Purchase of government bonds by a supranational central bank: its impact on business cycles

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Abstract

The euro area emerged from the euro crisis without meeting the conditions presented by the theory of optimum currency area. The decisive policy that ended this crisis was the Outright Monetary Transactions policy. Besides, the quantitative easing policy supports the economy of the euro area after this crisis. To examine the impact of these supranational monetary policies on the business cycles in the euro area, which is not covered by optimum currency area theory, we use a Kaldorian two-country model featuring a monetary union and imperfect capital mobility. We find that an increase in government bond purchases is a stabilizing factor, whereas an extreme increase in the degree of counter-cyclical fiscal policy is a destabilizing factor. Nevertheless, as long as the fiscal and monetary policies of two countries are not extremely active, but active to a certain extent, the equilibrium point becomes locally stable. Furthermore, even if the business cycles are not synchronized, the purchase of government bonds of a particular country is effective in stabilizing the business cycles of both countries. From these results, we suggest that the euro area satisfies the metacriteria of an optimum currency area through the implementation of a government bond purchase system by a supranational central bank system.

Keywords Optimum currency area · Business cycle stability · Kaldorian two-country model · Euro area · Outright monetary transactions · Quantitative easing

JEL Classification E12 · F15 · F41 · F42 · F44 · F45

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1 Introduction

The euro area took various measures to deal with the euro crisis, but the decisive policy that ended this crisis was the Outright Monetary Transactions (OMT) policy. The OMT was introduced by the European Central Bank (ECB) in 2012 and allows the ECB to make unlimited purchases of government bonds issued by crisis-hit countries. The introduction of the OMT has also helped to clarify the ECB's stance on the euro crisis. De Grauwe (2018) notes that the OMT addresses concerns that were destabilizing the system, and values the OMT as a lender of last resort.

Even as the euro area started to emerge from this crisis, it was pointed out that the region was not an optimum currency area (OCA). This was a set phrase that had been gaining ground even before the crisis began. The theory of optimum currency area is a summary of the pre-requisites for successful currency integration. Its foundation was developed by Mundell (1961), McKinnon (1963), and Kenen (1969). Since the 1960s, various economists have discussed the conditions for the establishment of an optimal currency area [see Mongelli (2002), De Grauwe (2018), and Baldwin and Wyplosz (2019)].

However, critics have alleged a problem in the OCA theory itself, in that the different criteria cannot be integrated within a uniform framework. According to Gächter et al. (2012), the discussion of this problem led to the development of a few “metacriteria” that implicitly subsume some of the individual conditions; the synchronization of business cycles has been established as a key metacriterion [(e.g., De Haan et al. (2008) and De Grauwe and Ji (2016, 2017)].

The euro has had a different effect on the synchronization of business cycles before and after its introduction. As the time of the introduction of the euro approached, the business cycle converged [e.g., Altavilla (2004), Camacho et al. (2006), and Darvas and Szapáry (2008)]. This is because it was necessary to converge economic fundamentals, such as the inflation rate and the interest rate, as a precondition for the introduction of the euro. However, since the establishment of the euro, the synchronicity of business cycles among countries within the euro area disappeared. It was under these circumstances that the euro crisis broke out and the possibility of a euro collapse was briefly discussed [see Krugman (2012)]. The euro area is not an OCA, and the need for a fiscal union, which is one of the conditions for an optimal currency area, was pointed out in particular.

However, the euro area survived this crisis without meeting the conditions presented by the OCA theory. The OCA theory has not discussed supranational monetary policy, such as the OMT that contributed to the exit from this crisis. One of the reasons for this is that countries lose their independent monetary policy in a single currency area considered to be a type of fixed exchange rate system, as described in the impossible trinity. Then, it is assumed that they will not be able to make monetary policies that are appropriate to their respective situations. The

loss of an independent monetary policy, as pointed out in the impossible trinity, occurs in a single currency area; however, it differs from a fixed exchange rate system such as the gold standard. The difference is the presence of a supranational central bank.

Regarding the supranational system of controlling the value of gold to maintain the gold standard, Keynes (1930) states:

“The ideal arrangement would surely be to set up a supranational bank to which the central banks of the world would stand in much the same relation as their own member banks stand to them.” [(Keynes (1930, p. 399))]

In the context of the euro area, this supranational bank is the ECB, as the current central bank. Keynes (1930) also notes important implications for the ECB’s monetary policy in response to the euro crisis.

“The supranational bank should also have a discretionary power to conduct open-market operations, by the purchase or sale on its own initiative either of long-term or short-term securities, with the assent, in the case of a purchase though not necessarily in the case of a sale, of the adherent central bank in whose national money the securities in question are payable.” [Keynes (1930, pp. 400–401)]

This idea is an important point that is also relevant to the policies of OMT and quantitative easing (QE). However, this suggestion about supranational central banks has been underutilized in OCA theory, even though it is related to how the ECB would eventually deal with the euro crisis.

Although the euro crisis ended, the euro area is still experiencing prolonged economic stagnation. In the context of the post-euro crisis, the ECB launched the Public Sector Purchase Programme in March 2015, which was a so-called quantitative easing initiative. Moreover, even as some countries struggle to emerge from the economic stagnation that followed the euro crisis, the COVID-19 epidemic has caused further economic stagnation since February 2020. Under these circumstances, the role of the ECB’s supranational monetary policy has become increasingly important in the euro area, where fiscal policy is constrained.

This study contributes to the construction of a uniform framework for the OCA theory by analyzing supranational monetary policy from the perspective of the synchronization of business cycles. Moreover, this study allows us to consider the conditions necessary for the euro area to survive.

Therefore, this study examines the impact of government bond purchases by the Eurosystem on the business cycles of countries. The Eurosystem is the monetary authority of the euro area and is headed by the ECB, which is a supranational central bank. In doing so, we carry out a stability analysis of the business cycle with a Kaldorian two-country model with imperfect capital movement. We are interested in how the demand-led economies under currency integration

with Keynesian underemployment fluctuate through time and whether the system is dynamically stable or not. Kaldorian business cycle model developed by Kaldor (1940) is a suitable Keynesian model that can analyze these issues. This paper extends the Kaldorian business cycle model of a closed economy to a two-country model under currency integration that is suitable for our purpose. If the purchase of government bonds of a particular country by a supranational central bank stabilizes the business cycles of that country as well as those of other euro area countries and converges them to an equilibrium point, then the synchronicity of business cycles is likely to increase.

The results of this study show the purchase of government bonds of a country that is facing a crisis can stabilize the business cycle of the country. The euro area satisfies the metacriteria of an OCA through the purchase of government bonds by the supranational central banking system. These results will help to build institutions for the successful implementation of ECO, which is a common currency that is scheduled to be introduced in the Economic Community of West African States (ECOWAS) in 2020.

The rest of this paper is organized as follows. In Sect. 2, we formalize the two-country Kaldorian model with currency integration, which consists of an eight-dimensional system of nonlinear differential equations. In Sect. 3, we investigate the conditions for local stability of the equilibrium point mathematically. In Sect. 4, we present the results of some numerical simulations that support the theoretical analysis in Sect. 3. Finally, Sect. 5 concludes this article.

2 Formulation of the model on government bond purchase

In this section, we analyze the effect of government bond purchases by central banks in the monetary union using the Kaldorian two-country model with currency integration. The two-country version of Kaldor (1940)'s model of business cycle has been discussed extensively in previous literature. Asada et al. (2001) analyzed the stability conditions of the business cycles in two regions using the two-country discrete-time Kaldorian model of business cycles under fixed exchange rates and imperfect capital mobility. Asada (2004) used a continuous-time Kaldorian imperfect capital movement two-country model to consider the stability conditions of business cycles in the two regions with fixed exchange rates, with a particular focus on the speed of adjustment in the goods market. Asada et al. (2003) performed a more complex business cycle analysis in the context of a multidimensional two-country model. Inaba and Asada (2020) considered the Kaldorian three-country model with fixed exchange rates. Nakao (2017) argued that an increase in capital mobility between countries in a capital markets union is a destabilizing factor, whereas an increase in fiscal transfers between such countries is a stabilizing factor, using Keynesian and Kaldorian two-country models. Nakao (2019) proved that a high degree of economic

openness can adjust a shock in the monetary union regardless of whether the shock is asymmetric, using Keynesian and Kaldorian two-country models with a monetary union system.

To analyze a monetary union, such as the euro area, we represent the exchange rate as follows:

$$E = E^e = \bar{E} = 1, \quad (1)$$

where E is the exchange rate and E^e is the expected exchange rate in the near future. We assume that the exchange rate and expected exchange rate are one because a single currency is used in a monetary union.

Furthermore, to simplify the analysis, we focus on a fixed price economy.

Assumption 1

$$p_i = 1. \quad (2)$$

The subscript i ($i = 1, 2$) is the index number of a country and p_i is the price level of country i . This assumption eliminates price fluctuations. This study considers an economy in the medium-term, in which the changes of prices are neglected although the changes of the real capital stocks are allowed for, and we do not deal with the problem of inflation and deflation in this paper.

Under these assumptions about the exchange rate and prices, our model consists of the following system of dynamic equations over the medium-term with flexible capital stock¹.

