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Multi-component Reliability Inference in Modified Weibull Extension Distribution and Progressive Censoring Scheme

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Abstract

The statistical inference of multi-component reliability stress–strength system with nonidentical-component strengths is considered for the modified Weibull extension distribution in the presence of progressive censoring samples. For this aim, we study the estimation of multi-component reliability parameter in classical and Bayesian inference. So we derive some point and interval estimates such as maximum likelihood estimation, asymptotic confidence intervals, uniformly minimum variance unbiased estimation, approximate and exact Bayes estimation and highest posterior density intervals. Comparing of different estimates is provided by employing the Monte Carlo simulation, the mean squared error and coverage probabilities. Finally, one real data is utilized to illustrate the applicability of this new model.

Keywords Multi-component reliability · Classical estimation · Bayes estimation · Progressive censored

Mathematics Subject Classification 62F10 · 62F15 · 62N02

1 Introduction

The Weibull distribution, which introduced by Swedish physicist [25], is one of the most commonly used lifetime distributions for modeling data in reliability, engineering, finance, hydrology, physics and environmental studies. This distribution is very

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flexible in modeling failure time data, as the corresponding failure rate function can be increasing, constant or decreasing. On the other hand, for complex system, the failure rate function can often be of bathtub shape and modeling of this system is so important in reliability analysis. For this reason, the modified Weinull extension (MWEx) with bathtub shaped failure rate function was proposed by Xie et al. [26]. Since its inception from 2002, the MWEx distribution has received a considerable amount of attention from the statistical community, with over 560 citations to date. Its versatility and effectiveness in a variety of situations have been portrayed in numerous books and papers. The probability density function (PDF) and cumulative distribution function (CDF) of this distribution are, respectively,

$$f(x) = \lambda\beta\left(\frac{x}{\alpha}\right)^{\beta-1} e^{\left(\frac{x}{\alpha}\right)^{\beta}+\lambda\alpha\left(1-e^{\left(\frac{x}{\alpha}\right)^{\beta}}\right)}, \tag{1}$$

$$F(x) = 1 - e^{\lambda\alpha\left(1-e^{\left(\frac{x}{\alpha}\right)^{\beta}}\right)}, \tag{2}$$

where $\alpha,\ \beta,\ \lambda > 0$. Herein we denote a MWEx distribution with the parameters α, β and λ by MWEx(α, β, λ). It is notable that MWEx is a general distribution where some most distributions can be obtained from it. In following, we explain about two of them: first one is Weibull distribution and the second one is Chen's distribution, see Chen [8]. The Weibull distribution can be obtained as a special case of this distribution when α is so small that $1-e^{(\frac{x}{\alpha})^{\beta}}$ is approximately equal to $-(\frac{x}{\alpha})^{\beta}$. The particular case of the MWEx distribution for $\alpha = 1$ is Chen's distribution. The failure rate function (FRF) of MWEx distribution is

$$h(x) = \lambda\beta\left(\frac{x}{\alpha}\right)^{\beta-1} e^{\left(\frac{x}{\alpha}\right)^{\beta}},$$

which depends only on the shape parameter β. The HF increases if $\beta \geq 1$ and is bathtub shaped if $\beta < 1$. Some possible shapes of the PDF and the FRF of MWEx distribution are shown in Fig. 1.

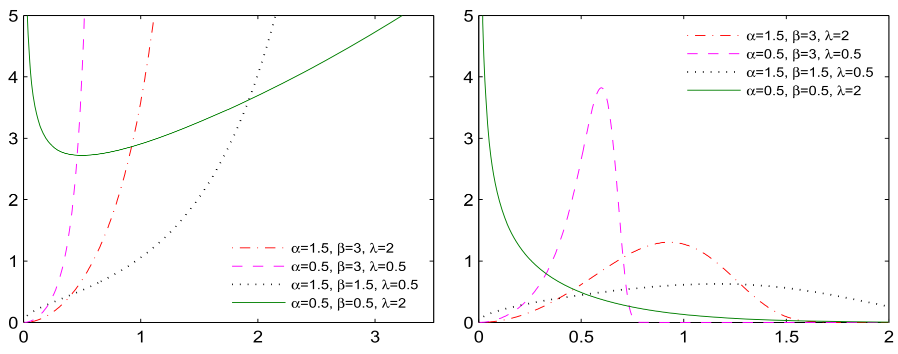

Fig. 1 Shape of failure rate (left) and probability density (right) functions of MWEx distribution

The main contribution in the paper is the study in the context of Weibull extension model, there are other related works. For example, a new five parameter distribution, known as the modified beta flexible Weibull extension distribution, is derived and studied by Abubakari et al. [1]. Kamal and Ismail [12] considered the flexible Weibull Extension-Burr XII distribution. Nassar et al. [19] introduced a new extension of Weibull distribution, called Alpha logarithmic transformed Weibull distribution that provides better fits than some of its known generalizations. Peng and Yan [20] introduced a new extended Weibull distribution with one scale parameter and two shape parameters. Also, a new distribution as the exponentiated modified Weibull extension distribution has been considered by Sarhan and Apaloo [24].

Type-I and Type-II censoring schemes are two most fundamental schemes in many different censoring schemes. In these schemes, removing of the active units during the tests is not possible, whereas the removal of surviving units during the test can be pre-planned and intentional in order to save time and cost associated with test. For this and many other reasons, the progressive censoring is introduced. Combining the Type-II and progressive censoring schemes leads to consider progressive Type-II censoring scheme. Recently, this scheme was very successful in applications. Under this scheme, on a life test, N units are placed and before hand the experimenter decides n be the number of failures to be observed. So, at the first failure time, R_1 units randomly are removed from $N-1$ surviving units. At the second failure time, from the $N-R_1-1$ remaining units, R_2 units randomly are removed from the experiment. By continuing this process, finally, at the $n-$th failure time, all the remaining surviving units $R_n = N-n-R_1-\cdots-R_{n-1}$ randomly are removed from the experiment. So, in a progressive Type-II censoring scheme n is the number of failure time observations, $\{X_1,\ldots,X_n\}$ is the censoring sample, and $\{N, n, R_1,\ldots,R_n\}$ is the progressive censoring scheme, such that $R_1+\cdots+R_n+n = N$. Clearly, Type-II progressive censoring scheme can be converted to the conventional Type-II right censoring scheme (by $R_1=\cdots=R_{n-1}=0$ and $R_n = N-n$) and complete sampling case (by $R_1=\cdots=R_n=0$ and $N=n$). We propose that the reader refers to the book of Balakrishnan and Aggarwala [4], for more details on progressively censoring and relevant references.

