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Surfaces in Non-flat 3-Space Forms Satisfying $\square\vec{H} = \lambda\vec{H}$

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Abstract

In this paper, we locally classify the surfaces immersed into the non-flat (Riemannian or Lorentzian) 3-space forms satisfying the condition $\square\vec{\mathbf{H}} = \lambda\vec{\mathbf{H}}$ for a real number λ, where $\vec{\mathbf{H}}$ is the mean curvature vector field and $\square$ denotes the Cheng–Yau operator of the surface. We obtain the classification result by proving, at a first step, that the mean curvature function must be constant and, in a second step, we complete the classification.

Keywords Mean curvature vector field · Spacelike surface · Timelike surface · B-scroll · Newton transformation · Cheng–Yau operator

Mathematics Subject Classification 53B25 · 53B30

1 Introduction and Statement of the Main Results

Let us denote by $\mathbb{M}^3_q(c)$ the standard model of a three-dimensional non-flat Riemannian (when the index is $q = 0$) or Lorentzian (when the index is $q = 1$) space form with constant curvature $c = \pm 1$, which are given as hyperquadrics of the corresponding pseudo-Euclidean 4-spaces. That is, when $q = 0$, $\mathbb{M}^3_0(c) = \mathbb{M}^3(c)$ will denote the

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unit 3-sphere as hyperquadric of the Euclidean 4-space,

$$\mathbb{S}^3 = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \langle x, x\rangle = 1\} \subset \mathbb{R}^4, \text{ if } c = 1,$$

or the hyperbolic 3-space as hyperquadric of the Lorentz–Minkowski 4-space,

$$\mathbb{H}^3 = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4_1 : \langle x, x\rangle_1 = -1\} \subset \mathbb{R}^4_1, \text{ if } c = -1,$$

where $\langle , \rangle_1 = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$ is the Lorentzian metric in $\mathbb{R}^4_1$. Similarly, when $q = 1$ $\mathbb{M}^3_1(c)$ will denote the de Sitter 3-space as hyperquadric of the Lorentz–Minkowski 4-space,

$$\mathbb{S}^3_1 = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4_1 : \langle x, x\rangle_1 = 1\} \subset \mathbb{R}^4_1, \text{ if } c = 1,$$

or the anti de Sitter 3-space as hyperquadric of the pseudo-Euclidean 4-space of index 2,

$$\mathbb{H}^3_1 = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4_2 : \langle x, x\rangle_2 = -1\} \subset \mathbb{R}^4_2, \text{ if } c = -1,$$

where $\langle , \rangle_2 = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2$ is the index 2 metric in $\mathbb{R}^4_2$.

Let $\mathbb{R}^4_t$ stand for the corresponding pseudo-Euclidean 4-space of index t where $\mathbb{M}^3_q(c)$ is lying, with $t \in \{0, 1, 2\}$, and consider $\psi : M^2_s \to \mathbb{M}^3_q(c) \subset \mathbb{R}^4_t$ an isometric immersion of a non-degenerate surface M^2_s of index s. As usual and to simplify the notation, when any of the indexes q, t or s vanishes, we will omit it. We will often omit the dimension of M, writing simply M or M_s if the index is relevant. We will also denote simply by $\langle , \rangle$ the corresponding pseudo-Euclidean metric in $\mathbb{R}^4_t$, without distinguish the index.

The well-known Laplace–Beltrami equation for the particular case of non-degenerate isometrically immersed surfaces $\psi : M^2_s \to \mathbb{M}^3_q(c) \subset \mathbb{R}^4_t$ states that

$$\Delta\psi = 2\vec{\mathbf{H}}, \tag{1}$$

where Δ is the Laplace–Beltrami operator of the surface and $\vec{\mathbf{H}}$ denotes the mean curvature vector field of the immersion in $\mathbb{R}^4_t$. Since $\vec{\mathbf{H}} = HN - c\psi$, where H is the mean curvature function of M^2_s into $\mathbb{M}^3_q(c)$ and N is a unit vector field normal to M^2_s in $\mathbb{M}^3_q(c)$, it follows from here that $\Delta\psi = \lambda\psi$ for a real constant λ if and only if $H = 0$, that is, if and only if M is a minimal surface in $\mathbb{M}^3_q(c)$, giving so a simple version of the Takahashi theorem for the particular case of surfaces in $\mathbb{M}^3_q(c)$ [14, 17]. It follows from (1) that every surface in $\mathbb{M}^3_q(c)$ satisfying $\Delta\psi = \lambda\psi$, with $\lambda \in \mathbb{R}$, trivially satisfies also the weaker condition $\Delta\vec{\mathbf{H}} = \lambda\vec{\mathbf{H}}$. Motivated by this fact, in [1], Alías, Ferrández and Lucas studied the condition $\Delta\vec{\mathbf{H}} = \lambda\vec{\mathbf{H}}$ for hypersurfaces in non-flat space forms. In particular, when $q = 0$ and as a consequence of Proposition 3.3 in [1], it follows that minimal surfaces and totally umbilical surfaces are the only surfaces in the non-flat Riemannian 3-space forms satisfying $\Delta\vec{\mathbf{H}} = \lambda\vec{\mathbf{H}}$. On the other hand, for the case of surfaces in the non-flat Lorentzian 3-space forms, $q = 1$, they also proved that minimal surfaces, totally umbilical surfaces and B-scrolls are the only non-degenerate surfaces in $\mathbb{M}^3_1(c)$ satisfying that condition.