Disequilibrium quantity adjustment process in the goods market in country i is

$$\dot{Y}_i = \alpha_i [C_i + I_i + G_i + J_i - Y_i]; \alpha_i > 0. \quad (3)$$

Keynesian consumption function in a country i is

$$C_i = c_i(Y_i + r_i B_i^i + r_j B_j^i - T_i) + C_{0i}; 0 < c_i < 1, C_{0i} \geq 0. \quad (4)$$

Standard Kaldorian investment function in a country i is:

$$I_i = I_i(Y_i, K_i, r_i); I_{Y_i}^i = \frac{\partial I_i}{\partial Y_i} > 0, I_{K_i}^i = \frac{\partial I_i}{\partial K_i} < 0, I_{r_i}^i = \frac{\partial I_i}{\partial r_i} < 0. \quad (5)$$

Government expenditure function in a country i is

$$G_i = G_{0i} + \gamma_i(\bar{Y}_i - Y_i); \gamma_i > 0. \quad (6)$$

Standard tax function in a country i is

¹ In the conventional dichotomy, the short-run model consists of fixed prices without capital accumulation, and the long-run model consists of flexible prices, capital accumulation, population growth, and technical progress. Thus, in this study, we assume that the medium-term model consists of a fixed price with capital accumulation and we omit growth factors because the economies in this model do not grow at the equilibrium point.

$$T_i = \tau_i(Y_i + r_i B_i^i + r_j B_j^i) - T_{0i}; 0 < \tau_i < 1, T_{0i} \geq 0. \quad (7)$$

Dynamic equation of capital stock in a country i is

$$\dot{K}_i = I_i. \quad (8)$$

Outstanding nominal government bond of a country i is

$$B_i = B_i^i + B_i^j + \theta_i. \quad (9)$$

Budget constraint of a country i is

$$\dot{B}_i = G_i + B_i r_i - T_i. \quad (10)$$

Dynamic equation of the outstanding nominal government bonds of a country i is

$$\dot{B}_i = \dot{B}_i^i + \dot{B}_i^j + \dot{\theta}_i. \quad (11)$$

Dynamic equation of the outstanding nominal government bonds of country i held by country j is

$$\dot{B}_i^j = -\dot{B}_j^i = \beta(r_i - r_j); \beta > 0. \quad (12)$$

From Eq. (12), the following equation holds.

$$B_1^2 + B_2^1 = \bar{D}. \quad (13)$$

Net export (current account) function in a country i is:

$$J_i = J_i(Y_i, Y_j); J_{Y_i}^i = \frac{\partial J_i}{\partial Y_i} < 0, J_{Y_j}^i = \frac{\partial J_i}{\partial Y_j} > 0. \quad (14)$$

Capital account functions in a country 1 is

$$\begin{aligned} Q_1 &= \dot{B}_1^2 - \dot{B}_2^1 + r_2 B_2^1 - r_1 B_1^2 \\ &= \beta(r_1 - r_2) - \beta(r_2 - r_1) + r_2 B_2^1 - r_1 B_1^2 \\ &= 2\beta(r_1 - r_2) + r_2 B_2^1 - r_1 B_1^2. \end{aligned} \quad (15)$$

The definition of total balance of payments of country i is

$$A_i = J_i + Q_i. \quad (16)$$

The relationship between the first and second countries with respect to net exports, capital account, and balance of payments is, respectively,

$$J_1 + J_2 = 0, \quad (17)$$

$$Q_1 + Q_2 = 0, \quad (18)$$

$$A_1 + A_2 = 0. \quad (19)$$

The following equation describes the total nominal money supply of two countries belonging to a monetary union that is equal to the money supply of the supranational central bank.

$$M = M_1 + M_2. \quad (20)$$

The money supply in the monetary union depends on purchasing government bonds of both countries by the supranational central bank.

$$\dot{M} = \dot{\theta}_1 + \dot{\theta}_2. \quad (21)$$

The nominal money supply of country i increases (decreases) according to the total balance of payment surplus (deficit) of country i .

$$\dot{M}_i = \dot{\theta}_i + A_i. \quad (22)$$

The equilibrium condition in the monetary market is

$$M_i = L_i(Y_i, r_i); \frac{\partial L_i}{\partial Y_i} > 0, \frac{\partial L_i}{\partial r_i} < 0. \quad (23)$$

The meanings of the symbols are as follows. Y_i is real net national income. C_i is real private consumption expenditure. c_i is the marginal propensity to consume. C_{0i} is basic consumption. I_i is real net private investment expenditure. G_i is real government expenditure. G_{0i} is basic government expenditure. K_i is real capital stock. B_i is the outstanding nominal government bonds. B_i^i is the outstanding nominal government bonds of country i held by the private sector in country i . B_i^j is the outstanding nominal government bonds of country i held by the private sector in country j . r_i is the nominal rate of interest.² $\bar{Y}_i$ is the level of real national income that helps a government determine the counter-cyclical government expenditure (this is not necessarily natural output). T_i is the real income tax. τ_i is the marginal tax rate. T_{0i} is negative income tax (or basic income). M is the nominal money supply in the whole monetary union. M_i is the nominal money supply. J_i is real net exports. Q_i is the real capital account balance. A_i is the real total balance of payments. Parameter α_i represents the adjustment speed of the goods market. Parameter β indicates the degree of mobility of international capital flows; the larger the value of β is, the higher is the degree of mobility of international capital flows.³ Parameter γ_i says the degree of counter-cyclical fiscal policy and a large γ_i implies a large counter-cyclical government expenditure. Parameter θ_i is the outstanding nominal government bond of country i held by the supranational central bank.

² In this study, for the sake of simplicity, public bonds and stock are treated as perfect substitute goods. Therefore, the interest rate r_i is an average of various kinds of interest rates.

³ The model of perfect capital mobility is a special case in which β is infinite, and in this case the following equation is always established in the case of a fixed exchange rate system: $r_1 = r_2$.

Equation (9) is the definitional equation of the outstanding nominal government bonds of country i , representing that a nominal government bond of country i is held by both i and j . Equation (11) reflects that the outstanding nominal government bond of country i depends on the amount of bonds held by both countries. Equations (17), (18), and (19) imply that the net export surplus, capital account balance surplus, and the total balance of payments surplus of a country must be accompanied by the same amount of the current account deficit, capital account balance deficit, and the total balance of payments deficit of another country, respectively.

We transform these equations into a more compact system. We obtain the following equation by solving Eq. (23) with respect to r_i .

$$r_i = r_i(Y_i, M_i); r_{Y_i}^i = \frac{\partial r_i}{\partial Y_i} = -\frac{L_{Y_i}^i}{L_{r_i}^i} > 0, r_{M_i}^i = \frac{\partial r_i}{\partial M_i} = \frac{1}{L_{r_i}^i} < 0. \quad (24)$$

Furthermore, we assume the following equations.

Assumption 2

$$\dot{\theta}_1 = \dot{\theta}_2 = 0, \quad (25)$$

$$\bar{M} = M_1 + M_2. \quad (26)$$

The central bank can increase $\bar{M}$. To simplify the discussion, we assume that θ_i is a parameter rather than a variable. However, since its magnitude affects performance, we perform a comparative dynamic analysis with θ_i as a policy parameter. On this assumption, the nominal money supply in the whole monetary union is fixed.

Therefore, we transform Eqs. (21) and (22) into the following equations.

$$\dot{M} = 0, \quad (27)$$

$$\dot{M}_i = A_i. \quad (28)$$

Combining Eqs. (3)–(28), we obtain the following eight-dimensional system of nonlinear differential equations, given policy parameters G_0 , τ_i , and $\bar{M}$.

$$\begin{aligned} \dot{Y}_1 = & \alpha_1 [\{c_1(1 - \tau_1) - 1\}Y_1 \\ & + c_1(1 - \tau_1)\{(B_1 - \theta_1 - B_1^2)r_1(Y_1, M_1) + (\bar{D} - B_1^2)r_2(Y_2, \bar{M} - M_1)\} \\ & + c_1T_{01} + C_{01} + G_{01} + \gamma_1(\bar{Y}_1 - Y_1) + I_1(Y_1, K_1, r_1(Y_1, M_1)) \\ & + J_1(Y_1, Y_2)] \\ = & F_1(Y_1, K_1, B_1, B_1^2, Y_2, M_1; \alpha_1, \gamma_1, \theta_1), \end{aligned} \quad (29)$$

$$\dot{K}_1 = I_1(Y_1, K_1, r_1(Y_1, M_1)) = F_2(Y_1, K_1, M_1), \quad (30)$$

$$\begin{aligned}\dot{B}_1 &= G_{01} + \gamma_1(\bar{Y}_1 - Y_1) + B_1 r_1(Y_1, M_1) \\ &\quad - \tau_1 \{Y_1 + (B_1 - \theta_1 - B_1^2) r_1(Y_1, M_1) + (\bar{D} - B_1^2) r_2(Y_2, \bar{M} - M_1)\} + T_{01} \\ &= F_3(Y_1, B_1, B_1^2, Y_2, M_1; \gamma_1, \theta_1),\end{aligned}\quad (31)$$

$$\begin{aligned}\dot{B}_1^2 &= \beta \{r_1(Y_1, M_1) - r_2(Y_2, \bar{M} - M_1)\} \\ &= F_4(Y_1, Y_2, M_1; \beta),\end{aligned}\quad (32)$$

$$\begin{aligned}\dot{Y}_2 &= \alpha_2 [\{c_2(1 - \tau_2) - 1\} Y_2 \\ &\quad + c_2(1 - \tau_2) \{(B_2 - \theta_2 - (\bar{D} - B_1^2)) r_2(Y_2, \bar{M} - M_1) + B_1^2 r_1(Y_1, M_1)\} \\ &\quad + c_2 T_{02} + C_{02} + G_{02} + \gamma_2(\bar{Y}_2 - Y_2) + I_2(Y_2, K_2, r_2(Y_2, \bar{M} - M_1)) \\ &\quad - J_1(Y_1, Y_2)] \\ &= F_5(Y_1, B_1^2, Y_2, K_2, B_2, M_1; \alpha_2, \gamma_2, \theta_2),\end{aligned}\quad (33)$$

