL(2,\mathbb{C})$ is called *reducible* if there exists a nontrivial subspace V of $\mathbb{C}^2$, such that for each $g \in G$, $\varphi(g)(V) \subset V$. Otherwise, we say that φ is *irreducible*.

By definition, a representation $\varphi : G \to SL(2,\mathbb{C})$ is reducible if all the matrices $\varphi(g)$, $g \in G$, have a common one-dimensional eigenspace. A representation is *abelian* if its image is an abelian subgroup of $SL(2,\mathbb{C})$, and *non-abelian* otherwise. From [8, Lemma 1.2.1], every abelian representation $\varphi : G \to SL(2,\mathbb{C})$ is reducible.

The *character* of a representation $\varphi \in R(G)$ is the function $\chi_\varphi : G \to \mathbb{C}$, such that $\chi_\varphi(g) = tr(\varphi(g))$, the set of all characters χ_φ, $\varphi \in R(G)$, is denoted by $\chi(G)$. For each $g \in G$, let $\tau_g : R(G) \to \mathbb{C}$, with $\tau_g(\varphi) = tr(\varphi(g))$. Let T be the ring generated by all the functions τ_g, for $g \in G$. It was proved in [8, Proposition 1.4.1] that if $h_1, \cdots, h_n$ are generators of G, then T is generated by the set of all functions $\tau_{h_{i_1} h_{i_2} \cdots h_{i_k}}$, where $i_1, \cdots, i_k$ are distinct numbers in $\{1, \cdots, n\}$. Let $g_1, \cdots, g_s$ be a set of elements of G such that, $\tau_{g_1}, \cdots, \tau_{g_s}$ generate T. Define the map $t : R(G) \to \mathbb{C}^s$, by

$$t(\varphi) = (\tau_{g_1}(\varphi), \cdots, \tau_{g_s}(\varphi)) = (tr(\varphi(g_1)), \cdots, tr(\varphi(g_s)))$$

and let $X(G) = t(R(G))$. The proof of the following theorem can be found in [4, Theorem A] and [8, Corollary 1.4.5].

Theorem 1 *$X(G)$ is a closed algebraic set.*

We also have that the function $\Lambda : \chi(G) \to X(G)$, given by $\Lambda(\chi_\varphi) = t(\varphi) = (tr(\varphi(g_1)), \cdots, tr(\varphi(g_s)))$ is an injective function. In fact, let $\varphi, \psi \in R(G)$ be two representations of G such that $t(\varphi) = t(\psi)$, then $\tau_{g_i}(\varphi) = \tau_{g_i}(\psi)$, for $i = 1, 2, ..., s$. Now, let $f \in T$. Since $\tau_{g_1}, \cdots, \tau_{g_s}$ is a generating set of the ring T, then $f = \sum \alpha_{n_1,n_2,...,n_s} \tau_{g_1}^{n_1} \cdots \tau_{g_s}^{n_s}$, where $n_1, ..., n_s$ are integer numbers and $\alpha_{n_1,n_2,...,n_s}$ are complex numbers, such that $\alpha_{n_1,n_2,...,n_s} = 0$ except for a finite number of them. Given that $\tau_{g_i}(\varphi) = \tau_{g_i}(\psi)$, for $i = 1, 2, ..., s$, then

$$f(\varphi) = \sum \alpha_{n_1,n_2,...,n_s} \tau_{g_1}^{n_1}(\varphi) \cdots \tau_{g_s}^{n_s}(\varphi) = \sum \alpha_{n_1,n_2,...,n_s} \tau_{g_1}^{n_1}(\psi) \cdots \tau_{g_s}^{n_s}(\psi) = f(\psi).$$

Due to the fact that, for every $g \in G$, $\tau_g \in T$,

$$\chi_\varphi(g) = \tau_g(\varphi) = \tau_g(\psi) = \chi_\psi(g),$$

we have, $\chi_\varphi = \chi_\psi$. Thereby, we identify the points of $X(G)$ with the corresponding character, and so, $X(G)$ is called the *space of characters* of the group G and $t(\varphi) = \chi_\varphi$, $\varphi \in R(G)$. The space $X(G)$ does not depend, up to canonical-isomorphism, of the generating set of the ring T, so $X(G)$ is well-defined. See [8] for more details.

Proposition 2 [8, *Proposition* 1.5.2] *If φ and φ' are irreducible representations of G in $SL(2, \mathbb{C})$. Then, φ and φ' are equivalent if and only if $\chi_\varphi = \chi_{\varphi'}$.*

A subset K of S^3 is called a *knot* if there is an orientation-preserving embedding $\phi : S^1 \to S^3$, such that $K = \phi(S^1)$. Two knots K and K' are said to be *equivalent* is there exists an orientation-preserving homeomorphism $\psi : S^3 \to S^3$, such that $\psi(K) = K'$. A knot $K \subset S^3$ is usually represented by a diagram in the plane or S^2, called *knot diagram*, which is the projection of it in some plane of S^3, such that their intersections are transverse and no more than two points of K have the same projection.

A *torus knot* is a knot isotopic to a simple, closed curve D that lies on the boundary $\partial(V)$ of an unknoted solid torus $V \subset S^3$. Since V is a solid torus, there are two simple closed curves μ (a meridian) and λ (a longitude) on the border $\partial(V)$ of V such that $D = m\mu + n\lambda$, where m and n are positive integer numbers. In this case, we say that the torus knot is of type (n, m) and denoted by $K_{n,m}$. When m and n are relative prime numbers, then $K_{n,m}$ has one component, see [3] and [7].

Let K be a knot in S^3, then the fundamental group $\pi_1(S^3 \setminus K)$ of the knot complement $S^3 \setminus K$ or shortly called *the knot group of K* has a *Wirtinger* presentation $\pi_1(S^3 \setminus K) = \langle x_1, x_2, \cdots, x_m \mid r_1, r_2, \cdots, r_m \rangle$, see [3, 26] and [40] for more details.

It is well known, see [25], that the group of the torus knot $K_{n,m}$ has the presentation $\Gamma_{n,m} = \langle x, y \mid x^n = y^m \rangle$. In this way, the ring T is generated by $\tau_x, \tau_y, \tau_{xy}$. This is because $\tau_{xy} = \tau_{yx}$. Therefore, the character variety of $\Gamma_{n,m}$ has the form:

$$X(\Gamma_{n,m}) = \{(\tau_x(\varphi), \tau_y(\varphi), \tau_{xy}(\varphi)) \in \mathbb{C}^3 \mid \varphi \in R(\Gamma_{n,m})\}.$$

In the last section we present a more explicit description of such algebraic variety.

Lemma 1 *With the above notation. The generators x and y are torsion free. Therefore, the center $Z(\Gamma_{n,m})$ of the group $\Gamma_{n,m}$ of the knot $K_{n,m}$ is not trivial.*

Proof Suppose that there is a positive integer p such that $y^p = 1$. Then $y^p \in \overline{\langle r \rangle}$, where $r = x^n y^{-m}$. Because, $\overline{\langle r \rangle} = \{wrw^{-1} \mid w \in F\}$, F is the free group on the set $\{x, y\}$, then $y^p = \prod_{i=1}^{k} w_i r^{\epsilon_i} w_i^{-1}$. Let us denote $l_y(w)$ and $l_x(w)$ the sum of the superscripts of the occurrences of the letters x and y, respectively, in the word $w \in F$. So,

$$p = l_y(y^p) = l_y\left(\prod_{i=1}^{k} w_i r^{\epsilon_i} w_i^{-1}\right) = l_y(r) \sum_{i=1}^{k} \epsilon_i.$$

On the other hand,

$$0 = l_x(y^p) = l_x\left(\prod_{i=1}^{k} w_i r^{\epsilon_i} w_i^{-1}\right) = l_x(r) \sum_{i=1}^{k} \epsilon_i = n \sum_{i=1}^{k} \epsilon_i.$$

Then, $\sum_{i=1}^{k} \epsilon_i = 0$. Thereby, $p = 0$. In a similar way, we prove that x is torsion free.

Since $\langle y^m \rangle \subset Z(\Gamma_{n,m})$ and $\langle y^m \rangle \neq \{1\}$, then $Z(\Gamma_{n,m})$ is not trivial.

3 Toroidal knot representations

In this section we prove some important properties about representations of torus knots groups. These results are necessaries for the understanding of the last section.