The Laplace–Beltrami operator of a surface in a 3-space form is an intrinsic second-order linear differential operator which arises naturally as the linearized operator of the first variation of the mean curvature of the surface for normal variations. Similarly, the Cheng–Yau operator, denoted here by $\Box$ and introduced in [6], arises naturally as the linearized operator of the first variation of the Gaussian curvature of the surface. Since that time, the Cheng–Yau operator has provided fruitful applications to the study of surfaces with constant Gaussian curvature (and, more generally, to the study of hypersurfaces with constant scalar curvature in the higher-dimensional setting), including maximum principles results like those in [2]. The Laplace–Beltrami operator and the Cheng-Yau operator can be seen as second-order linear differential operators in trace form given by $\mathrm{tr}(P_k \circ \nabla^2 f)$, with $k = 0$ and $k = 1$, respectively, where $\nabla^2 f : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ is the self-adjoint linear operator metrically equivalent to Hessian of f, $P_0 = I$ is the identity operator and $P_1 = P$ is the Newton transformation associated to the shape operator of M (for the details, see next section). Although in general the Cheng–Yau operator is not elliptic, even in the Riemannian case, it still shares nice properties with the Laplace–Beltrami operator.

From this point of view, and inspired by Garay's extension of Takahashi theorem in [9] and its subsequent generalizations and extensions [5, 8, 11, 12], the first author, jointly with Gürbüz in [3] and with Kashani in [4], initiated the study of the general condition $\Box\psi = A\psi + b$ for (hyper)surfaces in Riemannian space forms, where A is a constant matrix and b is a constant vector. In particular, as a consequence of [4, Corollary 1.5] it follows that the only surfaces in the non-flat Riemannian space forms satisfying the condition $\Box\psi = \lambda\psi$ for a real constant λ are the totally geodesic surfaces. This can be seen also as a consequence of Eq. (10) in next section, which is the equivalent to the Laplace–Beltrami Eq. (1) for the Cheng–Yau operator. However, in this case Eq. (10) does not imply a direct relation between condition $\Box\psi = \lambda\psi$ and condition $\Box\vec{\mathbf{H}} = \lambda\vec{\mathbf{H}}$, like it happens between conditions $\Delta\psi = \lambda\psi$ and $\Delta\vec{\mathbf{H}} = \lambda\vec{\mathbf{H}}$.

In this paper, and motivated by the previous results on the study of condition $\Delta\vec{\mathbf{H}} = \lambda\vec{\mathbf{H}}$, we consider the study of the condition $\Box\vec{\mathbf{H}} = \lambda\vec{\mathbf{H}}$ for $\lambda \in \mathbb{R}$ for non-degenerate surfaces in non-flat 3-space forms. In particular, for the case of Riemannian 3-space forms we obtain the following classification result.

Theorem 1 *Let M be a surface immersed into the Euclidean sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ or into the hyperbolic space $\mathbb{H}^3 \subset \mathbb{R}^4_1$, and let $\Box$ be the Cheng–Yau operator of M. The surface M satisfies the condition $\Box\vec{\mathbf{H}} = \lambda\vec{\mathbf{H}}$ if and only if*

(1) $\lambda = 0$ *and*

- *(a) M is a totally geodesic surface in $\mathbb{S}^3$;*
- *(b) M is either a totally geodesic surface or a flat totally umbilical surface in $\mathbb{H}^3$.*

(2) $\lambda \neq 0$ *and*

- *(a) M is a totally umbilical, but not totally geodesic, surface in $\mathbb{S}^3$;*
- *(b) M is a non-flat totally umbilical, but not totally geodesic, surface in $\mathbb{H}^3$.*

See Examples 1 and 2 for the precise description of the totally geodesic and totally umbilical surfaces in $\mathbb{S}^3$ and $\mathbb{H}^3$.

On the other hand, for the case of Lorentzian 3-space forms the classification is much richer due to the fact that, in that case, the non-degenerate surface can be either spacelike ($s = 0$) or timelike ($s = 1$), and in the latter its shape operator is not necessarily diagonalizable. In this case, we state separately the case $\lambda = 0$ and the case $\lambda \neq 0$. For $\lambda = 0$, we obtain the following classification.

Theorem 2 *Let M_s be a non-degenerate surface immersed into the de Sitter space $\mathbb{S}^3_1 \subset \mathbb{R}^4_1$ or into the anti de Sitter space $\mathbb{H}^3_1 \subset \mathbb{R}^4_2$, and let $\Box$ be the Cheng–Yau operator of M_s. The surface M_s satisfies the condition $\Box\vec{\mathbf{H}} = \vec{\mathbf{0}}$ if and only if*

(a) M is one of the following surfaces: a totally geodesic (spacelike or timelike) surface, a flat totally umbilical (spacelike) surface or a minimal B-scroll in $\mathbb{S}^3_1$,
(b) M is one of the following surfaces: a totally geodesic (spacelike or timelike) surface, a flat totally umbilical (timelike) surface, a minimal B-scroll or a flat B-scroll in $\mathbb{H}^3_1$.

On the other hand, when $\lambda \neq 0$, the classification is as follows:

Theorem 3 *Let M_s be a non-degenerate surface immersed into the de Sitter space $\mathbb{S}^3_1 \subset \mathbb{R}^4_1$ or into the anti de Sitter space $\mathbb{H}^3_1 \subset \mathbb{R}^4_2$, and let $\Box$ be the Cheng–Yau operator of M_s. The surface M_s satisfies the condition $\Box\vec{\mathbf{H}} = \lambda\vec{\mathbf{H}}$, with $\lambda \neq 0$, if and only if*

(a) M is either a non-flat totally umbilical, but not totally geodesic, (spacelike or timelike) surface or a non-minimal B-scroll in $\mathbb{S}^3_1$.
(b) M is either a non-flat totally umbilical, but not totally geodesic, (spacelike or timelike) surface or a non-minimal and non-flat B-scroll in $\mathbb{H}^3_1$.

Again, see Examples 1 and 2 for the precise description of the totally geodesic and totally umbilical surfaces in $\mathbb{S}^3_1$ and $\mathbb{H}^3_1$, and see Example 3 for the definition of B-scrolls.

The paper is organized as follows. In Sect. 2, we collect the basic definitions and give detailed computations for the expression of $\Box\vec{\mathbf{H}}$. In Sect. 3, we show the examples of surfaces satisfying condition the $\Box\vec{\mathbf{H}} = \lambda\vec{\mathbf{H}}$. Finally, in Sect. 4 we locally characterize those examples as the only ones satisfying the required condition.

