São Paulo Journal of Mathematical Sciences (2023) 17:511–514
https://doi.org/10.1007/s40863-023-00372-w

ORIGINAL ARTICLE

On the characteristic polynomial of $\mathfrak{sl}(2, \mathbb{F})$: a corollary that Muir missed

K. Castillo[1]

Accepted: 18 July 2023 / Published online: 10 August 2023

Abstract
In this note we show how conjectures and current problems on determinants and eigenvalues of highly structured tridiagonal matrices can be solved using very classical results.

Keywords Hu–Zhan's conjecture · Sylvester's type determinants · Muir's theorem

Mathematics Subject Classification 15A15

In [4, p. 432], Z. Hu and P. B. Zhang conjectured that

$$Z_{n+1} = \begin{vmatrix} z_0+nz_1 & nz_2 & & & \\ z_3 & z_0+(n-2)z_1 & \ddots & & \\ & \ddots & \ddots & 2z_2 & \\ & & (n-1)z_3 & z_0-(n-2)z_1 & z_2 \\ & & & nz_3 & z_0-nz_1 \end{vmatrix} \tag{1}$$
$$= \prod_{k=0}^{n}\left(z_0-(n-2k)\sqrt{z_1^2+z_2z_3}\right).$$

The authors were only able to calculate the determinant in two cases: in the case when $z_0 = 0$ and in the case when $z_1 = 0$. In [1], Chen et al. prove the above conjecture in connection with the characteristic polynomial of a finite-dimensional Lie

Communicated by José Alberto Cuminato.

✉ K. Castillo
kenier@mat.uc.pt

1 CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal

algebra. In [3], the finite-dimensional Lie algebra is explored for the same purpose. However, this approach masks the simplicity of the problem being addressed, producing long proofs of simple results. Indeed, these determinants are implicitly contained in the elementary lore of the "Theory of Determinants", which finds its roots in a note published in Nouvelles Annales de Mathématiques in 1854 by J. J. Sylvester. For the reader's convenience, we reproduce below in their entirety (see [5, pp. 544–545]) two theorems from "A treatise on the Theory of Determinants" by T. Muir, in the edition revised and enlarged by W. H. Metzler, from which (1) trivially follows:

"576. The continuant

$$\Delta_n = \begin{vmatrix} a & b & & & & \\ -(n-1)c & a-(b+c) & 2b & & & \\ & -(n-2)c & a-2(b+c) & 3b & & \\ & & \ddots & \ddots & \ddots & \\ & & & -c & a-(n-1)(b+c) \end{vmatrix}_n$$

$$\equiv \phi_n(a,b,c) \text{ say,}$$

$$= (a-\overline{n-1}c)(a-\overline{n-2}c-b)(a-\overline{n-3}c-2b)\cdots(a-\overline{n-1}b).''$$

(This is $\Delta_n = (a-(n-1)c)(a-(n-2)c-b)(a-(n-3)c-2b)\cdots(a-(n-1)b)$.)

"577. The foregoing leads to the theorem that the value of the continuant Δ_n is not altered by adding to its matrix the matrix of the continuant

$$D_n = \begin{vmatrix} (n-1)x & x & & & & \\ (1-n)x & (n-3)x & 2x & & & \\ & (2-n)x & (n-5)x & & & \\ & & (3-n)x & & & \\ & & \ddots & \ddots & \ddots & \\ & & & & -(n-3)x & (n-1)x \\ & & & & -x & -(n-1)x \end{vmatrix}.''$$

We have never seen the above result applied in the literature. However such results are extremely flexible and useful. By Theorem 576, we see at once that $Z_n = \Delta_n$ for $a = z_0$, $b = -\sqrt{z_1^2 + z_2 z_3}$ and $c = -b$. For these values of a, b and c, add to the corresponding matrix of the determinant Δ_n the matrix of the determinant D_n with $x = z_1$ to get a matrix whose transpose is similar to the matrix of the determinant Z_n, and so Hu–Zhan's conjecture follows, because these operations, according to Theorem 577, have not altered Δ_n. Indeed,

$$\Delta_n = \begin{vmatrix} z_0 & -\sqrt{z_1^2+z_2z_3} & & \\ -(n-1)\sqrt{z_1^2+z_2z_3} & z_0 & -2\sqrt{z_1^2+z_2z_3} & \\ \ddots & \ddots & \ddots & \\ & & -\sqrt{z_1^2+z_2z_3} & z_0 \end{vmatrix}$$

$$= \prod_{k=0}^{n-1}\left(z_0-(n-2k-1)\sqrt{z_1^2+z_2z_3}\right)$$

$$= \begin{vmatrix} z_0+(n-1)z_1 & -\sqrt{z_1^2+z_2z_3}+z_1 & & \\ (n-1)(-\sqrt{z_1^2+z_2z_3}-z_1) & z_0+(n-3)z_1 & & \\ & \ddots & & \ddots \\ & \ddots & & \\ & -\sqrt{z_1^2+z_2z_3}-z_1 & & z_0-(n-1)z_1 \end{vmatrix}$$

$$= Z_n.$$

Naturally, according to [2, Lemma 7.2, p. 32], and taking into account the relation between the elements of the sub-diagonals of the considered matrices (regardless of the value by which they appear multiplied), we can make a direct connection with $\mathfrak{sl}(2,\mathbb{F})$. But, when calculating this and other related determinants that fall into what might be called Sylvester's type determinants, we only need a little trick to transform known results into new results. The reader can look for other recent results in the literature that can be easily proved with the help of Muir's theorems. Clearly, it is not our goal to cite them here, because it would not make this note stronger, only longer.

Acknowledgements This work was supported by the Centre for Mathematics of the University of Coimbra-UIDB/00324/2020, funded by the Portuguese Government through FCT/ MCTES.

Funding Open access funding provided by FCT|FCCN (b-on).

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Chen, Z., Chen, X., Ding, M.: On the characteristic polynomial of $\mathfrak{sl}(2,\mathbb{F})$. Linear Algebra Appl. **579**, 237–243 (2019)
2. Humphreys, J.E.: Introduction to Lie Algebras and Representation Theory. Third Printing, Revised. Graduate Texts in Mathematics, vol. 9. Springer, New York (1980)
3. Hu, Z.: Eigenvalues and eigenvectors of a class of irreducible tridiagonal matrices. Linear Algebra Appl. **619**, 328–337 (2021)
4. Hu, Z., Zhang, P.B.: Determinants and characteristic polynomials of Lie algebras. Linear Algebra Appl. **563**, 426–439 (2019)
5. Muir, T.: A Treatise on the Theory of Determinants. Revised and enlarged by William H. Metzler Dover Publications, Inc., New York (1960)