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ORIGINAL ARTICLE

A decomposition theorem for finite-dimensional Jordan superalgebras whose simple part is $\mathfrak{K}\mathrm{an}(2)$

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Abstract

We consider a finite-dimensional Jordan superalgebra $\mathcal{A}$ over a field of characteristic zero $\mathbb{F}$ such that $\mathcal{N}$ is the solvable radical of $\mathcal{A}$. We proved that if $\mathcal{N}^2 = 0$ and $\mathcal{A}/\mathcal{N}$ is isomorphic to simple Jordan superalgebra of Grassmann Poisson bracket $\mathfrak{K}\mathrm{an}(2)$, then an analogous to Wedderburn Principal Theorem (WPT) holds.

Keywords Jordan superalgebras · Wedderburn principal theorem · Decomposition theorem · Semisimple Jordan superalgebra · Second cohomology group · Grassmann poisson bracket

Mathematics Subject Classification 17C70 · 17A70 · 17C10

1 Introduction

This paper is a continuation of a series of papers where there were proven analogous versions to WPT for finite-dimensional Jordan superalgebras [1–3].

One of the most classical theorems in structure theory for finite-dimensional associative algebras was given by Wedderburn [4], proving that for all finite dimensional associative algebra $\mathcal{A}$ over an arbitrary field, with nilpotent radical $\mathcal{N}$, there exists a subalgebra $\mathcal{S} \subseteq \mathcal{A}$ such that $\mathcal{S} \cong \mathcal{A}/\mathcal{N}$ and $\mathcal{A} = \mathcal{S} \oplus \mathcal{N}$. This theorem is known in classical literature as the *Wedderburn Principal Theorem (WPT)*.

Communicated by Vyacheslav Futorny.

To Ivan Pavlovich Shestakov on the occasion of his 75th birthday.

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Analogous versions of the WPT were proved for finite-dimensional Jordan algebras by Albert [5], Penico [6], and Aškinuze [7].

Superalgebras are algebras that admit a decomposition as a direct sum of vector spaces $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ and it satisfies the multiplicative rule $\mathcal{A}_i\mathcal{A}_j \subseteq \mathcal{A}_{i+j \bmod 2}$. $\mathcal{A}_0$ and $\mathcal{A}_1$ are called respectively even and odd parts of $\mathcal{A}$. It is well known that all associative algebra is an associative superalgebra. However, it is not a general rule. For example, if $\mathfrak{J}$ is a Jordan superalgebra with odd part non-zero, then $\mathfrak{J}$ is not a Jordan algebra while $\mathfrak{J}_0$ is a Jordan algebra. It arises as a natural question to research the validity of the WPT for finite-dimensional non-associative superalgebras. Note that this is equivalent to investigating which superalgebras $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ with radical $\mathcal{N}$ there exists a $\mathcal{S}_0$-superbimodule $\mathcal{S}_1 \subseteq \mathcal{A}_1$ such that $\mathcal{S}_1 \cong \mathcal{A}_1/\mathcal{N}_1$, $\mathcal{A}_1 = \mathcal{S}_1 \oplus \mathcal{N}_1$, $\mathcal{A} = (\mathcal{S}_0 \oplus \mathcal{S}_1) \oplus (\mathcal{N}_0 \oplus \mathcal{N}_1)$, where $\mathcal{S}_0 \subseteq \mathcal{A}_0$ is a semisimple superalgebra, and $\mathcal{A}_0/\mathcal{N}_0 \cong \mathcal{S}_0$, $\mathcal{A}_0 = \mathcal{S}_0 \oplus \mathcal{N}_0$.

Additionally, it is known that on the variety of finite-dimensional Jordan and alternative algebras, solubility implies nilpotence. However, in finite-dimensional Jordan and alternative superalgebras Shestakov [8] proved that there exist solvable superalgebras that are not nilpotent. In this sense as a first step, we consider solvable radical to study the validity of WPT in Jordan superalgebras. The first author [1] proved that it is possible to have an analogous version to WPT for finite-dimensional Jordan superalgebras when some conditions are imposed over the irreducible superbimodules contained in the solvable radical. First, as in the case of Jordan algebras, he proved that it is possible to reduce the problem to Jordan superalgebra $\mathcal{A}$ with solvable radical $\mathcal{N}$ with $\mathcal{N}^2 = 0$ such that the quotient Jordan superalgebra $\mathcal{A}/\mathcal{N}$ is isomorphic to a simple Jordan superalgebra $\mathfrak{J}$. As a consequence of this result, it is possible to consider case by case of simple Jordan superalgebra. Second, he proved that if $\mathfrak{J}$ is a simple Jordan superalgebra then it is possible to reduce the proof of WPT considering the irreducible Jordan $\mathfrak{J}$-superbimodules that are contained in $\mathcal{N}$.

The classification of finite-dimensional simple Jordan superalgebras over an algebraically closed field of characteristic zero was given by Kac [9] and Kantor [10].

In this paper, Sect. 2 gives some preliminary results regarding the theory of Jordan superalgebras, including the definition of the superalgebra of the Grassmann Poisson bracket $\mathfrak{K}\text{an}(2)$, and the classification of irreducible Jordan $\mathfrak{K}\text{an}(2)$– superbimodules given by Folleco and Shestakov [11]. Section 3 presents the conditions over $\mathfrak{r}(\widetilde{x}_i\widetilde{x}_j) \in \mathcal{N}$, where $\widetilde{x}_i \in \mathcal{A}$ is a preimage under canonical homomorphism of $\bar{\bar{x}}_i$, $\bar{\bar{x}}_i \in \overline{\overline{X}}$ and $\overline{\overline{X}}$ is an additive basis of $\mathcal{A}/\mathcal{N}$ such that $\bar{S} = \text{alg}\,\langle \overline{\overline{X}}_i \rangle \cong \mathfrak{K}\text{an}(2)$. We assume that $\widetilde{x}_i\widetilde{x}_j = \sum_k \alpha_k \widetilde{x}_k + \mathfrak{r}(\widetilde{x}_i\widetilde{x}_j)_k$. Finally, Sect. 4 provides a proof of the main theorem of this paper. We shall prove that if $\mathcal{N}^2 = 0$ and $\mathcal{A}/\mathcal{N}$ is isomorphic to the simple Jordan superalgebra of Grassmann Poisson bracket $\mathfrak{K}\text{an}(2)$, then an analogous to WPT holds.

2 Basic concepts and notation

Throughout the paper, all algebras are considered over an algebraically closed field $\mathbb{F}$ of characteristic zero.

Recall that a superalgebra $\mathfrak{J} = \mathfrak{J}_0 \oplus \mathfrak{J}_1$ is said to be a *Jordan superalgebra*, if for every $a, b, c, d \in \mathfrak{J}_0 \dot{\cup} \mathfrak{J}_1$ the superalgebra satisfies the super identities

$$ab = (-1)^{|a||b|} ba, \tag{2.1}$$

$$\begin{aligned} &((ab)c)d + (-1)^{|d|(|c|+|b|)+|b||c|}((ad)c)b + (-1)^{|a|(|b|+|c|+|d|)+|d||c|}((bd)c)a \\ &= (ab)(cd) + (-1)^{|d|(|c|+|b|)}(ad)(bc) + (-1)^{|c|(|a|+|b|)}(ac)(bd). \end{aligned} \tag{2.2}$$

We denote the parity of a by $|a| = i$ if $a \in \mathfrak{J}_i$.

Let $\mathcal{N} = \mathcal{N}_0 \oplus \mathcal{N}_1$ be a superbimodule over $\mathfrak{J}$. We say that $\mathcal{N}$ is a Jordan $\mathfrak{J}$-superbimodule if the split null extension $\mathcal{E} = \mathfrak{J} \oplus \mathcal{N}$ is a Jordan superalgebra. A *regular* $\mathfrak{J}$*-superbimodule*, denoted as $\mathcal{R}\mathrm{eg}\,\mathfrak{J}$, is defined on the vector superspace $\mathfrak{J}$ with an action coinciding with the multiplication in $\mathfrak{J}$. Besides, if $\mathcal{N}$ is a $\mathfrak{J}$-superbimodule, then the superbimodule $\mathcal{N}^{\mathrm{op}}$ is defined as a copy of $\mathcal{N}$ where $\mathcal{N}_0^{\mathrm{op}} = \mathcal{N}_1$, $\mathcal{N}_1^{\mathrm{op}} = \mathcal{N}_0$, and the action is defined via

$$am^{\mathrm{op}} = (-1)^{|a|}(am)^{\mathrm{op}}, \qquad m^{\mathrm{op}}a = (ma)^{\mathrm{op}} \tag{2.3}$$

for all $a \in \mathfrak{J}_0 \dot{\cup} \mathfrak{J}_1$ and $m \in \mathcal{N}_0 \dot{\cup} \mathcal{N}_1$. $\mathcal{N}^{\mathrm{op}}$ is called the opposite of the superbimodule $\mathcal{N}$.

Now, we consider the simple Jordan superalgebra of type Poisson Grassmann bracket $\mathfrak{K}\mathrm{an}(2) := (\mathbb{F} \cdot 1 + \mathbb{F} \cdot f_1 + \mathbb{F} \cdot f_2 + \mathbb{F} \cdot e_{12}) \oplus (\mathbb{F} \cdot u + \mathbb{F} \cdot e_1 + \mathbb{F} \cdot e_2 + \mathbb{F} \cdot f_{12})$ where 1 is a unity, $f_1 \bullet f_1 = f_2 \bullet f_2 = 1$, and nonzero products are given by

$$f_2 \bullet e_1 = -f_1 \bullet e_2 = e_{12} \bullet u = f_{12}, \quad f_1 \bullet f_{12} = -e_2, \quad f_2 \bullet f_{12} = e_1. \tag{2.4}$$

$$e_1 \bullet e_2 = e_{12}, \quad e_i \bullet u = f_i, \tag{2.5}$$

for $i = 1, 2$. Products in (2.4) and (2.5) are symmetric and skew-symmetric respectively. Irreducible Jordan superbimodules over the simple Jordan superalgebra Poisson Grassmann Bracket $\mathfrak{K}\mathrm{an}(n)$ were classified by O. Folleco and I. Shestakov [11]. In particular, it was proved that if $\mathcal{V}$ is an unital irreducible Jordan superbimodule over $\mathfrak{K}\mathrm{an}(2)$, then there exists a special element v in $\mathcal{V}$ such that $v \cdot e_I = v \cdot f_I = 0$ where $I \in \{1, 2, 12\}$. Besides, they proved that the action of $\mathfrak{K}\mathrm{an}(2)$ on $\mathcal{V}(v, \alpha)$ depends on the choice of a special element $v \in \mathcal{V}(v, \alpha)$ and a parameter $\alpha = \mathcal{R}_1^2 \in \mathbb{F}$, where $\mathcal{R}_x$ denotes the right multiplication operator. Without loss of generality, we can assume that v is an even element. Then, for an additive basis for $\mathcal{V}$ given by $v, w_1, w_2, v_{12}, w,\ v_1,\ v_2,$ and w_{12} where v, w_1, w_2, v_{12} are even elements and w, v_1, v_2, w_{12} are odd elements, the nonzero right action of $\mathfrak{K}\mathrm{an}(2)$ over $\mathcal{V}(v, \alpha)$ is given by the following relations:

$$
\begin{aligned}
&v\cdot u = v_1\cdot f_1 = v_2\cdot f_2 = -w_{12}\cdot e_{12} = -w_1\cdot e_1 = -w_2\cdot e_2 = -v_{12}\cdot f_{12} = w,\\
&w\cdot f_1 = w_2\cdot f_{12} = v_{12}\cdot e_2 = v_1, w\cdot f_2 = -w_1\cdot f_{12} = -v_{12}\cdot e_1 = v_2,\\
&v_{12}\cdot f_2 = v_1\cdot u = -w_{12}\cdot e_2 = w_1, -v_{12}\cdot f_1 = v_2\cdot u = w_{12}\cdot e_1 = w_2,\\
&w_1\cdot f_2 = -w_2\cdot f_1 = v_{12}, v_{12}\cdot u = w_{12},\\
&w_1\cdot u = \alpha v_1,\quad w_2\cdot u = \alpha v_2,\quad w_{12}\cdot u = \alpha v_{12},\qquad w\cdot u = w_{12}\cdot f_{12} = \alpha v,\\
&v_1\cdot e_1 = v_2\cdot e_2 = -v_{12}\cdot e_{12} = v,
\end{aligned}
\tag{2.6}
$$

where $\alpha \in \mathbb{F}$. Note that if $\alpha = 0$, then $\mathcal{V}(v, 0) = \mathcal{R}\mathrm{eg}\,\mathfrak{K}\mathrm{an}(2)$.