Inference about stress–strength parameter is one of the interest and fundamental problems in reliability analysis. This parameter is shown by $R = P(Y < X)$, where Y and X are known as stress and strength and they are two independent random variables. Obviously, the system is reliable so long as the strength X is greater than its stress Y. This parameter has many applicable in different fields. For example, in clinical studies in medicine, if X and Y are the response of the control group to a therapeutic approach and the response of the treated group, respectively (see [11]), then R can be seen as the measure of treatment effect. The link between statistics and reliability theory leads to estimate of the stress–strength parameter, starting with the pioneering work of [6]. From that time, many researchers have studied inference on the reliability parameter from the classical and Bayesian points of views. Very recently, Al-Babtain et al. [2] considered Bayesian and non-Bayesian reliability estimation of stress–strength model for power-modified Lindley distribution. Sabry et al. [23] discussed Monte Carlo simulation of stress–strength model and reliability estimation for extension of the exponential distribution. Also, Metwally et al. [18] studied reliability analysis of the new exponential inverted Topp–Leone distribution with applications. About the multi- stress–strength reliability, Yousef and Almetwally [28] investigated multi-stress–strength reliability based on progressive first failure for Kumaraswamy model in Bayesian and non-Bayesian approaches. Also, Almetwally et al. [3] studied optimal plan of multi-stress–strength reliability Bayesian and non-Bayesian methods for the alpha power exponential model using progressive first failure. Moreover, about the fuzzy reliability approach, Sabry et al. [22] considered inference of fuzzy reliability model for inverse Rayleigh distribution. Also, Meriem et al. [17] introducing the Power XLindley distribution studied statistical inference, fuzzy reliability and COVID-19 application.

Reliability scientists, a system with more than one component, have called a multi-component system. In such system, there is one common stress component and k strength independent and identical components. Obviously, the system is reliable so long as at least s from k strength components exceed its stress. This model has received a great deal of attention, in recently years and is known as G system: s-out-of-k. Many examples can be given of multi-component systems. For instance, consider one G system: the 4 out of 8 as functioning V-8 engine of an automobile. In this system, the car can be derived if only four cylinders are firing. However, the automobile cannot be driven, if less than four cylinders are fired. Also, for another example, consider a suspension bridge. In this case, heavy traffic, wind loading, corrosion, etc., can be considered as stresses and the k number of vertical cable pairs can be considered as strengths. In this situation, the bridge breaks down, if a minimum s number of vertical cables are damaged. A more homely but complicated example of a multi-component system would be a music (stero Hi-Fi) system consisting of an FM tuner and record changer in parallel; connected in series with an amplifier and speakers (with the two speakers, say A and B) connected in parallel. Bhattacharyya and Johnson [5] firstly have presented this model as follows:

$$R_{s,k} = \sum_{p=s}^{k} \binom{k}{p} \int_{-\infty}^{\infty} \big(1 - F_X(y)\big)^p \big(F_X(y)\big)^{k-p} \mathrm{d}F_Y(y), \tag{3}$$

where the strength variables $(X_1, \ldots, X_k)$ are independent and identically distributed with the CDF $F_X(\cdot)$ and stress variable Y has the CDF $F_Y(\cdot)$. Recently, this model has attracted a lot of attention and has been considered for complete and censored samples by some authors, for instance [14, 15].

As we saw, this assumption that the strengths are of i.i.d should be considered in every applications and this condition is not available in many practical situations, when the system component structures are different. The readers can see Farahmand et al. [10], for more details. So, in following, we try to focus on multi-component stress–strength models with nonidentical random strengths.

In following, one system with $\mathbf{k} = (k_1, \ldots, k_m)$ strength components is studied. In such systems, the components have nonidentical distributions so that the i-th component, $i = 1, \ldots, m$, follows a distribution with CDF $F_X(\cdot)$ and all of this strength variables are affected by a common stress Y with CDF $F_Y(\cdot)$. Obviously, this system is reliable so long as at least $\mathbf{s} = (s_1, \ldots, s_m)$ from $\mathbf{k}$ strength components exceed its stress. Rasethuntsa and Nadar [21] have improved 3 to derive one suitable model as follows:

$$R_{\mathbf{s},\mathbf{k}} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \left(\prod_{l=1}^{m} \binom{k_l}{p_l} \right) \int_{-\infty}^{\infty} \prod_{l=1}^{m} \left(\left(1 - F_l(y)\right)^{p_l} \left(F_l(y)\right)^{k_l - p_l} \right) \mathrm{d}F_Y(y). \tag{4}$$

In very recently paper, Kohansal et al. [16] studied this model, for two components of strength variables. But, now, we consider this model for m components strength variables. So, we assume that the i-th strength component follows a MWEx$(\alpha, \beta, \lambda_i)$, for $i = 1, \ldots, m$, and the stress Y follows a MWEx(α, β, λ) distribution.

Accordingly, we continue the paper as follows. In Sect. 2, when the common parameters are unknown, the point statistical estimation of $R_{\mathbf{s},\mathbf{k}}$ is considered, so that, we obtain the MLE and Bayesian estimation are obtained. Because the lack of explicit from, we approximate the Bayesian estimation by Markov Chain Monte Carlo (MCMC) method. Also, in view of interval estimation, we studied the asymptotic and HPD intervals. In Sect. 3, when the common parameters are known, the statistical estimation of $R_{\mathbf{s},\mathbf{k}}$ is considered, so that, we obtain the MLE, exact Bayesian estimation, UMVUE, asymptotic and HPD intervals. In Sect. 4, we use the Numerical simulation results to compare the theoretical methods and Sect. 5, we consider the general case, when all parameters are different and unknown. In Sect. 6, one real data is utilized to illustrative the applicability of this new model. Finally, we conclude the paper in Sect. 7.

2 Inference on $R_{\mathbf{s},\mathbf{k}}$ with Unknown Common Parameters

In many empirical data analysis, the common parameters values of strengths and stress variables are approximately same. So, in such situations, we assume that they are equal. Also, some estimations, in other cases spatially the case with known common parameters, can be obtained from this case. In other advantages of this case, we can pointed the extent of estimations.

2.1 MLE of $R_{\mathbf{s},\mathbf{k}}$

Now, we suppose that $X_1 \sim$ MWEx$(\alpha, \beta, \lambda_1)$, $X_2 \sim$ MWEx$(\alpha, \beta, \lambda_2)$, …, $X_m \sim$ MWEx$(\alpha, \beta, \lambda_m)$ and $Y \sim$ MWEx(α, β, λ) are independent random variables. Using Eqs. (1) and (2), we can obtain the multi-component reliability with nonidentical-component strengths in (4) as follows:

$$R_{\mathbf{s},\mathbf{k}} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \left(\prod_{l=1}^{m} \binom{k_l}{p_l} \right) \int_{0}^{\infty} e^{\left(\sum_{l=1}^{m} \lambda_l p_l\right)\alpha(1-e^{(\frac{y}{\alpha})^{\beta}})} \prod_{l=1}^{m} \left(1 - e^{\lambda_l \alpha(1-e^{(\frac{y}{\alpha})^{\beta}})}\right)^{k_l - p_l}$$
$$\times \lambda\beta(\frac{y}{\alpha})^{\beta-1} e^{\lambda\alpha(1-e^{(\frac{y}{\alpha})^{\beta}})+(\frac{y}{\alpha})^{\beta}} \mathrm{d}y \quad Put : t = e^{\alpha(1-e^{(\frac{y}{\alpha})^{\beta}})}$$