Lemma 2 *Let us consider the infinite sequence of polynomials in $\mathbb{Z}[x]$, $\{\phi_k(x)\}_{k=0}^{\infty}$, where $\phi_0(x) = 1$, $\phi_1(x) = x$ and $\phi_k(x) = x\phi_{k-1}(x) - \phi_{k-2}(x)$, for $k > 1$. Let A be a matrix in $SL(2, \mathbb{C})$ and $\alpha = tr(A)$. Then, for each positive integer $k > 1$,*

$$A^k = \phi_{k-1}(\alpha)A - \phi_{k-2}(\alpha)I.$$

Proof We prove this by induction on the number k. If $k = 2$, then

$$A^2 = \alpha A - \det(A)I = \phi_{2-1}(\alpha)A - \phi_{2-2}(\alpha)I.$$

Thus, the lemma holds, when $k = 2$. Now suppose that

$$A^k = \phi_{k-1}(\alpha)A - \phi_{k-2}(\alpha)I.$$

We have that

$$\begin{aligned} A^{k+1} &= \phi_{k-1}(\alpha)A^2 - \phi_{k-2}(\alpha)A \\ &= \phi_{k-1}(\alpha)(\alpha A - I) - \phi_{k-2}(\alpha)A \\ &= (\alpha\phi_{k-1}(\alpha) - \phi_{k-2}(\alpha))A - \phi_{k-1}(\alpha)I \\ &= \phi_k(\alpha)A - \phi_{k-1}(\alpha)I. \end{aligned}$$

Therefore, the lemma holds.

Theorem 3 *For each pair of positive integers m and n, with $n, m > 1$, the set $R(\Gamma_{n,m})$ of representations over $SL(2, \mathbb{C})$ can be endowed with the structure of an affine algebraic set. Indeed, there exists a bijective function from $R(\Gamma_{n,m})$ to the algebraic set*

$$\left\{ (A,B) : \begin{array}{ll} \phi_{n-1}(\alpha)a_{ij} - \phi_{m-1}(\beta)b_{ij} - (\phi_{n-2}(\alpha) - \phi_{m-2}(\beta))\delta_{ij} & = 0, \quad i,j = 1,2 \\ a_{11}a_{22} - a_{12}a_{21} - 1 = 0, \quad b_{11}b_{22} - b_{12}b_{21} - 1 & = 0 \end{array} \right\}$$

where $A = (a_{ij})$, $B = (b_{ij})$ are matrices in $SL(2, \mathbb{C})$ and $I = (\delta_{ij})$ is the identity matrix. Moreover, $\alpha = tr(A)$ and $\beta = tr(B)$ are polynomials in the diagonal entries of the matrices A and B, respectively.

Proof We map each representation $\varphi \in R(\Gamma_{n,m})$ to a 8-tuple $(\varphi(x), \varphi(y)) = (A, B) \in \mathbb{C}^8$ of matrices, consisting of the image of the generators, and embed $SL(2, \mathbb{C})$ in $\mathbb{C}^4$ by using the entries of the matrices. Then $R(\Gamma_{n,m})$ is in bijection with the zero set in $\mathbb{C}^8$ of the polynomials given by the matrix entries of the group relation $x^n = y^m$ and by the determinant equal to one. That is, $A^n = B^m$, $\det A = 1$ and $\det B = 1$. On the other hand, by Lemma 2, we have that:

$$A^n = \phi_{n-1}(\alpha)A - \phi_{n-2}(\alpha)I \ \text{ and } \ B^m = \phi_{m-1}(\beta)B - \phi_{m-2}(\beta)I.$$

Thus, $A^n = B^m$, if and only if, $\phi_{n-1}(\alpha)A - \phi_{m-1}(\beta)B - (\phi_{m-2}(\beta) - \phi_{n-2}(\alpha))I = 0$, if and only if $\phi_{n-1}(\alpha)a_{ij} - \phi_{m-1}(\beta)b_{ij} - (\phi_{n-2}(\alpha) - \phi_{m-2}(\beta))\delta_{ij} = 0$ for $i, j = 1, 2$

From Theorem 3, the ideal $J_{n,m}$ corresponding to $R(\Gamma_{n,m})$, can be generated by $x_{11}x_{22} - x_{12}x_{21} - 1$, $y_{11}y_{22} - y_{12}y_{21} - 1$, $\phi_{n-1}(x_{11} + x_{22})x_{ij} - \phi_{m-1}(y_{11} + y_{22})y_{ij} - (\phi_{n-2}(x_{11} + x_{22}) - \phi_{m-2}(y_{11} + y_{22}))\delta_{ij}$ for $i, j = 1, 2$ in the ring of polynomials $\mathbb{C}[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]$. Thus we can identify the affine algebraic set $V(J_{n,m})$ with $R(\Gamma_{n,m})$, the set of all representations of $\Gamma_{n,m}$ into $SL(2, \mathbb{C})$.

Abelian representations in $R(\Gamma_{n,m})$ also define affine algebraic sets, moreover they correspond to a closed subset of $V(J_{n,m})$, with the Zariski topology. Thus, the non-abelian representations in $R(\Gamma_{n,m})$ correspond to an open subset of $V(J_{n,m})$. We denote the abelian representations by $A(\Gamma_{n,m})$ and the non-abelian ones by $N(\Gamma_{n,m})$. Then, $R(\Gamma_{n,m}) = A(\Gamma_{n,m}) \cup N(\Gamma_{n,m})$. We study these two subsets in a more detailed way. We can obtain a decomposition of $R(\Gamma_{n,m})$ into closed subsets just by taking closures, but the unions are no longer disjoint. That is, $R(\Gamma_{n,m}) = A(\Gamma_{n,m}) \cup \overline{N(\Gamma_{n,m})}$.

4 Decomposition of $R(\Gamma_{n,m})$ as irreducible closed subsets

This section is devoted to find the irreducible components of $R(\Gamma_{n,m})$ and some of their ideals. We prove that the closed subset $A(\Gamma_{n,m})$ is irreducible. Then, since the set of non-abelian representations in $R(\Gamma_{n,m})$ can be decomposed as a disjoint union of the set of reducible non-abelian representations and the set of irreducible non-abelian representations, in this section we also study the irreducible components of each of these subsets.

First we have that $A(\Gamma_{n,m}) \cong R((\Gamma_{n,m})_{ab})$. Moreover, if $gcd(n,m) = 1$, then $(\Gamma_{n,m})_{ab}$ is a cyclic group isomorphic to $\mathbb{Z}$, see [26] Proposition 4.2. for an algebraic proof. Thus,

$$A(\Gamma_{n,m}) \cong R((\Gamma_{n,m})_{ab}) \cong R(\mathbb{Z}). \tag{1}$$

Any one of the structures in (1) is isomorphic to $SL(2,\mathbb{C})$ as it will be proved in the following proposition.

Proposition 4 *Let n, m be two positive integers such that $n, m > 1$ and $\gcd(n,m) = 1$. Then the map $f : SL(2,\mathbb{C}) \to A(\Gamma_{n,m})$, defined by*

$$f(X) := (X^m, X^n), \text{ for each } X \in SL(2,\mathbb{C}),$$

is a bijective morphism of algebraic sets. Thereby, $A(\Gamma_{n,m})$ is an irreducible affine algebraic variety of dimension 3.

Proof Let X, Y be two matrices in $SL(2,\mathbb{C})$. Assume that $(X^m, X^n) = (Y^m, Y^n)$, that is, $X^m = Y^m$ and $X^n = Y^n$. By Bezout's identity there exist integer numbers r and s such that $1 = rm + sn$. Thus, $X = X^{rm+sn} = X^{rm}X^{sn} = Y^{rm}Y^{sn} = Y^{rm+sn} = Y$. So, f is an injective function. On the other hand, since $\gcd(n,m) = 1$ and hence $\langle x, y : x^n = y^m, 1 = [x,y]\rangle$ is a cyclic group, it follows that for each pair of matrices A, B in $SL(2,\mathbb{C})$ such that $AB = BA$ and $A^n = B^m$, there exists $X \in SL(2,\mathbb{C})$ such that $A = X^m$ and $B = X^n$. Therefore, the function f is onto.

Let n and m be two positive integers greater than or equal to 2. The following lemma will be useful to prove the non existence of non-abelian representations of $\Gamma_{n,m}$ in $SL(2,\mathbb{C})$ such that the images under it of x and y in $SL(2,\mathbb{C})$ are both parabolic matrices.

Lemma 3 *Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ be a parabolic matrix. If $B^m = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, for some $k \neq 0$ and $m \geq 2$, then $B = \begin{pmatrix} 1 & \frac{k}{m} \\ 0 & 1 \end{pmatrix}$ or $B = \begin{pmatrix} -1 & -\frac{k}{m} \\ 0 & -1 \end{pmatrix}$.*

Proof Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ be a parabolic matrix, that is $tr^2(B) = 4$.