$$\dot{K}_2 = I_2(Y_2, K_2, r_2(Y_1, M_1)) = F_6(Y_2, K_2, M_1),\quad (34)$$

$$\begin{aligned}\dot{B}_2 &= G_{02} + \gamma_2(\bar{Y}_2 - Y_2) + B_2 r_2(Y_2, \bar{M} - M_1) \\ &\quad - \tau_2 \{Y_2 + (B_2 - \theta_2 - (\bar{D} - B_1^2)) r_2(Y_2, \bar{M} - M_1) + B_1^2 r_1(Y_1, M_1)\} + T_{02} \\ &= F_7(Y_1, B_1^2, Y_2, B_2, M_1; \gamma_2, \theta_2),\end{aligned}\quad (35)$$

$$\begin{aligned}\dot{M}_1 &= J_1(Y_1, Y_2) + 2\beta \{r_1(Y_1, M_1) - r_2(Y_2, \bar{M} - M_1)\} \\ &\quad + (\bar{D} - B_1^2) r_2(Y_2, \bar{M} - M_1) - B_1^2 r_1(Y_1, M_1) \\ &= F_8(Y_1, B_1^2, Y_2, M_1; \beta).\end{aligned}\quad (36)$$

3 Analysis of local stability

In this section, assuming the existence and uniqueness of an equilibrium solution $(Y_1^*, K_1^*, B_1^*, B_1^{2*}, Y_2^*, K_2^*, B_2^*, M_1^*) > (0, 0, 0, 0, 0, 0, 0, 0)$, we analyze the stability of such as an equilibrium solution. The numerical simulation in the next section provides a case in which such an equilibrium point exists.

We assume the following inequality for the interest rate at the equilibrium point.

Assumption 3

$$r_i > 0.$$

This assumption implies that a negative interest rate at the equilibrium point should not be considered.

The Jacobian matrix of the system of Eqs. (29)–(36) that is evaluated at the equilibrium point can be written as follows.

$$J = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} & 0 & 0 & F_{18} \\ F_{21} & F_{22} & 0 & 0 & 0 & 0 & 0 & F_{28} \\ F_{31} & 0 & F_{33} & F_{34} & F_{35} & 0 & 0 & F_{38} \\ F_{41} & 0 & 0 & 0 & F_{45} & 0 & 0 & F_{48} \\ F_{51} & 0 & 0 & F_{54} & F_{55} & F_{56} & F_{57} & F_{58} \\ 0 & 0 & 0 & 0 & F_{65} & F_{66} & 0 & F_{68} \\ F_{71} & 0 & 0 & F_{74} & F_{75} & 0 & F_{77} & F_{78} \\ F_{81} & 0 & 0 & F_{84} & F_{85} & 0 & 0 & F_{88} \end{bmatrix}, \quad (37)$$

where the following relationships are satisfied.

$$\begin{aligned} F_{11} &= \alpha_1 \Phi_{11}(\gamma_1, \theta_1) \\ &= \alpha_1 [\underbrace{\{c_1(1 - \tau_1) - 1\}}_{(-)} + \underbrace{c_1(1 - \tau_1)}_{(+)} \underbrace{(B_1 - \theta_1 - B_1^2)}_{(+)} + \underbrace{r_{Y_1}^1}_{(+)} - \underbrace{\gamma_1}_{(+)} + \underbrace{I_{Y_1}^1}_{(+)} \\ &\quad + \underbrace{I_{r_1}^1 r_{Y_1}^1}_{(-)(+)} + \underbrace{J_{Y_1}^1}_{(-)}], \\ F_{12} &= \alpha_1 \Phi_{12} = \alpha_1 \underbrace{I_{K_1}^1}_{(-)} < 0, \\ F_{13} &= \alpha_1 \Phi_{13} = \alpha_1 \underbrace{c_1(1 - \tau_1)}_{(+)} r_1 > 0, \\ F_{14} &= \alpha_1 \Phi_{14} = -\alpha_1 \underbrace{c_1(1 - \tau_1)}_{(+)} (r_1 + r_2) < 0, \\ F_{15} &= \alpha_1 \Phi_{15} = \alpha_1 [\underbrace{c_1(1 - \tau_1)}_{(+)} \underbrace{(\bar{D} - B_1^2)}_{(+)} \underbrace{r_{Y_2}^2}_{(+)} + \underbrace{J_{Y_2}^1}_{(+)}] > 0, \\ F_{18} &= \alpha_1 \Phi_{18}(\theta_1) \\ &= \alpha_1 [\underbrace{c_1(1 - \tau_1)}_{(+)} \{ \underbrace{(B_1 - \theta_1 - B_1^2)}_{(+)} \underbrace{r_{M_1}^1}_{(-)} - \underbrace{(\bar{D} - B_1^2)}_{(+)} \underbrace{r_{M-M_1}^2}_{(-)} \} + \underbrace{I_{r_1}^1 r_{M_1}^1}_{(-)(-)}] > 0, \\ F_{21} &= \underbrace{I_{Y_1}^1}_{(+)} > 0, \\ F_{22} &= \Phi_{12}, \\ F_{28} &= \underbrace{I_{r_1}^1 r_{M_1}^1}_{(-)(-)} > 0, \end{aligned}$$

$$F_{31} = F_{31}(\gamma_1, \theta_1) = \underbrace{-\gamma_1}_{(+)} + \underbrace{B_1 r_{Y_1}^1}_{(+)(+)} - \underbrace{\tau_1 \{1 + (B_1 - \theta_1 - B_1^2) r_{Y_1}^1\}}_{(+)(+)},$$

$$F_{33} = \underbrace{(1 - \tau_1) r_1}_{(+)} > 0,$$

$$F_{34} = \tau_1 (r_1 + r_2) > 0,$$

$$F_{35} = -\tau_1 \underbrace{(\bar{D} - B_1^2)}_{(+)} \underbrace{r_{Y_2}^2}_{(+)} < 0,$$

$$F_{38} = F_{38}(\theta_1) = \underbrace{B_1 r_{M_1}^1}_{(+)(-)} - \underbrace{\tau_1 (B_1 - \theta_1 - B_1^2) r_{M_1}^1}_{(+)(-)} - \underbrace{(\bar{D} - B_1^2) r_{\bar{M}-M_1}^2}_{(+)(-)} < 0,$$

$$F_{41} = \underbrace{\beta r_{Y_1}^1}_{(+)(+)} > 0,$$

$$F_{45} = -\underbrace{\beta r_{Y_2}^2}_{(+)(+)} < 0,$$

$$F_{48} = \underbrace{\beta (r_{M_1}^1)}_{(+)(-)} + \underbrace{r_{\bar{M}-M_1}^2}_{(-)} < 0,$$

$$F_{51} = \alpha_2 \Phi_{51} = \alpha_2 \underbrace{[c_2(1 - \tau_2) B_1^2 r_{Y_1}^1]}_{(+)(+)(+)} - \underbrace{J_{Y_1}^1}_{(-)} > 0,$$

$$F_{54} = \alpha_2 \Phi_{54} = \alpha_2 \underbrace{c_2(1 - \tau_2)(r_1 + r_2)}_{(+)} > 0,$$

$$F_{55} = \alpha_2 \Phi_{55}(\gamma_2, \theta_2) \\ = \alpha_2 [\underbrace{\{c_2(1 - \tau_2) - 1\}}_{(-)} + \underbrace{c_2(1 - \tau_2)}_{(+)} \underbrace{(B_2 - \theta_2 - (\bar{D} - B_1^2))}_{(+)} \underbrace{r_{Y_2}^2}_{(+)} - \underbrace{\gamma_2}_{(+)} + \underbrace{I_{Y_2}^2}_{(+)} + \underbrace{I_{r_2}^2 r_{Y_2}^2}_{(-)(+)} - \underbrace{J_{Y_2}^1}_{(+)}],$$

$$F_{56} = \alpha_2 \Phi_{56} = \alpha_2 \underbrace{I_{K_2}^2}_{(-)} < 0,$$

$$F_{57} = \alpha_2 \Phi_{57} = \alpha_2 \underbrace{c_2(1 - \tau_2) r_2}_{(+)} > 0,$$

$$F_{58} = \alpha_2 \Phi_{58}(\theta_2) = -\alpha_2 \underbrace{[c_2(1 - \tau_2)]}_{(+)} \underbrace{\{(B_2 - \theta_2 - (\bar{D} - B_1^2))\}}_{(+)} \underbrace{r_{\bar{M}-M_1}^2}_{(-)} - \underbrace{B_1^2 r_{M_1}^1}_{(+)(-)} - \underbrace{I_{r_2}^2 r_{\bar{M}-M_1}^2}_{(-)(-)}],$$

$$F_{65} = \underbrace{I_{Y_2}^2}_{(+)} > 0,$$

$$F_{66} = \Phi_{56} = \underbrace{I_{K_2}^2}_{(-)} < 0,$$

$$F_{68} = \underbrace{I_{r_2}^2}_{(-)} \underbrace{r_{\bar{M}-M_1}^2}_{(-)} > 0,$$