$$
\begin{aligned}
&= \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \Big(\prod_{l=1}^{m} \binom{k_l}{p_l}\Big)\lambda \int_0^1 t^{\sum_{l=1}^{m}\lambda_l p_l+\lambda-1} \prod_{l=1}^{m}\Big((1-t_l^{\lambda})^{k_l-p_l}\Big)\mathrm{d}t \\
&= \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \Big(\prod_{l=1}^{m} \binom{k_l}{p_l}\Big)\lambda \int_0^1 t^{\sum_{l=1}^{m}\lambda_l p_l+\lambda-1} \prod_{l=1}^{m}\Big(\sum_{q_l=0}^{k_l-p_l}\binom{k_l-p_l}{q_l}(-1)^{q_l}t^{\lambda_l q_l}\Big)\mathrm{d}t \\
&= \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \sum_{q_1=0}^{k_1-p_1} \cdots \sum_{q_m=0}^{k_m-p_m} \Big(\prod_{l=1}^{m} \binom{k_l}{p_l}\Big) \times \Big(\prod_{l=1}^{m} \binom{k_l-p_l}{q_l}\Big) \\
&\quad \times (-1)^{\sum_{l=1}^{m} q_l} \lambda \int_0^1 t^{\sum_{l=1}^{m}\lambda_l(p_l+q_l)+\lambda-1} \mathrm{d}t \\
&= \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \sum_{q_1=0}^{k_1-p_1} \cdots \sum_{q_m=0}^{k_m-p_m} \Big(\prod_{l=1}^{m} \binom{k_l}{p_l}\Big) \times \Big(\prod_{l=1}^{m} \binom{k_l-p_l}{q_l}\Big) \\
&\quad \times (-1)^{\sum_{l=1}^{m} q_l} \frac{\lambda}{\sum_{l=1}^{m}\lambda_l(p_l+q_l)+\lambda}. \qquad (5)
\end{aligned}
$$

Now, to derive the MLE of $R_{\mathbf{s},\mathbf{k}}$, we use the invariance property, so, first, we obtain the MLE's unknown parameters $\alpha,\ \beta,\ \lambda_1,\ ,\ldots,\ \lambda_m,\ \lambda$. With n system on the life-testing experiment, we construct the likelihood function. So, the observed samples can be provided as follows:

$$
\overset{\text{Observed stress variables}}{Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}} \quad \text{and} \quad \overset{\text{Observed strength variables}}{X_l = \begin{bmatrix} X_{11}^{(l)} & \ldots & X_{1k_l}^{(l)} \\ \vdots & \ddots & \vdots \\ X_{n1}^{(l)} & \ldots & X_{nk_l}^{(l)} \end{bmatrix}},\ l = 1, \ldots, m.
$$

In following, we assume that $\{Y_1, \ldots, Y_n\}$ is progressive censoring sample from $\mathrm{MWEx}(\alpha, \beta, \lambda)$ with the $\{N, n, S_1, \ldots, S_n\}$ censoring scheme. Also, $\{X_{i1}^{(l)}, \ldots, X_{ik_l}^{(l)}\}$, $i = 1, \ldots, n$, $l = 1, \ldots, m$ is progressive censoring sample from $\mathrm{MWEx}(\alpha, \beta, \lambda_i)$ with the $\{K_l, k_l, R_{i1}^{(l)}, \ldots, R_{ik_l}^{(l)}\}$ censoring scheme, where $i = 1, \ldots, n, l = 1, \ldots, m$. Therefore, we write the likelihood function of $\lambda_1,\ ,\ldots,\ \lambda_m$, and λ, α, β as

$$
\begin{aligned}
L(\lambda_1, \ldots, \lambda_m, \lambda, \alpha, \beta|\text{data}) \propto \prod_{i=1}^{n}\Bigg(\prod_{l=1}^{m}\Bigg(\prod_{j_l=1}^{k_l} f_l(x_{ij_l}^{(l)})\big(1-F_l(x_{ij_l}^{(l)})\big)^{R_{ij_l}^{(l)}}\Bigg)\Bigg) \\
\times f_Y(y_i)\big(1-F_Y(y_i)\big)^{S_i}.
\end{aligned}
$$

About the advantage of this likelihood function, we can say that this is a general function, so that, some other likelihood functions case can be obtained from it as follows:

- $R_{ij_l}^{(l)} = 0,\ S_i = 0 \Rightarrow R_{\mathbf{s},\mathbf{k}}$ in complete sample case.

- $\mathbf{k} = (k_1, k_2, 0, \ldots, 0) \Rightarrow R_{\mathbf{s},\mathbf{k}}$ with two nonidentical-component in the progressive censoring case.
- $\mathbf{k} = (k, 0, \ldots, 0) \Rightarrow R_{s,k}$ in the progressive censoring case.
- $\mathbf{k} = (k_1, k_2, 0, \ldots, 0)$, $R^{(l)}_{ij_l} = 0$, $S_i = 0 \Rightarrow R_{\mathbf{s},\mathbf{k}}$ with two nonidentical-component in complete sample case.
- $\mathbf{k} = (k, 0, \ldots, 0)$, $R^{(l)}_{ij_l} = 0$, $S_i = 0 \Rightarrow R_{s,k}$ in complete sample case.
- $\mathbf{k} = (1, 0, \ldots, 0) \Rightarrow R = P(X < Y)$ in the progressive censoring case.
- $\mathbf{k} = (1, 0, \ldots, 0)$, $R^{(l)}_{ij_l} = 0$, $S_i = 0 \Rightarrow R = P(X < Y)$ in complete sample case.

We can obtain the likelihood function, based on observed data, as

$$L(\lambda_1, \ldots, \lambda_m, \lambda, \alpha, \beta|\text{data}) \propto \Big(\prod_{l=1}^{m} \lambda_l^{nk_l}\Big)\beta^{n(\sum_{l=1}^{m} k_l+1)} \lambda^n \times \Big(\prod_{i=1}^{n}\prod_{l=1}^{m}\prod_{j_l=1}^{k_l} (\frac{x^{(l)}_{ij_l}}{\alpha})^{\beta-1}\Big)$$
$$\times \Big(\prod_{i=1}^{n} (\frac{y_i}{\alpha})^{\beta-1}\Big) \times e^{\sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j_l=1}^{k_l} (\frac{x^{(l)}_{ij_l}}{\alpha})^{\beta} + \sum_{i=1}^{n} (\frac{y_i}{\alpha})^{\beta}} \times e^{\alpha\big(\sum_{l=1}^{m} \lambda_l A_l(\alpha,\beta) + \lambda B(\alpha,\lambda)\big)}, \tag{6}$$

where

$$A_l(\alpha, \beta) = \sum_{i=1}^{n}\sum_{j_l=1}^{k_l} \big(R^{(l)}_{ij_l} + 1\big)\big(1 - e^{(\frac{x^{(l)}_{ij_l}}{\alpha})^{\beta}}\big), \; l = 1, \ldots, m, \tag{7}$$

$$B(\alpha, \beta) = \sum_{i=1}^{n} \big(S_i + 1\big)\big(1 - e^{(\frac{y_i}{\alpha})^{\beta}}\big). \tag{8}$$