Now, let $\mathcal{A}$ be a finite-dimensional Jordan superalgebra with radical $\mathcal{N}$ such that $\mathcal{N}^2 = 0$, and $\mathcal{A}/\mathcal{N} \cong \mathfrak{K}\mathrm{an}(2)$. By [1] it follows that an analog to WPT is valid for $\mathcal{A}$ if it is valid for each irreducible $\mathfrak{K}\mathrm{an}(2)$-superbimodule. Due to [11], we just need to consider two cases, $\mathcal{N} \cong \mathcal{V}(v, \alpha)$, and $\mathcal{N} \cong \mathcal{V}(v, \alpha)^{\mathrm{op}}$.

As a consequence of WPT for Jordan algebras it follows that there exist $\widetilde{1}$, $\widetilde{f}_1$, $\widetilde{f}_2$, $\widetilde{e}_{12} \in \mathcal{A}_0$ such that $\mathcal{S}_0 = \mathrm{alg}\langle\widetilde{1}, \widetilde{f}_1, \widetilde{f}_2, \widetilde{e}_{12}\rangle \cong (\mathfrak{K}\mathrm{an}(2))_0$, and $\mathcal{A}_0 = \mathcal{S}_0 \oplus \mathcal{N}_0$. Since $(\mathcal{A}/\mathcal{N})_1 \cong \mathcal{A}_1/\mathcal{N}_1 \cong (\mathfrak{K}\mathrm{an}(2))_1$, then there exist elements $\widetilde{u}$, $\widetilde{e}_1$, $\widetilde{e}_2$ and $\widetilde{f}_{12} \in \mathcal{A}_1$ such that $\mathrm{vec}\langle\overline{\widetilde{u}}, \overline{\widetilde{e}_1}, \overline{\widetilde{e}_2}, \overline{\widetilde{f}_{12}}\rangle \cong (\mathfrak{K}\mathrm{an}(2))_1$, where $\overline{\widetilde{x}}$ denotes the image of $\widetilde{x}$ under canonical homomorphism. In the following, we identify elements $\widetilde{x} \in \mathcal{A}$ with elements $x \in \mathfrak{K}\mathrm{an}(2)$. In the sequel, we denote the products in $\mathcal{A}$ by juxtaposition, i.e., for $\widetilde{x}$ and $\widetilde{y} \in \mathcal{A}$ we write $\widetilde{x}\widetilde{y}$. Note that products $\widetilde{x}\widetilde{y} \in \mathcal{A}$ can be written as $\widetilde{x}\widetilde{y} = \widetilde{x\bullet y} + \mathfrak{r}(\widetilde{x}\widetilde{y})$ where $\mathfrak{r}(\widetilde{x}\widetilde{y}) \in \mathcal{N}$. $\mathfrak{r}(\widetilde{x}\widetilde{y})$ is called *the radical component of the product* $\widetilde{x}\widetilde{y}$. So, given $\widetilde{n} \in \mathcal{N}$ and $\widetilde{x} \in \mathcal{A}$, we denote $\widetilde{n}\widetilde{x} = \widetilde{n\cdot x}$, where $\widetilde{n}$ is identified with $n \in \mathcal{V}(v, \alpha)$ ($n \in \mathcal{V}(v, \alpha)^{\mathrm{op}}$). Without loss of generality, we can assume the following products in $\mathcal{A}$: skew-symmetric products are given by

$$
\begin{aligned}
\widetilde{u}\widetilde{u} &= \mathfrak{r}(\widetilde{u}\widetilde{u}), & \widetilde{e}_i\widetilde{e}_i &= \mathfrak{r}(\widetilde{e}_i\widetilde{e}_i), \widetilde{f}_{12}\widetilde{f}_{12} = \mathfrak{r}(\widetilde{f}_{12}\widetilde{f}_{12}),\\
\widetilde{e}_1\widetilde{f}_{12} &= \mathfrak{r}(\widetilde{e}_1\widetilde{f}_{12}), & \widetilde{e}_2\widetilde{f}_{12} &= \mathfrak{r}(\widetilde{e}_2\widetilde{f}_{12}), \widetilde{f}_{12}\widetilde{u} = \mathfrak{r}(\widetilde{f}_{12}\widetilde{u}),\\
\widetilde{e}_1\widetilde{u} &= \widetilde{f}_1 + \mathfrak{r}(\widetilde{e}_1\widetilde{u}), & \widetilde{e}_2\widetilde{u} &= \widetilde{f}_2 + \mathfrak{r}(\widetilde{e}_2\widetilde{u}), \widetilde{e}_1\widetilde{e}_2 = \widetilde{e}_{12} + \mathfrak{r}(\widetilde{e}_1\widetilde{e}_2),
\end{aligned}
\tag{2.7}
$$

while symmetric products are given by

$$
\begin{aligned}
\widetilde{f}_i\widetilde{f}_i &= \widetilde{1}\widetilde{1} = \widetilde{1}, & \widetilde{1}\widetilde{f}_i &= \widetilde{f}_i, \widetilde{1}\widetilde{e}_{12} = \widetilde{e}_{12},\\
\widetilde{1}\widetilde{u} &= \widetilde{u} + \mathfrak{r}(\widetilde{1}\widetilde{u}), & \widetilde{1}\widetilde{e}_i &= \widetilde{e}_i + \mathfrak{r}(\widetilde{1}\widetilde{e}_i), \widetilde{1}\widetilde{f}_{12} = \widetilde{f}_{12} + \mathfrak{r}(\widetilde{1}\widetilde{f}_{12}),\\
\widetilde{f}_i\widetilde{u} &= \mathfrak{r}(\widetilde{f}_i\widetilde{u}), & \widetilde{f}_1\widetilde{e}_1 &= \mathfrak{r}(\widetilde{f}_1\widetilde{e}_1), \widetilde{f}_2\widetilde{e}_2 = \mathfrak{r}(\widetilde{f}_2\widetilde{e}_2),\\
\widetilde{e}_{12}\widetilde{f}_{12} &= \mathfrak{r}(\widetilde{e}_{12}\widetilde{f}_{12}), & \widetilde{f}_1\widetilde{f}_{12} &= -\widetilde{e}_2 + \mathfrak{r}(\widetilde{f}_1\widetilde{f}_{12}), \widetilde{f}_2\widetilde{f}_{12} = \widetilde{e}_1 + \mathfrak{r}(\widetilde{f}_2\widetilde{f}_{12}),\\
\widetilde{f}_1\widetilde{e}_2 &= -\widetilde{f}_{12} + \mathfrak{r}(\widetilde{f}_1\widetilde{e}_2), & \widetilde{f}_2\widetilde{e}_1 &= \widetilde{f}_{12} + \mathfrak{r}(\widetilde{f}_2\widetilde{e}_1), \widetilde{e}_{12}\widetilde{u} = \widetilde{f}_{12} + \mathfrak{r}(\widetilde{e}_{12}\widetilde{u}),
\end{aligned}
\tag{2.8}
$$

for $i = 1, 2$, where $\mathfrak{r}(\widetilde{x}\widetilde{y}) \in \mathcal{N}_0\dot{\cup}\mathcal{N}_1$, and $\mathfrak{r}(\widetilde{x}\widetilde{y}) = (-1)^{|\widetilde{x}||\widetilde{y}|}\mathfrak{r}(\widetilde{y}\widetilde{x})$.

Now, using the identity (2.2) and the fact that $\mathcal{N}$ is a unitary irreducible bimodule over $\mathcal{A}$, it follows that for all $\widetilde{a} \in \mathcal{A}_1$ holds $2((\widetilde{1}a)\widetilde{1})\widetilde{1} + \widetilde{1}\widetilde{a} = 3(\widetilde{1}\widetilde{a})\widetilde{1}$. So, using this equation and assuming $\widetilde{1}\widetilde{a} = \widetilde{a} + \widetilde{r}$ where $\widetilde{r} \in \mathcal{N}$, as a consequence, we conclude that $\widetilde{r} = 0$. Then, we write $\widetilde{1}\widetilde{u} = \widetilde{u}$, $\widetilde{1}\widetilde{e}_i = \widetilde{e}_i$, and $\widetilde{1}\widetilde{f}_{12} = \widetilde{f}_{12}$ for $i = 1, 2$. Therefore, we obtain that $\widetilde{1}$ is the unity in $\mathcal{A}$.

3 Conditions on the radical component of products in $\mathcal{A}$

In this section, we investigate the conditions that are satisfied by the radical components of products $\widetilde{x}\widetilde{y} \in \mathcal{A}$. We need to consider two cases $\mathcal{N} \cong \mathcal{V}(v, \alpha)$, and $\mathcal{N} \cong \mathcal{V}(v, \alpha)^{\mathrm{op}}$. From now on, let $\widetilde{1}$, $\widetilde{f}_1$, $\widetilde{f}_2$, $\widetilde{e}_{12}$, $\widetilde{u}$, $\widetilde{e}_1$, $\widetilde{e}_2$ and $\widetilde{f}_{12} \in \mathcal{A}$ such that (2.7) and (2.8) holds.

3.1 Case 1

Assume that $\mathcal{N}_0 = \mathrm{span}\langle v, w_1, w_2, v_{12}\rangle$, and $\mathcal{N}_1 = \mathrm{span}\langle w, v_1, v_2, w_{12}\rangle$ such that $\mathcal{N} \cong \mathcal{V}(v, \alpha)$ with nonzero actions given by (2.6).

Initially, we present some lemmas used mainly to make short the proof of WPT in this case.

Lemma 3.1 *Let $\widetilde{f}_1, \widetilde{f}_2, \widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_{12}, \widetilde{f}_{12}$, and $\widetilde{u}$ be as equations (2.7) and (2.8) given in 2. Then there exist scalars ξ_0, η_0, α_0, δ_0 τ_0, σ_0, ϵ_0 ξ_1, α_1, δ_2, such that $\widetilde{f}_1\widetilde{u} = \xi_0 w + \xi_1 v_1$, $\widetilde{f}_2\widetilde{u} = \eta_0 w + \xi_1 v_2$, $\widetilde{f}_1\widetilde{e}_1 = \alpha_0 w + \alpha_1 v_1$, $\widetilde{f}_2\widetilde{e}_2 = \delta_0 w + \delta_2 v_2$, $\widetilde{e}_{12}\widetilde{e}_1 = \tau_0 w$, $\widetilde{e}_{12}\widetilde{e}_2 = \sigma_0 w$, and $\widetilde{e}_{12}\widetilde{f}_{12} = \epsilon_0 w$.*

Proof Let us consider $a = c = d = \widetilde{f}_i$, and $b = \widetilde{x} \in \mathcal{A}_1$ in the equation (2.2), we conclude that $((\widetilde{f}_i\widetilde{x})\widetilde{f}_i)\widetilde{f}_i = \widetilde{f}_i\widetilde{x}$. Since $\widetilde{f}_i\widetilde{x} \in \mathcal{N}_1$, we write $\widetilde{f}_i\widetilde{x} = Aw + B_1 v_1 + B_2 v_2 + Cw_{12}$ where A, B_i, and $C \in \mathbb{F}$ are arbitrary scalars for $i = 1, 2$. By linearly independence of elements w, and v_i we obtain $\widetilde{f}_i\widetilde{x} = Aw + B_i v_i$ for $i = 1, 2$. Thus, we have that there exist scalars ξ_i, α_i, η_j, $\delta_j \in \mathbb{F}$ such that $\widetilde{f}_1\widetilde{u} = \xi_0 w + \xi_1 v_1$, $\widetilde{f}_1\widetilde{e}_1 = \alpha_0 w + \alpha_1 v_1$, $\widetilde{f}_2\widetilde{u} = \eta_0 w + \eta_2 v_2$, and $\widetilde{f}_2\widetilde{e}_2 = \delta_0 w + \delta_2 v_2$.