$$\begin{aligned}
F_{71} &= -\tau_2 B_1^2 r_{Y_1}^1 < 0, \\
&\quad (+)(+)(+) \\
F_{74} &= -\tau_2 (r_1 + r_2) < 0, \\
&\quad (+) \\
F_{75} &= F_{75}(\gamma_2, \theta_2) = -\gamma_2 + B_2 r_{Y_2}^2 - \tau_2 \underbrace{\{1 + (B_2 - \theta_2 - (\bar{D} - B_1^2))r_{Y_2}^2\}}_{(+)}, \\
&\quad (+) \quad (+)(+) \quad (+) \\
F_{77} &= \underbrace{(1 - \tau_2)}_{(+)} r_2 > 0, \\
F_{78} &= F_{78}(\theta_2) \\
&= -B_2 r_{M-M_1}^2 + \tau_2 \underbrace{\{(B_2 - \theta_2 - (\bar{D} - B_1^2))r_{M-M_1}^2\}}_{(+)} + B_1^2 r_{M_1}^1 > 0, \\
&\quad (+) \quad (-) \quad (+) \quad (-) \quad (+) \quad (-) \\
F_{81} &= F_{81}(\beta) = J_{Y_1}^1 + 2\beta r_{Y_1}^1 - B_1^2 r_{Y_1}^1 > 0, \\
&\quad (-) \quad (+)(+) \quad (+)(+) \\
F_{84} &= -(r_1 + r_2) < 0, \\
F_{85} &= F_{85}(\beta) \\
&= J_{Y_2}^1 - 2\beta r_{Y_2}^2 + B_2^1 r_{Y_2}^2 < 0, \\
&\quad (+) \quad (+)(+) \quad (+)(+) \\
F_{88} &= F_{88}(\beta) \\
&= 2\beta \underbrace{(r_{M_1}^1 + r_{M-M_1}^2)}_{(+)} - (\bar{D} - B_1^2) r_{M-M_1}^2 - B_1^2 r_{M_1}^1 < 0. \\
&\quad (+) \quad (-) \quad (-) \quad (+) \quad (-) \quad (+) \quad (-)
\end{aligned}$$

We shall study the local stability/instability of the equilibrium point under the following set of assumptions.

Assumption 4

- (i) $\Phi_{11}(0, 0) > 0$ and $\Phi_{55}(0, 0) > 0$.
- (ii) Parameter β is fixed at such a sufficiently large value that we have $F_{81}(\beta) > 0$, $F_{85}(\beta) < 0$, and $F_{88}(\beta) < 0$.

Assumption 4 (i) implies that the marginal propensities of investment of both countries ($I_{Y_1}^1$ and $I_{Y_2}^2$) are sufficiently large at the equilibrium point, which is a standard assumption of the Kaldorian business cycle theory, as proposed by Kaldor (1940). Assumption 4 (ii) implies that the capital movement between countries is sufficiently active. We can write the characteristic equation of this system at the equilibrium point as follows.

$$f(\lambda) \equiv |\lambda \mathbf{I} - \mathbf{J}|$$

$$= \lambda^8 + a_1 \lambda^7 + a_2 \lambda^6 + a_3 \lambda^5 + a_4 \lambda^4 + a_5 \lambda^3 + a_6 \lambda^2 + a_7 \lambda + a_8 = 0. \quad (38)$$

We can express the coefficients a_m ($m = 1, 2, \dots, 8$) as follows, where v_n , ϕ_n , χ_n , ψ_n and ω_n ($n = 1, 2, \dots, 12$) may be functions of the parameters γ_1 , θ_1 , γ_2 , and θ_2 .

$$a_1 = -\text{Tr} \mathbf{J}$$

$$= -\alpha_1 \underbrace{\Phi_{11}(\gamma_1, \theta_1)}_{(?) } - \underbrace{\Phi_{12}}_{(-)} - \underbrace{(1 - \tau_1)r_1}_{(+)} - \alpha_2 \underbrace{\Phi_{55}(\gamma_2, \theta_2)}_{(?) } - \underbrace{\Phi_{56}}_{(-)}$$

$$- \underbrace{(1 - \tau_2)r_2}_{(+)} - \underbrace{F_{88}(\beta)}_{(-)}, \quad (39)$$

$$a_2 = \text{sum of all principal second-order minors of } \mathbf{J}$$

$$= \alpha_1 \alpha_2 \begin{vmatrix} \Phi_{11}(\gamma_1, \theta_1) & \Phi_{15} \\ \Phi_{51} & \Phi_{55}(\gamma_2, \theta_2) \end{vmatrix} + \alpha_1 v_1 + \alpha_2 v_2 + v_3$$

$$= \alpha_1 \alpha_2 \left\{ \underbrace{\Phi_{11}(\gamma_1, \theta_1)}_{(?) } \underbrace{\Phi_{55}(\gamma_2, \theta_2)}_{(?) } - \underbrace{\Phi_{15}}_{(+)} \underbrace{\Phi_{51}}_{(+)} \right\} + \alpha_1 v_1 + \alpha_2 v_2 + v_3 \quad (40)$$

$$= \alpha_1 \alpha_2 \psi_1 + \alpha_1 v_1 + \alpha_2 v_2 + v_3,$$

$$a_3 = -(\text{sum of all principal third-order minors of } \mathbf{J})$$

$$= \alpha_1 \alpha_2 \left\{ -\Phi_{12} \begin{vmatrix} \Phi_{11}(\gamma_1, \theta_1) & 1 & \Phi_{15} \\ F_{21} & 1 & 0 \\ \Phi_{51} & 0 & \Phi_{55}(\gamma_2, \theta_2) \end{vmatrix} - \Phi_{56} \begin{vmatrix} \Phi_{11}(\gamma_1, \theta_1) & \Phi_{15} & 0 \\ \Phi_{51} & \Phi_{55}(\gamma_2, \theta_2) & 1 \\ 0 & F_{65} & 1 \end{vmatrix} \right.$$

$$\left. - \begin{vmatrix} \Phi_{11}(\gamma_1, \theta_1) & \Phi_{15} & \Phi_{18}(\theta_1) \\ \Phi_{51} & \Phi_{55}(\gamma_2, \theta_2) & \Phi_{58}(\theta_2) \\ F_{81}(\beta) & F_{85}(\beta) & F_{88}(\beta) \end{vmatrix} + r_1 \chi_1 + (r_1 + r_2) \chi_2 + r^2 \chi_3 \right\}$$

$$+ \alpha_1 v_1 + \alpha_2 v_2 + v_3$$

$$= \alpha_1 \alpha_2 \left\{ \underbrace{(-\Phi_{12})}_{(-)} - \underbrace{\Phi_{56}}_{(-)} - \underbrace{F_{88}(\beta)}_{(-)} \right\} \underbrace{\Phi_{11}(\gamma_1, \theta_1)}_{(?) } \underbrace{\Phi_{55}(\gamma_2, \theta_2)}_{(?) } + \underbrace{\phi_1}_{(?) } \underbrace{\Phi_{11}(\gamma_1, \theta_1)}_{(?) }$$

$$+ \underbrace{\phi_2}_{(?) } \underbrace{\Phi_{55}(\gamma_2, \theta_2)}_{(?) } + \phi_3 + r_1 \chi_1 + (r_1 + r_2) \chi_2 + r_2 \chi_3 \} + \alpha_1 v_4 + \alpha_2 v_5 + v_6$$

$$= \alpha_1 \alpha_2 \psi_2 + \alpha_1 v_4 + \alpha_2 v_5 + v_6, \quad (41)$$

a_4 = sum of all principal fourth-order minors of J

$$\begin{aligned}
 &= \alpha_1 \alpha_2 \left\{ \Phi_{12} \Phi_{56} \begin{vmatrix} \Phi_{11}(\gamma_1, \theta_1) & 1 & \Phi_{15} & 0 \\ F_{21} & 1 & 0 & 0 \\ \Phi_{51} & 0 & \Phi_{55}(\gamma_2, \theta_2) & 1 \\ 0 & 0 & F_{65} & 1 \end{vmatrix} \right. \\
 &\quad + \Phi_{12} \begin{vmatrix} \Phi_{11}(\gamma_1, \theta_1) & 1 & \Phi_{15} & \Phi_{18} \\ F_{21} & 1 & 0 & F_{28} \\ \Phi_{51} & 0 & \Phi_{55}(\gamma_2, \theta_2) & \Phi_{58}(\theta_2) \\ F_{81}(\beta) & 0F_{81}(\beta) & F_{88}(\beta) & \end{vmatrix} \\
 &\quad \left. + \Phi_{56} \begin{vmatrix} \Phi_{11}(\gamma_1, \theta_1) & \Phi_{15} & 0 & \Phi_{18} \\ \Phi_{51} & \Phi_{55}(\gamma_2, \theta_2) & 1 & \Phi_{58}(\theta_2) \\ 0 & F_{65} & 1 & F_{68} \\ F_{81}(\beta) & F_{85}(\beta) & 0 & F_{88}(\beta) \end{vmatrix} + r_1 \chi_4 + (r_1 + r_2) \chi_5 + r_2 \chi_6 \right\} \\
 &\quad + \alpha_1 v_7 + \alpha_2 v_8 + v_9 \\
 &= \alpha_1 \alpha_2 \{ \underset{(-)}{\Phi_{12}} \underset{(-)}{\Phi_{56}} + \underset{(-)}{\Phi_{12}} \underset{(-)}{\Phi_{88}(\beta)} + \underset{(-)}{\Phi_{56}} \underset{(-)}{\Phi_{88}(\beta)} \} \underset{(?)}{\Phi_{11}(\gamma_1, \theta_1)} \underset{(?)}{\Phi_{55}(\gamma_2, \theta_2)} \\
 &\quad + \underset{(?)}{\phi_4} \underset{(?)}{\Phi_{11}(\gamma_1, \theta_1)} + \underset{(?)}{\phi_5} \underset{(?)}{\Phi_{11}(\gamma_1, \theta_1)} + \phi_6 + r_1 \chi_4 + (r_1 + r_2) \chi_5 + r_2 \chi_6 \} \\
 &\quad + \alpha_1 v_7 + \alpha_2 v_8 + v_9 \\
 &= \alpha_1 \alpha_2 \psi_3 + \alpha_1 v_7 + \alpha_2 v_8 + v_9,
 \end{aligned} \tag{42}$$