Assume that $B^m = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$. Since $B^m B^{-1} = B^{m-1} = B^{-1}B^m$ and $B^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, then

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} d-ck & -b+ak \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b+dk \\ -c & a-ck \end{pmatrix}.$$

Thus, we obtain that $c = 0, -b + ak = -b + dk$ and hence

$$B = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

Now, we can prove by induction in m that

$$B^m = \begin{pmatrix} a^m & ma^{m-1}b \\ 0 & a^m \end{pmatrix}.$$

Thus, $a^m = 1$ and $k = ma^{m-1}b$. Given that $\det(B) = 1$, it follows that $a^2 = 1$, that is, $a \in \{1, -1\}$. Therefore, if m is odd, then $a = 1$ and $b = \frac{k}{m}$. Besides, if m is even, then either $a = 1$ or $a = -1$, so $b = \frac{k}{m}$ or $b = -\frac{k}{m}$, respectively. In any case, $B = \begin{pmatrix} \pm 1 & \pm\frac{k}{m} \\ 0 & \pm 1 \end{pmatrix}$.

Theorem 5 *If $\varphi : \Gamma_{n,m} \to SL(2, \mathbb{C})$ is a non-abelian representation and $A = \varphi(x)$, $B = \varphi(y)$, then A and B both are not parabolic matrices.*

Proof Assume that $tr^2(A) = tr^2(B) = 4$. Then there exists a non singular matrix P such that $P^{-1}AP = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, see [2].

Thus $P^{-1}A^nP = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, and hence $P^{-1}B^mP = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

From Lemma 3, $P^{-1}BP = \begin{pmatrix} a & \lambda \\ 0 & a \end{pmatrix}$, where $a \in \{1, -1\}$ and $n = \pm m\lambda$.

Since the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} a & \lambda \\ 0 & a \end{pmatrix}$ commute, then A and B commute, which contradicts the fact that φ is a non-abelian representation.

Theorem 6 *There are not non-abelian representations $\varphi : \Gamma_{n,m} \to SL(2, \mathbb{C})$ such that $\varphi(x)$ or $\varphi(y)$ are parabolic matrices.*

Proof We reason by contradiction, let us assume that there exists a non-abelian representation $\varphi : \Gamma_{n,m} \to SL(2, \mathbb{C})$ such that $A := \varphi(x)$ is a parabolic matrix. From Theorem 5, the matrix $B := \varphi(y)$ is not parabolic. Then there exist non singular matrices P and Q such that

$$P^{-1}AP = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$Q^{-1}BQ = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix},$$

see [2].

Since $A^n = B^m$, then $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} u^m & 0 \\ 0 & u^{-m} \end{pmatrix}$ are conjugates, and hence $(u^m + u^{-m})^2 = 4$. Thus, $u^{4m} + 2u^{2m} + 1 = 4u^{2m}$. By factoring we have that $(u^{2m} - 1)^2 = 0$ and $u^{2m} = 1$. From this, it follows that $B^{2m} = I$, and hence $A^{2n} = I$, which is a contradiction.

Similarly we can prove that there does not exist a non-abelian representation φ of $\Gamma_{n,m}$ in $SL(2, \mathbb{C})$ such that $B := \varphi(y)$ is a parabolic matrix.

Corollary 1 *Let $\varphi : \Gamma_{n,m} \to SL(2, \mathbb{C})$ be a non-abelian representation of $\Gamma_{n,m}$ in $SL(2, \mathbb{C})$. Then, $\varphi(x)$ and $\varphi(y)$ are not parabolic matrices.*

Proof From Theorem 5, $\varphi(x)$ and $\varphi(y)$ both are not parabolic matrices. Thus, from Theorem 6 the result holds.

Theorem 7 *There exist non-abelian representations of $\Gamma_{n,m}$ in $SL(2, \mathbb{C})$, for each $n, m \geq 2$.*

Proof We consider three cases

Case 1: $n = 2$ and $m = 2$. Let us consider

$$\varphi : \langle x, y : x^2 = y^2 \rangle \to SL(2, \mathbb{C}),$$

defined by $\varphi(x) = A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\varphi(y) = B = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$. We have that $A^2 = B^2$, then φ is a representation. On the other hand, $AB \neq BA$, thus φ is non abelian.

Case 2: $n = 2$ and $m = 3$ or $n = 3$ and $m = 2$. Let us consider

$$\varphi : \langle x, y : x^2 = y^3 \rangle \to SL(2, \mathbb{C}),$$

defined by $\varphi(x) = A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\varphi(y) = B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. We have that $A^2 = B^3$, then φ is a representation. On the other hand, $AB \neq BA$, thus φ is non abelian.

Similarly we can prove that there exist non-abelian representations of $G_{3,2} = \left\langle x, y : x^3 = y^2 \right\rangle$ in $SL(2, \mathbb{C})$.

Case 3: n and m are positive integers $n, m \geq 3$.

Let $\omega_n = e^{\frac{2p\pi}{n} i}$ and $\omega_m = e^{\frac{2q\pi}{m} i}$ be two primitive roots of the unity, with $\omega_n \neq \pm 1$ and $\omega_m \neq \pm 1$. Let us take

$$A_n = \begin{pmatrix} \omega_n 0 \\ 0\omega_n^{-1} \end{pmatrix} \text{and } B_m = \begin{pmatrix} \omega_m 0 \\ 0\omega_m^{-1} \end{pmatrix}.$$

Then A_n and $B_m \in SL(2, \mathbb{C})$. Now, taking $P = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\varphi : \langle x, y : x^n = y^m \rangle \to SL(2, \mathbb{C})$, defined by

$$\varphi(x) = PA_nP^{-1} = \begin{pmatrix} \frac{1}{\omega_n} & 0 \\ \frac{\omega_n^2 - 1}{\omega_n} & \omega_n \end{pmatrix} \text{ and } \varphi(y) = QB_mQ^{-1} = \begin{pmatrix} \omega_m^{-1} & 0 \\ 0 & \omega_m \end{pmatrix},$$

we obtain that φ is a representation. Now,

$$\begin{pmatrix} \frac{1}{\omega_n} & 0 \\ \frac{\omega_n^2 - 1}{\omega_n} & \omega_n \end{pmatrix} \begin{pmatrix} \omega_m^{-1} & 0 \\ 0 & \omega_m \end{pmatrix} = \begin{pmatrix} \omega_m^{-1} & 0 \\ 0 & \omega_m \end{pmatrix} \begin{pmatrix} \frac{1}{\omega_n} & 0 \\ \frac{\omega_n^2 - 1}{\omega_n} & \omega_n \end{pmatrix},$$

if and only if, $\omega_n^2 = 1$ or $\omega_m^2 = 1$. Therefore φ is a non abelian representation.

The equalities $\omega_n^2 = 1$ or $\omega_m^2 = 1$ described in the previous proof hold whenever $m \in \{1, -1, 2, -2\}$. Moreover, we observe that $tr^2(A_n) = 4\cos^2(\frac{2p\pi}{n})$ and $tr^2(B_m) = 4\cos^2(\frac{2q\pi}{m})$. Hence, A_n and B_m are parabolic matrices if and only if $n, m \in \{1, -1, 2, -2\}$. Furthermore, $tr(PA_nP^{-1}) = tr(A_n)$ and $tr(AB_mQ^{-1}) = tr(B_m)$. Thus, PA_nP^{-1} and AB_mQ^{-1} are parabolic matrices if and only if $n, m \in \{1, -1, 2, -2\}$. On the other hand, if $n = 2$ or $m = 2$, then $PA_nP^{-1} = \pm I$ or $QB_mQ^{-1} = \pm I$. So that, case 1 can not be included in case 2.

Example 1 There exist representations of $G_{\alpha,2} = \left\langle z, h : z^\alpha = h^2 \right\rangle$ in $SL(2, \mathbb{C})$, with α an odd positive integer. Moreover, $G_{\alpha,2} \cong G(\alpha, 1)$, where $G(\alpha, 1)$ is the group of the 2-bridge knot $S(\alpha, 1)$, see [26, Theorem 6.6].

As we can see, representations of groups of toroidal knots are not faithful.

Lemma 4 *Let A, B be two matrices in $SL(2, \mathbb{C})$, with $\alpha = tr(A)$ and $\beta = tr(B)$. If $A^n = B^m$ and $AB \neq BA$. Then, $\phi_{n-1}(\alpha) = 0$, $\phi_{m-1}(\beta) = 0$, $\phi_{n-2}(\alpha) = \phi_{m-2}(\beta) = \pm 1$, $A^n = B^m = \pm I$.*

Proof We have that:

$$A^n = \phi_{n-1}(\alpha)A - \phi_{n-2}(\alpha)I \text{ and } B^m = \phi_{m-1}(\beta)B - \phi_{m-2}(\beta)I, \tag{2}$$

so

$$\begin{aligned}\phi_{n-1}(\alpha)BA &= BA^n + \phi_{n-2}(\alpha)B\\ &= BB^m + \phi_{n-2}(\alpha)B\\ &= A^nB + \phi_{n-2}(\alpha)B\\ &= (A^n + \phi_{n-2}(\alpha)I)B\\ &= \phi_{n-1}(\alpha)AB.\end{aligned}$$

Thus, $\phi_{n-1}(\alpha) = 0$. Similarly, we can prove that $\phi_{m-1}(\beta) = 0$. By substituting $\phi_{n-1}(\alpha)$ and $\phi_{m-1}(\beta)$ in (2), we obtain that $\phi_{n-2}(\alpha) = \phi_{m-2}(\beta)$, and consequently, by taking determinants in (2), $(-\phi_{n-2}(\alpha))^2 = 1$ and $(-\phi_{m-2}(\beta))^2 = 1$. Thus, $\phi_{n-2}(\alpha) = \phi_{m-2}(\beta) = \pm 1$ and hence $A^n = B^m = \pm I$.