Now, taking $a = \widetilde{f}_1$, $b = \widetilde{u}$, and $c = d = \widetilde{f}_2$ in the equation (2.2), we get that $(\mathfrak{r}(\widetilde{f}_1\widetilde{u})\widetilde{f}_2)\widetilde{f}_2 + (\mathfrak{r}(\widetilde{f}_2\widetilde{u})\widetilde{f}_2)\widetilde{f}_1 - \mathfrak{r}(\widetilde{f}_1\widetilde{u}) = 0$. Consequently, we conclude that $\eta_2 = \xi_1$. It is easy to see that if $r \in \mathcal{N}_1$, then $(r\widetilde{f}_i)\widetilde{e}_{12} = 0$. So, substituting $a = c = \widetilde{f}_1$, $b = \widetilde{e}_i$, and $d = \widetilde{e}_{12}$ in the equation (2.2), we have that $((\widetilde{f}_i\widetilde{e}_i)\widetilde{f}_i)\widetilde{e}_{12} + ((\widetilde{e}_{12}\widetilde{e}_i)\widetilde{f}_i)\widetilde{f}_i = \widetilde{e}_{12}\widetilde{e}_i$. Noting that $\widetilde{f}_i\widetilde{e}_i \in \mathcal{N}_1$, it is clear that equality $\mathfrak{r}((\widetilde{e}_{12}\widetilde{x})\widetilde{f}_i)\widetilde{f}_i - \mathfrak{r}(\widetilde{e}_{12}\widetilde{x}) = 0$ holds, and therefore we have scalars τ_0, τ_1, σ_0, and σ_2 such that $\widetilde{e}_{12}\widetilde{e}_1 = \tau_0 w + \tau_1 v_1$, and $\widetilde{e}_{12}\widetilde{e}_2 = \sigma_0 w + \sigma_2 v_2$.

Considering $a = d = \widetilde{e}_{12}$, $b = \widetilde{u}$, and $c = \widetilde{f}_1$ in the equation (2.2), we obtain $((\widetilde{e}_{12}\widetilde{u})\widetilde{f}_1)\widetilde{e}_{12} = 0$. Expanding this equation, we have that $(\mathfrak{r}(\widetilde{e}_{12}\widetilde{u})\widetilde{f}_1)\widetilde{e}_{12} + \mathfrak{r}(\widetilde{f}_1\widetilde{f}_{12})\widetilde{e}_{12} - \mathfrak{r}(\widetilde{e}_{12}\widetilde{e}_2) = 0$. From this, and using the fact that $(\mathfrak{r}(\widetilde{e}_{12}\widetilde{u})\widetilde{f}_1)\widetilde{e}_{12} = 0$, we conclude that $\mathfrak{r}(\widetilde{f}_1\widetilde{f}_{12})\widetilde{e}_{12} - \mathfrak{r}(\widetilde{e}_{12}\widetilde{e}_2) = 0$. Now, let $\widetilde{f}_j\widetilde{f}_{12} = A_j w + B_{j1} v_1 + B_{j2} v_2 + C_j w_{12}$ with $A_j, B_{ji}, C_j \in \mathbb{F}$, for $i, j = 1, 2$; substituting this in the last equality with $j = 1$, we determine that $-C_1 w - (\sigma_0 w + \sigma_2 v_2) = 0$.

Then, by linear independence of elements, we conclude that $\sigma_2 = 0$, $C_1 = -\sigma_0$. Similarly, with $j = 2$, we deduce that $\tau_1 = 0$ and $C_2 = \tau_0$. Thus, we can write

$$\widetilde{e}_{12}\widetilde{e}_1 = \tau_0 w, \quad \text{and} \quad \widetilde{e}_{12}\widetilde{e}_2 = \sigma_0 w. \tag{3.1}$$

Putting $a = \widetilde{f}_i$, $b = \widetilde{u}$, and $c = d = \widetilde{e}_{ij}$ in the equation (2.2), we have that $((\widetilde{f}_i\widetilde{u})\widetilde{e}_{ij})\widetilde{e}_{ij} + ((\widetilde{e}_{ij}\widetilde{u})\widetilde{e}_{ij})\widetilde{f}_i = 0$, and it follows that

$$\mathfrak{r}((\widetilde{e}_{ij}\widetilde{u})\widetilde{e}_{ij})\widetilde{f}_i + \mathfrak{r}(\widetilde{e}_{ij}\widetilde{f}_{ij})\widetilde{f}_i = 0. \tag{3.2}$$

Let $\mathfrak{r}(\widetilde{e}_{12}\widetilde{u}) = \varphi_0 w + \varphi_1 v_1 + \varphi_2 v_2 + \varphi_{12} w_{12}$, and $\mathfrak{r}(\widetilde{e}_{12}\widetilde{f}_{12}) = \epsilon_0 w + \epsilon_1 v_1 + \epsilon_2 v_2 + \epsilon_{12} w_{12}$, with φ_i, $\epsilon_i \in \mathbb{F}$ for $i \in \{0, 1, 2, 12\}$. Replacing these equalities in the equation (3.2) and using the fact that w, v_1, v_2, and w_{12} are linearly independent, we get that $\epsilon_1 = \epsilon_2 = 0$, and $\epsilon_0 = \varphi_{12}$. Finally, substituting $a = c = d = \widetilde{e}_{12}$, and $b = \widetilde{u}$ in the equation (2.2), we obtain that $((\widetilde{e}_{12}\widetilde{u})\widetilde{e}_{12})\widetilde{e}_{12} = 0$. Thus, by simplifying the last equation, we get that $\mathfrak{r}(\widetilde{e}_{12}\widetilde{f}_{12})\widetilde{e}_{12} = 0$, hence $\epsilon_{12} = 0$. Therefore, we can write

$$\widetilde{e}_{12}\widetilde{f}_{12} = \epsilon_0 w, \tag{3.3}$$

which completes the proof. □

Lemma 3.2 *If* $\widetilde{x} \in \{\widetilde{e}_1, \widetilde{e}_2, \widetilde{f}_{12}\}$, *then* $\widetilde{x}\widetilde{x} = 0$. *If* $\alpha \neq 0$ *then* $\widetilde{u}\widetilde{u} = 0$; *otherwise, there exists* $\Lambda_u \in \mathbb{F}$ *such that* $\widetilde{u}\widetilde{u} = \Lambda_u v_{12}$.

Proof Assume that $\widetilde{x}\widetilde{x} = A_x v + B_x w_1 + C_x w_2 + D_x v_{12}$, where A_x, B_x, C_x, and $D_x \in \mathbb{F}$. Putting $a = b = \widetilde{x} \in \mathcal{A}_1$, $c = d = \widetilde{f}_i$ in the equation (2.2) using linear independence of v, w_i, and v_{ij}, we get $((\widetilde{x}\widetilde{x})\widetilde{f}_i)\widetilde{f}_i = \widetilde{x}\widetilde{x}$. Then, we write $\mathfrak{r}(\widetilde{x}\widetilde{x}) = D_x v_{12}$. Now, taking $a = d = \widetilde{e}_{ij}$, $b = \widetilde{u}$, and $c = \widetilde{f}_{ij}$ in the equation (2.2), we conclude that $((\widetilde{e}_{ij}\widetilde{u})\widetilde{f}_{ij})\widetilde{e}_{ij} - (\widetilde{e}_{ij}\widetilde{u})(\widetilde{e}_{ij}\widetilde{f}_{ij}) = 0$. Thus, it follows that $(\mathfrak{r}(\widetilde{e}_{ij}\widetilde{u})\widetilde{f}_{ij})\widetilde{e}_{ij} + \mathfrak{r}(\widetilde{f}_{ij}\widetilde{f}_{ij})\widetilde{e}_{ij} + \mathfrak{r}(\widetilde{e}_{ij}\widetilde{f}_{ij})\widetilde{f}_{ij} = 0$. Observe that $(\mathfrak{r}(\widetilde{e}_{ij}\widetilde{u})\widetilde{f}_{ij})\widetilde{e}_{ij} = \mathfrak{r}(\widetilde{e}_{ij}\widetilde{f}_{ij})\widetilde{f}_{ij} = 0$. Therefore $\mathfrak{r}(\widetilde{f}_{ij}\widetilde{f}_{ij})\widetilde{e}_{ij} = 0$ and using the linear independence of v, w_1, w_2, and v_{12} we obtain that $D_{12} = 0$. Besides, replacing $a = \widetilde{f}_i$, $b = \widetilde{f}_{ij}$, $c = \widetilde{e}_j$, and $d = \widetilde{e}_{ij}$ in (2.2) we get $((\widetilde{f}_i\widetilde{f}_{ij})\widetilde{e}_j)\widetilde{e}_{ij} + ((\widetilde{e}_{ij}\widetilde{f}_{ij})\widetilde{e}_j)\widetilde{f}_i - (\widetilde{f}_i\widetilde{f}_{ij})(\widetilde{e}_{ij}\widetilde{e}_j) - (\widetilde{e}_{ij}\widetilde{f}_{ij})(\widetilde{f}_i\widetilde{e}_j) = 0$. Solving this equation and using equations (3.1) and (3.3), we conclude that $\mathfrak{r}(\widetilde{e}_j\widetilde{e}_j)\widetilde{e}_{ij} = 0$. Then $D_j = 0$ for $j = 1, 2$.

Finally, assume that $\widetilde{u}\widetilde{u} = \Lambda_u v_{12}$. Let us consider $a = b = c = d = \widetilde{u}$ in the equation (2.2), we obtain $((\widetilde{u}\widetilde{u})\widetilde{u})\widetilde{u} = 0$. Then, it follows from this that $(\mathfrak{r}(\widetilde{u}\widetilde{u})\widetilde{u})\widetilde{u} = 0$. Therefore $\alpha\Lambda_u = 0$. Note that if $\alpha \neq 0$, then $\Lambda_u = 0$, which proves the Lemma. □

Lemma 3.3 *Let* $\widetilde{e}_1$, $\widetilde{e}_2$, $\widetilde{f}_{12}$, *and* $\widetilde{e}_{12}$ *as equations* (2.7) *and* (2.8) *in Sect.* 2. *Let* ϵ_0 *be a scalar as Lemma* 3.1. *Then there exist* ∇_0 *and* $\Omega_0 \in \mathbb{F}$ *such that* $\widetilde{e}_1\widetilde{f}_{12} = \Omega_0 v + \epsilon_0 w_2$, *and* $\widetilde{e}_2\widetilde{f}_{12} = \nabla_0 v - \epsilon_0 w_1$.