Remark 1 Certainly, it is an interesting problem to study the condition $\Box\vec{\mathbf{H}} = \lambda\vec{\mathbf{H}}$ for general hypersurfaces of any dimension in Riemannian or Lorentzian space forms. This was actually the first intention of the authors, but the computations became much more difficult and overall conclusive results were not possible. This is still a work in progress; the authors hope to obtain in the near future the corresponding classification results at least for hypersurfaces in four-dimensional Riemannian space forms. According to the authors' calculations, in the case of higher dimension it does not seem feasible to give a unified approach to the problem which can work in general, as in the two-dimensional case.

2 Preliminaries

Let $\psi : M_s^2 \longrightarrow \mathbb{M}_q^3(c) \subset \mathbb{R}_t^4$ be an isometric immersion of a non-degenerate surface M_s^2 of index s into $\mathbb{M}_q^3(c)$ and let N be a (locally defined) unit vector field normal to M_s^2 in $\mathbb{M}_q^3(c)$ where $\langle N, N\rangle = \varepsilon = \pm 1$ (in the case $q = 0$, $\varepsilon \equiv 1$). Let ∇^0, $\overline{\nabla}$ and ∇ denote the Levi-Civita connections on $\mathbb{R}_t^4$, $\mathbb{M}_q^3(c)$ and M_s^2, respectively. Then, the Gauss and Weingarten formulas are given by

$$\nabla^0_X Y = \nabla_X Y + \varepsilon \langle SX, Y\rangle N - c\langle X, Y\rangle \psi, \tag{2}$$

and

$$S(X) = -\overline{\nabla}_X N = -\nabla^0_X N,$$

for all tangent vector fields $X, Y \in \mathfrak{X}(M)$, where $S : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ stands for the shape operator (or Weingarten endomorphism) of M, with respect to the chosen orientation N.

Let $\mathcal{B} = \{E_1, E_2, N\}$ be an adapted (local) frame on M, with E_1, E_2 tangent to M. We will say that $\mathcal{B}$ is an orthonormal frame when

$$\langle E_1, E_1\rangle = \varepsilon_1, \quad \langle E_2, E_2\rangle = \varepsilon_2 \quad \langle N, N\rangle = \varepsilon, \quad \text{and}$$
$$\langle E_1, E_2\rangle = \langle E_1, N\rangle = \langle E_2, N\rangle = 0,$$

where $\varepsilon_1, \varepsilon_2, \varepsilon \in \{-1, 1\}$ and $\varepsilon_1 + \varepsilon_2 + \varepsilon = 1$, if $q = 1$, while that $\varepsilon_1 + \varepsilon_2 + \varepsilon = 3$ if $q = 0$, and we will say that $\mathcal{B}$ is a pseudo-orthonormal frame when:

$$\langle E_1, E_2\rangle = -1, \quad \langle N, N\rangle = \varepsilon \quad \text{and } \langle E_1, E_1\rangle = \langle E_2, E_2\rangle = 0,$$
$$\langle E_1, N\rangle = \langle E_2, N\rangle = 0.$$

It is well known (see, for instance, [15, pp. 261–262]) that the shape operator S of a non-degenerate surface M_s^2 can be expressed, in an appropriate frame, in one of the following types:

$$\text{I. } S \approx \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}; \qquad \text{II. } S \approx \begin{bmatrix} \kappa & -b \\ b & \kappa \end{bmatrix}, \quad b \neq 0; \qquad \text{III. } S \approx \begin{bmatrix} \kappa & 0 \\ 1 & \kappa \end{bmatrix}. \tag{3}$$

In cases I and II, S is represented with respect to an orthonormal frame, whereas in case III, the frame is pseudo-orthonormal. If $s = 0$, it only appears case I. The characteristic polynomial $Q_S(t)$ of the shape operator S is given by

$$Q_S(t) = \det(tI - S) = t^2 - \mathrm{tr}(S)t + \det(S).$$

Then, the mean and the Gaussian curvatures of M are given by

$$H = \frac{\varepsilon}{2}\mathrm{tr}(S) \qquad \text{and} \qquad K = c + \varepsilon \det(S),$$

respectively. The Newton transformation of M is the operator $P : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ defined by

$$P = (-1)^q (2\varepsilon HI - S). \tag{4}$$

Note that by Cayley–Hamilton theorem, we have $S \circ P = (-1)^q \varepsilon (K - c)I$. Recall that the Newton transformations were introduced by Reilly [16] in the Riemannian context.

In the next result, we collect the main algebraic and analytic properties of the Newton transformation P that will be necessary in the rest of the paper. The first four claims are simply direct algebraic computations; the proof of claims (e) and (f) can be easily adapted from [3] (for the Riemannian case) and [13] (for the Lorentzian case). Recall here that, in our notation, the divergence of a vector field $X \in \mathfrak{X}(M)$ is the smooth function defined as the trace of operator ∇X, where $\nabla X(Y) := \nabla_Y X$. That is,

$$\mathrm{div}(X) = \mathrm{tr}(\nabla X) = \sum_{i,j} g^{ij} \left\langle \nabla_{E_i} X, E_j \right\rangle,$$

$\{E_i\}$ being any local frame of tangent vectors fields, where (g^{ij}) represents the inverse of the metric $(g_{ij}) = (\langle E_i, E_j \rangle)$. Analogously, the divergence of an operator $T : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ is the vector field $\mathrm{div}(T) \in \mathfrak{X}(M)$ defined as the trace of ∇T, that is,

$$\mathrm{div}(T) = \mathrm{tr}(\nabla T) = \sum_{i,j} g^{ij} (\nabla_{E_i} T) E_j,$$

where $\nabla T(E_i, E_j) = (\nabla_{E_i} T) E_j$.