A direct consequence of the previous lemma is the following corollary.

Corollary 2 *Let $\varphi : \Gamma_{n,m} \to SL(2,\mathbb{C})$ be a non-abelian representation of $\Gamma_{n,m}$ in $SL(2,\mathbb{C})$. If $A = \varphi(x)$ and $B = \varphi(y)$, then A and B have finite order.*

The following notation will be useful in order to simplify the proof and certain definitions given in the rest of this paper. For $c \in \mathbb{C}$, with $c \neq 0$, we denote by $D(c) := \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}$ and $J(c) := \begin{pmatrix} c & 1 \\ 0 & c^{-1} \end{pmatrix}$,

Lemma 5 *Let A be a matrix in $SL(2,\mathbb{C})$ and n be a positive integer such that $A^n = \pm I$. Then there exists a non singular matrix $P \in SL(2,\mathbb{C})$ and a complex number $u \in \mathbb{C}$ such that $A = PD(u)P^{-1}$ and $u^n = \pm 1$.*

Proof If $A = I$ or $A = -I$, the result holds. Let us assume that $A \neq I$ and $A \neq -I$. It is well known that the characteristic polynomial of the matrix A is $ch_A(x) = x^2 - tr(A)x + \det(A)$. Thus, according to the number of different eigenvalues of A, we have the following cases:

Case 1: If $ch_A(x) = (x-u)^2$, for some $u \in \mathbb{C}$. Then, $tr(A) = 2u$, $det(A) = u^2 = 1$, and hence $u^{-1} = u$. Given that $A \neq I$ and $A \neq -I$, its minimal polynomial must be equal to $ch_A(x)$ and, from the canonical Jordan form, the matrix A is similar to a matrix in Jordan canonical form $\begin{pmatrix} u & 1 \\ 0 & u \end{pmatrix} = \begin{pmatrix} u & 1 \\ 0 & u^{-1} \end{pmatrix} = J(u)$ i.e. there exists a matrix $P \in SL(2,\mathbb{C})$ such that $A = PJ(u)P^{-1}$. We have that

$$J(u)^n = (P^{-1}AP)^n = P^{-1}A^nP = \pm P^{-1}IP = \pm I$$

and, by induction, that $J(u)^n = \begin{pmatrix} u^n & \phi_{n-1}(u+u^{-1}) \\ 0 & u^{-n} \end{pmatrix}$ and $\phi_{n-1}(u+u^{-1}) = u^{-n}(u + u^3 + u^5 + \cdots + u^{2n-1})$. Thus, $u^n = \pm 1$ and $u^{-n}(u + u^3 + u^5 + \cdots + u^{2n-1}) = 0$. Then $u + u^3 + u^5 + \cdots + u^{2n-1} = 0$ which contradicts $u^2 = 1$ and $u \neq 0$. Therefore, this case is not possible.

Case 2: If $ch_A(x) = (x - r_1)(x - r_2)$, for some $r_1 \neq r_2$ in $\mathbb{C}$. Then, $tr(A) = r_1 + r_2$ and $det(A) = r_1 r_2 = 1$. Thus, A is a diagonalizable matrix, and hence the matrix A is similar to a matrix in Jordan canonical form $D(u)$, with $u = r_1$ and $u^{-1} = r_2$ i.e. there exists a matrix $P \in SL(2, \mathbb{C})$ such that $A = PD(u)P^{-1}$. We have, that

$$D(u)^n = (P^{-1}AP)^n = \pm P^{-1}A^nP = P^{-1}IP = \pm I$$

and $D(u)^n = \begin{pmatrix} u^n & 0 \\ 0 & u^{-n} \end{pmatrix}$. Therefore, $u^n = \pm 1$.

Lemma 6 *Let $u \in \mathbb{C}$ be a complex number such that $u \neq 0$, $u^2 \neq 1$ and let $A = \begin{pmatrix} a & b \\ c & u + u^{-1} - a \end{pmatrix} \in SL(2, \mathbb{C})$ be a matrix. Then,*

1. *$bc = 0$ if and only if $a = u$ or $a = \frac{1}{u}$.*
2. *$P^{-1}AP = D(u)$, where $P \in SL(2, \mathbb{C})$ and*

(a) *If $a \neq u$, then*

$$P = \begin{pmatrix} \frac{b(a-u)}{\left(bc+(a-u)^2\right)y} & y \\ \frac{-(a-u)^2}{\left(bc+(a-u)^2\right)y} & \frac{c}{a-u}y \end{pmatrix}, \qquad y \in \mathbb{C} - \{0\} \tag{3}$$

(b) *If $a = u$, then*

$$P = \begin{cases} \begin{pmatrix} w^{-1} & \frac{-buw}{u^2-1} \\ 0 & w \end{pmatrix} & \text{if} \quad c = 0 \quad \text{and} \quad w \in \mathbb{C} - \{0\} \\ \begin{pmatrix} w^{-1} & 0 \\ \frac{cu}{(u^2-1)w} & w \end{pmatrix} & \text{if} \quad b = 0 \quad \text{and} \quad w \in \mathbb{C} - \{0\}. \end{cases} \tag{4}$$

Proof First, we prove (1). We have that $\det(A) = 1$ i.e. $a\left(-a + u + \frac{1}{u}\right) - bc = 1$ i.e. $a\left(-au + u^2 + 1\right) - ubc = u$ i.e. $(a - u)(1 - au) = ubc$. Thus, it follows that $bc = 0$ if and only if $a = u$ or $a = \frac{1}{u}$.

Let us assume that $a \neq u$. In order to prove Equation (3), we consider the matrix P defined by

$$P := \begin{pmatrix} \frac{b(a-u)}{\left(bc+(a-u)^2\right)y} & y \\ \frac{-(a-u)^2}{\left(bc+(a-u)^2\right)y} & \frac{c}{a-u}y \end{pmatrix},$$

for each $y \in \mathbb{C} - \{0\}$. First we observe that $bc + (a - u)^2 \neq 0$ and, hence, $P \in SL(2, \mathbb{C})$. In fact, by contradiction, let us assume that $bc + (a - u)^2 = 0$ i.e.

$bc = -(a-u)^2$. It is well known that $ch_A(x) = x^2 - (u+u^{-1})x + 1 = (x-u)(x-u^{-1})$ and $ch_A(A) = 0$. Then, $(A - uI)(A - u^{-1}I) = 0$ i.e.

$$\begin{pmatrix} a-u & b \\ c & u^{-1}-a \end{pmatrix}\begin{pmatrix} a-u^{-1} & b \\ c & u-a \end{pmatrix} = \begin{pmatrix} \frac{(au-1)(a-u)+bcu}{u} & 0 \\ 0 & \frac{bcu+(1-au)(u-a)}{u} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which implies $\frac{(au-1)(a-u)+bcu}{u} = 0$ and from this, $(au-1)(a-u)+bcu = 0$. Since, $bc = -(a-u)^2$, it follows that $(au-1)(a-u)-(a-u)^2u = 0$ i.e. $(a-u)((a-u)u + (au-1)) = 0$ and then $(a-u)(u^2-1) = 0$, which is a contradiction, because $u^2 \neq 1$ and $a - u \neq 0$.

Now, by expanding the indicated operations on the left hand, we have that

$$AP - PD(u) = \begin{pmatrix} 0 & \frac{u(ay(a-u)+bcy)-y(-u+a)}{u(-u+a)} \\ \frac{au^3-2a^2u^2-bcu^2-u^2+a^3u+2au+abcu-a^2}{uy\left(bc+(-u+a)^2\right)} & 0 \end{pmatrix}.$$

Since $\det\begin{pmatrix} a & b \\ c & u+u^{-1}-a \end{pmatrix} = 1$, it follows that $a\left(u+\frac{1}{u}-a\right) - bc = 1$ i.e. $au^2 + a - a^2u - bcu - u = 0$. Thus,

$$\frac{u(ay(a-u)+bcy)-y(-u+a)}{u(-u+a)} = 0$$

and

$$\frac{au^3-2a^2u^2-bcu^2-u^2+a^3u+2au+abcu-a^2}{uy\left(bc+(-u+a)^2\right)} = 0.$$

Therefore, Equation (3) holds.