Proof By Lemma 3.1, we have that $\widetilde{e}_{12}\widetilde{f}_{12} = \epsilon_0 w$. Let $\widetilde{e}_i\widetilde{f}_{12} = A_i v + B_{i1}w_1 + B_{i2}w_2 + C_i v_{12}$ with A_i, B_{i1}, B_{i2}, $C_i \in \mathbb{F}$ for $i = 1, 2$. Now, putting $a = d = \widetilde{f}_i$, and $b = c = e_j$ in the equation (2.2), we get that $((\widetilde{f}_i\widetilde{e}_j)\widetilde{e}_j)\widetilde{f}_i - (\widetilde{f}_i\widetilde{e}_j)(\widetilde{f}_i\widetilde{e}_j) = 0$ for $i = 1, 2$. From this, it follows that $(\mathfrak{r}(\widetilde{f}_i\widetilde{e}_j)\widetilde{e}_j)\widetilde{f}_i + \mathfrak{r}(\widetilde{e}_j\widetilde{f}_{ij})\widetilde{f}_i = 0$ and it is clear that $(\mathfrak{r}(\widetilde{f}_i\widetilde{e}_j)\widetilde{e}_j)\widetilde{f}_i = 0$. Consequently, we conclude that $\mathfrak{r}(\widetilde{e}_j\widetilde{f}_{ij})\widetilde{f}_i = 0$, i.e., $(A_j v + B_{j1}w_1 + B_{j2}w_2 + C_i v_{12})\widetilde{f}_i = 0$. Now, applying action on the last equation, we obtain that $\pm B_{ii}v_{12} \pm C_i w_j = 0$, and using the linear independence of w_j and v_{12}, we get that $B_{ii} = C_i = 0$ for $i = 1, 2$. So, we write $\widetilde{e}_i\widetilde{f}_{12} = A_i v + B_{ij}w_j$. Further, substituting $a = \widetilde{f}_i$, $b = \widetilde{e}_i$, and $c = d = \widetilde{f}_{ij}$ in identities (2.2) we obtain that $((\widetilde{f}_i\widetilde{e}_i)\widetilde{f}_{ij})\widetilde{f}_{ij} - ((\widetilde{f}_i\widetilde{f}_{ij})\widetilde{f}_{ij})\widetilde{e}_i - ((\widetilde{e}_i\widetilde{f}_{ij})\widetilde{f}_{ij})\widetilde{f}_i = 0$. From this, it is clear to see that $\mathfrak{r}(\widetilde{e}_j\widetilde{f}_{ij})\widetilde{e}_i - (\mathfrak{r}(\widetilde{e}_i\widetilde{f}_{ij})\widetilde{f}_{ij})\widetilde{f}_i = 0$. Therefore, we obtain that $B_{12} = -B_{21}$.

Finally, substituting $a = \widetilde{e}_{12}$, $b = \widetilde{e}_1$, $c = \widetilde{f}_1$, and $d = \widetilde{f}_{12}$ in the equation (2.2) and making the calculations, we obtain that $(\mathfrak{r}(\widetilde{e}_{12}\widetilde{f}_{12})\widetilde{f}_1)\widetilde{e}_1 + (\mathfrak{r}(\widetilde{e}_1\widetilde{f}_{12})\widetilde{f}_1)\widetilde{e}_{12} = 0$. Therefore, we conclude that $\epsilon_0 = B_{12}$, and the proof is complete. □

Note that, the following identities (3.4)–(3.15) hold as a consequence of Lemma's 3.1, 3.2, and 3.3.

$$\mathfrak{r}(\widetilde{e}_i\widetilde{e}_j)\widetilde{e}_{ij} = 0. \tag{3.4}$$

$$(\mathfrak{r}(\widetilde{f}_i\widetilde{e}_j)\widetilde{f}_i)\widetilde{e}_i - \mathfrak{r}(\widetilde{f}_i\widetilde{f}_{ij})\widetilde{e}_i = 0 \tag{3.5}$$

$$(\mathfrak{r}(\widetilde{e}_{ij}\widetilde{u})\widetilde{f}_i)\widetilde{f}_i + \mathfrak{r}(\widetilde{f}_i\widetilde{f}_{ij})\widetilde{f}_i - \mathfrak{r}(\widetilde{f}_i\widetilde{e}_j) - \mathfrak{r}(\widetilde{e}_{ij}\widetilde{u}) = 0. \tag{3.6}$$

$$(\mathfrak{r}(\widetilde{f}_i\widetilde{e}_j)\widetilde{e}_{ij})\widetilde{u} - \mathfrak{r}(\widetilde{e}_{ij}\widetilde{f}_{ij})\widetilde{u} - \mathfrak{r}(\widetilde{e}_{ij}\widetilde{u})\widetilde{f}_{ij} = 0 \tag{3.7}$$

$$(\mathfrak{r}(\widetilde{e}_{ij}\widetilde{u})\widetilde{u})\widetilde{f}_i + \mathfrak{r}(\widetilde{f}_{ij}\widetilde{u})\widetilde{f}_i = 0 \tag{3.8}$$

$$(\mathfrak{r}(\widetilde{e}_{ij}\widetilde{u})\widetilde{f}_i)\widetilde{f}_j + \mathfrak{r}(\widetilde{f}_i\widetilde{f}_{ij})\widetilde{f}_j - \mathfrak{r}(\widetilde{f}_j\widetilde{e}_j) = 0. \tag{3.9}$$

$$(\mathfrak{r}(\widetilde{e}_{ij}\widetilde{u})\widetilde{f}_i)\widetilde{e}_i + \mathfrak{r}(\widetilde{f}_i\widetilde{f}_{ij})\widetilde{e}_i + \mathfrak{r}(\widetilde{e}_i\widetilde{e}_j) - (\mathfrak{r}(\widetilde{e}_{ij}\widetilde{e}_i)\widetilde{f}_i)\widetilde{u} - (\mathfrak{r}(\widetilde{e}_i\widetilde{u})\widetilde{f}_i)\widetilde{e}_{ij} = 0. \tag{3.10}$$

$$\mathfrak{r}(\widetilde{f}_i\widetilde{f}_{ij})\widetilde{e}_j - (\mathfrak{r}(\widetilde{e}_{ij}\widetilde{e}_j)\widetilde{f}_i)\widetilde{u} - (\mathfrak{r}(\widetilde{e}_j\widetilde{u})\widetilde{f}_i)\widetilde{e}_{ij} + (\mathfrak{r}(\widetilde{e}_{ij}\widetilde{u}) + \mathfrak{r}(\widetilde{f}_i\widetilde{e}_j)\widetilde{f}_{ij} = 0. \tag{3.11}$$

$$(\mathfrak{r}(\widetilde{e}_{ij}\widetilde{u})\widetilde{e}_i)\widetilde{f}_{ij} - 2\mathfrak{r}(\widetilde{e}_i\widetilde{f}_{ij})\widetilde{f}_{ij} + \mathfrak{r}(\widetilde{e}_{ij}\widetilde{f}_{ij})\widetilde{f}_i = 0. \tag{3.12}$$

$$2\mathfrak{r}(\widetilde{f}_i\widetilde{f}_{ij})\widetilde{f}_{ij} - 2\mathfrak{r}(\widetilde{e}_j\widetilde{f}_{ij}) - (\mathfrak{r}(\widetilde{e}_{ij}\widetilde{f}_{ij})\widetilde{f}_i)\widetilde{u} - (\mathfrak{r}(\widetilde{f}_{ij}\widetilde{u})\widetilde{f}_i)\widetilde{e}_{ij} + \mathfrak{r}(\widetilde{e}_{ij}\widetilde{u})\widetilde{e}_j = 0. \tag{3.13}$$

$$(\mathfrak{r}(\widetilde{f}_i\widetilde{e}_j)\widetilde{u})\widetilde{u} - \mathfrak{r}(\widetilde{f}_{ij}\widetilde{u})\widetilde{u} - (\mathfrak{r}(\widetilde{e}_j\widetilde{u})\widetilde{u})\widetilde{f}_i - \mathfrak{r}(\widetilde{f}_j\widetilde{u})\widetilde{f}_i + \mathfrak{r}(\widetilde{u}\widetilde{u})\widetilde{f}_{ij} = 0. \tag{3.14}$$

$$\begin{aligned}&(\mathfrak{r}(\widetilde{f}_i\widetilde{f}_{ij})\widetilde{u})\widetilde{u} - \mathfrak{r}(\widetilde{e}_j\widetilde{u})\widetilde{u} - \mathfrak{r}(\widetilde{f}_j\widetilde{u}) - (\mathfrak{r}(\widetilde{f}_i \quad \widetilde{u})\widetilde{u})\widetilde{f}_{ij}\\ &- (\mathfrak{r}(\widetilde{f}_{ij}\widetilde{u})\widetilde{u})\widetilde{f}_i + \mathfrak{r}(\widetilde{u}\widetilde{u})\widetilde{e}_j = 0.\end{aligned} \tag{3.15}$$

The proof of these identities is not completely presented. Identities (3.4)–(3.15) are obtained via appropriated replacements in the equation (2.2). The proofs of identities (3.4) and (3.6) are only sketched. First, putting $a = \widetilde{f}_i$, $b = \widetilde{f}_{ij}$, $c = \widetilde{e}_i$, and $d = \widetilde{e}_{ij}$ in the equation (2.2), we get $((\widetilde{f}_i\widetilde{f}_{ij}\widetilde{e}_i)\widetilde{e}_{ij}) + ((\widetilde{e}_{ij}\widetilde{f}_{ij})\widetilde{e}_i)\widetilde{f}_i - (\widetilde{f}_i\widetilde{f}_{ij})(\widetilde{e}_{ij}\widetilde{e}_i) - (\widetilde{e}_{ij}\widetilde{f}_{ij})(\widetilde{f}_i\widetilde{e}_i) = 0$. An easy computation shows that $\mathfrak{r}(\widetilde{e}_i\widetilde{e}_j)\widetilde{e}_{ij} = 0$, which completes the proof of the identity (3.4). Now, taking $a = \widetilde{e}_{ij}$, $b = \widetilde{u}$, and $c = d = \widetilde{f}_i$ in the equation (2.2) we obtain that $((\widetilde{e}_{ij}\widetilde{u})\widetilde{f}_i)\widetilde{f}_i + ((\widetilde{f}_i\widetilde{u})\widetilde{f}_i)\widetilde{u}_{ij} - \widetilde{e}_{ij}\widetilde{u} = 0$. By Lemma 3.1, we have that $(\mathfrak{r}(\widetilde{f}_i\widetilde{u})\widetilde{f}_i)\widetilde{e}_{ij} = 0$, then $(\mathfrak{r}(\widetilde{e}_{ij}\widetilde{u})\widetilde{f}_i)\widetilde{f}_i + \mathfrak{r}(\widetilde{f}_i\widetilde{f}_{ij})\widetilde{f}_i - \mathfrak{r}(\widetilde{f}_i\widetilde{e}_j) - \mathfrak{r}(\widetilde{e}_{ij}\widetilde{u}) = 0$; i.e., identity (3.6) is proved. Similarly, identities (3.5), (3.7)–(3.15) are proved.