a_5 = -(sum of all principal fifth-order minors of J)

$$\begin{aligned}
 &= -\alpha_1 \alpha_2 \Phi_{12} \Phi_{56} \begin{vmatrix} \Phi_{11}(\gamma_1, \theta_1) & 1 & \Phi_{15} & 0 & \Phi_{18}(\theta_1) \\ F_{21} & 1 & 0 & 0 & F_{28} \\ \Phi_{51} & 0 & \Phi_{55}(\gamma_2, \theta_2) & 1 & \Phi_{58}(\theta_2) \\ 0 & 0 & F_{65} & 1 & F_{68} \\ F_{81}(\beta) & 0 & F_{85}(\beta) & 0 & F_{88}(\beta) \end{vmatrix} + r_1 \omega_1 + (r_1 + r_2) \omega_2 \\
 &\quad + r_2 \omega_3 + r_1(r_1 + r_2) \omega_4 + r_1 r_2 \omega_5 + (r_1 + r_2) r_2 \omega_6 \\
 &\quad + r_1(r_1 + r_2) r_2 \omega_7 \\
 &= \alpha_1 \alpha_2 \{ - \underset{(-)}{\Phi_{12}} \underset{(-)}{\Phi_{56}} \underset{(-)}{F_{88}(\beta)} \underset{(?)}{\Phi_{11}(\gamma_1, \theta_1)} \underset{(?)}{\Phi_{55}(\gamma_2, \theta_2)} + \underset{(?)}{\phi_7} \underset{(?)}{\Phi_{11}(\gamma_1, \theta_1)} \\
 &\quad + \underset{(?)}{\phi_8} \underset{(?)}{\Phi_{55}(\gamma_2, \theta_2)} + \underset{(?)}{\phi_9} \} + r_1 \omega_1 \\
 &\quad + (r_1 + r_2) \omega_2 + r_2 \omega_3 + r_1(r_1 + r_2) \omega_4 + r_1 r_2 \omega_5 + r_2(r_1 + r_2) \omega_6 \\
 &\quad + r_1 r_2(r_1 + r_2) \omega_7 \\
 &= \alpha_1 \alpha_2 \psi_4 + r_1 \omega_1 + (r_1 + r_2) \omega_2 + r_2 \omega_3 + r_1(r_1 + r_2) \omega_4 + r_1 r_2 \omega_5 \\
 &\quad + r_2(r_1 + r_2) \omega_6 + r_1 r_2(r_1 + r_2) \omega_7,
 \end{aligned} \tag{43}$$

$$\begin{aligned}
 a_6 &= \text{sum of all principal sixth-order minors of } J \\
 &= \alpha_1 \alpha_2 \Phi_{12} \Phi_{56} \{ \beta(r_1 + r_2) \underset{(+)}{F_{28}} \underset{(?)}{\Phi_{11}}(\gamma_1, \theta_1) \underset{(?)}{\Phi_{55}}(\gamma_2, \theta_2) + \phi_{10} \underset{(?)}{\Phi_{11}}(\gamma_1, \theta_1) \\
 &\quad + \phi_{11} \underset{(?)}{\Phi_{55}}(\gamma_2, \theta_2) + \phi_{12} \} + r_1 \chi_7 + r_2 \chi_8 \} r_1(r_1 + r_2) \omega_8 + r_1 r_2 \omega_9 \\
 &\quad + r_2(r_1 + r_2) \omega_{10} + r_1 r_2(r_1 + r_2) \omega_{11} \\
 &= \alpha_1 \alpha_2 \psi_5 + r_1(r_1 + r_2) \omega_8 + r_1 r_2 \omega_9 + r_2(r_1 + r_2) \omega_{10} + r_1 r_2(r_1 + r_2) \omega_{11},
 \end{aligned} \tag{44}$$

$$\begin{aligned}
 a_7 &= -(\text{sum of all principal seventh-order minors of } J) \\
 &= \alpha_1 \alpha_2 [- \underset{(-)}{\Phi_{12}} \underset{(-)}{\Phi_{56}} \underset{(-)}{\Phi_{48}} (1 - \tau_1) r_1(r_1 + r_2) \beta \{ \underset{(?)}{\Phi_{11}}(\gamma_1, \theta_1) \\
 &\quad - c_1 \underset{(?)}{F_{31}}(\gamma_1, \theta_1) \underset{(?)}{\Phi_{55}}(\gamma_2, \theta_2) + \phi_{13} \underset{(?)}{\Phi_{11}}(\gamma_1, \theta_1) + \phi_{14} \underset{(?)}{\Phi_{55}} + \phi_{15} \} \\
 &\quad - \underset{(-)}{\Phi_{12}} \underset{(-)}{\Phi_{56}} \underset{(-)}{\Phi_{48}} (1 - \tau_2) (r_1 + r_2) r_2 \beta \{ \underset{(?)}{\Phi_{11}}(\gamma_1, \gamma_1) \underset{(?)}{\Phi_{55}}(\gamma_2, \theta_2) \\
 &\quad + \phi_{16} \underset{(?)}{\Phi_{11}}(\gamma_1, \theta_1) + \phi_{17} \underset{(?)}{\Phi_{55}}(\gamma_2, \theta_2) + \phi_{18} \} + r_1 r_2 \chi_9] + r_1 r_2(r_1 + r_2) \omega_{12} \\
 &= \alpha_1 \alpha_2 \psi_6 + r_1 r_2(r_1 + r_2) \omega_{12},
 \end{aligned} \tag{45}$$

$$\begin{aligned}
 a_8 &= \det J \\
 &= \alpha_1 \alpha_2 \beta \underset{(-)}{\Phi_{12}} \underset{(-)}{\Phi_{56}} (1 - \tau_1)(1 - \tau_2) r_1(r_1 + r_2) r_2 \psi_7,
 \end{aligned} \tag{46}$$

where

$$\psi_7 = \begin{vmatrix} \Phi_{11}(\gamma_1, \theta_1) & 1 & c_1 & -c_1(1 - \tau_1) & \Phi_{15} & 0 & 0 & \Phi_{18}(\theta_1) \\ F_{21} & 1 & 0 & 0 & 0 & 0 & 0 & F_{28} \\ F_{31}(\gamma_1, \theta_1) & 0 & 1 & \tau_1 & F_{35} & 0 & 0 & F_{38}(\theta_1) \\ \Phi_{41} & 0 & 0 & 0 & \Phi_{45} & 0 & 0 & \Phi_{48} \\ \Phi_{51} & 0 & 0 & c_2(1 - \tau_2) & \Phi_{55}(\gamma_2, \theta_2) & 1 & c_2 & \Phi_{58}(\theta_2) \\ 0 & 0 & 0 & 0 & F_{65} & 1 & 0 & F_{68} \\ F_{71} & 0 & 0 & -\tau_2 & F_{75}(\gamma_2, \theta_2) & 0 & 1 & F_{78}(\theta_2) \\ F_{81}(\beta) & 0 & 0 & -1 & F_{85}(\beta) & 0 & 0 & F_{88}(\beta) \end{vmatrix}. \tag{47}$$

We can write the Routh–Hurwitz terms as follows.

$$\Delta_1 = a_1, \tag{48a}$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} = a_1 a_2 - a_3, \tag{48b}$$

$$\Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = a_3\Delta_2 + a_1(a_5 - a_1a_4) = a_3\Delta_2 - a_1^2a_4 + a_1a_5, \quad (48c)$$

$$\Delta_4 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ 1 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & 1 & a_2 & a_4 \end{vmatrix} = a_4\Delta_3 - a_5 \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & 1 & a_2 \end{vmatrix} + a_6 \begin{vmatrix} a_1 & a_3 & a_5 \\ 0 & a_1 & a_3 \\ 0 & 1 & a_2 \end{vmatrix} - a_7 \begin{vmatrix} 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \\ 0 & 1 & a_2 \end{vmatrix}, \quad (48d)$$

$$\Delta_5 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 & 0 \\ 1 & a_2 & a_4 & a_6 & a_8 \\ 0 & a_1 & a_3 & a_5 & a_7 \\ 0 & 1 & a_2 & a_4 & a_6 \\ 0 & 0 & a_1 & a_3 & a_5 \end{vmatrix} \quad (48e)$$

$$= a_5\Delta_4 - a_6 \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ 1 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & 0 & a_1 & a_3 \end{vmatrix} + a_7 \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ 1 & a_2 & a_4 & a_6 \\ 0 & 1 & a_2 & a_4 \\ 0 & 0 & a_1 & a_3 \end{vmatrix} - a_8 \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ 0 & a_1 & a_3 & a_5 \\ 0 & 1 & a_2 & a_4 \\ 0 & 0 & a_1 & a_3 \end{vmatrix},$$

