Bull. Malays. Math. Sci. Soc. (2023) 46:151
https://doi.org/10.1007/s40840-023-01545-8

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

On a Simple Sufficient Condition for the Uniform Starlikeness

Agnieszka Wiśniowska-Wajnryb[1]

Received: 9 February 2023 / Revised: 8 June 2023 / Accepted: 20 June 2023 /
Published online: 4 July 2023

Abstract

The aim of this paper is to prove the theorem which generates many examples of functions belonging to a geometrically defined class of uniformly starlike functions introduced by Goodman in 1991. Only a very few explicit uniformly starlike functions were known until now. Next we obtain inclusion relations between some subclasses of convex functions and the class of uniformly starlike functions.

Keywords Starlike functions · Convex functions · Uniformly starlike functions

Mathematics Subject Classification 30C45

1 Introduction

Let $\mathcal{S}$ denote the class of all functions f that are analytic and univalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$.

A set $D \subset \mathbb{C}$ is said to be starlike with respect to w_0, an interior point of D, if the intersection of each half-line beginning at w_0 with the interior of D is connected.

We denote by $\mathcal{ST}$ the class of all starlike functions, i.e., the subclass of $\mathcal{S}$ consisting of functions that map U onto domains starlike with respect to $w_0 = 0$ (briefly starlike domains). Recall that a function $f \in \mathcal{S}$ is starlike if and only if

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > 0 \quad \text{for all } z \in U.$$

Communicated by See Keong Lee.

✉ Agnieszka Wiśniowska-Wajnryb
agawis@prz.edu.pl

[1] Faculty of Mathematics and Applied Physics, Rzeszów University of Technology, Al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland

Let $\mathcal{CV}$ denote the class of all functions $f \in \mathcal{S}$ that are convex in U, i.e., such that $f(U)$ is a convex domain.

Let $\gamma : z = z(t), t \in [a, b]$, be a smooth, directed arc and suppose that a function f is analytic on γ. Then the arc $f(\gamma)$ is said to be
- starlike with respect to $w_0 \notin f(\gamma)$ if $\arg(f(z(t)) - w_0)$ is a nondecreasing function of t,
- convex if the argument of the tangent to $f(\gamma)$ is a nondecreasing function of t.

In 1991, Goodman ([1, 2]) introduced geometrically defined classes $\mathcal{UCV}$ and $\mathcal{UST}$ of uniformly convex and uniformly starlike functions, respectively.

Recall that a function $f \in \mathcal{S}$ is in the class $\mathcal{UCV}$ ($\mathcal{UST}$) if for every circular arc $\gamma \subset U$ with center $\zeta \in U$, the arc $f(\gamma)$ is convex (starlike with respect to $f(\zeta)$).

If we take ζ such that $|\zeta| \leq k, 0 \leq k \leq 1$ we obtain the following natural extension of the concept of uniform starlikeness (see [11] and also [10, 12]). Namely, a function $f \in \mathcal{S}$ is said to be k-uniformly starlike in U, if the image of every circular arc γ contained in U with center at ζ, where $|\zeta| \leq k$, is starlike with respect to $f(\zeta)$.

We denote by k-$\mathcal{UST}$ the class of all k-uniformly starlike functions. Notice that 0-$\mathcal{UST} = \mathcal{ST}$ and 1-$\mathcal{UST} = \mathcal{UST}$. Moreover, it is clear that $\mathcal{UST} \subset k$-$\mathcal{UST} \subset \mathcal{ST}$ for every $k \in [0, 1]$.

Goodman obtained the analytic conditions for $\mathcal{UCV}$ and $\mathcal{UST}$ expressed by two complex variables. For an arbitrary k such that $0 \leq k \leq 1$, the class k-$\mathcal{UST}$ can be characterized (see [11]) as follows

$$f \in k\text{-}\mathcal{UST} \Leftrightarrow \operatorname{Re} \frac{f(z) - f(\zeta)}{(z - \zeta) f'(z)} \geq 0 \text{ for all } z \in U, |\zeta| \leq k.$$

For $k = 1$, we get Goodman's condition for uniform starlikeness.