Lemma 3.4 *Let* $\widetilde{1}$, $\widetilde{f}_1$, $\widetilde{f}_2$, $\widetilde{e}_{12}$, $\widetilde{u}, \widetilde{e}_1, \widetilde{e}_2, \widetilde{f}_{12}$ *be as equations* (2.7) *and* (2.8) *in Sect.* 2. *Let* ξ_0, α_0, δ_0, η_0, τ_0, σ_0, ϵ_0, ξ_1, α_1, δ_2, Ω_0, ∇_0, $\Lambda_u \in \mathbb{F}$ *as Lemmas* 3.1, 3.2, *and* 3.3. *Then*, *there exist scalars* β_0, θ_0, φ_0, λ_0, γ_0, χ_0 *such that the following equalities hold*: *symmetry products*

$$\begin{aligned}\widetilde{f}_1\widetilde{f}_{12} &= -\widetilde{e}_2 + \gamma_0 w + \beta_0 v_1 + \delta_0 v_2 - \sigma_0 w_{12},\\ \widetilde{f}_2\widetilde{f}_{12} &= \widetilde{e}_1 + \lambda_0 v - \alpha_0 w_1 - \theta_0 w_2 + \tau_0 w_{12},\\ \widetilde{e}_{12}\widetilde{u} &= \widetilde{f}_{12} + \varphi_0 w + (\Omega_0 + \alpha\tau_0)v_1 + (\nabla_0 + \alpha\sigma_0)v_2 + \epsilon_0 w_{12},\\ \widetilde{f}_1\widetilde{e}_2 &= -\widetilde{f}_{12} + \beta_0 w + \gamma_0 v_1 - (\nabla_0 + \alpha\sigma_0)v_2 - \epsilon_0 w_{12},\\ \widetilde{f}_2\widetilde{e}_1 &= \widetilde{f}_{12} + \theta_0 w + (\Omega_0 + \alpha\tau_0)v_1 - \lambda_0 v_2 + \epsilon_0 w_{12},\end{aligned}$$

and skew-symmetric products

$$\begin{aligned}\widetilde{e}_1\widetilde{e}_2 &= \widetilde{e}_{12} + \chi_0 v + \tau_0 w_1 + \sigma_0 w_2,\\ \widetilde{e}_1\widetilde{u} &= \widetilde{f}_1 + (\alpha(-\lambda_0 - \nabla_0 - \alpha\sigma_0) - \xi_0)v + \alpha_0 w_1 + (\chi_0 + \varphi_0 + \beta_0)w_2 - \alpha\tau_0 v_{12},\\ \widetilde{e}_2\widetilde{u} &= \widetilde{f}_2 + (\alpha(\gamma_0 + \Omega_0 + \alpha\tau_0) - \eta_0)v - (\chi_0 + \varphi_0 - \theta_0)w_1 + \delta_0 w_2 - \alpha\sigma_0 v_{12},\\ \widetilde{f}_{12}\widetilde{u} &= \alpha(\beta_0 - \theta_0 + \varphi_0 + \chi_0)v - (\Omega_0 + \alpha\tau_0)w_1 + (\nabla_0 + \alpha\sigma_0)w_2,\end{aligned}$$

and $\Lambda_u = 0$, $\delta_2 = \gamma_0 + \Omega_0 + \alpha\tau_0$, $\alpha_1 = -\lambda_0 - \nabla_0 - \alpha\sigma_0$, $\alpha\epsilon_0 = 0$.

Proof Let ξ_i, α_i for $i = 0, 1$; δ_j, σ_j, η_j for $j = 0, 2$; φ_k for $k = 0, 1, 2$; ϵ_0, τ_0, σ_0, Λ_u, ∇_0, and Ω_0 be as in Lemmas 3.1, 3.2, and 3.3.

Assume that there exist some scalars β_i, θ_i, γ_i, λ_i, Δ_i, κ_i, ρ_i, φ_i, and $\chi_i \in \mathbb{F}$, $i = 0, 1, 2, 12$ such that

$$\begin{aligned}
\widetilde{f}_{12}\widetilde{u} &= \Delta_0 v + \Delta_1 w_1 + \Delta_2 w_2 + \Delta_{12} v_{12}, & \widetilde{e}_{12}\widetilde{u} &= \widetilde{f}_{12} + \varphi_0 w + \varphi_1 v_1 + \varphi_2 v_2 + \epsilon_0 w_{12} \\
\widetilde{f}_1\widetilde{e}_2 &= -\widetilde{f}_{12} + \beta_0 w + \beta_1 v_1 + \beta_2 v_2 + \beta_{12} w_{12}, & \widetilde{f}_2\widetilde{e}_1 &= \widetilde{f}_{12} + \theta_0 w + \theta_1 v_1 + \theta_2 v_2 + \theta_{12} w_{12}, \\
\widetilde{f}_1\widetilde{f}_{12} &= -\widetilde{e}_2 + \gamma_0 w + \gamma_1 v_1 + \gamma_2 v_2 - \sigma_0 w_{12}, & \widetilde{f}_2\widetilde{f}_{12} &= \widetilde{e}_1 + \lambda_0 w + \lambda_1 v_1 + \lambda_2 v_2 + \tau_0 w_{12}, \\
\widetilde{e}_1\widetilde{u} &= \widetilde{f}_1 + \rho_0 v + \rho_1 w_1 + \rho_2 w_2 + \rho_{12} v_{12}, & \widetilde{e}_2\widetilde{u} &= \widetilde{f}_2 + \kappa_0 v + \kappa_1 w_1 + \kappa_2 w_2 + \kappa_{12} v_{12}, \\
\widetilde{e}_1\widetilde{e}_2 &= \widetilde{e}_{12} + \chi_0 v + \chi_1 w_1 + \chi_2 w_2 + \chi_{12} v_{12}.
\end{aligned}$$

Now, we use identities (3.4)–(3.15) to complete our proof. By the identity (3.4), it is clear that $\chi_{12} = 0$. Now, considering the identity (3.5) we have that $\chi_1 = \tau_0$, $\chi_2 = \sigma_0$, $\lambda_2 = -\theta_0$, and $\gamma_1 = \beta_0$. Taking the identity (3.6) we have that $\gamma_0 = \beta_1$, $\varphi_2 = -\beta_2$, $\lambda_0 = -\theta_2$, $\varphi_1 = \theta_1$, and $\epsilon_0 = \theta_{12} = -\beta_{12}$. Note that the identity (3.7) implies that $\alpha\epsilon_0 = 0$. Observe that if $\alpha \neq 0$, then $\epsilon_0 = 0$. Later, using the identity (3.8) we obtain that $\Delta_i + \varphi_i = 0$, for $i = 1, 2$, and $\Delta_{12} = -\alpha\epsilon_0 = 0$. Combining these last two equalities, it is clear that $\varphi_1 = -\Delta_1 = \theta_1$, and $\varphi_2 = -\Delta_2 = -\beta_2$. Further, considering the identity (3.9), we get $\delta_2 = \gamma_0 + \varphi_1$, $\alpha_1 = -\lambda_0 - \varphi_2$, $\gamma_2 = \delta_0$, and $\lambda_1 = -\alpha_0$. Besides, taking the identity (3.10), we have that $\rho_2 = \chi_0 + \varphi_0 + \beta_0$, and $\kappa_1 = -(\chi_0 + \varphi_0 - \theta_0)$. Later, from the identity (3.11), it follows that $\kappa_2 = \gamma_2 = \delta_0$, and $\rho_1 = -\lambda_1 = \alpha_0$. Further, by the equation (3.13) we get that $\varphi_1 = \Omega_0 + \alpha\tau_0$, and $\varphi_2 = \nabla_0 + \alpha\sigma_0$. Thus, we conclude that $\delta_2 = \gamma_0 + \Omega_0 + \alpha\tau_0$, and $\alpha_1 = -\lambda_0 - \nabla_0 - \alpha\sigma_0$. Besides, using the identity (3.14), we obtain that $\kappa_0 = \alpha(\beta_1 - \Delta_1) - \eta_0 = \alpha(\gamma_0 + \Omega_0 + \alpha\tau_0) - \eta_0$, $\rho_0 = \alpha(\theta_2 + \Delta_2) - \xi_0 = \alpha(-\lambda_0 - \nabla_0 - \alpha\sigma_0)$, and $\Delta_0 = \alpha\beta_0 - \alpha\kappa_1 - \Lambda_u = \alpha\rho_2 - \alpha\theta_0 - \Lambda_u$. Thus, $\Delta_0 = \alpha(\chi_0 + \varphi_0 + \beta_0 - \theta_0) - \Lambda_u$. Finally, since the identity (3.15) we have that $\Delta_0 = \alpha\gamma_1 - \alpha\kappa_1 + \Lambda_u$, $\kappa_{12} = -\alpha\sigma_0$, and $\rho_{12} = -\alpha\tau_0$. Note that $\gamma_1 = \beta_0$ and comparing this with the identity (3.14), we conclude $\Lambda_u = 0$, and the proof is complete. □

Note that by Lemmas 3.1 and 3.4 we have proved that:

Theorem 3.5 *Let $\mathcal{A}$ be a finite-dimensional Jordan superalgebra with solvable radical $\mathcal{N}$ such that $\mathcal{N}^2 = 0$, $\mathcal{A}/\mathcal{N} \cong \mathfrak{K}\mathrm{an}(2)$, and $\mathcal{N}$ is isomorphic to irreducible $\mathfrak{K}\mathrm{an}(2)$-superbimodule $\mathcal{V}(v, \alpha)$. Then there exists $\widetilde{1} \in \mathcal{A}_0$ such that $\widetilde{1}$ is a unity of $\mathcal{A}$ and there exist elements $\widetilde{f}_1$, $\widetilde{f}_2$, $\widetilde{e}_{12} \in \mathcal{A}_0$, and $\widetilde{u}$, $\widetilde{e}_1$, $\widetilde{e}_2$, $\widetilde{f}_{12} \in \mathcal{A}_1$ such that $\mathrm{alg}\langle \widetilde{1}, \widetilde{f}_1, \widetilde{f}_2, \widetilde{e}_{12}\rangle \cong (\mathfrak{K}\mathrm{an}(2))_0$ and $\mathrm{span}\langle \overline{\widetilde{u}}, \overline{\widetilde{e}_1}, \overline{\widetilde{e}_2}, \overline{\widetilde{f}_{12}}\rangle \cong (\mathfrak{K}\mathrm{an}(2))_1$ where $\overline{\widetilde{x}}$ is the image of $\widetilde{x}$ under canonical homomorphism. Moreover, there exist scalars ξ_0, ξ_1, α_0, δ_0, σ_0, η_0, τ_0, σ_0, ϵ_0 β_0, θ_0, φ_0, λ_0, γ_0, χ_0 such that $\alpha\epsilon_0 = 0$ and the non-zero products are given by: symmetric products*

$$
\begin{aligned}
\widetilde{f_i}\widetilde{f_i} &= 1, \quad i = 1, 2;\\
\widetilde{f_1}\widetilde{u} &= \xi_0 w + \xi_1 v_1, \quad \widetilde{f_2}\widetilde{u} = \xi_0 w + \xi_1 v_2,\\
\widetilde{f_1}\widetilde{e}_1 &= \alpha_0 w + (-\lambda_0 - \nabla_0 - \alpha\sigma_0)v_1,\\
\widetilde{f_2}\widetilde{e}_2 &= \delta_0 w + (\gamma_0 + \Omega_0 + \alpha\tau_0)v_2,\\
\widetilde{e}_{12}\widetilde{e}_1 &= \tau_0 w, \quad \widetilde{e}_{12}\widetilde{e}_2 = \sigma_0 w, \quad \widetilde{e}_{12}\widetilde{f}_{12} = \epsilon_0 w\\
\widetilde{f_1}\widetilde{f}_{12} &= -\widetilde{e}_2 + \gamma_0 w + \beta_0 v_1 + \delta_0 v_2 - \sigma_0 w_{12},\\
\widetilde{f_2}\widetilde{f}_{12} &= \widetilde{e}_1 + \lambda_0 v - \alpha_0 w_1 - \theta_0 w_2 + \tau_0 w_{12},\\
\widetilde{e}_{12}\widetilde{u} &= \widetilde{f}_{12} + \varphi_0 w + (\Omega_0 + \alpha\tau_0)v_1 + (\nabla_0 + \alpha\sigma_0)v_2 + \epsilon_0 w_{12},\\
\widetilde{f_1}\widetilde{e}_2 &= -\widetilde{f}_{12} + \beta_0 w + \gamma_0 v_1 - (\nabla_0 + \alpha\sigma_0)v_2 - \epsilon_0 w_{12},\\
\widetilde{f_2}\widetilde{e}_1 &= \widetilde{f}_{12} + \theta_0 w + (\Omega_0 + \alpha\tau_0)v_1 - \lambda_0 v_2 + \epsilon_0 w_{12},
\end{aligned}
$$

and skew-symmetric products,

$$
\begin{aligned}
\widetilde{f}_{12}\widetilde{e}_1 &= \Omega_0 v + \epsilon_0 w_2, \quad \widetilde{f}_{12}\widetilde{e}_2 = \nabla_0 v - \epsilon_0 w_1\\
\widetilde{e}_1\widetilde{e}_2 &= \widetilde{e}_{12} + \chi_0 v + \tau_0 w_1 + \sigma_0 w_2,\\
\widetilde{e}_1\widetilde{u} &= \widetilde{f_1} + (\alpha(-\lambda_0 - \nabla_0 - \alpha\sigma_0) - \xi_0)v + \alpha_0 w_1 + (\chi_0 + \varphi_0 + \beta_0)w_2 - \alpha\tau_0 v_{12},\\
\widetilde{e}_2\widetilde{u} &= \widetilde{f_2} + (\alpha(\gamma_0 + \Omega_0 + \alpha\tau_0) - \eta_0)v - (\chi_0 + \varphi_0 - \theta_0)w_1 + \delta_0 w_2 - \alpha\sigma_0 v_{12},\\
\widetilde{f}_{12}\widetilde{u} &= \alpha(\beta_0 - \theta_0 + \varphi_0 + \chi_0)v - (\Omega_0 + \alpha\tau_0)w_1 + (\nabla_0 + \alpha\sigma_0)w_2.
\end{aligned}
$$

3.2 Case 2

Assume that $\mathcal{N}_0 = \text{span}\langle w, v_1, v_2, w_{12}\rangle$ and $\mathcal{N}_1 = \text{span}\langle v, w_1, w_2, v_{12}\rangle$ such that $\mathcal{N} \cong \mathcal{V}(v, \alpha)^{\text{op}}$. We present a series of lemmas to simplify the proof in this case.