Let us assume that $a = u$. Then $A = \begin{pmatrix} a & b \\ c & u^{-1} \end{pmatrix}$. Given that, $\det\begin{pmatrix} a & b \\ c & u^{-1} \end{pmatrix} = 1$, it follows that $a\left(\frac{1}{u}\right) - bc = 1$ i.e. $-bc = 0$. Now, we can prove directly the equations in (4) by expanding their right sides and then simplifying.

Finally, if $P = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, then any solution of the equation system $AP - PD(u) = 0$, $\det P = 1$, is equal to one of the forms given in Lemma 6, depending on $a \neq u$ or $a = u$.

Lemma 7 *Let $u \in \mathbb{C}$ be a complex number such that $u \neq 0$, $u^2 \neq 1$ and let $A = \begin{pmatrix} a & b \\ c & u+u^{-1}-a \end{pmatrix} \in SL(2,\mathbb{C})$ be a matrix. Then $P^{-1}AP = J(u)$, where*

1. *If $a \neq u$, then* $P = \begin{pmatrix} \frac{-bz}{a-u} & \frac{(a-u)^2-bz^2}{-\left(bc+(a-u)^2\right)z} \\ z & \frac{-(a-u)\left(c+z^2\right)}{\left(bc+(a-u)^2\right)z} \end{pmatrix}$, *for each* $z \in \mathbb{C}-\{0\}$.
2. *If $a = u$ then*

$$P = \begin{cases} \begin{pmatrix} \frac{1}{w} & \frac{u\left(1-bw^2\right)}{(u^2-1)w} \\ 0 & w \end{pmatrix} & \text{if} \quad c=0 \text{ and } w \in \mathbb{C}-\{0\} \\ \begin{pmatrix} x & \frac{ux}{\left(u^2-1\right)} \\ \frac{cux}{\left(u^2-1\right)} & \frac{\left(u^2-1\right)^2+cu^2x^2}{x\left(u^2-1\right)^2} \end{pmatrix} & \text{if} \quad b=0 \text{ and } x \in \mathbb{C}-\{0\}. \end{cases} \tag{5}$$

Proof The proof of this lemma is similar to the one of Lemma 6.

Theorem 8 *Let $\varphi \in R(\Gamma_{n,m})$ be a representation of $\Gamma_{n,m}$ in $SL(2,\mathbb{C})$. Then φ is reducible and a non-abelian representation if and only if there exist a matrix $M \in SL(2,\mathbb{C})$ and complex numbers $u, v \in \mathbb{C}$ such that $u^n = \pm 1$ and $v^m = \pm 1$ and $\varphi(x) = MJ(u)M^{-1}, \varphi(y) = MD(v)M^{-1}, u^2 \neq 1$ and $v^2 \neq 1$.*

Proof Let $\varphi \in R(\Gamma_{n,m})$ be a representation of $\Gamma_{n,m}$ in $SL(2,\mathbb{C})$. Let us assume that φ is reducible and non-abelian. Since φ is reducible, $\varphi(x) = Q\begin{pmatrix} u & a_{12} \\ 0 & u^{-1}\end{pmatrix}Q^{-1}$ and $\varphi(y) = Q\begin{pmatrix} v & b_{12} \\ 0 & v^{-1}\end{pmatrix}Q^{-1}$ for some matrix $Q \in SL(2,\mathbb{C})$ and $u, v, a_{12}, b_{12} \in \mathbb{C}$, with $u, v \neq 0$. Now, given that φ is non-abelian, by Lemma 4 we have that $(\varphi(x))^n = \pm I$ and $(\varphi(y))^m = \pm I$, and hence, by Lemma 5, u is a n-th root of unity and v is a m-th root of unity or $u^n = -1$ and $v^m = -1$. Then, using (4) and (5),

$$\begin{pmatrix} u & a_{12} \\ 0 & u^{-1}\end{pmatrix} = \begin{pmatrix} w^{-1} & \frac{u\left(1-a_{12}w^2\right)}{\left(u^2-1\right)w} \\ 0 & w \end{pmatrix} J(u) \begin{pmatrix} w^{-1} & \frac{u\left(1-a_{12}w^2\right)}{\left(u^2-1\right)w} \\ 0 & w \end{pmatrix}^{-1}, \quad w \in \mathbb{C}-\{0\}$$

and

$$\begin{pmatrix} v & b_{12} \\ 0 & v^{-1}\end{pmatrix} = \begin{pmatrix} w^{-1} & \frac{-b_{12}vw}{v^2-1} \\ 0 & w \end{pmatrix} D(v) \begin{pmatrix} w^{-1} & \frac{-b_{12}vw}{v^2-1} \\ 0 & w \end{pmatrix}^{-1}, \quad w \in \mathbb{C}-\{0\}.$$

Thus, taking $w = \sqrt{\frac{\frac{u}{u^2-1}}{\frac{ua_{12}}{u^2-1}-\frac{vb_{12}}{v^2-1}}}$, we have that

$$\begin{pmatrix} u & a_{12} \\ 0 & u^{-1}\end{pmatrix} = PJ(u)P^{-1} \quad \text{and} \quad \begin{pmatrix} v & b_{12} \\ 0 & v^{-1}\end{pmatrix} = PD(v)P^{-1}$$

for some matrix $P \in SL(2,\mathbb{C})$. Therefore, $\varphi(x) = QPJ(u)P^{-1}Q^{-1}$ and $\varphi(y) = QPD(v)P^{-1}Q^{-1}$.

Reciprocally, if $\varphi(x) = MJ(u)M^{-1}$, $\varphi(y) = MD(v)M^{-1}$, for some matrix $M \in SL(2,\mathbb{C})$, an n-th root of unity u and an m-th root of unity v, or $u^n = -1$ and $v^m = -1$, with $u^2 \neq 1$ and $v^2 \neq 1$, then $\varphi(x)$ and $\varphi(y)$ have a common eigenvector. Thus, the representation $\varphi \in R(\Gamma_{n,m})$ is reducible. Now, since $J(u)D(v) \neq D(v)J(u)$, it follows that φ is non-abelian.

Let n, m be two positive integers such that $n, m > 1$ and $\gcd(n,m) = 1$. We define $U^{(n,m)}$ as the set of pairs $(u,v) \in \mathbb{C}^2$ such that u is a n-th root of unity and v is a m-th root of unity, or $u^n = -1$ and $v^m = -1$; with $u^2 \neq 1$, $v^2 \neq 1$. Moreover, for each $(u,v) \in U^{(n,m)}$, we define

$$V_{(u,v)}^{(n,m)} := \left\{ (MJ(u)M^{-1}, MD(v)M^{-1}) \ : \ M \in SL(2,\mathbb{C}) \right\}.$$

Then, Theorem 8 shows that $V_{(u,v)}^{(n,m)} \subset V(J_{n,m}) = R(\Gamma_{n,m})$ and the subset of reducible non-abelian representations in $R(\Gamma_{n,m})$ corresponds bijectively to the subset $\bigcup V_{(u,v)}^{(n,m)} \subset V(J_{n,m})$ where the union is taken over all $(u,v) \in U^{(n,m)}$.

We observe that $(MJ(u)M^{-1}, MD(v)M^{-1}) = (NJ(u)N^{-1}, ND(v)N^{-1})$ if and only if $N^{-1}M \in \{-I, I\}$. In fact, $MJ(u)M^{-1} = NJ(u)N^{-1}$ and $MD(v)M^{-1} = ND(v)N^{-1}$ if and only if $J(u) = M^{-1}NJ(u)N^{-1}M$ and $D(v) = M^{-1}ND(v)N^{-1}M$, by Lemmas 4 and 5, if and only if, $N^{-1}M = \begin{pmatrix} \frac{1}{w} & \frac{u(1-w^2)}{(u^2-1)w} \\ 0 & w \end{pmatrix} = \begin{pmatrix} \frac{1}{w} & 0 \\ 0 & w \end{pmatrix}$ if and only if $N^{-1}M \in \{-I, I\}$. Therefore, each $V_{(u,v)}^{(n,m)}$ is birationally equivalent to $\frac{SL(2,\mathbb{C})}{\{-I,I\}} = PSL(2,\mathbb{C})$ and hence it is irreducible.

Example 2 Let $\varphi : \Gamma_{n,m} \to SL(2,\mathbb{C})$ be a non-abelian representation such that $\varphi(x) = A = \begin{pmatrix} \frac{1}{u} & 0 \\ \frac{1-u^2}{u} & u \end{pmatrix}$ and $\varphi(y) = B = \begin{pmatrix} \frac{1}{v} & \frac{1-v^2}{v} \\ 0 & v \end{pmatrix}$, where u is a n-th root of unity and v is a m-th root of unity, or $u^n = -1$ and $v^m = -1$, with $u^2 \neq 1$ and $v^2 \neq 1$.