It turned out that (see [3, 7])

$$f \in \mathcal{UCV} \Leftrightarrow \operatorname{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > \left| \frac{z f''(z)}{f'(z)} \right|, \quad z \in U.$$

Finding this analytic condition essentially simplified further investigations of uniformly convex functions ([3, 4, 7, 8]).

It is more difficult to investigate the class $\mathcal{UST}$ (k-$\mathcal{UST}$) because of its characterization in terms of two complex variables. In particular checking whether a function belongs to the class $\mathcal{UST}$ leads to very complicated computations. Hence only simple examples of uniformly starlike functions are known.

In this paper, we give many examples of members of the class of uniformly starlike functions $\mathcal{UST}$ and we establish its connections with some subclasses of convex functions. For instance, we get that all functions convex of order $3/4$ are uniformly starlike.

2 Some Members of the Class of Uniformly Starlike Functions

Goodman ([2]) gave some examples of uniformly starlike functions. He proved that

$$\frac{z}{1 - Az} \in \mathcal{UST} \Leftrightarrow |A| \leq \frac{1}{\sqrt{2}}$$

and

$$\left(|A| \leq \frac{\sqrt{2}}{2n}, n > 1\right) \Rightarrow z + Az^n \in \mathcal{UST}.$$

It was proven in [11] that

$$f(z) = z + Az^2 \in k\text{-}\mathcal{UST} \Leftrightarrow |A| \leq \sqrt{3}/\sqrt{4(k^2+3)}.$$

For $k = 1$ we get

$$f(z) = z + Az^2 \in \mathcal{UST} \Leftrightarrow |A| \leq \sqrt{3}/4.$$

This result was mentioned by Goodman, but without a proof.

We have also more general result (see [11]):

If $0 < k \leq 1$ and for some integer $n \geq 2$

$$|A| \leq \frac{1}{n}\sqrt{\frac{n+1}{n+1+(n-1)k^2}},$$

then $f(z) = z + Az^n \in k\text{-}\mathcal{UST}$.

For $k = 1$ we get the result of Merkes and Salmassi ([5]), which improves the bound $|A| \leq 1/(\sqrt{2}n)$ obtained by Goodman.

Merkes and Salmassi ([5]) proved the following result.

Theorem 2.1 *Let $f \in \mathcal{S}$. If for all $z \in U$, $w \in U$*

$$\operatorname{Re}\frac{f'(w)}{f'(z)} \geq 0,$$

then $f \in \mathcal{UST}$.

Using this, we get the following sufficient condition for uniform starlikeness

Theorem 2.2 *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in U$. If*

$$|\operatorname{Arg} f'(z)| \leq \frac{\pi}{4} \quad for \ \ z \in U,$$

then $f \in \mathcal{UST}$.

Proof Let f be analytic in U and normalized by $f(0) = f'(0) - 1 = 0$. Assume that $|\operatorname{Arg} f'(z)| \leq \pi/4$, $z \in U$. This condition implies that $\operatorname{Re} f'(z) > 0$ for $z \in U$, hence $f \in \mathcal{S}$. Moreover if $z \in U$, $w \in U$ then

$$\left|\operatorname{Arg}\frac{f'(w)}{f'(z)}\right| = |\operatorname{Arg} f'(w) - \operatorname{Arg} f'(z)| \leq |\operatorname{Arg} f'(w)| + |\operatorname{Arg} f'(z)| \leq \frac{\pi}{2}.$$

This is equivalent to

$$\operatorname{Re} \frac{f'(w)}{f'(z)} \geq 0 \text{ for all } z, w \in U.$$

Thus by Theorem 2.1, $f \in \mathcal{UST}$.

This result generates many examples of uniformly starlike functions. □

Example 2.1 Let

$$P_1(z) = \sqrt{\frac{1+z}{1-z}}, \quad z \in U.$$

Then $P_1(U) = \{w \in \mathbb{C} : |\operatorname{Arg} w| < \pi/4\}$.