To obtain the MLEs of unknown parameters, after deriving the log-likelihood function from (6), we should solve together the following equations:

$$\frac{\partial \ell}{\partial \lambda_l} = \frac{nk_l}{\lambda_l} + \alpha A_l(\alpha, \beta), \; l = 1, \ldots, m, \quad \frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + \alpha B(\alpha, \beta), \tag{9}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta}\left(\sum_{l=1}^{m} k_l + 1\right) + \sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j_l=1}^{k_l} \log\big(\frac{x^{(l)}_{ij_l}}{\alpha}\big) + \sum_{i=1}^{n} \log\big(\frac{y_i}{\alpha}\big)$$
$$+ \sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j_l=1}^{k_l} \big(\frac{x^{(l)}_{ij_l}}{\alpha}\big)^{\beta} \log\big(\frac{x^{(l)}_{ij_l}}{\alpha}\big)\Big(1 - \alpha\lambda_l\Big(R^{(l)}_{ij_l} + 1\Big)e^{\big(\frac{x^{(l)}_{ij_l}}{\alpha}\big)^{\beta}}\Big)$$
$$+ \sum_{i=1}^{n} \big(\frac{y_i}{\alpha}\big)^{\beta} \log\big(\frac{y_i}{\alpha}\big)\Big(1 - \alpha\lambda\Big(S_i + 1\Big)e^{\big(\frac{y_i}{\alpha}\big)^{\beta}}\Big), \tag{10}$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{-n(\beta - 1)}{\alpha}\big(\sum_{l=1}^{m} k_l + 1\big) - \frac{\beta}{\alpha}\Big(\sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j_l=1}^{k_l} \big(\frac{x^{(l)}_{ij_l}}{\alpha}\big)^{\beta} + \sum_{i=1}^{n} \big(\frac{y_i}{\alpha}\big)^{\beta}\Big)$$

$$+\sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j_l=1}^{k_l}\lambda_l\Big(R_{ij_l}^{(l)}+1\Big)\Big(1-e^{\left(\frac{x_{ij_l}^{(l)}}{\alpha}\right)^{\beta}}+\beta\big(\frac{x_{ij_l}^{(l)}}{\alpha}\big)^{\beta}e^{\left(\frac{x_{ij_l}^{(l)}}{\alpha}\right)^{\beta}}\Big)$$
$$+\lambda\sum_{i=1}^{n}\Big(S_i+1\Big)\Big(1-e^{\left(\frac{y_i}{\alpha}\right)^{\beta}}+\beta\big(\frac{y_i}{\alpha}\big)^{\beta}e^{\left(\frac{y_i}{\alpha}\right)^{\beta}}\Big) \tag{11}$$

The MLEs of $\lambda_1, \ \ldots, \lambda_m, \ \lambda, \ \alpha, \ \beta$, presented by $\widehat{\lambda}_1, \ \ldots, \widehat{\lambda}_m, \ \widehat{\lambda}, \ \widehat{\alpha}, \ \widehat{\beta}$, can be obtained from simultaneous solution of equations (9), (10) and (11), using one numerical method such as Newton–Raphson algorithm. Finally, the invariance property of MLE concludes that the MLE of $R_{\mathbf{s},\mathbf{k}}$, presented by $\widehat{R}_{\mathbf{s},\mathbf{k}}^{\text{MLE}}$, can be obtained as

$$\widehat{R}_{\mathbf{s},\mathbf{k}}^{\text{MLE}}=\sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\sum_{q_1=0}^{k_1-p_1}\cdots\sum_{q_m=0}^{k_m-p_m}\Big(\prod_{l=1}^{m}\binom{k_l}{p_l}\Big)\times\Big(\prod_{l=1}^{m}\binom{k_l-p_l}{q_l}\Big)$$
$$\times(-1)^{\sum_{l=1}^{m}q_l}\frac{\widehat{\lambda}}{\sum_{l=1}^{m}\widehat{\lambda}_l(p_l+q_l)+\widehat{\lambda}}. \tag{12}$$

2.2 Asymptotic Confidence Interval

We provide the asymptotic confidence interval of $R_{\mathbf{s},\mathbf{k}}$, in this section. For such aim, firstly, using the multivariate central limit theorem, we derive the asymptotic distribution of unknown parameters $\lambda_1, \ \ldots, \lambda_m, \ \lambda, \ \alpha, \ \beta$, and then later, using delta method, we derive the asymptotic distribution and asymptotic confidence interval of $R_{\mathbf{s},\mathbf{k}}$.

It is noted that the expected Fisher information matrix $J(\theta) = -E(I(\theta))$ prepares the asymptotic variances and covariances of the parameter vector θ. In this presented, $\theta = (\lambda_1, \ldots, \lambda_m, \lambda, \alpha, \beta)$ is a vector of unknown parameters and $I(\theta) = [I_{ij}] = [\partial^2\ell/(\partial\theta_i\partial\theta_j)], i, j = 1, \ldots, m+3$, is the observed Fisher information matrix. From the $I(\theta)$ elements, obviously, we cannot obtain $J(\theta)$ elements easily. So, we use the observed Fisher information matrix instead of expected Fisher information matrix. The elements of $I(\theta)$ matrix are as follows:

$$I_{l,l}=\frac{nk_l}{\lambda_l^2},\ l=1,\ldots,m,\quad I_{m+1,m+1}=\frac{n}{\lambda^2},\quad I_{l,k}=0,\ l,k=1,\ldots,m+1, l\neq k$$
$$I_{l,m+2}=-\sum_{i=1}^{n}\sum_{j_l=1}^{k_l}\Big(R_{ij_l}^{(l)}+1\Big)\Big(1-e^{\left(\frac{x_{ij_l}^{(l)}}{\alpha}\right)^{\beta}}+\beta\big(\frac{x_{ij_l}^{(l)}}{\alpha}\big)^{\beta}e^{\left(\frac{x_{ij_l}^{(l)}}{\alpha}\right)^{\beta}}\Big),\ l=1,\ldots,m,$$
$$I_{m+1,m+2}=-\sum_{i=1}^{n}\Big(S_i+1\Big)\Big(1-e^{\left(\frac{y_i}{\alpha}\right)^{\beta}}+\beta\big(\frac{y_i}{\alpha}\big)^{\beta}e^{\left(\frac{y_i}{\alpha}\right)^{\beta}}\Big),$$
$$I_{l,m+3}=\alpha\sum_{i=1}^{n}\sum_{j_l=1}^{k_l}\Big(R_{ij_l}^{(l)}+1\Big)\big(\frac{x_{ij_l}^{(l)}}{\alpha}\big)^{\beta}\log\big(\frac{x_{ij_l}^{(l)}}{\alpha}\big)e^{\left(\frac{x_{ij_l}^{(l)}}{\alpha}\right)^{\beta}},\ l=1,\ldots,m,$$

$$I_{m+1,m+3} = \alpha \sum_{i=1}^{n} \Big(S_i + 1\Big)(\frac{y_i}{\alpha})^{\beta} \log(\frac{y_i}{\alpha}) e^{\left(\frac{y_i}{\alpha}\right)^{\beta}},$$

$$\begin{aligned}I_{m+2,m+3} =& -\frac{n}{\alpha}(\sum_{l=1}^{m} k_l + 1) - \frac{1}{\alpha}\Bigg(\sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j_l=1}^{k_l}(\frac{x_{ij_l}^{(l)}}{\alpha})^{\beta}\Big(1 + \beta\log(\frac{x_{ij_l}^{(l)}}{\alpha})\Big)\\&+\sum_{i=1}^{n}(\frac{y_i}{\alpha})^{\beta}\Big(1 + \beta\log(\frac{y_i}{\alpha})\Big)\Bigg)\\&+\sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j_l=1}^{k_l}\lambda_l\Big(R_{ij_l}^{(l)} + 1\Big)(\frac{x_{ij_l}^{(l)}}{\alpha})^{\beta} e^{\left(\frac{x_{ij_l}^{(l)}}{\alpha}\right)^{\beta}}\Big(\beta(\frac{x_{ij_l}^{(l)}}{\alpha})^{\beta}\log(\frac{x_{ij_l}^{(l)}}{\alpha})\\&+\beta\log(\frac{x_{ij_l}^{(l)}}{\alpha}) - \log(\frac{x_{ij_l}^{(l)}}{\alpha}) + 1\Big)\\&+\lambda\sum_{i=1}^{n}\Big(S_i + 1\Big)(\frac{y_i}{\alpha})^{\beta} e^{\left(\frac{y_i}{\alpha}\right)^{\beta}}\Big(\beta(\frac{y_i}{\alpha})^{\beta}\log(\frac{y_i}{\alpha}) + \beta\log(\frac{y_i}{\alpha}) - \log(\frac{y_i}{\alpha}) + 1\Big),\end{aligned}$$