We have that $A = PD(u)P^{-1}$ and $B = QD(v)Q^{-1}$, with $P = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$. We have that $AB - BA \neq 0$. Then, φ is an irreducible non abelian representation.

Example 3 Let $\varphi : \Gamma_{n,m} \to SL(2,\mathbb{C})$ be a representation of $\Gamma_{n,m}$ in $SL(2,\mathbb{C})$, such that

$$\varphi(x) = A = \begin{pmatrix} \frac{1}{u} & 0 \\ z & u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \frac{u}{1-u^2}z \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix} \begin{pmatrix} \frac{u}{1-u^2}z & -1 \\ 1 & 0 \end{pmatrix}, \qquad z \neq 0$$

and

$$\varphi(y) = B = \begin{pmatrix} \frac{1}{v} & t \\ 0 & v \end{pmatrix} = \begin{pmatrix} -\frac{v}{v^2-1}t & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & \frac{1}{v} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -\frac{v}{v^2-1}t \end{pmatrix}, \quad t \neq 0.$$

Where u is a n-th root of unity and v is a m-th root of unity, or $u^n = -1$ and $v^m = -1$, with $u^2 \neq 1$ and $v^2 \neq 1$.

We have that φ is irreducible if and only if

$$tz\left(tz + \frac{(u^2-1)(v^2-1)}{uv} \right) \neq 0.$$

On the other hand,

$$AB - BA = \begin{pmatrix} tz & \frac{(u^2-1)t}{u} \\ \frac{(v^2-1)z}{v} & -tz \end{pmatrix}.$$

Thus, φ is a non abelian irreducible representation if and only if $tz\left(tz + \frac{(u^1-1)(v^2-1)}{uv} \right) \neq 0.$

Theorem 9 *Let $\varphi \in R(\Gamma_{n,m})$ be a non-abelian representation of $\Gamma_{n,m}$ in $SL(2,\mathbb{C})$. Then, φ is irreducible if and only if there exist matrices $P, Q \in SL(2,\mathbb{C})$ and complex numbers $u, v \in \mathbb{C}$ such that $u^n = \pm 1$ and $v^m = \pm 1$, $u^2 \neq 1$, $v^2 \neq 1$, $\varphi(x) = PD(u)P^{-1}$, $\varphi(y) = QD(v)Q^{-1}$ and the matrices P and Q satisfy one of the following properties:*

1. $P = \begin{pmatrix} 1 & 0 \\ \frac{a_{21}u}{u^2-1} & 1 \end{pmatrix}, Q = \begin{pmatrix} 1 & -\frac{b_{12}v}{v^2-1} \\ 0 & 1 \end{pmatrix}$ *and*
$$R_1 := a_{21}b_{12}\left(a_{21}b_{12} + \frac{(u^2-1)(v^2-1)}{uv} \right) \neq 0$$
2. $P = \begin{pmatrix} 1 & -\frac{a_{12}u}{u^2-1} \\ 0 & 1 \end{pmatrix}, Q = \begin{pmatrix} 1 & 0 \\ \frac{b_{21}v}{v^2-1} & 1 \end{pmatrix}$ *and*
$$R_2 := b_{21}a_{12}\left(b_{21}a_{12} + \frac{(v^2-1)(u^2-1)}{vu} \right) \neq 0$$
3. $P = \begin{pmatrix} 1 & \frac{-a_{12}u}{u^2-1} \\ 0 & 1 \end{pmatrix}, Q = \begin{pmatrix} \frac{-b_{12}v}{v^2-1} & 1 \\ \frac{(b_{11}-v)v}{v^2-1} & \frac{b_{21}}{b_{11}-v} \end{pmatrix}$ *and*
$$R_3 := a_{12}b_{12}b_{21}\left(\frac{u}{u^2-1}a_{12}(b_{11}-v) - b_{12} \right)\left(a_{12}b_{21} + \frac{u^2-1}{u}(b_{11}-v) \right) \neq 0$$
4. $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, Q = \begin{pmatrix} \frac{-b_{12}v}{v^2-1} & 1 \\ \frac{(b_{11}-v)v}{v^2-1} & \frac{b_{21}}{b_{11}-v} \end{pmatrix}$ *and* $R_4 := b_{12}b_{21} \neq 0$
5. $P = \begin{pmatrix} 1 & 0 \\ \frac{a_{21}u}{u^2-1} & 1 \end{pmatrix}, Q = \begin{pmatrix} \frac{-b_{12}v}{v^2-1} & 1 \\ \frac{(b_{11}-v)v}{v^2-1} & \frac{b_{21}}{b_{11}-v} \end{pmatrix}$ *and*

$$R_5 := a_{21}b_{12}b_{21}\left(\frac{u}{u^2-1}a_{21}b_{12} + (b_{11}-v)\right)\left(b_{21} - \frac{u}{u^2-1}a_{21}(b_{11}-v)\right) \neq 0$$

6. $P = \begin{pmatrix} \frac{-a_{12}u}{u^2-1} & 1 \\ \frac{(a_{11}-u)u}{u^2-1} & \frac{a_{21}}{a_{11}-u} \end{pmatrix}, Q = \begin{pmatrix} 1 & \frac{-b_{12}v}{v^2-1} \\ 0 & 1 \end{pmatrix}$ *and*

$$R_6 := b_{12}a_{12}a_{21}\left(\frac{v}{v^2-1}b_{12}(a_{11}-u) - a_{12}\right)\left(b_{12}a_{21} + \frac{v^2-1}{v}(a_{11}-u)\right) \neq 0$$

7. $P = \begin{pmatrix} \frac{-a_{12}u}{u^2-1} & 1 \\ \frac{(a_{11}-u)u}{u^2-1} & \frac{a_{21}1}{a_{11}-u} \end{pmatrix}, Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $R_7 := a_{12}a_{21} \neq 0$

8. $P = \begin{pmatrix} \frac{-a_{12}u}{u^2-1} & 1 \\ \frac{(a_{11}-u)u}{u^2-1} & \frac{a_{21}}{a_{11}-u} \end{pmatrix}, Q = \begin{pmatrix} 1 & 0 \\ \frac{b_{21}v}{v^2-1} & 1 \end{pmatrix}$ *and*

$$R_8 := b_{21}a_{12}a_{21}\left(\frac{v}{v^2-1}b_{21}a_{12} + (a_{11}-u)\right)\left(a_{21} - \frac{v}{v^2-1}b_{21}(a_{11}-u)\right) \neq 0$$

9. $P = \begin{pmatrix} \frac{-a_{12}u}{u^2-1} & 1 \\ \frac{(a_{11}-u)u}{u^2-1} & \frac{a_{21}}{a_{11}-u} \end{pmatrix}, Q = \begin{pmatrix} \frac{-b_{12}v}{v^2-1} & 1 \\ \frac{(b_{11}-v)v}{v^2-1} & \frac{b_{21}}{b_{11}-v} \end{pmatrix},$

$$\begin{aligned} R_9 := a_{12}a_{21}b_{12}b_{21}&\big(a_{21}b_{12} + (a_{11}-u)(b_{11}-v)\big)\big(a_{12}b_{21} + (a_{11}-u)(b_{11}-v)\big) \\ &\big(a_{12}(b_{11}-v) - b_{12}(a_{11}-u)\big)\big(a_{21}(b_{11}-v) - b_{21}(a_{11}-u)\big) \neq 0. \end{aligned}$$

Proof Let $\varphi \in R(\Gamma_{n,m})$ be a representation of $\Gamma_{n,m}$ in $SL(2,\mathbb{C})$. Let us assume that φ is irreducible and non-abelian. Since φ is non-abelian, by Lemma 4, we have that $(\varphi(x))^n = \pm I$ and $(\varphi(y))^m = \pm I$, and hence, by Lemma 5, there exist $P, Q \in SL(2,\mathbb{C})$ and $u, v \in \mathbb{C}$, such that

$$\varphi(x) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & u + u^{-1} - a_{11} \end{pmatrix} = PD(u)P^{-1}$$

and

$$\varphi(y) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & v + v^{-1} - b_{11} \end{pmatrix} = QD(v)Q^{-1},$$

with u a n-th root of unity and v a m-th root of unity or $u^n = -1$ and $v^m = -1$. Moreover, the matrices P and Q satisfy (3) or (4). For each pair $P = (p_{ij})$ and $Q = (q_{ij})$ of these matrices we define

$$R(P,Q) = R_{11}(P,Q) \cdot R_{12}(P,Q) \cdot R_{21}(P,Q) \cdot R_{22}(P,Q),$$

where

$$R_{jk}(P,Q) := \det\begin{pmatrix} p_{1j} & q_{1k} \\ p_{2j} & q_{2k} \end{pmatrix} = p_{1j}q_{2k} - p_{2j}q_{1k}, \quad j,k = 1,2.$$

Thus, each column of P is linearly independent with each column of Q if and only if $R(P,Q) \neq 0$. Given that φ is irreducible, it follows that each column of P is linearly independent with each column of Q. On the other hand, by Lemma 6, we have that $a_{12}a_{21} = 0$ if and only if $(a_{11} - u)(1 - ua_{11}) = 0$. Similarly, $b_{12}b_{21} = 0$ if and only if $(b_{11} - v)(1 - vb_{11}) = 0$. Thus, taking into account the above and according to the different possibilities that result for a_{12}, a_{21}, b_{12} and b_{21}, we can compute $R(P, Q)$ for all possible matrices P and Q satisfying (3) or (4). Then, the only possibilities for P and Q in order to φ be an irreducible representation correspond to one of the descriptions given in (1)- (9), in the statement of the theorem.