Lemma 4 *The Newton transformation P of a non-degenerate surface immersed in $\mathbb{M}^3_q(c)$ satisfies the following properties:*

(a) *P is self-adjoint and commutes with S,*
(b) *$tr(P) = (-1)^q\, 2\varepsilon H$,*
(c) *$tr(S \circ P) = (-1)^q\, 2\varepsilon (K - c)$,*
(d) *$tr(S^2 \circ P) = (-1)^q\, 2H(K - c)$.*
(e) *$tr(\nabla_X S \circ P) = (-1)^q \langle \varepsilon \nabla K, X \rangle$,*
(f) $\mathrm{div}(P) = 0$.

Using this lemma, we obtain $\mathrm{div}(P(\nabla f)) = \mathrm{tr}\big(P \circ \nabla^2 f\big)$, where $\nabla^2 f : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f, given by $\left\langle \nabla^2 f(X), Y \right\rangle = \langle \nabla_X(\nabla f), Y \rangle$, for all $X, Y \in \mathfrak{X}(M)$. Associated with the Newton transformation P, we can define the second-order linear differential operator $\square : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ given by

$$\square f = \mathrm{tr}\big(P \circ \nabla^2 f\big). \tag{5}$$

An interesting property of $\Box$ is the following. For every couple of smooth functions $f, g \in \mathcal{C}^\infty(M)$, we have

$$\Box(fg) = g\Box f + f\Box g + 2\langle P(\nabla f), \nabla g\rangle. \tag{6}$$

Note that the operator $\Box$ can be naturally extended to vector valued functions as follows: If $F = (f_1, f_2, f_3, f_4) : M_s^2 \to \mathbb{R}_t^4$ with $f_i \in \mathcal{C}^\infty(M)$, then $\Box F = (\Box f_1, \Box f_2, \Box f_3, \Box f_4)$.

Next, we are going to compute $\Box$ acting on the coordinate components of the immersion ψ, that is, a function given by $\langle a, \psi\rangle$, where $a \in \mathbb{R}_t^4$ is a fixed arbitrary vector. A direct computation shows that

$$\nabla\langle a, \psi\rangle = a^\top = a - \varepsilon\langle a, N\rangle N - c\langle a, \psi\rangle\psi, \tag{7}$$

where $a^\top \in \mathfrak{X}(M)$ denotes the tangential component of a along the immersion. Taking covariant derivative in (7), and using that $\nabla^0_X a = 0$, jointly with the Gauss and Weingarten formulae, we obtain

$$\nabla_X\nabla\langle a, \psi\rangle = \nabla_X a^\top = \varepsilon\langle a, N\rangle S(X) - c\langle a, \psi\rangle X, \tag{8}$$

for every vector field $X \in \mathfrak{X}(M)$. Finally, from (5) and Lemma 4, we find that

$$\begin{aligned}\Box\langle a, \psi\rangle &= \varepsilon\langle a, N\rangle \operatorname{tr}(S\circ P) - c\langle a, \psi\rangle \operatorname{tr}(P)\\ &= (-1)^q\Big(2(K-c)\langle a, N\rangle - 2\varepsilon cH\langle a, \psi\rangle\Big).\end{aligned} \tag{9}$$

Then, we can compute $\Box\psi$ as follows:

$$\Box\psi = (-1)^q\Big(2(K-c)N - 2\varepsilon cH\psi\Big). \tag{10}$$

On the other hand, a straightforward computation yields

$$\nabla\langle a, N\rangle = -S(a^\top).$$

From Weingarten formula and (8), we find that

$$\begin{aligned}\nabla_X\nabla\langle a, N\rangle &= -\nabla_X(Sa^\top) = -(\nabla_X S)(a^\top) - S(\nabla_X a^\top)\\ &= -(\nabla_{a^\top}S)(X) - \varepsilon\langle a, N\rangle S^2(X) + c\langle a, \psi\rangle S(X),\end{aligned}$$

for every tangent vector field X. This equation, jointly with Lemma 4 and (5), yields

$$\begin{aligned}\Box\langle a, N\rangle &= -\operatorname{tr}\left(P\circ\nabla_{a^\top}S\right) - \varepsilon\langle a, N\rangle\operatorname{tr}\left(P\circ S^2\right) + c\langle a, \psi\rangle\operatorname{tr}(P\circ S)\\ &= (-1)^q\Big(-\varepsilon\Big\langle\nabla K, a^\top\Big\rangle - 2H\varepsilon(K-c)\langle a, N\rangle + 2c\varepsilon(K-c)\langle a, \psi\rangle\Big).\end{aligned} \tag{11}$$

In other words,

$$\square N = (-1)^q\Big(-\varepsilon\nabla K - 2H\varepsilon(K-c)N + 2c\varepsilon(K-c)\psi\Big). \tag{12}$$

Let us consider now $\vec{\mathbf{H}}$ the mean curvature vector field of M_s^2 in the pseudo-Euclidean space $\mathbb{R}_t^4$ where $\mathbb{M}_q^3(c)$ is lying. Then, it is easy to show that $\vec{\mathbf{H}}$ is given by

$$\vec{\mathbf{H}} = HN - c\psi. \tag{13}$$

An easy computation from (13), and using (6), (9) and (11), yields the following

$$\begin{aligned}
\square\langle a, \vec{\mathbf{H}}\rangle &= \square(H\langle a, N\rangle) - c\,\square\langle a, \psi\rangle \\
&= \langle a, N\rangle\,\square H + H\square\langle a, N\rangle + 2\Big\langle P(\nabla H), \nabla\langle a, N\rangle\Big\rangle \\
&\quad - c(-1)^q\big[2(K-c)\langle a, N\rangle - 2\varepsilon c H\langle a, \psi\rangle\big] \\
&\quad = (-1)^q\Big(-\varepsilon H\langle a, \nabla K\rangle - 2(-1)^q\langle a, S\circ P(\nabla H)\rangle \\
&\qquad + \big[(-1)^q\square H - 2H^2\varepsilon(K-c) - 2c(K-c)\big]\langle a, N\rangle \\
&\qquad + \big[2c\varepsilon H(K-c) + 2\varepsilon H\big]\langle a, \psi\rangle\Big).
\end{aligned}$$

In other words,

$$\begin{aligned}
(-1)^q\square\vec{\mathbf{H}} = &-\varepsilon H\nabla K - 2\varepsilon(K-c)\nabla H \\
&+\big[(-1)^q\square H - 2(K-c)\left(c+\varepsilon H^2\right)\big]N + 2c\varepsilon HK\psi.
\end{aligned} \tag{14}$$

3 Examples

Our goal in this section is to give some examples of surfaces in $\mathbb{M}_q^3(c)$ satisfying the condition

$$\square\vec{\mathbf{H}} = \lambda\vec{\mathbf{H}}, \qquad \lambda\in\mathbb{R}. \tag{15}$$

In the next section, we will show that they are the only ones.