$$\begin{aligned}I_{m+2,m+2} =& \frac{n(\beta-1)}{\alpha^2}(\sum_{l=1}^{m} k_l + 1) + \frac{\beta}{\alpha^2}(1 + \frac{1}{\alpha})\Bigg(\sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j_l=1}^{k_l}(\frac{x_{ij_l}^{(l)}}{\alpha})^{\beta} + \sum_{i=1}^{n}(\frac{y_i}{\alpha})^{\beta}\Bigg)\\&-\frac{\beta}{\alpha}\sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j_l=1}^{k_l}\lambda_l\Big(R_{ij_l}^{(l)} + 1\Big)(\frac{x_{ij_l}^{(l)}}{\alpha})^{\beta} e^{\left(\frac{x_{ij_l}^{(l)}}{\alpha}\right)^{\beta}}\Big(\beta(\frac{x_{ij_l}^{(l)}}{\alpha})^{\beta} + \beta - 1\Big)\\&-\frac{\lambda\beta}{\alpha}\sum_{i=1}^{n}\Big(S_i + 1\Big)(\frac{y_i}{\alpha})^{\beta} e^{\left(\frac{y_i}{\alpha}\right)^{\beta}}\Big(\beta(\frac{y_i}{\alpha})^{\beta} + \beta - 1\Big),\end{aligned}$$

$$\begin{aligned}I_{m+3,m+3} =& -\frac{n}{\beta^2}(\sum_{l=1}^{m} k_l + 1)\\&+\sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j_l=1}^{k_l}(\frac{x_{ij_l}^{(l)}}{\alpha})^{\beta}\log^2(\frac{x_{ij_l}^{(l)}}{\alpha})\Big(1 - \alpha\lambda_l\Big(R_{ij_l}^{(l)} + 1\Big)e^{\left(\frac{x_{ij_l}^{(l)}}{\alpha}\right)^{\beta}}\Big((\frac{x_{ij_l}^{(l)}}{\alpha}) + 1\Big)\Big)\\&+\sum_{i=1}^{n}(\frac{y_i}{\alpha})^{\beta}\log^2(\frac{y_i}{\alpha})\Big(1 - \alpha\lambda\Big(S_i + 1\Big)e^{\left(\frac{y_i}{\alpha}\right)^{\beta}}\Big((\frac{y_i}{\alpha}) + 1\Big)\Big).\end{aligned}$$

From the multivariate central limit theorem, it can be concluded that

$$(\widehat{\lambda}_1, \widehat{\lambda}_2, \ldots, \widehat{\lambda}_m, \widehat{\lambda}, \widehat{\alpha}, \widehat{\beta}) \sim N_{m+3}((\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda, \alpha, \beta), \mathbf{I}^{-1}(\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda, \alpha, \beta)),$$

where $\mathbf{I}(\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda, \alpha, \beta) = \big[I_{i,j}\big],\ i, j = 1, \ldots, m+3$, is a symmetric matrix, in which $I_{i,j}$ are given in the above equations. Also, $\mathbf{I}^{-1}(\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda, \alpha, \beta) = \frac{[b_{i,j}]}{\det\big(\mathbf{I}(\lambda_1,\lambda_2,\ldots,\lambda_m,\lambda,\alpha,\beta)\big)},\ i, j = 1, \ldots, m+3$, in which b_{ij} is the elements of $adj(\mathbf{I}(\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda, \alpha, \beta))$.

Now, from the delta method, it can be concluded that

$$\widehat{R}_{\mathbf{s},\mathbf{k}}^{\text{MLE}} \sim N(R_{\mathbf{s},\mathbf{k}}, B),$$

where, $B = \mathbf{b^T I^{-1}}(\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda, \alpha, \beta)\mathbf{b}$, in which

$$\mathbf{b} = \left[\frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \lambda_1} \ \cdots \ \frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \lambda_m} \ \frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \lambda} \ \frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \alpha} \ \frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \beta} \right]^T = \left[\frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \lambda_1} \ \cdots \ \frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \lambda_m} \ \frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \lambda} \ 0 \ 0 \right]^T,$$

with

$$\frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \lambda_l} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \sum_{q_1=0}^{k_1-p_1} \cdots \sum_{q_m=0}^{k_m-p_m} \Big(\prod_{l=1}^{m} \binom{k_l}{p_l}\Big) \times \Big(\prod_{l=1}^{m} \binom{k_l - p_l}{q_l}\Big) \times (-1)^{\sum_{l=1}^{m} q_l + 1} \frac{\lambda (p_l + q_l)}{\big(\sum_{l=1}^{m} \lambda_l (p_l + q_l) + \lambda\big)^2}, \ l = 1, \ldots, m, \tag{13}$$

$$\frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \lambda} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \sum_{q_1=0}^{k_1-p_1} \cdots \sum_{q_m=0}^{k_m-p_m} \Big(\prod_{l=1}^{m} \binom{k_l}{p_l}\Big) \times \Big(\prod_{l=1}^{m} \binom{k_l - p_l}{q_l}\Big) \times (-1)^{\sum_{l=1}^{m} q_l} \frac{\sum_{l=1}^{m} \lambda_l (p_l + q_l)}{\big(\sum_{l=1}^{m} \lambda_l (p_l + q_l) + \lambda\big)^2}. \tag{14}$$

So,

$$B = \Big(\det\big(\mathbf{I}(\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda, \alpha, \beta)\big)\Big)^{-1} \left[\frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \lambda_1} \ \cdots \ \frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \lambda_m} \ \frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \lambda} \ 0 \ 0 \right] \times \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,m} & b_{1,m+1} & b_{1,m+2} & b_{1,m+3} \\ & b_{2,2} & \cdots & b_{2,m} & b_{2,m+1} & b_{2,m+2} & b_{2,m+3} \\ & & \ddots & \vdots & \vdots & \vdots & \vdots \\ & & & b_{m,m} & b_{m,m+1} & b_{m,m+2} & b_{m,m+3} \\ & & & & b_{m+1,m+1} & b_{m+1,m+2} & b_{m+1,m+3} \\ & & & & & b_{m+2,m+2} & b_{m+2,m+3} \\ & & & & & & b_{m+3,m+3} \end{bmatrix} \begin{bmatrix} \frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \lambda_1} \\ \vdots \\ \frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \lambda_m} \\ \frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \lambda} \\ 0 \\ 0 \end{bmatrix}$$

$$= \Big(\det\big(\mathbf{I}(\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda, \alpha, \beta)\big)\Big)^{-1} \times \Big(\sum_{j=1}^{m} \sum_{i=1}^{m} \frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \lambda_j} \frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \lambda_i} b_{j,i} + \frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \lambda} \big(\sum_{i=1}^{m} \frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \lambda_i} b_{i,m+1} \big)^2 + \big(\frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \lambda} \big)^2 b_{m+1,m+1} \Big).$$

Consequently, we construct a $100(1-\eta)\%$ asymptotic confidence interval for $R_{\mathbf{s},\mathbf{k}}$ as

$$(\widehat{R}^{\text{MLE}}_{\mathbf{s},\mathbf{k}} - z_{1-\frac{\eta}{2}} \sqrt{\widehat{B}}, \ \widehat{R}^{\text{MLE}}_{\mathbf{s},\mathbf{k}} + z_{1-\frac{\eta}{2}} \sqrt{\widehat{B}}),$$

where z_η is 100η-th percentile of $N(0, 1)$.