$$\Delta_6 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 & 0 & 0 \\ 1 & a_2 & a_4 & a_6 & a_8 & 0 \\ 0 & a_1 & a_3 & a_5 & a_7 & 0 \\ 0 & 1 & a_2 & a_4 & a_6 & a_8 \\ 0 & 0 & a_1 & a_3 & a_5 & a_7 \\ 0 & 0 & 1 & a_2 & a_4 & a_6 \end{vmatrix} = a_6\Delta_5 - a_7 \begin{vmatrix} a_1 & a_3 & a_5 & a_7 & 0 \\ 1 & a_2 & a_4 & a_6 & a_8 \\ 0 & a_1 & a_3 & a_5 & a_7 \\ 0 & 1 & a_2 & a_4 & a_6 \\ 0 & 0 & 1 & a_2 & a_4 \end{vmatrix} + a_8 \begin{vmatrix} a_1 & a_3 & a_5 & a_7 & 0 \\ 1 & a_2 & a_4 & a_6 & a_8 \\ 0 & a_1 & a_3 & a_5 & a_7 \\ 0 & 0 & a_1 & a_3 & a_5 \\ 0 & 0 & 1 & a_2 & a_4 \end{vmatrix}, \quad (48f)$$

$$\Delta_7 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 & 0 & 0 & 0 \\ 1 & a_2 & a_4 & a_6 & a_8 & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & a_7 & 0 & 0 \\ 0 & 1 & a_2 & a_4 & a_6 & a_8 & 0 \\ 0 & 0 & a_1 & a_3 & a_5 & a_7 & 0 \\ 0 & 0 & 1 & a_2 & a_4 & a_6 & a_8 \\ 0 & 0 & 0 & a_1 & a_3 & a_5 & a_7 \end{vmatrix} = a_7\Delta_6 - a_8 \begin{vmatrix} a_1 & a_3 & a_5 & a_7 & 0 & 0 \\ 1 & a_2 & a_4 & a_6 & a_8 & 0 \\ 0 & a_1 & a_3 & a_5 & a_7 & 0 \\ 0 & 1 & a_2 & a_4 & a_6 & a_8 \\ 0 & 0 & a_1 & a_3 & a_5 & a_7 \\ 0 & 0 & 0 & a_1 & a_3 & a_5 \end{vmatrix}, \quad (48g)$$

$$\Delta_8 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 & 0 & 0 & 0 & 0 \\ 1 & a_2 & a_4 & a_6 & a_8 & 0 & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & a_7 & 0 & 0 & 0 \\ 0 & 1 & a_2 & a_4 & a_6 & a_8 & 0 & 0 \\ 0 & 0 & a_1 & a_3 & a_5 & a_7 & 0 & 0 \\ 0 & 0 & 1 & a_2 & a_4 & a_6 & a_8 & 0 \\ 0 & 0 & 0 & a_1 & a_3 & a_5 & a_7 & 0 \\ 0 & 0 & 0 & 1 & a_2 & a_4 & a_6 & a_8 \end{vmatrix} = a_8\Delta_7. \quad (48h)$$

A set of necessary and sufficient conditions for the local stability of the eight-dimensional nonlinear system given by Eqs. (29)–(36) is provided by the following “Routh–Hurwitz theorem” [cf. Gandolfo (2009: p. 239)]:

$$\Delta_1 > 0, \Delta_2 > 0, \dots, \Delta_8 > 0. \quad (49)$$

The following set of inequalities is a set of necessary (but not sufficient) conditions for the local stability of this dynamic system, which is a corollary of the Liénard–Chipart theorem [cf. Gandolfo (2009: p. 240)].

$$a_1 > 0, a_2 > 0, \dots, a_8 > 0. \quad (50)$$

In other words, a set of inequalities (49) implies a set of inequalities (50), but not vice versa. The following proposition provides a set of sufficient conditions for dynamic instability of the equilibrium point.

Proposition 1 *Suppose that the adjustment speeds of the goods market in two countries, α_1 and α_2 , are sufficiently large and all of the fiscal and monetary policy parameter values γ_1 , θ_1 , γ_2 , and θ_2 are close to zero. Then, the equilibrium point of the dynamic system (29)–(36) becomes unstable under Assumptions 2–4.*

Proof See Appendix 1.

This proposition means that the equilibrium point of the system becomes unstable if the adjustment speeds of the goods market in the two countries are sufficiently high, and the fiscal and monetary policies of both countries are sufficiently inactive. On the other hand, the following proposition provides an essential set of necessary (but not sufficient) conditions for the local stability of the equilibrium point of the system. $\square$

Proposition 2 *Suppose that the parameter values α_1 , α_2 , β , γ_1 , and γ_2 are sufficiently large, and both θ_1 and θ_2 are close to their economically meaningful upper bounds, ($\bar{\theta}_1$ and $\bar{\theta}_2$). Then, the following set of inequalities are satisfied for all positive but sufficiently small values of r_1 and r_2 under Assumption 4.*

$$a_1 > 0, a_2 > 0, \dots, a_7 > 0, \Delta_1 > 0, \Delta_2 > 0, \dots, \Delta_7 > 0. \quad (51)$$

Proof See Appendix 3.

We can make use of the following lemma to prove this proposition. $\square$

Lemma 1 *Suppose that the parameter values α_1 , α_2 , γ_1 , and γ_2 are sufficiently large, and both θ_1 and θ_2 are close to their economically meaningful upper bounds ($\bar{\theta}_1$ and $\bar{\theta}_2$). Then, the following set of conditions are satisfied under Assumption 4 if $r_1 = r_2 = 0$.*

$$\begin{aligned} a_1 > 0, a_2 > 0, \dots, a_5 > 0, a_6 = a_7 = a_8 = 0, \\ \Delta_1 > 0, \Delta_2 > 0, \dots, \Delta_5 > 0, \Delta_6 = \Delta_7 = \Delta_8 = 0. \end{aligned} \quad (52)$$

Proof See Appendix 2.

The only missing inequality in Proposition 2 is $a_8 > 0$. If we have this inequality in addition to a set of inequalities (51), we also have $\Delta_8 = a_8 \Delta_7 > 0$. In this case, all of the Routh–Hurwitz conditions for the local stability of the equilibrium point (49) are satisfied. However, we can show that ψ_7 in Eq. (46) becomes negative so that we have $a_8 < 0$ when the fiscal parameter values γ_1 and γ_2 are too large.

This result means that the equilibrium point of this dynamic system becomes unstable if the fiscal policies of two countries are extremely active, because of the “overshooting” phenomenon. Nevertheless, it may be possible that the equilibrium point of this system becomes locally stable when the fiscal and monetary policies of two countries are considerably, but not extremely, active. Kaldorian investment function holds instability. When the government increases government spending excessively to prevent a recession, the policy makes the economy too good, and when the government suppresses the economy excessively, it makes it more recessionary. To avoid overshooting, policies that are neither overly passive nor overly aggressive are necessary. In this case, the destabilizing ‘overshooting’ fluctuations occur. In the next section, we shall present some numerical examples that support this conjecture. In these examples, cyclical fluctuations occur not only when the equilibrium point becomes locally stable, but also when it is unstable. $\square$

4 Numerical simulations

In this section, we present numerical simulations that support the theoretical analysis regarding relationships between α_i , β , θ_i , and γ_i from the previous sections. The setting of the parameters and the functional forms of the numerical simulations are contained in Appendix 4.

Figures 1, 2, 3, 4, 5, 6, 7 and 8 illustrate the trajectories of Y_1 and Y_2 by numerical simulation. Figure 1 shows a time pattern of Y_1 and Y_2 when a central bank does not purchase government bonds ($\theta_1 = 0$ and $\theta_2 = 0$), and the economies are unstable because business cycles diverge. Furthermore, an increase in the adjustment speed of goods markets α_i intensifies the instability of business cycles. Figure 2 shows the case of a higher adjustment speed of the goods market.

However, when only country 1 performs QE under the same fiscal policy parameters ($\theta_1 = 0.6$, $\theta_2 = 0$, and $\alpha_i = 1.2$), as shown in Fig. 3, there are large fluctuations immediately after the shock, but then the economy converges to the equilibrium point. This is also true in Fig. 4 ($\theta_1 = 0$, $\theta_2 = 0.9$, and $\alpha_i = 1.2$), which shows the case where only the second country buys government bonds.

The critical point is that the central bank stabilizes not only the economies of the countries that have purchased government bonds but also the economies of countries that have not purchased the bonds. This shows that the OMT does work, as it purchased unlimited amounts of government bonds of crisis-stricken countries during the euro crisis. It also implies that OMT can stabilize the business cycle in the overall euro area.

Fig. 1 High degree of capital movement and unsynchronized business cycles ($\alpha_i = 1.2$, $\beta = 5$, $\theta_i = 0$, $\gamma_i = 0.5$)

Fig. 2 High adjustment speed of goods market and unsynchronized business cycles ($\alpha_i = 1.5$, $\beta = 5$, $\theta_i = 0$, $\gamma_i = 0.5$)

Figure 5 shows the case where the central bank purchases government bonds of both countries. By purchasing government bonds throughout the single currency area, it is possible to stabilize the business cycle with smaller parameters θ_i than if only country 1 or country 2 were to purchase government bonds individually. Belke et al. (2017) points out that different monetary policies are required when the amplitude of the business cycle is different, even when the correlation of business cycle factors is high between the countries. However, a single monetary policy of

Fig. 3 Convergence by purchasing government bond of country 1 ($\alpha_i = 1.2$, $\beta = 5$, $\theta_1 = 0.6$, $\theta_2 = 0$, $\gamma_i = 0.5$)

Note: $\alpha_i = 1.2$, $\beta = 5$, $\theta_1 = 0$, $\theta_2 = 0.9$, $\gamma_i = 0.5$

Fig. 4 Convergence by purchasing government bond of country 2 ($\alpha_i = 1.2$, $\beta = 5$, $\theta_1 = 0$, $\theta_2 = 0.9$, $\gamma_i = 0.5$)

purchasing the same amount of government bonds stabilizes the business cycle even though the amplitudes of the two countries' cycle are not synchronized, as shown in Fig. 5. In other words, the ECB's supranational monetary policy affects the euro area, even though it is a single monetary policy.