If $f_1'(z) = P_1(z)$, then in view of Theorem 2.2, $f_1 \in \mathcal{UST}$. Thus the function

$$f_1(z) = 2 \arctan \sqrt{\frac{1+z}{1-z}} - \sqrt{1-z^2} + 1 - \frac{\pi}{2}, \quad z \in U,$$

is uniformly starlike in U.

Example 2.2 Let

$$P_2(z) = 1 + \frac{2}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2, \quad z \in U.$$

This function plays an important role in the class $\mathcal{UCV}$ of uniformly convex functions. It is known (see [3, 7]) that $f \in \mathcal{UCV}$ if and only if

$$1 + \frac{zf''(z)}{f'(z)} \prec P_2(z) \text{ in } U.$$

The image of the unit disk U under P_2 is bounded by the parabola

$$\{w = u + iv : v^2 = 2u - 1\}.$$

Moreover $|\operatorname{Arg} P_2(z)| < \pi/4$ for z in U. Thus $f_2'(z) = P_2(z)$ for $z \in U$, implies $f_2 \in \mathcal{UST}$.

Using standard methods of integration after some calculations we get that

$$f_2(z) = z + \frac{2}{\pi^2} \left[(z-1) \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2 + 4\sqrt{z} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right) + 4 \log(1-z) \right],$$
$$z \in U,$$

belongs to the class $\mathcal{UST}$.

Example 2.3 Let

$$P_3(z) = \frac{1+Az}{1-Az}, \quad z \in U,$$

where $0 < A \le \sqrt{2} - 1$.

Then P_3 maps U onto the disk

$$\left\{ w \in \mathbb{C} : \left| w - \frac{1+A^2}{1-A^2} \right| < \frac{2A}{1-A^2} \right\}.$$

For $0 < A \le \sqrt{2} - 1$ the disk $P_3(U)$ is contained in the region $\{w \in \mathbb{C} : |\mathrm{Arg}\, w| < \pi/4\}$. Hence if $f_3'(z) = P_3(z)$ then $f_3 \in \mathcal{UST}$, since $|\mathrm{Arg} f_3'(z)| < \pi/4$ for $z \in U$.

Thus the function

$$f_3(z) = -z - \frac{2}{A} \log(1 - Az), \; z \in U \; (0 < A \le \sqrt{2} - 1),$$

is in the class $\mathcal{UST}$.

Example 2.4 Let

$$P_4(z) = \frac{1}{\sqrt{1-z}}, \quad z \in U.$$

This function maps the unit disk onto the domain bounded by the hyperbola

$$\{w = u + iv : u^2 - v^2 = 1/2, u > 0\}$$

and clearly $P_4(U) \subset \{w \in \mathbb{C} : |\mathrm{Arg}\, w| < \pi/4\}$. Thus

$$f_4(z) = 2 - 2\sqrt{1-z} = z + \frac{1 \cdot 1}{1 \cdot 4} z^2 + \frac{1 \cdot 1 \cdot 3}{1 \cdot 4 \cdot 6} z^3 + \frac{1 \cdot 1 \cdot 3 \cdot 5}{1 \cdot 4 \cdot 6 \cdot 8} z^4 + \cdots$$

is uniformly starlike in U, because of $f_4'(z) = P_4(z)$ for $z \in U$.

Example 2.5 Let

$$P_5(z) = \sqrt{1+z}, \quad z \in U.$$

The image of the unit disk U under P_5 is bounded by the lemniscate

$$\{w = u + iv : (u^2 + v^2)^2 - 2(u^2 - v^2) = 0, u > 0\}$$

and is contained in the region $\{w \in \mathbb{C} : |\mathrm{Arg}\, w| < \pi/4\}$. Hence the function

$$f_5(z) = \frac{2}{3}\left[(1+z)^{3/2} - 1\right] = z + \frac{1}{4} z^2 - \frac{1 \cdot 1}{4 \cdot 6} z^3 + \frac{1 \cdot 1 \cdot 3}{4 \cdot 6 \cdot 8} z^4 - \ldots, \quad z \in U,$$

belongs to the class $\mathcal{UST}$, since $f_5'(z) = P_5(z)$ for $z \in U$.