2.3 Bayes Estimation of $R_{\mathbf{s},\mathbf{k}}$

In this section, under the squared error loss function, we infer the Bayesian estimation and corresponding credible intervals for $R_{\mathbf{s},\mathbf{k}}$, assuming that the unknown parameters are the independent gamma random variables. So, we consider the prior distributions of the parameters as

$$\begin{aligned}
\lambda_l &\sim \Gamma(a_l, b_l) : \ \pi_l(\lambda_l) \propto \lambda_l^{a_l-1} e^{-b_l \lambda_l}, \ l = 1, \ldots, m, \\
\lambda &\sim \Gamma(a_{m+1}, b_{m+1}) : \ \pi_{m+1}(\lambda) \propto \lambda^{a_{m+1}-1} e^{-b_{m+1}\lambda}, \\
\alpha &\sim \Gamma(a_{m+2}, b_{m+2}) : \ \pi_{m+2}(\alpha) \propto \alpha^{a_{m+2}-1} e^{-b_{m+2}\alpha}, \\
\beta &\sim \Gamma(a_{m+3}, b_{m+3}) : \ \pi_{m+3}(\beta) \propto \beta^{a_{m+3}-1} e^{-b_{m+3}\beta}.
\end{aligned}$$

By this selection, we write the joint posterior density function of $\lambda_1, \ \cdots, \lambda_m, \ \lambda, \ \alpha, \ \beta$ by

$$\begin{aligned}
\pi(\lambda_1, \ldots, \lambda_m, \lambda, \alpha, \beta|\text{data}) &\propto L(\lambda_1, \ldots, \lambda_m, \lambda, \alpha, \beta|\text{data}) \\
&\qquad \Big(\prod_{l=1}^{m} \pi_l(\lambda_l)\Big) \pi_{m+1}(\lambda) \pi_{m+2}(\alpha) \pi_{m+3}(\beta). \qquad (15)
\end{aligned}$$

After some calculations, we conclude that the Bayes estimations of the unknown parameters cannot be obtained in a closed form, from (15), so that we should approximate the Bayesian estimations. For this, we propose the MCMC method. For this aim, by simplifying equation (15), we can rewrite it as follows:

$$\begin{aligned}
\pi(\lambda_1, \ldots, \lambda_m, \lambda, \alpha, \beta|\text{data}) &\propto \Big(\prod_{l=1}^{m} \lambda_l^{nk_l + a_l - 1} e^{-\lambda_l \left(b_l - \alpha A_l(\alpha, \beta)\right)} \Big) \\
&\times \Big(\lambda^{n + a_{m+1} - 1} e^{-\lambda \left(b_{m+1} - \alpha B(\alpha, \beta)\right)} \Big) \\
&\times \Big(\prod_{i=1}^{n} \prod_{l=1}^{m} \prod_{j_l=1}^{k_l} \big(\frac{x_{ij_l}^{(l)}}{\alpha}\big)^{\beta-1} \Big) \times \Big(\prod_{i=1}^{n} \big(\frac{y_i}{\alpha}\big)^{\beta-1} \Big) \times \alpha^{a_{m+2}-1} e^{-b_{m+2}\alpha} \\
&\times e^{\sum_{i=1}^{n} \sum_{l=1}^{m} \sum_{j_l=1}^{k_l} \left(\frac{x_{ij_l}^{(l)}}{\alpha}\right)^{\beta} + \sum_{i=1}^{n} \left(\frac{y_i}{\alpha}\right)^{\beta}} \times \beta^{a_{m+3} + n(\sum_{l=1}^{m} k_l + 1) - 1} e^{-b_{m+3}\beta},
\end{aligned}$$

where $A_l(\cdot, \cdot), l = 1, \ldots, m$ and $B(\cdot, \cdot)$ are given in (7) and (8), respectively. So, we can obtain the posterior PDFs of the parameters as

$$\begin{aligned}
&\lambda_l|\alpha, \beta, \text{data} \sim \Gamma(nk_l + a_l, b_l - \alpha A_l(\alpha, \beta)), \ l = 1, \ldots, m, \\
&\lambda|\alpha, \beta, \text{data} \sim \Gamma(n + a_{m+1}, b_{m+1} - \alpha B(\alpha, \beta)),
\end{aligned}$$

$$\pi(\alpha|\lambda_1,\ldots,\lambda_m,\lambda,\beta,\text{data}) \propto \alpha^{a_{m+2}-1} \times \Big(\prod_{i=1}^{n}\prod_{l=1}^{m}\prod_{j_l=1}^{k_l}\big(\frac{x_{ij_l}^{(l)}}{\alpha}\big)^{\beta-1}\Big)$$
$$\times \Big(\prod_{i=1}^{n}\big(\frac{y_i}{\alpha}\big)^{\beta-1}\Big) \times e^{\sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j_l=1}^{k_l}\big(\frac{x_{ij_l}^{(l)}}{\alpha}\big)^{\beta}+\sum_{i=1}^{n}\big(\frac{y_i}{\alpha}\big)^{\beta}} \times e^{-b_{m+2}\alpha+\sum_{l=1}^{m}\lambda_l\alpha_l A_l(\alpha,\beta)+\lambda\alpha B(\alpha,\lambda)}$$
$$\pi(\beta|\lambda_1,\ldots,\lambda_m,\lambda,\alpha,\text{data}) \propto \beta^{a_{m+3}+n(\sum_{l=1}^{m}k_l+1)-1} \times \Big(\prod_{i=1}^{n}\prod_{l=1}^{m}\prod_{j_l=1}^{k_l}\big(\frac{x_{ij_l}^{(l)}}{\alpha}\big)^{\beta-1}\Big)$$
$$\times \Big(\prod_{i=1}^{n}\big(\frac{y_i}{\alpha}\big)^{\beta-1}\Big) \times e^{\sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j_l=1}^{k_l}\big(\frac{x_{ij_l}^{(l)}}{\alpha}\big)^{\beta}+\sum_{i=1}^{n}\big(\frac{y_i}{\alpha}\big)^{\beta}} \times e^{-b_{m+3}\beta+\sum_{l=1}^{m}\lambda_l\alpha_l A_l(\alpha,\beta)+\lambda\alpha B(\alpha,\lambda)}.$$

Because the posterior PDFs of $\lambda_l,\ l=1,\ldots,m$ and λ are gamma distributions, we generate random samples from them, easily. But, the posterior PDFs of α and β are not well-known distributions. So, we use Metropolis–Hastings method, to generate random samples from them. Therefore, the Gibbs sampling algorithm can be implemented by the following:

1. Begin with initial values $(\lambda_{1(0)},\ldots,\lambda_{m(0)},\ \lambda_{(0)},\ \alpha_{(0)},\ \beta_{(0)})$.
2. Set $t=1$.
3. Generate $\alpha_{(t)}$ from $\pi(\alpha|\lambda_{1(t-1)},\ldots,\lambda_{m(t-1)},\lambda_{(t-1)},\beta_{(t-1)},\text{data})$ using Metropolis–Hastings method, with $N(\alpha_{(t-1)},1)$ as proposal distribution.
4. Generate $\beta_{(t)}$ from $\pi(\beta|\lambda_{1(t-1)},\ldots,\lambda_{m(t-1)},\lambda_{(t-1)},\alpha_{(t-1)},\text{data})$ using Metropolis–Hastings method, with $N(\beta_{(t-1)},1)$ as proposal distribution.
5 : m + 4. Generate $\lambda_{l(t)}$ from $\Gamma(nk_l+a_l, b_l-\alpha_{(t-1)}A_l(\alpha_{(t-1)},\beta_{(t-1)})),\ l=1,\ldots,m$.
m + 5. Generate $\lambda_{(t)}$ from $\Gamma(n+a_{m+1}, b_{m+1}-\alpha_{(t-1)}B(\alpha_{(t-1)},\beta_{(t-1)}))$.
m + 6. Evaluate the value

$$R_{(t)\mathbf{s},\mathbf{k}} = \sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\sum_{q_1=0}^{k_1-p_1}\cdots\sum_{q_m=0}^{k_m-p_m}\Big(\prod_{l=1}^{m}\binom{k_l}{p_l}\Big)\times\Big(\prod_{l=1}^{m}\binom{k_l-p_l}{q_l}\Big)$$
$$\times(-1)^{\sum_{l=1}^{m}q_l}\frac{\lambda_{(t)}}{\sum_{l=1}^{m}\lambda_{l(t)}(p_l+q_l)+\lambda_{(t)}}.$$

m + 7. Set $t=t+1$.
m + 8. Repeat T times in Steps 3 : m+7.

Finally, we obtain the Bayesian estimation of $R_{\mathbf{s},\mathbf{k}}$ as follows:

$$\widehat{R}_{\mathbf{s},\mathbf{k}}^{\text{MC}} = \frac{1}{T}\sum_{t=1}^{T}R_{(t)\mathbf{s},\mathbf{k}}. \tag{16}$$

Moreover, a $100(1-\eta)\%$ HPD credible interval of $R_{\mathbf{s},\mathbf{k}}$ can be provided, using the idea of Chen and Shao [9] as follows. First, sort $R_{(1)\mathbf{s},\mathbf{k}},\ldots,R_{(T)\mathbf{s},\mathbf{k}}$ as $R_{((1)\mathbf{s},\mathbf{k})},\ldots,R_{((T)\mathbf{s},\mathbf{k})}$ and then construct all the $100(1-\eta)\%$ confidence intervals of $R_{\mathbf{s},\mathbf{k}}$, as:

$$(R_{((1)\mathbf{s},\mathbf{k})}, R_{(([T(1-\eta)])\mathbf{s},\mathbf{k})}),\ldots,(R_{(([T\eta])\mathbf{s},\mathbf{k})}, R_{((T)\mathbf{s},\mathbf{k})}),$$

where $[T]$ symbolizes the largest integer less than or equal to T. The HPD credible interval of $R_{\mathbf{s},\mathbf{k}}$ is the shortest length interval.

3 Inference on $R_{\mathbf{s},\mathbf{k}}$ Known Common Parameters

When the common parameters values of strengths and stress variables are known, obtaining the estimations have less computational complexity than the case which considered in previous section. Moreover, due to the diverse and nice estimations, this case is very popular with researchers.

3.1 MLE of $R_{\mathbf{s},\mathbf{k}}$

Suppose that $\{Y_1,\ldots,Y_n\}$ is progressive censoring sample from MWEx(α,β,λ) with $\{N,n,S_1,\ldots,S_n\}$ censoring scheme. Also, $\{X_{i1}^{(l)},\ldots,X_{ik_l}^{(l)}\}$, $i=1,\ldots,n$, $l=1,\ldots,m$ is progressive censoring sample from MWEx(α,β,λ_i) with $\{K_l,k_l,R_{i1}^{(l)},\ldots,R_{ik_l}^{(l)}\}$ censoring scheme, where $i=1,\ldots,n$, $l=1,\ldots,m$. Now, assuming that α and β are known, from Sect. 2.1, we obtain the MLE of $R_{\mathbf{s},\mathbf{k}}$ by

$$\begin{aligned}\widehat{R}^{\text{MLE}}_{\mathbf{s},\mathbf{k}} &= \sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\sum_{q_1=0}^{k_1-p_1}\cdots\sum_{q_m=0}^{k_m-p_m}\left(\prod_{l=1}^{m}\binom{k_l}{p_l}\right)\times\left(\prod_{l=1}^{m}\binom{k_l-p_l}{q_l}\right)\\ &\quad\times(-1)^{\sum_{l=1}^{m}q_l}\left(1+\sum_{l=1}^{m}(p_l+q_l)\frac{k_l B(\alpha,\beta)}{A_l(\alpha,\beta)}\right)^{-1}.\end{aligned}\tag{17}$$

About the asymptotic confidence interval, when the common parameters α and β are known, the Fisher information matrix can be obtained by

$$\mathbf{I}(\lambda_1,\lambda_2,\ldots,\lambda_m,\lambda)=\begin{cases}I_{ij} & i=j,\\ 0 & i\neq j,\end{cases}\quad i,j=1,\ldots,m+1.$$

As $\mathbf{I}(\lambda_1,\lambda_2,\ldots,\lambda_m,\lambda)$ is a diagonal square matrix, so, similar to Sect. 2.2, we can obtain the asymptotic distribution of $R_{\mathbf{s},\mathbf{k}}$ as $\widehat{R}^{\text{MLE}}_{\mathbf{s},\mathbf{k}}\sim N(R_{\mathbf{s},\mathbf{k}},C)$, where

$$C=\sum_{j=1}^{m}\left(\frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial\lambda_j}\right)^2\frac{1}{I_{j,j}}+\left(\frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial\lambda}\right)^2\frac{1}{I_{m+1,m+1}},$$

in which $\frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \lambda_j}$ and $\frac{\partial R_{\mathbf{s},\mathbf{k}}}{\partial \lambda}$ are given in (13) and (14), respectively. Consequently, we construct a $100(1-\eta)\%$ asymptotic confidence interval for $R_{\mathbf{s},\mathbf{k}}$ as

$$(\widehat{R}^{\text{MLE}}_{\mathbf{s},\mathbf{k}} - z_{1-\frac{\eta}{2}}\sqrt{\widehat{C}},\ \widehat{R}^{\text{MLE}}_{\mathbf{s},\mathbf{k}} + z_{1-\frac{\eta}{2}}\sqrt{\widehat{C}}),$$

where z_η is 100η-th percentile of $N(0, 1)$.