Fig. 5 Convergence by purchasing government bonds of both countries ($\alpha_i = 1.2$, $\beta = 5$, $\theta_1 = 0.3$, $\theta_2 = 0.3$, $\gamma_i = 0.5$)

Fig. 6 Counter-cyclical fiscal policy stabilizing business cycle fluctuations ($\alpha_i = 1.2$, $\beta = 5$, $\theta_1 = 0$, $\theta_2 = 0$, $\gamma_i = 0.7$)

Figure 6 shows the business cycle for the entire single currency area when the central bank does not purchase government bonds, and the counter-cyclical fiscal policy parameters γ_i are larger than those in Figs. 1, 2, 3, 4 and 5. The economy converges to the equilibrium point and stabilizes in this case as well. This effect means that expansionary fiscal policies, instead of austerity policies, are needed to stabilize the business cycles in the euro area. Furthermore, when both bond purchases

Fig. 7 Combination of purchasing government bond and counter-cyclical fiscal policy further stabilizing business cycle fluctuations ($\alpha_i = 1.2$, $\beta = 5$, $\theta_1 = 0.3$, $\theta_2 = 0.3$, $\gamma_i = 0.7$)

Fig. 8 Extreme active fiscal policy diverges from an equilibrium point ($\alpha_i = 1.2$, $\beta = 5$, $\theta_i = 0$, $\gamma_i = 1.4$)

and expansionary fiscal policies are combined, the economy reaches the equilibrium point in a shorter period and stabilizes, as shown in Fig. 7.

Figure 8 illustrates how extremely active fiscal policies of the two countries diverge business cycles from the equilibrium point. Nevertheless, if the fiscal and monetary policies of the two countries are considerably active, then the equilibrium point becomes locally stable, as indicated in Figs. 5 and 6.

Figures 1, 2, 3, 4, 5, 6, 7 and 8 show that the entire single currency area has moved away from its equilibrium point due to shocks. Therefore, the shocks are symmetric. Figures 9, 10, 11 and 12 show the case of asymmetric shocks, where one country is near the equilibrium point, and the other country is relatively farther from the equilibrium point due to the shock.

Fig. 9 The shock occurred in country 1 only ($\alpha_i = 1.2$, $\beta = 5$, $\theta_i = 0$, $\gamma_i = 0.5$)

Fig. 10 Convergence by purchasing government bond of country 1 ($\alpha_i = 1.2$, $\beta = 5$, $\theta_1 = 0.6$, $\theta_2 = 0$, $\gamma_i = 0.5$)

Fig. 11 The shock occurred in country 2 only ($\alpha_i = 1.2$, $\beta = 5$, $\theta_i = 0$, $\gamma_i = 0.5$)

Fig. 12 Convergence by purchasing government bond of country 2 ($\alpha_i = 1.2$, $\beta = 5$, $\theta_1 = 0$, $\theta_2 = 0.9$, $\gamma_i = 0.5$)

Figure 9 illustrates a case where country 1 with a relatively small economy deviates from the equilibrium point due to shock, while country 2, with a relatively large economy, is less affected by the shock. In this case, the amplitude of the business cycles diverges, as in Figs. 1, 2, 3, 4, 5, 6, 7 and 8. Figure 10 describes the case where the central bank purchases the government bonds of country 1 in response to the same shock as in Fig. 9. The business cycle is stabilized in this case as well.

Figure 11 illustrates the opposite case of Fig. 9, where country 2 deviates from the equilibrium point due to a shock. Figure 12 describes the case where the central bank purchases the government bonds of country 2 in the event of a shock, as in Fig. 11. As a result, the business cycle is stabilized by purchasing government bonds.

Table 1 shows the equilibrium national income for each of the cases shown in Figs. 1, 2, 3, 4, 5, 6, 7, 10 and 12. There are two points worth discussing here. First, the purchase of government bonds raises the equilibrium national income. Second, when either country purchases government bonds, not only does the equilibrium national income of that country increase, but the equilibrium national income of the other country also increases.

The result that bond purchases achieve both a stabilization of the business cycle and an increase in equilibrium national income has important implications for the economic recovery of peripheral countries that were severely shocked by the euro crisis. The results that show the business cycle stabilizes and the equilibrium national income are raised when the central banks of euro member countries purchase their government bonds show the positive effect of the OMT. This has also had a positive effect on other euro member states. Moreover, this means that the QE is an essential policy for the euro area because the uniform purchasing of government bonds within the euro area also contributes to stabilizing the business cycle as well as increasing the equilibrium national income. The existence of the ECB, which manages policies regarding government bond purchase policies, is also significant for the euro area, and its supranational monetary policy will determine the future of the eurozone's economy. It is an essential factor in maintaining the eurozone as a monetary union.

5 Conclusion

We found the following results. First, the purchase of government bonds of a country that is facing a crisis, through policies such as OMT or others, can stabilize the business cycle of the country. This means that setting up a government bond purchasing system like the OMT can help end the crisis. Second, a steep increase in

Table 1 Equilibrium national income

Figure	α_i	β	θ_1	θ_2	γ_i	Y_1^*	Y_2^*	$Y_1^* + Y_2^*$
1	1.2	5	0	0	0.5	262.698	343.968	606.666
3	1.2	5	0.5	0	0.5	296.480	350.725	647.205
4	1.2	5	0	0.9	0.5	270.309	400.288	670.597
5	1.2	5	0.3	0.3	0.5	285.081	368.589	653.67
6	1.2	5	0	0	0.7	288.363	364.578	652.941
7	1.2	5	0.3	0.3	0.7	303.768	381.383	685.151
10	1.2	5	0.6	0	0.5	301.512	351.731	670.597
12	1.2	5	0	0.9	0.5	270.309	400.288	685.151

the degree of counter-cyclical fiscal policy is a destabilizing factor. However, if the fiscal and monetary policies of the two countries are active to a certain degree only, and not extremely active, the equilibrium point becomes locally stable. Third, the simulations show that even if the business cycles are not synchronized, the purchase of government bonds of a particular country is effective in stabilizing the business cycles of both countries. We found that the implementation of the OMT not only helped the crisis-stricken countries, but also reduced the negative impact on the euro area as a whole.

As a result, the euro area satisfies the metacriteria of an OCA that Gächter et al. (2012) point out through the purchase of government bonds by the supranational central banking system. This result is essential because it helps to construct a uniform framework for the OCA theory. Therefore, this construction also helps to introduce ECO in the ECOWAS.

The ECB is the only organization capable of supporting the macroeconomy of the euro area as a whole, and the importance of the ECB in the euro area has increased relatively due to fiscal constraints. The ECB has implemented a QE policy to fulfill its role. This study shows that the implementation of QE in the euro area can stabilize the business cycle and increase the equilibrium national incomes of euro area countries as well as can be achieved through the OMT policy.

In this study, there is no discussion of how price fluctuations and risk premium might have affected the stability of models. Given that the inflation rate, the expected inflation rate, and risk premium play important roles in the stability, this limitation must be borne in mind. This means that we abstracted from the effect of potential instability that is due to the changes of the prices, risk premium, and price expectation for simplicity's sake. Furthermore, we did not take into account the presence of countries outside the euro area. Therefore, in reality, capital flows to third countries may affect the effectiveness of the policy. Asada (2018) formulated a three-country Mundell–Fleming model with mixed types of fixed and flexible exchange rates. This model assumed that an integrated currency region and a third country are economically connected through a flexible exchange rate. However, the analysis becomes extremely complicated and does not yield precise results without making some limiting assumptions.

Even though QE is a necessary condition for the euro area economy to turn around and stabilize, it is not a sufficient condition. The euro area needs to continue its attempts to achieve solutions to its structural problems. These attempts include structural reforms to meet other OCA conditions, appropriate fiscal policy management, efforts toward fiscal union, and reform of the euro system. To counteract the short-term adverse effects of structural reforms, these reforms need to be carried out while the stimulus measures through QE policies are in effect. In time, through these efforts, the euro area can become an optimal currency area. These efforts may prove to be extremely important in saving the economies of the euro area countries from the turmoil caused by COVID-19.

Appendix 1: Proof of Proposition 1

Proof It follows from Assumption 4 that we have $\Phi_{11}(\gamma_1, \theta_1) > 0$ and $\Phi_{55}(\gamma_2, \theta_2) > 0$ if all of the parameter values γ_1 , θ_1 , γ_2 , and θ_2 are sufficiently close to zero. In this case, we have $a_1 < 0$ for all sufficiently large values of the parameters α_1 and α_2 from Eq. (39), which violates one of the necessary conditions for the local stability of the equilibrium point [a set of inequalities (50)]. $\square$

Appendix 2: Proof of Lemma 1

Proof Suppose that $r_1 = r_2 = 0$. In this case, it is obvious that $a_6 = a_7 = a_8 = \Delta_6 = \Delta_7 = \Delta_8 = 0$ from Eqs. (44), (45), (46), (48f), (48g), and (48h). In addition, we have $\Phi_{11}(\gamma_1, \theta_1) < 0$ and $\Phi_{55}(\gamma_2, \theta_2) < 0$ if γ_1 and γ_2 are sufficiently large and both θ_1 and θ_2 are close to their economically meaningful upper bounds. Then, we have $a_1 = \Delta_1 > 0$ for all sufficiently large values of $\alpha_1 > 0$ and $\alpha_2 > 0$ from Eqs. (39) and (48a).