Example 1 (Totally geodesic surfaces) If M is totally geodesic in $\mathbb{M}_q^3(c)$, then $S\equiv 0$. Then, $H = 0 = \det(S)$ and $K = c$, and it follows easily from (14) that $\square\vec{\mathbf{H}} = \vec{\mathbf{0}}$, which gives $\square\vec{\mathbf{H}} = \lambda\vec{\mathbf{H}}$ with $\lambda = 0$. Recall here that totally geodesic surfaces in $\mathbb{M}_q^3(c)$ are obtained as intersections of $\mathbb{M}_q^3(c)$ with hyperplanes through the origin of $\mathbb{R}_t^4$ and they are open pieces of the following surfaces:

1. Round spheres $\mathbb{S}^2\subset\mathbb{S}^3$, given by the equation $\langle x, a\rangle = 0$ for an arbitrary fixed unit vector $a\in\mathbb{R}^4$;

2. Hyperbolic planes $\mathbb{H}^2 \subset \mathbb{H}^3$, given by the equation $\langle x, a\rangle = 0$ for an arbitrary fixed spacelike unit vector $a \in \mathbb{R}^4_1$;
3. Spacelike round spheres $\mathbb{S}^2 \subset \mathbb{S}^3_1$, given by the equation $\langle x, a\rangle = 0$ for an arbitrary fixed timelike unit vector $a \in \mathbb{R}^4_1$;
4. Timelike de Sitter planes $\mathbb{S}^2_1 \subset \mathbb{S}^3_1$, given by the equation $\langle x, a\rangle = 0$ for an arbitrary fixed spacelike unit vector $a \in \mathbb{R}^4_1$;
5. Spacelike hyperbolic planes $\mathbb{H}^2 \subset \mathbb{H}^3_1$, given by the equation $\langle x, a\rangle = 0$ for an arbitrary fixed timelike unit vector $a \in \mathbb{R}^4_2$;
6. Timelike anti de Sitter planes $\mathbb{H}^2_1 \subset \mathbb{H}^3_1$, given by the equation $\langle x, a\rangle = 0$ for an arbitrary fixed spacelike unit vector $a \in \mathbb{R}^4_2$.

Example 2 (Totally umbilical (non-totally geodesic) surfaces) If M is a totally umbilical (but non-totally geodesic) surface in $\mathbb{M}^3_q(c)$, then $S = \mu I$ for $\mu \in \mathbb{R}$ and I the identity operator, with $H = \varepsilon\mu$ and $K = c + \varepsilon\mu^2 = c + \varepsilon H^2$ both constant, $H \neq 0$. Bearing in mind (13), it then follows from (14) that

$$\begin{aligned}\Box\vec{\mathbf{H}} &= (-1)^q\Big(-2\varepsilon H^2 \left(c+\varepsilon H^2\right) N + 2c\varepsilon H \left(c+\varepsilon H^2\right)\psi\Big)\\ &= (-1)^{q+1} 2\varepsilon H \left(c+\varepsilon H^2\right)\left(HN - c\psi\right)\\ &= (-1)^{q+1} 2\varepsilon H \left(c+\varepsilon H^2\right)\vec{\mathbf{H}}.\end{aligned}$$

Therefore, all totally umbilical surfaces in $\mathbb{M}^3_q(c)$ satisfy condition (15) with

$$\lambda = (-1)^{q+1} 2\varepsilon H \left(c+\varepsilon H^2\right) = (-1)^{q+1} 2\varepsilon HK.$$

Since $H \neq 0$, we have that $\lambda = 0$ if and only if $K = c + \varepsilon H^2 = 0$, which can happen only when $c = 1$ and $\varepsilon = -1$ or when $c = -1$ and $\varepsilon = 1$. They all have $H = \pm 1$ and correspond to open pieces of Euclidean planes in hyperbolic space $\mathbb{H}^3$, spacelike Euclidean planes in de Sitter space $\mathbb{S}^3_1$ and timelike Lorentz–Minkowski spaces in anti-de Sitter space $\mathbb{H}^3_1$ (see below for details).

As is well known, totally umbilical (but non-totally geodesic) surfaces in $\mathbb{M}^3_q(c)$ are obtained as intersections of $\mathbb{M}^3_q(c)$ with hyperplanes (not passing through the origin) of $\mathbb{R}^4_t$ and they are open pieces of the following surfaces:

1. Round spheres $\mathbb{S}^2(r) \subset \mathbb{S}^3$ of radius $0 < r < 1$, given by the equation $\langle x, a\rangle = \tau$ for an arbitrary fixed unit vector $a \in \mathbb{R}^4$ and $0 < |\tau| = \sqrt{1-r^2} < 1$, with $K = 1/r^2$;
2. Round spheres $\mathbb{S}^2(r) \subset \mathbb{H}^3$ of radius $r > 0$, given by the equation $\langle x, a\rangle = \tau$ for an arbitrary fixed timelike unit vector $a \in \mathbb{R}^4_1$ and $|\tau| = \sqrt{1+r^2} > 1$, with $K = 1/r^2$;
3. Hyperbolic planes $\mathbb{H}^2(r) \subset \mathbb{H}^3$ of radius $r > 1$, given by the equation $\langle x, a\rangle = \tau$ for an arbitrary fixed spacelike unit vector $a \in \mathbb{R}^4_1$ and $|\tau| = \sqrt{r^2-1} > 0$, with $K = -1/r^2$;
4. Euclidean planes $\mathbb{R}^2 \subset \mathbb{H}^3$, given by the equation $\langle x, a\rangle = \tau$ for an arbitrary fixed lightlike vector $a \in \mathbb{R}^4_1$ and $\tau \neq 0$, with $K = 0$;