Lemma 3.6 *Let $\widetilde{f_i}$, $\widetilde{e}_i$, $\widetilde{e}_{12}$, $\widetilde{f}_{12}$, and $\widetilde{u}$ be as Eqs. (2.7) and (2.8) in Sect. 2. Then there exist scalars ξ_2, ξ_{12}, η_{12}, α_2, α_{12}, δ_1, δ_{12}, and ϵ_0 such that $\widetilde{f_1}\widetilde{u} = \xi_2 w_2 + \xi_{12} v_{12}$, $\widetilde{f_2}\widetilde{u} = -\xi_2 w_1 + \eta_{12} v_{12}$, $\widetilde{f_1}\widetilde{e}_1 = \alpha_2 w_2 + \alpha_{12} v_{12}$, $\widetilde{f_2}\widetilde{e}_2 = \delta_1 w_1 + \delta_{12} v_{12}$, $\widetilde{e}_{12}\widetilde{e}_1 = \alpha_2 v$, $\widetilde{e}_{12}\widetilde{e}_2 = -\delta_1 v$, and $\widetilde{e}_{12}\widetilde{f}_{12} = \epsilon_0 v$.*

Proof We can now proceed analogously to the proof of Lemma 3.1 using the same replacements. □

Lemma 3.7 *For all $\widetilde{x} \in \mathcal{A}_1$, $\widetilde{x}\widetilde{x} = 0$.*

Proof Let $\widetilde{x} \in \mathcal{A}_1$ be. Using the Eq. (2.2), it is easy to see that $((\widetilde{x}\widetilde{x})\widetilde{f_i})\widetilde{f_i} = \widetilde{x}\widetilde{x}$. Then, we get that $((\widetilde{x}\widetilde{x})\widetilde{f_i})\widetilde{e}_i = (\widetilde{x}\widetilde{x})(\widetilde{f_i}\widetilde{e}_i)$, and, we conclude that $\widetilde{x}\widetilde{x} = 0$. □

By Lemmas 3.6 and 3.7, we are now in a position to show valid equations in $\mathcal{A}$.

$$(\mathfrak{r}(\widetilde{e}_{ij}\widetilde{u})\widetilde{f}_j)\widetilde{f}_i + \mathfrak{r}(\widetilde{f}_j\widetilde{f}_{ij})\widetilde{f}_i + \mathfrak{r}(\widetilde{f}_i\widetilde{e}_i) = 0. \tag{3.16}$$

$$(\mathfrak{r}(\widetilde{f}_i\widetilde{e}_j)\widetilde{f}_i)\widetilde{e}_i - \mathfrak{r}(\widetilde{f}_j\widetilde{f}_{ij})\widetilde{e}_i - (\mathfrak{r}(\widetilde{f}_i\widetilde{e}_i)\widetilde{f}_i)\widetilde{e}_j - (\mathfrak{r}(\widetilde{e}_i\widetilde{e}_j)\widetilde{f}_i)\widetilde{f}_i - 2\mathfrak{r}(\widetilde{f}_i\widetilde{e}_i)\widetilde{f}_{ij} = 0. \tag{3.17}$$

$$(\mathfrak{r}(\widetilde{f}_i\widetilde{u})\widetilde{f}_i)\widetilde{e}_{ij} + (\mathfrak{r}(\widetilde{e}_{ij}\widetilde{u})\widetilde{f}_i)\widetilde{f}_i + \mathfrak{r}(\widetilde{f}_j\widetilde{f}_{ij})\widetilde{f}_i - \mathfrak{r}(\widetilde{f}_i\widetilde{e}_j) - \mathfrak{r}(\widetilde{e}_{ij}\widetilde{u}) = 0. \tag{3.18}$$

$$(\mathfrak{r}(\widetilde{f}_i\widetilde{u})\widetilde{f}_i)\widetilde{e}_i - (\mathfrak{r}(\widetilde{e}_i\widetilde{u})\widetilde{f}_i)\widetilde{f}_i - (\mathfrak{r}(\widetilde{f}_i\widetilde{e}_i)\widetilde{f}_i)\widetilde{u} + \mathfrak{r}(\widetilde{e}_i\widetilde{u}) = 0. \tag{3.19}$$

$$(\mathfrak{r}(\widetilde{e}_j\widetilde{u})\widetilde{f}_j)\widetilde{f}_i + (\mathfrak{r}(\widetilde{f}_i\widetilde{e}_j)\widetilde{f}_j)\widetilde{u} - \mathfrak{r}(\widetilde{f}_j\widetilde{f}_{ij})\widetilde{u} - \mathfrak{r}(\widetilde{e}_i\widetilde{u}) - \mathfrak{r}(\widetilde{f}_j\widetilde{u})\widetilde{f}_{ij} = 0. \tag{3.20}$$

$$(\mathfrak{r}(\widetilde{f}_i\widetilde{u})\widetilde{f}_i)\widetilde{f}_{ij} - (\mathfrak{r}(\widetilde{f}_j\widetilde{f}_{ij})\widetilde{f}_i)\widetilde{u} + \mathfrak{r}(\widetilde{f}_i\widetilde{e}_j)\widetilde{u} - (\mathfrak{r}(\widetilde{f}_{ij}\widetilde{u})\widetilde{f}_i)\widetilde{f}_i + 2\mathfrak{r}(\widetilde{f}_i\widetilde{u})\widetilde{e}_j = 0. \tag{3.21}$$

$$(\mathfrak{r}(\widetilde{f}_i\widetilde{e}_j)\widetilde{e}_{ij})\widetilde{u} - \mathfrak{r}(\widetilde{e}_{ij}\widetilde{f}_{ij})\widetilde{u} + (\mathfrak{r}(\widetilde{e}_j\widetilde{u})\widetilde{e}_{ij})\widetilde{f}_i - \mathfrak{r}(\widetilde{f}_i\widetilde{e}_j)\widetilde{f}_{ij} - \mathfrak{r}(\widetilde{e}_{ij}\widetilde{u})\widetilde{f}_{ij} = 0 \tag{3.22}$$

$$\begin{aligned}&(\mathfrak{r}(\widetilde{f}_j\widetilde{f}_{ij})\widetilde{e}_i)\widetilde{u} + \mathfrak{r}(\widetilde{e}_i\widetilde{e}_j)\widetilde{u} + \mathfrak{r}(\widetilde{e}_{ij}\widetilde{u}) - (\widetilde{f}_{ij}\widetilde{u})\widetilde{e}_i)\widetilde{f}_i - \mathfrak{r}(\widetilde{f}_j\widetilde{f}_{ij})\widetilde{f}_i\\ &- \mathfrak{r}(\widetilde{e}_i\widetilde{u})\widetilde{e}_j + \mathfrak{r}(\widetilde{f}_i\widetilde{e}_j) = 0\end{aligned} \tag{3.23}$$

$$\begin{aligned}&(\mathfrak{r}(\widetilde{f}_{ij}\widetilde{u})\widetilde{e}_{ij})\widetilde{f}_i + (\mathfrak{r}(\widetilde{f}_j\widetilde{f}_{ij})\widetilde{e}_{ij})\widetilde{u} - \mathfrak{r}(\widetilde{e}_{ij}\widetilde{e}_j)\widetilde{u} - \mathfrak{r}(\widetilde{f}_j\widetilde{f}_{ij})\widetilde{f}_{ij}\\ &- \mathfrak{r}(\widetilde{e}_{ij}\widetilde{u})\widetilde{e}_j - \mathfrak{r}(\widetilde{e}_j\widetilde{f}_{ij}) = 0\end{aligned} \tag{3.24}$$

In the same manner as equalities (3.4)–(3.15) with suitable replacement in the identity (2.2), we can see those equations (3.16)–(3.24) hold in $\mathcal{A}$.

Lemma 3.8 *Let* $\widetilde{1}$, $\widetilde{f}_1$, $\widetilde{f}_2$, $\widetilde{e}_{12}$, $\widetilde{u}, \widetilde{e}_1, \widetilde{e}_2$, *and* $\widetilde{f}_{12}$ *be as relations eqrefpro*: *A*1*A*1 *and* (2.8) *in Sect*. 2. *Let* ξ_2, ξ_{12}, η_{12}, α_2, α_{12}, δ_1, δ_{12}, $\epsilon_0 \in \mathbb{F}$ *be as Lemma* 3.6. *Then*, *there exist scalars* γ_0, λ_0, β_{12}, θ_{12}, φ_0, φ_1, φ_2 *such that the following equalities hold*: *with symmetric products*

$$\begin{aligned}\widetilde{e}_{12}\widetilde{u} &= \widetilde{f}_{12} + \varphi_0 v + \varphi_1 w_1 + \varphi_2 w_2 + \epsilon_0 v_{12},\\ \widetilde{f}_1\widetilde{e}_2 &= -\widetilde{f}_{12} + (\xi_2 - \varphi_0)v - \varphi_1 w_1 - (\delta_1 + \varphi_2)w_2 + \beta_{12}v_{12},\\ \widetilde{f}_2\widetilde{e}_1 &= \widetilde{f}_{12} + (-\xi_0 + \varphi_0)v + (-\alpha_2 + \varphi_1)w_1 + \varphi_2 w_2 + \theta_{12}v_{12},\\ \widetilde{f}_1\widetilde{f}_{12} &= -\widetilde{e}_2 + \gamma_0 v + \delta_{12}w_1 - \beta_{12}w_2 + (\delta_1 + \varphi_2)v_{12},\\ \widetilde{f}_2\widetilde{f}_{12} &= \widetilde{e}_1 + \lambda_0 v - \theta_{12}w_1 + \alpha_{12}w_2 + (\alpha_2 - \varphi_1)v_{12},\end{aligned}$$

and skew-symmetric products

$$
\begin{aligned}
\widetilde{e}_1\widetilde{f}_{12} =& (\alpha_2 - \varphi_1)w - \alpha_{12}v_1 - \theta_{12}v_2,\\
\widetilde{e}_2\widetilde{f}_{12} =& -(\delta_1 + \varphi_1)w + \beta_{12}v_1 - \delta_{12}v_2,\\
\widetilde{f}_{12}\widetilde{u} =& -\varphi_0 w + (\xi_{12} - \alpha\varphi_1)v_1 + (\eta_{12} - \alpha\varphi_2)v_2 - \epsilon_0 w_{12},\\
\widetilde{e}_1\widetilde{e}_2 =& \widetilde{e}_{12} + (\alpha_{12} + \delta_{12})w + \alpha_2 v_1 + \delta_1 v_2\\
\widetilde{e}_1\widetilde{u} =& \widetilde{f}_1 + (-\lambda_0 + \eta_{12})w - \alpha(\epsilon_0 - \theta_{12})v_1 - (\epsilon_2 + \alpha\alpha_{12})v_2 - \alpha_2 w_{12},\\
\widetilde{e}_2\widetilde{u} =& \widetilde{f}_2 + (-\gamma_0 + \xi_{12})w + (\epsilon_2 + \alpha\delta_{12})v_1 - \alpha(\epsilon_0 + \beta_{12})v_2 + \delta_1 w_{12}.
\end{aligned}
$$