3 Uniformly Starlike Functions and Subclasses of Convex Functions

Let φ be an analytic univalent function which satisfies the following conditions: $\operatorname{Re}\varphi(z) > 0$ for $z \in U$, $\varphi(0) = 1, \varphi'(0) > 0$ and $\varphi(U)$ is convex and symmetric with respect to the real axis.

By $\mathcal{CV}(\varphi)$, we denote the subclass of convex functions defined by

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \text{ in } U.$$

In particular

$$\mathcal{CV} = \mathcal{CV}(\varphi_0), \quad \text{where } \varphi_0(z) = \frac{1+z}{1-z},$$
$$\mathcal{UCV} = \mathcal{CV}(\varphi_1), \quad \text{where } \varphi_1(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2,$$

and for

$$\varphi_2(z) = \frac{1+(1-2\alpha)z}{1-z}, \quad 0 \le \alpha < 1,$$

the class $\mathcal{CV}(\varphi_2)$ coincides with the class $\mathcal{CV}(\alpha)$ consisting of all functions that are convex of order α, thus

$$\mathcal{CV}(\varphi_2) = \mathcal{CV}(\alpha) = \left\{ f \in \mathcal{S} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ z \in U \right\}.$$

We need the following result of Suffridge [9]

Theorem 3.1 *Let h be starlike in U with $h(0) = 0$ and let $p(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots$ be analytic in U. If*

$$zp'(z) \prec h(z),$$

then

$$p(z) \prec q(z) = a + n^{-1}\int_0^z h(t)t^{-1}\,dt.$$

The function q is convex and is the best (a, n)-dominant.

Corollary 3.1 (Corollary 3.1d.1 in [6]) *Let h be starlike in U, with $h(0) = 0$ and let $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$ be analytic in U with $a \neq 0$. If*

$$\frac{zp'(z)}{p(z)} \prec h(z),$$

then

$$p(z) \prec q(z) = a \exp\left[n^{-1} \int_0^z h(t) t^{-1}\, dt\right]$$

and q is the best (a, n)-dominant.

Theorem 3.2 *Let $f \in \mathcal{CV}(\varphi)$. If*

$$\left|\operatorname{Arg}\left\{\exp \int_0^z \frac{\varphi(t) - 1}{t} dt\right\}\right| \leq \frac{\pi}{4} \quad for \;\; z \in U,$$

then $f \in \mathcal{UST}$.

Proof If $f \in \mathcal{CV}(\varphi)$ then

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \;\text{ in }\; U$$

or

$$z(\log f'(z))' \prec \varphi(z) - 1 \;\text{ in }\; U.$$

Since $\varphi(z) - 1$ is convex, therefore is starlike, we get by Theorem 3.1

$$\log f'(z) \prec \int_0^z \frac{\varphi(t) - 1}{t} \mathrm{d}t \;\text{ in }\; U.$$

From Corollary 3.1, we may also conclude that

$$f'(z) \prec \exp \int_0^z \frac{\varphi(t) - 1}{t} \mathrm{d}t \;\text{ in }\; U.$$

Thus from Theorem 2.2, if

$$\left|\operatorname{Arg}\left\{\exp \int_0^z \frac{\varphi(t) - 1}{t} \mathrm{d}t\right\}\right| \leq \frac{\pi}{4} \quad \text{for } z \in U,$$

then $f \in \mathcal{UST}$. □

Corollary 3.2

$$\mathcal{CV}(\varphi) \subset \mathcal{UST},$$

provided that

$$\left| \mathrm{Im} \int_0^z \frac{\varphi(t)-1}{t} dt \right| \le \frac{\pi}{4}, \quad z \in U.$$

Theorem 3.3

$$\mathcal{CV}(3/4) \subset \mathcal{UST}.$$

Proof Let $f \in \mathcal{CV}(\alpha),\ 0 \le \alpha < 1$. Then from the subordination

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \quad \text{in } U$$

we get

$$f'(z) \prec \frac{1}{(1-z)^{2(1-\alpha)}} \quad \text{in } U.$$

We have

$$\left| \mathrm{Arg} \frac{1}{(1-z)^{2(1-\alpha)}} \right| = 2(1-\alpha)|\mathrm{Arg}(1-z)| \le 2(1-\alpha)\frac{\pi}{2}.$$