5. Spacelike round spheres $\mathbb{S}^2(r) \subset \mathbb{S}^3_1$ of radius $r > 1$, given by the equation $\langle x, a\rangle = \tau$ for an arbitrary fixed timelike unit vector $a \in \mathbb{R}^4_1$ and $|\tau| = \sqrt{r^2-1} > 0$, with $K = 1/r^2$;
6. Spacelike hyperbolic planes $\mathbb{H}^2(r) \subset \mathbb{S}^3_1$ of radius $r > 0$, given by the equation $\langle x, a\rangle = \tau$ for an arbitrary fixed spacelike unit vector $a \in \mathbb{R}^4_1$ and $|\tau| = \sqrt{1+r^2} > 1$, with $K = -1/r^2$;
7. Timelike de Sitter planes $\mathbb{S}^2_1(r) \subset \mathbb{S}^3_1$ of radius $0 < r < 1$, given by the equation $\langle x, a\rangle = \tau$ for an arbitrary fixed spacelike unit vector $a \in \mathbb{R}^4_1$ and $0 < |\tau| = \sqrt{1-r^2} < 1$, with $K = 1/r^2$;
8. Spacelike Euclidean planes $\mathbb{R}^2 \subset \mathbb{S}^3_1$, given by the equation $\langle x, a\rangle = \tau$ for an arbitrary fixed lightlike vector $a \in \mathbb{R}^4_1$ and $\tau \neq 0$, with $K = 0$;
9. Timelike de Sitter planes $\mathbb{S}^2_1(r) \subset \mathbb{H}^3_1$ of radius $0 < r < 1$, given by the equation $\langle x, a\rangle = \tau$ for an arbitrary fixed timelike unit vector $a \in \mathbb{R}^4_2$ and $0 < |\tau| = \sqrt{1-r^2} < 1$, with $K = 1/r^2$;
10. Spacelike hyperbolic planes $\mathbb{H}^2(r) \subset \mathbb{H}^3_1$ of radius $r > 0$, given by the equation $\langle x, a\rangle = \tau$ for an arbitrary fixed timelike unit vector $a \in \mathbb{R}^4_2$ and $|\tau| = \sqrt{1+r^2} > 1$, with $K = -1/r^2$;
11. Timelike anti de Sitter planes $\mathbb{H}^2_1(r) \subset \mathbb{H}^3_1$ of radius $r > 1$, given by the equation $\langle x, a\rangle = \tau$ for an arbitrary fixed spacelike unit vector $a \in \mathbb{R}^4_2$ and $\tau \neq 0$, with $K = -1/r^2$;
12. Timelike Lorentz–Minkowski planes $\mathbb{R}^2_1 \subset \mathbb{H}^3_1$, given by the equation $\langle x, a\rangle = \tau$ for an arbitrary fixed lightlike vector $a \in \mathbb{R}^4_2$ and $\tau \neq 0$, with $K = 0$.

The situation becomes more interesting when the metric induced on the surface is a Lorentzian metric, which allows the possibility of non-diagonalizable shape operators. Let us start by looking for new examples of surfaces that satisfy the required condition and whose shape operator is not diagonalizable.

Example 3 (B-scrolls) Let $\gamma(s)$ be a null curve in $\mathbb{M}^3_1(c) \subset \mathbb{R}^4_t$ with an associated Cartan frame $\{A, B, C\}$; that is, $\gamma : I \subseteq \mathbb{R} \leftarrow \mathbb{M}^3_1(c) \subset \mathbb{R}^4_t$ is a curve with $\left\langle \gamma'(s), \gamma'(s)\right\rangle = 0$ and $\{A, B, C\}$ is a pseudo-orthonormal frame of vector fields along $\gamma(s)$ with

$$\langle A, A\rangle = \langle B, B\rangle = 0, \quad \langle A, B\rangle = -1,$$
$$\langle A, C\rangle = \langle B, C\rangle = 0, \quad \langle C, C\rangle = 1,$$

such that

$$\gamma'(s) = A(s),$$
$$C'(s) = -aA(s) - \kappa(s)B(s),$$

where a is a real number and $\kappa(s) \neq 0$ for all s. Then, the map $\psi : I \times \mathbb{R} \to \mathbb{M}^3_1(c) \subset \mathbb{R}^4_t$ given by $\psi(s, u) = \gamma(s) + uB(s)$ parametrizes a Lorentzian surface in $\mathbb{M}^3_1(c)$ which, following the usual terminology, is called a B-scroll (see [7] and [10]).

It is not difficult to see that $N(s, u) = -aB(s)u + C(s)$ defines a unit normal vector field along the surface, obviously with $\varepsilon = 1$, and the shape operator is given by the matrix

$$S = \begin{pmatrix} a & 0 \\ k(s) & a \end{pmatrix}$$

with respect the usual tangent frame $\left\{\frac{\partial \psi}{\partial s}, \frac{\partial \psi}{\partial u}\right\}$. Observe that S is not diagonalizable with minimal polynomial $m_S(t) = (t-a)^2$. Moreover, $H = a$ and $K = c + a^2$. In particular, a B-scroll is minimal if and only if $a = 0$, while a B-scroll is flat if and only if $c = -1$ and $a = \pm 1$. It then follows from (13) and (14) that

$$\Box \vec{\mathbf{H}} = (-1)^{q+1} 2a \left(c + a^2\right) \vec{\mathbf{H}} = \lambda \vec{\mathbf{H}}$$

so that B-scrolls in $\mathbb{M}^3_1(c)$ satisfy condition (15) with

$$\lambda = (-1)^{q+1} 2a \left(c + a^2\right).$$

In particular, $\lambda = 0$ if and only if it is either a minimal B-scroll in $\mathbb{S}^3_1$ $(a = 0)$ or it is either a minimal B-scroll $(a = 0)$ or a flat B-scroll $(a = \pm 1)$ in $\mathbb{H}^3_1$.