Proof Let $\beta_i, \theta_i, \gamma_i, \lambda_i, \Delta_i, \kappa_i, \rho_i, \varphi_i$, and $\chi_i \in \mathbb{F}$, for $i = 0, 1, 2, 12$ be such that

$$
\begin{aligned}
\widetilde{f}_{12}\widetilde{u} =& \Delta_0 w + \Delta_1 v_1 + \Delta_2 v_2 + \Delta_{12} w_{12}, & \widetilde{e}_{12}\widetilde{u} =& \widetilde{f}_{12} + \varphi_0 v + \varphi_1 w_1 + \varphi_2 w_2 + \epsilon_0 v_{12}\\
\widetilde{f}_1\widetilde{e}_2 =& -\widetilde{f}_{12} + \beta_0 v + \beta_1 w_1 + \beta_2 w_2 + \beta_{12} v_{12}, & \widetilde{f}_2\widetilde{e}_1 =& \widetilde{f}_{12} + \theta_0 v + \theta_1 w_1 + \theta_2 w_2 + \theta_{12} v_{12},\\
\widetilde{f}_1\widetilde{f}_{12} =& -\widetilde{e}_2 + \gamma_0 v + \gamma_1 w_1 + \gamma_2 w_2 + \gamma_{12} v_{12}, & \widetilde{f}_2\widetilde{f}_{12} =& \widetilde{e}_1 + \lambda_0 v + \lambda_1 w_1 + \lambda_2 w_2 + \lambda_{12} v_{12},\\
\widetilde{e}_1\widetilde{u} =& \widetilde{f}_1 + \rho_0 w + \rho_1 v_1 + \rho_2 v_2 + \rho_{12} w_{12}, & \widetilde{e}_2\widetilde{u} =& \widetilde{f}_2 + \kappa_0 w + \kappa_1 v_1 + \kappa_2 v_2 + \kappa_{12} w_{12},\\
\widetilde{e}_1\widetilde{f}_{12} =& \Omega_0 w + \Omega_1 v_1 + \Omega_2 v_2 + \Omega_{12} w_{12}, & \widetilde{e}_2\widetilde{f}_{12} =& \nabla_0 w + \nabla_1 v_1 + \nabla_2 v_2 + \nabla_{12} w_{12},\\
\widetilde{e}_1\widetilde{e}_2 =& \widetilde{e}_{12} + \chi_0 w + \chi_1 v_1 + \chi_2 v_2 + \chi_{12} w_{12}.
\end{aligned}
$$

Using the identity (3.4) we get that $\chi_{12} = 0$. Later, considering the identity (3.16), we have that $\lambda_2 = \alpha_{12}$, and $\gamma_1 = \delta_{12}$. The following identities hold as a consequence of identity (3.17): $\chi_0 = \alpha_{12} + \delta_{12}$, $\chi_1 = \alpha_2$, $\chi_2 = \delta_1$, and $\gamma_{12} = -\beta_2$, $\lambda_{12} = -\theta_1$. Now, considering the Eq. (3.18) it follows that $\beta_0 = \xi_2 - \varphi_0 = -\theta_0$, $\beta_1 = -\varphi_1$, $\gamma_2 = \beta_{12}$, $\theta_2 = \varphi_2$, and $\lambda_1 = -\theta_{12}$. Later, taking the identity (3.19) we conclude that $\rho_2 = -\xi_2 - \alpha\alpha_{12}$, $\kappa_1 = \xi_2 + \alpha\delta_{12}$, $\rho_{12} = -\alpha_2$, and $\kappa_{12} = \delta_1$. On the other hand, the Eq. (3.20) implies that $\kappa_0 = -\xi_{12} + \gamma_0$, $\rho_0 = \eta_{12} - \lambda_0$, $\rho_{12} = -\lambda_{12} + \beta_1$, and $\kappa_{12} = -\theta_2 + \gamma_{12}$ and, in consequence, $\theta_1 = -\alpha_2 + \varphi_1$, and $\beta_2 = -\delta_1 - \varphi_2$. Moreover, substituting $a = \widetilde{f}_i$, $b = c = \widetilde{u}$, and $d = \widetilde{e}_j$ in the identity (2.2), we get that $\rho_1 = \alpha(\Delta_{12} + \theta_{12})$, and $\kappa_2 = \alpha(\Delta_{12} - \beta_{12})$. Our next claim is obtained from the Eq. (3.21) concluding that $\Delta_0 = -\varphi_0$, $\Delta_1 = \xi_{12} - \alpha\varphi_1$, and $\Delta_2 = \eta_{12} - \alpha\varphi_2$. Further, replacing $a = \widetilde{f}_i$, $b = \widetilde{u}$, $c = \widetilde{e}_{ij}$, and $d = \widetilde{e}_j$ in the Eq. (2.2) and rewriting this replacement, we obtain $\varphi_{12} = \epsilon_0$. Besides, using the Eq. (3.23) we have that $\Delta_{12} = -\epsilon_0$. Finally, taking the Eq. (3.24) we obtain that $\Omega_{12} = \nabla_{12} = 0$, $\Omega_0 = \alpha_2 - \varphi_1$, $\nabla_0 = -(\delta_1 + \varphi_2)$, $\Omega_1 = -\alpha_{12}$, $\nabla_2 = -\delta_{12}$, $\Omega_2 = -\theta_{12}$, and $\nabla_1 = \beta_{12}$, which completes the proof. □

By Lemmas 3.6–3.8, we have proved the following theorem:

Theorem 3.9 *Let $\mathcal{A}$ be a finite-dimensional Jordan superalgebra with solvable radical $\mathcal{N}$ such that $\mathcal{N}^2 = 0$, $\mathcal{A}/\mathcal{N} \cong \mathfrak{K}\mathrm{an}(2)$, and $\mathcal{N}$ is isomorphic to irreducible $\mathfrak{K}\mathrm{an}(2)$-superbimodule $\mathcal{V}(v, \alpha)^{\mathrm{op}}$. Then there exists $\widetilde{1} \in \mathcal{A}_0$ such that $\widetilde{1}$ is a unity of $\mathcal{A}$ and there exist elements $\widetilde{f}_1$, $\widetilde{f}_2$, $\widetilde{e}_{12} \in \mathcal{A}_0$, and $\widetilde{u}$, $\widetilde{e}_1$, $\widetilde{e}_2$, $\widetilde{f}_{12} \in \mathcal{A}_1$ such that $\mathrm{alg}\langle\widetilde{1},\widetilde{f}_1,\widetilde{f}_2,\widetilde{e}_{12}\rangle \cong (\mathfrak{K}\mathrm{an}(2))_0$ and $\mathrm{span}\langle\overline{\widetilde{u}},\overline{\widetilde{e}}_1,\overline{\widetilde{e}}_2,\overline{\widetilde{f}}_{12}\rangle \cong (\mathfrak{K}\mathrm{an}(2))_1$ where $\overline{\widetilde{x}}$ is the image of $\widetilde{x}$ under canonical homomorphism. Moreover, there exist scalars ξ_2,*

ξ_{12}, η_{12}, α_2, α_{12} δ_1, δ_{12}, ϵ_0, γ_0, λ_0, β_{12}, θ_{12}, φ_0, φ_1, φ_2, *non-zero products are given by:symmetric products*

$$
\begin{aligned}
\widetilde{f_i}\widetilde{f_i} &= 1, \quad i = 1, 2;\\
\widetilde{f_1}\widetilde{u} &= \xi_2 w_2 + \xi_{12} v_{12}, \quad \widetilde{f_2}\widetilde{u} = -\xi_2 w_1 + \eta_{12} v_{12},\\
\widetilde{f_1}\widetilde{e}_1 &= \alpha_2 w_2 + \alpha_{12} v_{12}, \quad \widetilde{f_2}\widetilde{e}_2 = \delta_1 w_1 + \delta_{12} v_{12},\\
\widetilde{e}_{12}\widetilde{e}_1 &= \alpha_2 v, \quad \widetilde{e}_{12}\widetilde{e}_2 = -\delta_1 v, \quad \widetilde{e}_{12}\widetilde{f}_{12} = \epsilon_0 v,\\
\widetilde{e}_{12}\widetilde{u} &= \widetilde{f}_{12} + \varphi_0 v + \varphi_1 w_1 + \varphi_2 w_2 + \epsilon_0 v_{12},\\
\widetilde{f_1}\widetilde{e}_2 &= -\widetilde{f}_{12} + (\epsilon_2 - \varphi_0) v - \varphi_1 w_1 - (\delta_1 + \varphi_2) w_2 + \beta_{12} v_{12},\\
\widetilde{f_2}\widetilde{e}_1 &= \widetilde{f}_{12} + (-\epsilon_0 + \varphi_0) v + (-\alpha_2 + \varphi_1) w_1 + \varphi_2 w_2 + \theta_{12} v_{12},\\
\widetilde{f_1}\widetilde{f}_{12} &= -\widetilde{e}_2 + \gamma_0 v + \delta_{12} w_1 - \beta_{12} w_2 + (\delta_1 + \varphi_2) v_{12}\\
\widetilde{f_2}\widetilde{f}_{12} &= \widetilde{e}_1 + \lambda_0 v - \theta_{12} w_1 + \alpha_{12} w_2 + (\alpha_2 - \varphi_1) v_{12}
\end{aligned}
$$

and skew-symmetric products,

$$
\begin{aligned}
\widetilde{e}_1\widetilde{e}_2 &= \widetilde{e}_{12} + (\alpha_{12} + \delta_{12}) w + \alpha_2 v_1 + \delta_1 v_2\\
\widetilde{e}_1\widetilde{u} &= \widetilde{f_1} + (-\lambda_0 + \eta_{12}) w - \alpha(\epsilon_0 - \theta_{12}) v_1 - (\epsilon_2 + \alpha\alpha_{12}) v_2 - \alpha_2 w_{12}\\
\widetilde{e}_2\widetilde{u} &= \widetilde{f_2} + (-\gamma_0 + \xi_{12}) w + (\epsilon_2 + \alpha\delta_{12}) v_1 - \alpha(\epsilon_0 + \beta_{12}) v_2 + \delta_1 w_{12}\\
\widetilde{f}_{12}\widetilde{u} &= -\varphi_0 w + (\xi_{12} - \alpha\varphi_1) v_1 + (\eta_{12} - \alpha\varphi_2) v_2 - \epsilon_0 w_{12},\\
\widetilde{e}_1\widetilde{f}_{12} &= (\alpha_2 - \varphi_1) w - \alpha_{12} v_1 - \theta_{12} v_2,\\
\widetilde{e}_2\widetilde{f}_{12} &= -(\delta_1 + \varphi_1) w + \beta_{12} v_1 - \delta_{12} v_2.
\end{aligned}
$$

4 WPT for Jordan superalgebras of type $\mathfrak{Kan}(2)$

In this section, We apply Theorems 3.5 and 3.9 proved in Sect. 3 to prove our main result.