Reciprocally, suppose that $\varphi(x) = A = PD(u)P^{-1}$ and $\varphi(y) = B = QD(v)Q^{-1}$, where u is a n-th root of unity and v is a m-th root of unity or $u^n = -1$ and $v^m = -1$. Also, P and Q are matrices in $SL(2,\mathbb{C})$ that satisfy any of the properties (1) to (9). Then, $R(P,Q) \neq 0$, and hence, each column of P is linearly independent with each column of Q. Thus, φ is irreducible.

It is clear that the non-abelian irreducible representations form an open subset of $V(J_{n,m})$. Furthermore, we should note that each of the properties (1)-(9) defines a subset of $V(J_{n,m})$ which is a principal open set of some closed subset of $V(J_{n,m})$, with the Zariski topology. Thus, by Theorem 9 we have a decomposition of the set of irreducible non-abelian representations as an union of open principal sets.

5 Decomposition of $X(\Gamma_{n,m})$ as irreducible closed subsets

In this section, we use the results developed in the previous one in order to get a complete decomposition of the character variety $X(\Gamma_{n,m})$.

We have that $\tau_g(\varphi) = tr(\varphi(g))$, for each $\varphi \in R(\Gamma_{n,m})$ and $g \in \Gamma_{n,m}$. Then,

$$X(\Gamma_{n,m}) = \left\{(tr(\varphi(x)),\ tr(\varphi(y)),\ tr(\varphi(xy))) \in \mathbb{C}^3 \mid \varphi \in R(\Gamma_{n,m})\right\}.$$

Now, since $R(\Gamma_{n,m}) = A(\Gamma_{n,m}) \cup N(\Gamma_{n,m})$, we can decompose the character variety $X(\Gamma_{n,m})$ as follows:

$$X(\Gamma_{n,m}) = X_A(\Gamma_{n,m}) \cup X_N(\Gamma_{n,m}),$$

where

$$X_A(\Gamma_{n,m}) = \{(tr(\phi(x)),\ tr(\phi(y)),\ tr(\phi(xy))) \in \mathbb{C}^3 \mid \varphi \in A(\Gamma_{n,m})\}$$

and

$$X_N(\Gamma_{n,m}) = \{(tr(\phi(x)),\ tr(\phi(y)),\ tr(\phi(xy))) \in \mathbb{C}^3 \mid \varphi \in N(\Gamma_{n,m})\}.$$

Proposition 10 *Let n, m be two positive integers such that $n, m > 1$ and $\gcd(n,m) = 1$. Then, $X_A(\Gamma_{n,m})$ is isomorphic to the algebraic set of all triplets*

$$(t\phi_{m-1}(t) - 2\phi_{m-2}(t), t\phi_{n-1}(t) - 2\phi_{n-2}(t),\ t\phi_{m+n-1}(t) - 2\phi_{m+n-2}(t)) \in \mathbb{C}^3,$$

such that $t = tr(X)$ and $X \in SL(2,\mathbb{C})$. Each $\phi_k(x)$ is the polynomial defined in (2).

Proof From Proposition 4, we have that the function $f : SL(2,\mathbb{C}) \to A(\Gamma_{n,m})$, defined by

$$f(X) := (X^m, X^n),\ \text{for each}\ X \in SL(2,\mathbb{C}),$$

is a bijective morphism of algebraic sets. Then,

$$X_A(\Gamma_{n,m}) \cong \{(tr(X^m),\ tr(X^n),\ tr(X^{m+n})) \in \mathbb{C}^3 \mid X \in SL(2,\mathbb{C})\}.$$

On the other hand, by Lemma 2 it follows that $X^m = \phi_{m-1}(t)X - \phi_{m-2}(t)I$, $X^m = \phi_{n-1}(t)X - \phi_{n-2}(t)I$ and $X^{m+n} = \phi_{m+n-1}(t)X - \phi_{m+n-2}(t)I$, where $t := tr(X)$. Thus, $tr(X^m) = t\phi_{m-1}(t) - 2\phi_{m-2}(t)$, $tr(X^n) = t\phi_{n-1}(t) - 2\phi_{n-2}(t)$ and $tr(X^{m+n}) = t\phi_{m+n-1}(t) - 2\phi_{m+n-2}(t)$. Thereby, $X_A(\Gamma_{n,m})$ has the desired form.

We denote by $N_R(\Gamma_{n,m})$ the set of reducible non-abelian representations and by $N_I(\Gamma_{n,m})$ the irreducible non-abelian ones. Then,

$$N(\Gamma_{n,m}) = N_R(\Gamma_{n,m}) \cup N_I(\Gamma_{n,m})$$

and hence,

$$X_N(\Gamma_{n,m}) = X_{N_R}(\Gamma_{n,m}) \cup X_{N_I}(\Gamma_{n,m}),$$

where,

$$X_{N_R}(\Gamma_{n,m}) := \{(tr(\varphi(x)),\ tr(\varphi(y)),\ tr(\varphi(xy))) \in \mathbb{C}^3 \mid \varphi \in N_R(\Gamma_{n,m})\}$$

and

$$X_{N_I}(\Gamma_{n,m}) := \{(tr(\varphi(x)),\ tr(\varphi(y)),\ tr(\varphi(xy))) \in \mathbb{C}^3 \mid \varphi \in N_I(\Gamma_{n,m})\}.$$

We study these two subsets in a more detailed form.

Let $\varphi \in R(\Gamma_{n,m})$ be a representation. First, from Theorem 8, we have that $\varphi \in N_R(\Gamma_{n,m})$ if and only if there exists $(u,v) \in U^{(n,m)}$, such that $(\varphi(x), \varphi(y)) \in V^{(n,m)}_{(u,v)}$. When, $\Gamma_{n,m} = \langle x, y \mid x^n = y^m \rangle$,

$$V^{(n,m)}_{(u,v)} := \left\{ \left(MJ(u)M^{-1}, MD(v)M^{-1}\right) :\ M \in SL(2,\mathbb{C}) \right\}.$$

Moreover, $\left(MJ(u)M^{-1}, MD(v)M^{-1}\right) = \left(NJ(u)N^{-1}, ND(v)N^{-1}\right)$ if and only if $N^{-1}M \in \{-I, I\}$.

Proposition 11 *Let n, m be two positive integers such that $n, m > 1$ and* $\gcd(n,m) = 1$. *Then, $X_{N_R}(\Gamma_{n,m})$ is equal to the algebraic set of all triplets*

$$\left(\frac{u^2+1}{u}, \frac{v^2+1}{v}, \frac{u^2v^2+1}{uv}\right) \in \mathbb{C}^3,$$

such that $(u,v) \in U^{(n,m)}$. *That is, u is a n-th root of unity and v is a m-th root of unity, or* $u^n = -1$ *and* $v^m = -1$; *with* $u^2 \neq 1$ *and* $v^2 \neq 1$, *in both cases.*

Proof Let $(t_1,t_2,t_3) \in X_{N_R}(\Gamma_{n,m})$. Then there exists $\varphi \in N_R(\Gamma_{n,m})$ such that $(t_1,t_2,t_3) = (tr(\varphi(x)), tr(\varphi(y)), tr(\varphi(xy)))$. Now, there exist a matrix $M \in SL(2,\mathbb{C})$ and $(u,v) \in U^{(n,m)}$ such that $(\varphi(x), \varphi(y)) = \left(MJ(u)M^{-1}, MD(v)M^{-1}\right)$. Thus,

$$(t_1,t_2,t_3) = (tr(J(u)), tr(D(v)), tr(J(u)D(v)) = \left(\frac{u^2+1}{u}, \frac{v^2+1}{v}, \frac{u^2v^2+1}{uv}\right).$$

On the other hand, let $(t_1,t_2,t_3) \in \mathbb{C}^3$ such that $(t_1,t_2,t_3) = \left(\frac{u^2+1}{u}, \frac{v^2+1}{v}, \frac{u^2v^2+1}{uv}\right)$ where, $(u,v) \in U^{(n,m)}$. Then, we define $\varphi \in R(\Gamma_{n,m})$ by $\varphi(x) = J(u)$ and $\varphi(y) = D(v)$. It is straightforward to prove that $(t_1,t_2,t_3) = (tr(\varphi(x)), tr(\varphi(y)), tr(\varphi(xy)))$ and $\varphi \in N_R(\Gamma_{n,m})$. Therefore, $(t_1,t_2,t_3) \in X_{N_R}(\Gamma_{n,m})$.