The following result characterizes, at least locally, B-scrolls as the only Lorentzian surfaces in $\mathbb{M}^3_1(c)$ whose shape operator S has minimal polynomial $m_S(t) = (t-a)^2$, where $a \in \mathbb{R}$.

Proposition 5 *Let M^2_1 be a Lorentzian surface into $\mathbb{M}^3_1(c) \subset \mathbb{R}^4_t$, let a be a real number and let $(t-a)^2$ be the minimal polynomial of its shape operator S. Then, in a neighborhood of any point, M^2_1 is a B-scroll over a null curve.*

Proof A proof of this proposition for $a \neq 0$ can be found in [1, Theorem 4.2], but the proof is also valid for $a = 0$. For the reader convenience, let us do this proof for the remaining case $a = 0$, following the same idea that in [1].

Let us consider $p \in M^2_1$ and a pseudo-orthonormal frame $\{A, B\}$ of tangent vector fields in a neighborhood of p such that

$$S(A) = kB,$$
$$S(B) = 0,$$

where $k \neq 0$. Let N be a unit vector field normal to M^2_1 into $\mathbb{M}^3_1(c)$. Since M^2_1 is locally an embedded surface into $\mathbb{M}^3_1(c)$, we can take an integral curve $\gamma(s)$ of A starting from p. Namely, we can write $A(s) = A(\gamma(s))$, $B(s) = B(\gamma(s))$, $C(s) = N(\gamma(s))$ and $k(s) = k(\gamma(s))$ so that the covariant derivate of C is given by

$$\frac{DC}{ds}(s) = -k(s)B(s).$$

For each s, let $x_s(t)$ denote an integral curve of B starting from $\gamma(s)$. Then,

$$\frac{DB}{dt}(x_s(t)) = \nabla^0_{\dot{x}_s(t)} B(x_s(t)) = \nabla^0_B B(x_s(t)) = \nabla_B B(x_s(t)).$$

By Codazzi equation, we know that $\nabla_B B \in \text{span}\{B\}$. Thus, there exists a differentiable function f such that

$$\frac{DB}{dt}(x_s(t)) = f(x_s(t))B(x_s(t))$$

It is not hard to see that the solution of previous differential equation is equal to

$$B(x_s(t)) = g_s(t)B(s)$$

for a certain positive function $g_s(t)$ with $g_s(0) = 1$. Therefore,

$$x_s(t) = \gamma(s) + \int_0^t g_s(v)dv\, B(s),$$

that is, in a neighborhood of p, M_1^2 is a B-scroll with $a = 0$. □

4 Proof of the Main Results

If M is a surface in $\mathbb{M}_q^3(c)$ satisfying condition (15), we can use Eqs. (13) and (14) to obtain the following formulae

$$0 = H\nabla K + 2(K - c)\nabla H, \tag{16}$$

$$\Box H = (-1)^q 2(K - c)\left(c + \varepsilon H^2\right) + \lambda H, \tag{17}$$

$$HK = (-1)^{q+1}\varepsilon\frac{\lambda}{2}. \tag{18}$$

We state and prove our first result.

Proposition 6 *Let M be a surface immersed into $\mathbb{M}_q^3(c)$ satisfying the condition $\Box\vec{\mathbf{H}} = \lambda\vec{\mathbf{H}}$. Then, the mean curvature H is constant.*

Proof Let us assume that H is non-constant and consider the set $\mathcal{U} = \left\{p \in M : \nabla H(p) \neq 0\right\}$. On this set, we have from (16)

$$\nabla\left(H^2(K - c)\right) = H\left(2(K - c)\nabla H + H\nabla K\right) = 0.$$

In other words, $H^2K - cH^2$ is constant on $\mathcal{U}$, say $H^2K - cH^2 = C$. This, jointly with (18), yields

$$(-1)^{q+1}\varepsilon\frac{\lambda}{2}H - cH^2 = C \quad \text{is constant on } \mathcal{U},$$

which implies that H is root of the follow second degree polynomial equation

$$cX^2 + (-1)^q \varepsilon \frac{\lambda}{2} X + C = 0.$$

Then, H is locally constant on $\mathcal{U}$, which is a contradiction. Thus, $\mathcal{U}$ must be an empty set and H is constant. □

In particular, since we already know that H is constant, when $\lambda = 0$ Eqs. (16), (17) and (18) reduce to

$$H\nabla K = 0, \tag{19}$$

$$(K - c)\left(c + \varepsilon H^2\right) = 0, \tag{20}$$

$$HK = 0. \tag{21}$$

Thus, we distinguish the following cases:

Case I $H = 0$. That is, $\mathrm{tr}(S) = 0$. From (20), we have $K = c$, that is, $\det(S) = 0$. Therefore, if S is diagonalizable, it must be $S = 0$, and we have that M is totally geodesic in $\mathbb{M}^3_q(c)$. On the other hand, if S is not diagonalizable (which can happen only when $q = s = 1$ and hence $\varepsilon = 1$), we have that its minimal polynomial is given by $m_S(t) = t^2$, that is, zero is a double real eigenvalue of M^2_1. Thus, by Proposition 5 with $a = 0$ we conclude that M^2_1 must be a minimal B-scroll in $\mathbb{M}^3_1(c)$.