We have $\psi_1 > 0$, $\psi_2 > 0$, $\psi_3 > 0$, and $\psi_4 > 0$ in Eqs. (40)–(43) for all sufficiently large values of $\gamma_1 > 0$ and $\gamma_2 > 0$ because of $\lim_{\gamma_1 \rightarrow +\infty} \Phi_{11}(\gamma_1, \bar{\theta}_1) = \lim_{\gamma_2 \rightarrow +\infty} \Phi_{55}(\gamma_2, \bar{\theta}_2) = +\infty$. Then, we have $a_2 > 0$, $a_3 > 0$, $a_4 > 0$, and $a_5 > 0$ for all sufficiently large values of $\alpha_1 > 0$, $\alpha_2 > 0$, $\gamma_1 > 0$, and $\gamma_2 > 0$ from Eqs. (40) to (43).

If $\alpha_1 > 0$ and $\alpha_2 > 0$ are sufficiently large, the term $\alpha_1 \alpha_2 > 0$ dominates other term in Eq. (48b), so that we have $\Delta_2 > 0$. Next, the term $\alpha_3 \Delta_2 > 0$ dominates the term $-a_1^2 a_4 < 0$ in Eq. (48c), so that we have $\Delta_3 > 0$. In addition, we obtain $\Delta_4 = a_4 \Delta_3 > 0$ and $\Delta_5 = a_5 \Delta_4 > 0$ from Eqs. (48d) and (48e) because of the assumption that $r_1 = r_2 = 0$. This proves Lemma 1. $\square$

Appendix 3: Proof of Proposition 2

Proof Suppose that the parameter values α_1 , α_2 , β , γ_1 , and γ_2 are sufficiently large, and both θ_1 and θ_2 are close to their economically meaningful upper bounds, in addition to Assumption 4. Furthermore, suppose that r_1 and r_2 are positive, but they are sufficiently close to zero. In this case, we have the following set of inequalities from Lemma 1 because of the continuities of the functions with respect to the changes in the parameter values.

$$a_1 > 0, a_2 > 0, \dots, a_5 > 0, \Delta_1 > 0, \Delta_2 > 0, \dots, \Delta_5 > 0. \quad (53)$$

In Eq. (44), the term $\alpha_1 \alpha_2 \psi_5$ dominates other terms and ψ_5 becomes positive if r_1 and r_2 are positive but sufficiently close to zero and the parameter values $\alpha_1 > 0$, $\alpha_2 > 0$, $\gamma_1 > 0$ and $\gamma_2 > 0$ are sufficiently large. In this case, we have $a_6 > 0$. The same reasoning applies to Eq. (45) so that we have $a_7 > 0$ because

$$\frac{\partial\{\Phi_{11}(\gamma_1, \theta_1) - c_1 F_1(\gamma_1, \theta_1)\}}{\partial\gamma_1} = -(1 - c_1) < 0. \quad (54)$$

Furthermore, we can see that

$$\lim_{(r_1, r_2) \rightarrow (0,0)} (a_7/a_6) = \lim_{(r_1, r_2) \rightarrow (0,0)} (a_8/a_6) = \lim_{(r_1, r_2) \rightarrow (0,0)} (a_8/a_7) = 0. \quad (55)$$

Therefore, we obtain the following results:

$$\lim_{(r_1, r_2) \rightarrow (0,0)} (\Delta_6/a_6) = \Delta_5 > 0, \quad \lim_{(r_1, r_2) \rightarrow (0,0)} (\Delta_7/a_7) = \Delta_6 > 0. \quad (56)$$

In this case, we obtain $\Delta_6 > 0$ and $\Delta_7 > 0$. This proves Proposition 2. $\square$

Appendix 4: setting in numerical simulations

Based on the work of Asada (2004), we assume the following parameter values.

$$c_i = 0.8, \quad \tau_i = 0.2, \quad T_{0i} = 10, \quad C_{01} = 20, \quad C_{02} = 40, \\ G_{01} = 50, \quad G_{02} = 60, \quad \bar{M} = 600, \quad \bar{Y}_i = 500, \quad \bar{D} = 2, \quad \beta = 5.$$

Furthermore, we assume that the functional forms of the LM equation, investment function, and current account function are as follows:

$$r_i = 15\sqrt{Y_i} - M_i, \quad (57)$$

$$I_i = 20\sqrt{Y_i} - 0.3K_i - r_i, \quad (58)$$

$$J_1 = -0.35Y_1 + 0.25Y_2. \quad (59)$$

In this case, the eight-dimensional dynamical system (29)–(36) becomes as follows:

$$\begin{aligned} \dot{Y}_1 = & 70 - 0.3K_1 + M_1 + \gamma_1(500 - Y_1) + 5\sqrt{Y_1} - 1.3Y_1 \\ & + 0.8[10 + (-\theta_1 + B_1 - B_1^2)(-M_1 + 15\sqrt{Y_1}) + Y_1 \\ & - 0.2\{(-\theta_1 + B_1 - B_1^2)(-M_1 + 15\sqrt{Y_1}) + Y_1 + (2 - B_1^2)(-600 + M_1 + 15\sqrt{Y_2})\} \\ & + (2 - B_1^2)(-600 + M_1 + 15\sqrt{Y_2})] + 0.3Y_2, \end{aligned} \quad (60)$$

$$\dot{K}_1 = -0.3K_1 + M_1 + 5\sqrt{Y_1}, \quad (61)$$

$$\begin{aligned} \dot{B}_1 = & 60 + B_1(-M_1 + 15\sqrt{Y_1}) + \gamma_1(500 - Y_1) - 0.2\{(-\theta_1 + B_1 - B_1^2)(-M_1 + 15\sqrt{Y_1}) \\ & + Y_1 + (2 - B_1^2)(-600 + M_1 + 15\sqrt{Y_2})\}, \end{aligned} \quad (62)$$

$$\dot{B}_1^2 = 5(600 - 2M_1 + 15\sqrt{Y_1} - 15\sqrt{Y_2}), \quad (63)$$

$$\begin{aligned} \dot{Y}_2 = & 700 - 0.3K_2 - M_1 + 0.3Y_1 + \gamma_2(500 - Y_2) + 5\sqrt{Y_2} - 1.3Y_2 \\ & + 0.8\{10 + B_1^2(-M_1 + 15\sqrt{Y_1}) + (-2 - \theta_2 + B_1^2 + B_2)(-600 + M_1 + 15\sqrt{Y_2}) + Y_2 \\ & - 0.2(B_1^2(-M_1 + 15\sqrt{Y_1}) + (-2 - \theta_2 + B_1^2 + B_2)(-600 + M_1 + 15\sqrt{Y_2}) + Y_2)\}, \end{aligned} \quad (64)$$

$$\dot{K}_2 = 600 - 0.3K_2 - M_1 + 5\sqrt{Y_2}, \quad (65)$$

$$\begin{aligned} \dot{B}_2 = & 70 + B_2(-600 + M_1 + 15\sqrt{Y_2}) + \gamma_2(500 - Y_2) - 0.2\{B_1^2(-M_1 + 15\sqrt{Y_1}) \\ & + (-2 - \theta_2 + B_1^2 + B_2)(-600 + M_1 + 15\sqrt{Y_2}) + Y_2\}, \end{aligned} \quad (66)$$

$$\begin{aligned} \dot{M}_1 = & -B_1^2(-M_1 + 15\sqrt{Y_1}) - 0.3Y_1 + 10(600 - 2M_1 + 15\sqrt{Y_1} - 15\sqrt{Y_2}) \\ & + (2 - B_1^2)(-600 + M_1 + 15\sqrt{Y_2}) + 0.3Y_2. \end{aligned} \quad (67)$$

The equilibrium values of Eqs. (60)–(67) depend on the size of the parameters γ_1 , γ_2 , θ_1 , and θ_2 .

Now, we compute the trajectories produced by Eqs. (60)–(67) by selecting several values of γ_1 , γ_2 , θ_1 , and θ_2 and the following initial conditions of variables in Figs. 1, 2, 3, 4, 5, 6, 7 and 8:

$$\begin{aligned} Y_1(0) &= 260, K_1(0) = 1200, B_1(0) = 3.9, B_1^2(0) = 1, \\ Y_2(0) &= 340, K_2(0) = 1360, B_2(0) = 2.5, M_1(0) = 280. \end{aligned}$$

Initial conditions of variables in Figs. 9 and 10 are as follows.

$$\begin{aligned} Y_1(0) &= 200, K_1(0) = 1180, B_1(0) = 4.5, B_1^2(0) = 1, \\ Y_2(0) &= 344, K_2(0) = 1367, B_2(0) = 2.5, M_1(0) = 280. \end{aligned}$$

Furthermore, initial conditions of variables in Figs. 11 and 12 are as follows.

$$\begin{aligned} Y_1(0) &= 262, K_1(0) = 1212, B_1(0) = 3.9, B_1^2(0) = 1, \\ Y_2(0) &= 300, K_2(0) = 1360, B_2(0) = 2.9, M_1(0) = 280. \end{aligned}$$

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Conflict of interest The authors have no conflict of interest to declare that is relevant to the content of this article.

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