Proposition 12 *Let n, m be two positive integers such that* $n, m > 1$ *and* $\gcd(n,m) = 1$. *Then,* $X_{N_I}(\Gamma_{n,m}) = \bigcup_{k=1}^{9} X_{N_I^k}(\Gamma_{n,m})$, *where* $X_{N_I^k}(\Gamma_{n,m})$ *is equal to the algebraic set of all triplets*

$$\left(\frac{u^2+1}{u}, \frac{v^2+1}{v}, a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + \frac{\left(u^2+1-a_{11}u\right)\left(v^2+1-vb_{11}\right)}{uv}\right) \in \mathbb{C}^3,$$

such that $(u,v) \in U^{(n,m)}$ *and* $R_k(a_{11}, a_{12}, a_{21}, b_{11}, b_{12}, b_{21}, u, v) \neq 0$, *where each* R_k *is like in Theorem 9. Moreover, since u, v lie in a finite set, the closure of this set is given by*

$$\left\{\left(\frac{u^2+1}{u}, \frac{v^2+1}{v}, \lambda\right) : \lambda \in \mathbb{C}\right\} \subset \mathbb{C}^3.$$

Proof Since $N_I(\Gamma_{n,m}) = \bigcup_{k=1}^{9} N_I^k(\Gamma_{n,m})$, where $N_I^k(\Gamma_{n,m})$ is the component that correspond to each one of the nine cases described in Theorem 9, then

$$X_{N_I}(\Gamma_{n,m}) = \bigcup_{k=1}^{9} X_{N_I^k}(\Gamma_{n,m}),$$

where

$$X_{N_I^k}(\Gamma_{n,m}) = \{(tr(\varphi(x)), tr(\varphi(y), tr(\varphi(xy))) \mid \varphi \in N_I^k(\Gamma_{n,m})\},$$

for $k = 1, ..., 9$. From a direct calculation, we prove that $tr(\varphi(x)) = u + u^{-1}$, $tr(\varphi(y)) = v + v^{-1}$ and $tr(\varphi(xy)) = a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + \frac{(u^2+1-a_{11}u)(v^2+1-vb_{11})}{uv}$. Thereby, the proposition holds.

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References

1. Alexander, J.W.: Note on Riemann spaces. Bull. Amer. Math. Soc **26**, 370–372 (1920)
2. Beardon, A.: The Geometry of Discrete Groups. Springer-Verlag, New York (1983)
3. Burde, G., Zieschang, H.: Knots. Walter de Guyter, New York, NY (1985)
7. Crowell, R., Fox, R.: Introduction to Knot Theory. Blaisdell Publishing Company, New York (1963)
8. Culler, M., Shalen, P.: Varieties of group representations and splitting of 3-manifolds. Ann. of Math. **117**, 109–146 (1983)
4. González-Acuña, F.J., Montesinos-Amilibia, J.M.: On the character variety of group representations in $SL(2, \mathbb{C})$ and $PSL(2, \mathbb{C})$. Math. Z. **214**(4), 627–652 (1993)
5. Gordon, C., Luecke, J.: Knots are determined by their complements. J. Amer. Math. Soc. **2**(2), 371–415 (1989)
6. Gordon, C.: Riley's conjecture on SL(2, R) representations of 2-bridge knots. J. Knot Theory Ramifications **26**, 1740003 (2017)
9. Hardy, G.H., Wrigth, E.M.: An Introduction to the Theory of Numbers. Oxford University Press, U. S. A (1975)
10. Hoste, J., Thistlethwaite, M., Weeks, J.: The first 1,701,936 knots. Math. Intelligencer **20**(4), 33–48 (1998)
11. Kauffman, L.: Virtual Knot theory. Europ. J. Combinatorics **20**, 663–691 (1999)
12. Kawauchi, A.: A survery of Knot Theory. Birkhauser, Bassel, Boston, Berlin (1996)
13. Kinsey, L.C.: Topology of surfaces. Springer-Verlag, New York (1993)
14. Kim, S.: Virtual knot groups and their peripheral structure. J. Knot Theory and its Ramifications **9**(6), 797–812 (2000)
15. Klasen, E.P.: Representation of Knot groups in $SU(2)$. Trans. Am. Math. Soc. **326**(2), 795–828 (1991)
16. Liriano, S.: Algebraic geometric invariants for a class of one-relator groups. J. Pure Appl. Algebra **132**, 105–118 (1998)

17. Liriano, S.: Irreducible components in an algebraic variety of representations of a class of one relator groups. Internat. J. Algebra Comput. **9**(1), 129–133 (1999)
18. Lyndon, R.C., Schupp, P.E.: Combinatorial group theory. Springer, Berlin (1977)
19. Maclachlan, C., Reid, A.W.: The arithmetcis of hyperbolic 3-manifolds. Springer, New York (2003)
20. Magnus, W., Karrass, A., Solitor, D.: Combinatorial group theory 2nd, revised Dover Publ, New York (1966)
21. Marden, A.: The geometry of finite generated Kleinian groups. Ann. of Math. **99**, 383–462 (1974)
22. Martín-Morales, J. and Oller-Marcen, A. M.: On the varieties of representations and characters of a family of one-relator subgroups. Their irreducible components, preprint, arXiv:0805.4716v1 [math. AG]
23. Martín-Morales, J., Oller-Marcen, A.M.: Combinatorial aspects of the character variety of a family of one-relator groups. Topol. and its Appl. **156**, 2376–2389 (2009)
24. Martín-Morales, J., Oller-Marcen, A.M.: On the number of irreducible components of the representation variety of a family of one-relator groups. Internat. J. Algebra Comput. **20**(1), 77–87 (2010)
25. Massey, W.: A Basic Course in Algebraic Topology. Springer-Verlag, New York (1991)
26. Mira-Albanés, J.J., Rodríguez-Nieto, J.G., Salazar-Díaz, O.P.: Some Baumslag Solitar groups are two-bridge virtual knot groups. J. Knot Theory and Its Ramifications **26**(4), 1750019 (2017)
27. Mira-Albanés, J.J., Rodríguez-Nieto, J.G., Salazar-Díaz, O.P.: The special rank of virtual knot groups. J. Knot Theory and Its Ramifications **29**(12), 2050088 (2020)
28. Muñoz, V., Porti, J.: Geometry of the $SL(2,\mathbb{C})$-character variety of torus knots, Algebraic and Geometric. Topology **16**, 397–426 (2016)
29. Muñoz, V.: The $SL(2,\mathbb{C})$-character varieties of torus Knots. Rev. Mat. Complut. **22**(2), 489–497 (2009)
30. Oller-Marcén, A.M.: The $SL(2,\mathbb{C})$ character Variety of a Class of torus knots. Extracta Mathematicae. **23**(2), 163–172 (2008)
31. Oller-Marcén, A.M.: $SU(2)$ and $SL(2,\mathbb{C})$ representations of a class of Torus Knots. Extracta Math. **27**(1), 135–144 (2012)
32. Prasad, G.: Strong rigidity of Q-rank 1 lattices. Invent. Math. **21**, 255–286 (1973)
33. Pommerenke, C., Toro, M.: On the two-parabolic subgroups of SL(2, C). Rev. Colombiana Mat. **45**(1), 37–50 (2011)
34. Riley, R.: A quadratic parabolic group. Math. Proc. Cambridge Philos. Soc. **77**(2), 281–288 (1975)
35. Riley, R.: Discrete parabolic representations of link groups. Mathematika **22**(2), 141–150 (1975)
36. Riley, R.: Parabolic representations of knot groups I. Proc. London Math. Soc. (3) **24**, 217–242 (1972)
37. Riley, R.: Parabolic representations of knot groups. II. Proc. London Math. Soc. **31**, 495–512 (1975)
38. Riley, R.: Nonabelian representations of 2-bridge knot groups. Quart. J. Math. Oxford (2) **35**, 191 (1984)
39. Riley, R.: Holomorphically parametrized families of subgroups of $SL(2,\mathbb{C})$. Mathematika **32**, 248–252 (1985)
40. Rodríguez, J., Toro, M.: Virtual Knot groups and combinatorial knots. Sao Paulo J. Math. Sci. **3**(1), 297–314 (2009)
41. Schubert, H.: Knoten mit zwei Brcken. Math. Z. **65**, 133–170 (1956)
42. Thurston, W.P.: Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. Bull. Amer. Math. Soc. (N.S.) **6**(3), 357–381 (1982)