Case II $H \neq 0$. From (21), we conclude that

$$K = c + \varepsilon \det(S) = 0 \tag{22}$$

and using this in (20), we get $H^2 = -\varepsilon c$. Since ε and c only take values ± 1, it implies that

$$H^2 = -\varepsilon c = 1. \tag{23}$$

By (22), $\det(S) = -\varepsilon c = H^2 = (\varepsilon H)^2$. Therefore, the characteristic polynomial of S is given by

$$Q_S(t) = t^2 - \mathrm{tr}(S)t + \det(S) = t^2 - 2\varepsilon Ht + (\varepsilon H)^2 = (t - \varepsilon H)^2.$$

Therefore, its minimal polynomial is either $m_S(t) = t - \varepsilon H$ or $m_S(t) = (t - \varepsilon H)^2 = (t - H)^2$, with $H^2 = 1$. In the first case, the surface must be a flat totally umbilical surface and $\mathbb{M}^3_q(c) \neq \mathbb{S}^3$, while in the second case, and using Proposition 5 with $a = \pm 1$, M must be a flat B-scroll in $\mathbb{H}^3_1$.

Conversely, from Examples 1 and 2 we already know that totally geodesic surfaces in $\mathbb{M}^3_q(c)$ and flat totally umbilical surfaces in $\mathbb{M}^3_q(c) \neq \mathbb{S}^3$ verify condition (15) with $\lambda = 0$. Furthermore, from Example 3 we also know that minimal B-scrolls in $\mathbb{S}^3_1$ and in $\mathbb{H}^3_1$ and flat B-scrolls in $\mathbb{H}^3_1$ verify condition (15) with $\lambda = 0$. This reasoning completes the proof of Theorem 1 for the case $\lambda = 0$ and the proof of Theorem 2.

On the other hand, when $\lambda \neq 0$ and since H is constant, we know from (18) that $H \neq 0$ and $K \neq 0$ is also constant. Replacing (18) in (17) leads us to $K = c + \varepsilon H^2$, which implies $\det(S) = H^2$. Then, the characteristic polynomial of S is given by

$$Q_S(t) = t^2 - \mathrm{tr}(S)t + \det(S) = t^2 - 2\varepsilon H t + \varepsilon^2 H^2 = (t - \varepsilon H)^2.$$

Therefore, its minimal polynomial is either $m_S(t) = t - \varepsilon H$ or $m_S(t) = (t - H)^2$, in both case with $H \neq 0$ and $K \neq 0$. In the first case, the surface must be a non-flat totally umbilical, but not totally geodesic, surface in $\mathbb{M}^3_q(c)$, while in the second case, and using Proposition 5 M must be either a non-minimal B-scroll in $\mathbb{S}^3_1$ $(a \neq 0)$ or a non-minimal and non-flat B-scroll in $\mathbb{H}^3_1$ $(a \neq 0, \pm 1)$.

Conversely, from Example 2 we already know that non-flat totally umbilical surfaces in $\mathbb{M}^3_q(c)$ with $H \neq 0$ satisfy condition (15) with $\lambda \neq 0$. Moreover, from Example 3 we also know that non-minimal B-scrolls in $\mathbb{S}^3_1$ and non-minimal and non-flat B-scrolls in $\mathbb{H}^3_1$ also satisfy condition (15) with $\lambda \neq 0$. This reasoning completes the proof of Theorem 1 for the case $\lambda \neq 0$ and the proof of Theorem 3.

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Declarations

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References

1. Alías, L.J., Ferrández, A., Lucas, P.: Hypersurfaces in the non-flat Lorentzian space forms with a characteristic eigenvector field. J. Geom. **52**, 10–24 (1995)
2. Alías, L.J., García-Martínez, S.C., Rigoli, M.: A maximum principle for hypersurfaces with constant scalar curvature and applications. Ann. Glob. Anal. Geom. **41**, 307–320 (2012)

3. Alías, L.J., Gürbüz, N.: An extension of Takahashi theorem for the linearized operators of the higher order mean curvatures. Geom. Dedicata **121**, 113–127 (2006)
4. Alías, L.J., Kashani, S.M.B.: Hypersurfaces in space forms satisfying the condition $L_k = Ax + b$. Taiwanese J. Math. **14**, 1957–1977 (2010)
5. Chen, B.-Y., Petrovic, M.: On spectral decomposition of immersions of finite type. Bull. Austral. Math. Soc. **44**, 117–129 (1991)
6. Cheng, S.Y., Yau, S.T.: Hypersurfaces with constant scalar curvature. Math. Ann. **225**, 195–204 (1977)
7. Dajczer, M., Nomizu, K.: On flat surfaces in $\mathbb{S}_1^3$ and $\mathbb{H}_1^3$, Manifolds and Lie Groups, pp. 71–108. Univ. Notre Dame, Birkhäuser (1981)
8. Dillen, F., Pas, J., Verstraelen, L.: On surfaces of finite type in Euclidean 3-space. Kodai Math. J. **13**, 10–21 (1990)
9. Garay, O.J.: An extension of Takahashi's theorem. Geom. Dedicata **34**, 105–112 (1990)
10. Graves, L.: Codimension one isometric immersions between Lorentz spaces. Trans. Amer. Math. Soc. **252**, 367–392 (1979)
11. Hasanis, T., Vlachos, T.: Coordinate finite-type submanifolds. Geom. Dedicata **37**, 155–165 (1991)
12. Hasanis, T., Vlachos, T.: Hypersurfaces of E^{n+1} satisfying $\Delta x = Ax + B$. J. Austral. Math. Soc. Ser. A **53**, 377–384 (1992)
13. Lucas, P., Ramírez-Ospina, H.F.: Hypersurfaces in the Lorentz–Minkowski space satisfying $L_k\psi = A\psi + b$. Geom. Dedicata **153**, 151–175 (2011)
14. Markvorsen, S.: A characteristic eigenfunction for minimal hypersurfaces in space forms. Math. Z. **202**, 375–382 (1989)
15. O'Neill, B.: Semi-Riemannian Geometry with Applications to Relativity. Academic Press, New York London (1983)
16. Reilly, R.: Variational properties of functions of the mean curvatures for hypersurfaces in space forms. J. Diff. Geom. **8**, 465–477 (1973)
17. Takahashi, T.: Minimal immersions of Riemannian manifolds. J. Math. Soc. Jpn. **18**, 380–385 (1966)