Hence for $3/4 \le \alpha < 1$ the function f' satisfies the condition

$$|\mathrm{Arg}\, f'(z)| \le \sup_{|z|=1} \left| \mathrm{Arg} \frac{1}{(1-z)^{2(1-\alpha)}} \right| \le \pi/4, \quad z \in U.$$

By Theorem 3.2 we get

$$\mathcal{CV}(\alpha) \subset \mathcal{CV}(3/4) \subset \mathcal{UST} \quad \text{for } 3/4 \le \alpha < 1.$$

□

Theorem 3.4

$$\mathcal{CV}(\varphi_3) \subset \mathcal{UST},$$

where

$$\varphi_3(z) = 1 + Az \ \ for \ \ z \in U \ \ and \ \ 0 < A \le \pi/4.$$

Proof If $f \in \mathcal{CV}(\varphi_3)$ then

$$1 + \frac{zf''(z)}{f'(z)} \prec 1 + Az \ \text{ in } \ U.$$

Hence

$$f'(z) \prec \exp(Az) \ \text{ in } \ U$$

and since

$$\left|\text{Arg}\left[\exp\left(Ae^{i\theta}\right)\right]\right| = |A \sin\theta| \le A \le \frac{\pi}{4}$$

we get

$$|\text{Arg} f'(z)| \le \frac{\pi}{4}.$$

Thus by Theorem 3.2, f is uniformly starlike in U.

Theorem 3.4 can be written in an equivalent form as follows: □

Corollary 3.3 *Let $f \in \mathcal{S}$. If*

$$\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{\pi}{4}, \quad z \in U,$$

then $f \in \mathcal{UST}$.

Proof Let $f \in \mathcal{S}$. Then for $z \in U$

$$\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{\pi}{4} \Rightarrow \frac{zf''(z)}{f'(z)} \prec \frac{\pi}{4} z \ \text{ in } \ U.$$

Hence $f \in \mathcal{CV}(\varphi_3)$ with $A = \pi/4$, so by Theorem 3.4 the result follows. □

Declarations

Conflict of interest The author has no relevant financial or non-financial interests to disclose.

References

1. Goodman, A.W.: On uniformly convex functions. Ann. Polon. Math. **56**(1), 87–92 (1991)
2. Goodman, A.W.: On uniformly starlike functions. J. Math. Anal. Appl. **155**, 364–370 (1991)
3. Ma, W., Minda, D.: Uniformly convex functions. Ann. Polon. Math. **57**(2), 165–175 (1992)
4. Ma, W., Minda, D.: Uniformly convex functions II. Ann. Polon. Math. **58**, 275–285 (1993)
5. Merkes, E., Salmassi, M.: Subclasses of uniformly starlike functions. Int. J. Math. Math. Sci. **15**, 449–454 (1992)
6. Miller, S.S., Mocanu, P.T.: Differential Subordinations—Theory and Applications, CRC Press, (2000)
7. Rønning, F.: Uniformly convex functions and a corresponding class of starlike functions. Proc. Am. Math. Soc. **118**, 189–196 (1993)
8. Rønning, F.: On starlike functions associated with parabolic regions. Ann. Univ. Mariae Curie-Skłodowska Sect. A **45**, 117–122 (1991)
9. Suffridge, T.J.: Some remarks on convex maps of the unit disk. Duke Math. J. **37**, 755–777 (1970)
10. Wiśniowska-Wajnryb, A.: A Survey on k-Uniformly Starlike Functions, Current Research in Mathematical and Computer Sciences, Publisher UWM, Olsztyn, pp. 225–232. (2017)
11. Wiśniowska-Wajnryb, A.: On classes of uniformly starlike functions. Ann. Polon. Math. **108**(1), 11–19 (2013)
12. Wiśniowska-Wajnryb, A.: The order of starlikeness of the class of uniformly starlike functions. Math. Nachr. **289**(16), 2083–2088 (2016)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.