Theorem 4.1 *Let* $\mathcal{A}$ *be a finite-dimensional Jordan superalgebra over an algebraically closed field, with solvable radical* $\mathcal{N}$ *such that* $\mathcal{N}^2 = 0$ *and* $\mathcal{A}/\mathcal{N} \cong \mathfrak{K}\mathrm{an}(2)$. *Then, there exists* $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1$ *subsuperalgebra of* $\mathcal{A}$ *such that* $\mathcal{A} = \mathcal{N} \oplus \mathcal{S}$, *and* $\mathcal{S} \cong \mathfrak{K}\mathrm{an}(2)$, *i.e., an analogous to WPT holds for* $\mathcal{A}$.

Proof By results of the first author [1], the proof has two parts according to O. Folleco and I. Shestakov [11] results, we need to consider the cases $\mathcal{N} \cong \mathcal{V}(v, \alpha)$, and $\mathcal{N} \cong \mathcal{V}(v, \alpha)^{\mathrm{op}}$.

Case 1. Let $\mathcal{N}$ be an irreducible superbimodule over $\mathfrak{K}\mathrm{an}(2)$ such that $\mathcal{N}$ is isomorphic to $\mathcal{V}(v, \alpha)$. Assume that $\widetilde{1}$, $\widetilde{f_1}$, $\widetilde{f_2}$, $\widetilde{e}_{12}$, $\widetilde{u}$, $\widetilde{e}_1$, $\widetilde{e}_2$, $\widetilde{f}_{12} \in \mathcal{A}$ such that conditions of Theorem 3.5 hold. Let $\hat{1} = \widetilde{1}$, $\hat{f}_1 = \widetilde{f_1}$, $\hat{f}_2 = \widetilde{f_2}$, and $\hat{e}_{12} = \widetilde{e}_{12} \in \mathcal{A}_0$, and let $\mathcal{S}_0 = \text{alg}\langle \hat{1}, \hat{f}_1, \hat{f}_2, \hat{e}_{12}\rangle$, thus we have $\mathcal{S}_0 \cong (\mathfrak{K}\text{an}(2))_0$. Now, let $\hat{u} = \widetilde{u} + \mathfrak{r}(\hat{u})$, $\hat{e}_1 = \widetilde{e}_1 + \mathfrak{r}(\hat{e}_1)$, $\hat{e}_2\widetilde{e}_1 + \mathfrak{r}(\hat{e}_2)$, and $\hat{f}_{12} = \widetilde{f}_{12} + \mathfrak{r}(\hat{f}_{12}) \in \mathcal{A}_1$. From this, note that $\text{span}\langle \overline{\hat{u}}, \overline{\hat{e}_1}, \overline{\hat{e}_2}, \overline{\hat{f}_{12}}\rangle \cong (\mathfrak{K}\text{an}(2))_1$.

Let A, B_i, C, D, E_i, F, G, H_i, L, J, K_i, $M \in \mathbb{F}$ for $i = 1, 2$ be such that $\hat{u} = \widetilde{u} + Aw + B_1v_1 + B_2v_2 + Cw_{12}$, $\hat{f}_{12} = \widetilde{f}_{12} + Dw + E_1v_1 + E_2v_2 + Fw_{12}$, $\hat{e}_1 = \widetilde{e}_1 + Gw + H_1v_1 + H_2v_2 + Lw_{12}$, and $\hat{e}_2 = \widetilde{e}_2 + Jw + K_1v_1 + K_2v_2 + Mw_{12}$. Computing the products of elements $\hat{x}\hat{y}$, we have the following equalities hold:

$$\begin{aligned}
\hat{e}_1\hat{u} &= \widetilde{e}_1\widetilde{u} + (\alpha G - B_1)v + H_1w_1 + (H_2 - C)w_2 + \alpha Lv_{12},\\
\hat{e}_2\hat{u} &= \widetilde{e}_2\widetilde{u} + (\alpha J - B_2)v + (K_1 + C)w_1 + K_2w_2 + \alpha Mv_{12},\\
\hat{f}_{12}\hat{u} &= \widetilde{f}_{12}\widetilde{u} + \alpha(D - C)v + E_1w_1 + E_2w_2 + \alpha Fv_{12},\\
\hat{e}_1\hat{e}_2 &= \widetilde{e}_1\widetilde{e}_2 + (H_2 - K_1)v - Lw_1 - Mw_2,\\
\hat{e}_1\hat{f}_{12} &= \widetilde{e}_1\widetilde{f}_{12} + (\alpha L - E_1)v - Fw_2,\\
\hat{e}_1\hat{f}_{12} &= \widetilde{e}_1\widetilde{f}_{12} + (\alpha M - E_2)v - Fw_1.
\end{aligned} \tag{4.1}$$

Now, by Theorem 3.5, we have that $\hat{e}_1\hat{e}_2 = \hat{e}_{12}$, $\hat{e}_1\hat{u} = \hat{f}_1$, $\hat{e}_2\hat{u} = \hat{f}_2$, and $\hat{f}_{12}\hat{u} = \hat{f}_{12}\hat{e}_1 = \hat{f}_{12}\hat{e}_2 = 0$ if and only if $H_1 = -\alpha_0$, $K_2 = -\delta_0$, $L = \tau_0$, $M = \sigma_0$, $F = \epsilon_0$, $E_1 = \Omega_0 + \alpha\tau_0$, $E_2 = -\nabla_0 - \alpha\sigma_0$, and

$$\begin{aligned}
\alpha(G - \lambda_0 - \nabla_0 - \alpha\sigma_0) - B_1 - \xi_0 = 0, \alpha(J + \gamma_0 + \Omega_0 + \alpha\tau_0) - \eta_0 - B_2 = 0,\\
-(\chi_0 + \varphi_0 - \theta_0) + K_1 + C = 0, \chi_0 + \varphi_0 + \beta_0 + H_2 - C = 0,\\
\chi_0 + H_2 - K_1 = 0, \alpha(\beta_0 - \theta_0 + \varphi_0 + \chi_0 + D - C) = 0.
\end{aligned} \tag{4.2}$$

Hence, it is clear that $2K_1 = \chi_0 - (\theta_0 + \beta_0)$, $2H_2 = -(\theta_0 + \beta_0 + \chi_0)$, $2C = \chi_0 + \beta_0 - \theta_0 + 2\varphi_0$, and $\alpha(2D + (\chi_0 + \beta_0 - \theta_0)) = 0$. Further, using Lemma 3.1, we have that $\hat{f}_i\hat{u} = 0$ if and only if $A = -\xi_1$, $B_1 = \xi_0$, and $B_2 = \eta_0$; and $\hat{f}_i\hat{e}_i = 0$ if and only if $J = -(\gamma_0 + \Omega_0 + \alpha\tau_0)$, and $G = (\lambda_0 + \nabla_0 + \alpha\sigma_0)$. Observe that all these conditions are consistent with the first line of the equation (4.2).

It is easy to verify that $\hat{e}_{12}\hat{e}_1 = \hat{e}_{12}\hat{e}_1 = \hat{e}_{12}\hat{f}_2 = 0$. To complete the proof in this case, we need to prove that $\hat{e}_{12}\hat{u} = \hat{e}_1\hat{f}_2 = -\hat{e}_2\hat{f}_1 = \hat{f}_{12}$. A computation shows that

$$\begin{aligned}
\hat{e}_{12}\hat{u} &= \widetilde{e}_{12}\widetilde{u} - Cw\\
&= \widetilde{f}_{12} + \varphi_0 w + (\Omega_0 + \alpha\tau_0)v_1 + (\nabla + \alpha\sigma_0)v_2 + \epsilon_0 w_{12} - (\frac{1}{2}(\chi_0 + \beta_0 - \theta_0) - \varphi_0)w\\
&= \widetilde{f}_{12} + (\Omega_0 + \alpha\tau_0)v_1 + (\nabla + \alpha\sigma_0)v_2 + \epsilon_0 w_{12} - \frac{1}{2}(\chi_0 + \beta_0 - \theta_0)w\\
&= \widetilde{f}_{12} + Dw + E_1v_1 + E_2v_2 + Fw_{12}\\
&= \hat{f}_{12},
\end{aligned}$$

and

$$\begin{aligned}\hat{e}_1\hat{f}_2 &= \widetilde{e}_1\widetilde{f}_2 + Gv_2 + H_2w\\ &= \widetilde{f}_{12} + \theta_0 w + (\Omega_0 + \alpha\tau_0)v_1 - \lambda_0 v_2 + \epsilon_0 w_{12} + Gv_2 + H_2 w\\ &= \hat{f}_{12}.\end{aligned}$$

Similarly to the last computation, it is clear that $\hat{e}_1\hat{f}_2 = -\hat{f}_{12}$, $\hat{f}_1\hat{f}_{12} = -\hat{e}_2$, and $\hat{f}_2\hat{f}_{12} = \hat{e}_1$. Finally, taking $\mathcal{S}_1 = \text{alg}\langle \hat{u}, \hat{e}_1, \hat{e}_2, \hat{f}_{12}\rangle$, it follows that $\mathcal{S}_0 \oplus \mathcal{S}_1 \cong \mathfrak{K}\text{an}(2)$, and $\mathcal{A} = \mathcal{N} \oplus \mathcal{S}$, as a consequence, we have that an analogue to WPT holds for $\mathcal{A}$ when $\mathcal{N} \cong \mathcal{V}(v, \alpha)$, which completes the proof of the first case.

Case 2. Assume that $\mathcal{N}$ is an irreducible bimodule isomorphic to $\mathcal{N} \cong \mathcal{V}(v, \alpha)^{\text{op}}$, and let $\widetilde{1}$, $\widetilde{f}_1$, $\widetilde{f}_2$, $\widetilde{e}_{12}$, $\widetilde{u}$, $\widetilde{e}_1$, $\widetilde{e}_2$, and $\widetilde{f}_{12}$ be as Theorem 3.9. In the same manner, as in case 1, we can choose elements $\hat{1}$, $\hat{f}_1$, $\hat{f}_2$, $\hat{e}_{12} \in \mathcal{A}_0$ such that $\mathcal{S}_0 = \langle \hat{1}, \hat{f}_1, \hat{f}_2, he_{12}\rangle \cong (\mathfrak{K}\text{an}(2)_0)$. Taking elements $\hat{u}, \hat{e}_1, \hat{e}_2, \hat{f}_{12} \in \mathcal{A}_1$ such that

$$\begin{aligned}\hat{u} &= \widetilde{u} + Av - \eta_{12}w_1 + \xi_{12}w_2 + \xi_2 v_{12},\\ \hat{f}_{12} &= \widetilde{f}_{12} + (\varphi_0 - \xi_2)v + \varphi_1 w_1 + \varphi_2 w_2 + \epsilon_0 v_{12},\\ \hat{e}_1 &= \widetilde{e}_1 + \lambda_0 v + (\epsilon_0 - \theta_{12})w_1 + \alpha_{12}w_2 + \alpha_2 v_{12},\\ \hat{e}_2 &= \widetilde{e}_2 - \gamma_0 v - \delta_{12}w_1 + (\epsilon_0 + \beta_{12})w_2 - \delta_1 v_{12}.\end{aligned}$$

It is easy to see that $\mathcal{S}_1 = \text{vec}\langle \hat{u}, \hat{e}_1, \hat{e}_2, \hat{f}_{12}\rangle \cong (\mathfrak{K}\text{an}(2))_1$, and consequently, we conclude that there exists a subsuperalgebra $\mathcal{S}$ in $\mathcal{A}$ such that $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1 \cong \mathfrak{K}\text{an}(2)$ and $\mathcal{A} \cong \mathcal{S} \oplus \mathcal{N}$, which completes the proof. □

Observe that proof of the Theorem 4.1 is independent of α.

Let us mention one important consequence of the Theorem 4. This result implies that the second cohomology group (SCG)

$\mathcal{H}^2(\mathfrak{K}\text{an}(2), \mathcal{V}(v, \alpha)) = 0$, and $\mathcal{H}^2(\mathfrak{K}\text{an}(2), \mathcal{V}(v, \alpha)^{\text{op}}) = 0$.

The study of nontrivial SCG for finite-dimensional Jordan superalgebra was made by the authors in [12].

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