3.2 UMVUE of $R_{\mathbf{s},\mathbf{k}}$

Suppose that $\{Y_1, \ldots, Y_n\}$ is progressive censoring sample from MWEx(α, β, λ) with $\{N, n, S_1, \ldots, S_n\}$ censoring scheme. Also, $\{X^{(l)}_{i1}, \ldots, X^{(l)}_{ik_l}\}$, $i = 1, \ldots, n$, $l = 1, \ldots, m$ is progressive censoring sample from MWEx$(\alpha, \beta, \lambda_i)$ with $\{K_l, k_l, R^{(l)}_{i1}, \ldots, R^{(l)}_{ik_l}\}$ censoring scheme, where $i = 1, \ldots, n$, $l = 1, \ldots, m$. Now, assuming that α and β are known, we write the likelihood function by

$$L(\lambda_1, \ldots, \lambda_m, \lambda, \alpha, \beta|\text{data}) \propto \Big(\prod_{l=1}^{m} \lambda_l^{nk_l}\Big)\beta^{n(\sum_{l=1}^{m} k_l+1)} \lambda^n \times \Big(\prod_{i=1}^{n}\prod_{l=1}^{m}\prod_{j_l=1}^{k_l} (\frac{x^{(l)}_{ij_l}}{\alpha})^{\beta-1}\Big)$$
$$\times \Big(\prod_{i=1}^{n} (\frac{y_i}{\alpha})^{\beta-1}\Big) \times e^{\sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j_l=1}^{k_l}\left(\frac{x^{(l)}_{ij_l}}{\alpha}\right)^{\beta} + \sum_{i=1}^{n}\left(\frac{y_i}{\alpha}\right)^{\beta}} \times e^{\alpha\left(\sum_{l=1}^{m} \lambda_l A_l(\alpha,\beta) + \lambda B(\alpha,\lambda)\right)}, \quad (18)$$

where $A_l(\alpha, \beta)$, $l = 1, \ldots, m$ and $B(\alpha, \beta)$ are given in (7) and (8), respectively. We note that, from (18), when the parameters α and β are known, $A_l(\alpha, \beta)$, $l = 1, \ldots, m$ and $B(\alpha, \beta)$ are complete sufficient statistics for λ_l, $l = 1, \ldots, m$ and λ, respectively.

We obtain one progressive censoring samples from the exponential distribution with mean $\frac{1}{\lambda}$, by considering the transformation $Y_i^* = \alpha\Big(e^{\left(\frac{Y_i}{\alpha}\right)^{\beta}} - 1\Big)$, $i = 1, \ldots, n$. Now, using them, define the following variables:

$$Z_1 = NY_1^*,$$
$$Z_2 = (N - S_1 - 1)(Y_2^* - Y_1^*),$$
$$\vdots$$
$$Z_n = (N - \sum_{i=1}^{n} S_i - n + 1)(Y_n^* - Y_{n-1}^*).$$

We conclude that, from Cao and Cheng [7], $Z_1, \ldots, Z_n$ are independent and identically distributed, with mean of $\frac{1}{\lambda}$, from the exponential distribution and so, $B(\alpha, \beta) = \sum_{i=1}^{n} Z_i$ follow one gamma distribution with parameters n and λ, symbolically $B(\alpha, \beta) \sim \Gamma(n, \lambda)$.

Lemma 1 *Let* $X_{ij_l}^{(l)*} = \alpha\left(e^{\left(\frac{X_{ij_l}^{(l)}}{\alpha}\right)^{\beta}} - 1\right)$, $j_l = 1, \ldots, k_l$, $l = 1, \ldots, m$, $i = 1, \ldots, n$. *The conditional PDFs of* Y_1^* *given* $B(\alpha, \beta) = b$, $X_{11}^{(l)*}$ *given* $A_l(\alpha, \beta) = a_l$ *are, respectively, as follows:*

$$f_{Y_1^*|B(\alpha,\beta)=b}(y) = N(n-1)\frac{(b-Ny)^{n-2}}{b^{n-1}}, \quad 0 < y < b/N,$$
$$f_{X_{11}^{(l)*}|A_l(\alpha,\beta)=a_l}(x_l) = K_l(nk_l-1)\frac{(a_l-K_l x_l)^{nk_l-2}}{a^{nk_l-1}}, \quad 0 < x_l < a_l/K_l,\ l = 1, \ldots, m.$$

Proof Just like the method provided in [15], the lemma can be proved. □

Theorem 1 *Applying the complete sufficient statistics of* $A_l(\alpha, \beta)$, $l = 1, \ldots, m$ *and* $B(\alpha, \beta)$ *for* λ_l, $l = 1, \ldots, m$ *and* λ, *respectively, we obtain the UMVUE of* $\psi(\lambda_1, \ldots, \lambda_m, \lambda) = \frac{\lambda}{\sum_{l=1}^{m}(p_l+q_l)\lambda_l+\lambda}$, *which presented by* $\widehat{\psi}_U(\lambda_1, \ldots, \lambda_m, \lambda)$, *is as*

$$\begin{cases}
\sum_{j_1=0}^{nk_1-1}\cdots\sum_{j_m=0}^{nk_m-1}(-1)^{\sum_{l=1}^{m} j_l}\left(\prod_{l=1}^{m}\left(\frac{p_l+q_l}{a_l}\right)^{j_l}\right)\times\frac{\binom{nk_1-1}{j_1}\cdots\binom{nk_m-1}{j_m}}{\binom{n+\sum_{l=1}^{m} j_l-1}{\sum_{l=1}^{m} j_l}} & Case\ I, \\
\frac{n-1}{nk_1}\sum_{j_1=0}^{n-2}\sum_{j_2=0}^{nk_2-1}\cdots\sum_{j_m=0}^{nk_m-1}(-1)^{\sum_{l=1}^{m} j_l}\left(\frac{a_1}{p_1+q_1}\right)^{\sum_{l=1}^{m} j_l+1}\left(\prod_{\substack{l=1\\l\neq 1}}^{m}\left(\frac{p_l+q_l}{a_l}\right)^{j_l}\right) & \\
\qquad\times\frac{\binom{n-2}{j_1}\binom{nk_2-1}{j_2}\cdots\binom{nk_m-1}{j_m}}{\binom{nk_1+\sum_{l=1}^{m} j_l}{\sum_{l=1}^{m} j_l}} & Case\ II, \\
\frac{n-1}{nk_2}\sum_{j_1=0}^{nk_1-1}\sum_{j_2=0}^{n-2}\sum_{j_3=0}^{nk_3-1}\cdots\sum_{j_m=0}^{nk_m-1}(-1)^{\sum_{l=1}^{m} j_l}\left(\frac{a_2}{p_2+q_2}\right)^{\sum_{l=1}^{m} j_l+1}\left(\prod_{\substack{l=1\\l\neq 2}}^{m}\left(\frac{p_l+q_l}{a_l}\right)^{j_l}\right) & \\
\qquad\times\frac{\binom{nk_1-1}{j_1}\binom{n-2}{j_2}\binom{nk_3-1}{j_3}\cdots\binom{nk_m-1}{j_m}}{\binom{nk_2+\sum_{l=1}^{m} j_l}{\sum_{l=1}^{m} j_l}} & Case\ III, \\
\vdots & \\
\frac{n-1}{nk_m}\sum_{j_1=0}^{nk_1-1}\cdots\sum_{j_{m-1}=0}^{nk_{m-1}-1}\sum_{j_m=0}^{nk_m-1}(-1)^{\sum_{l=1}^{m} j_l}\left(\frac{a_m}{p_m+q_m}\right)^{\sum_{l=1}^{m} j_l+1}\left(\prod_{\substack{l=1\\l\neq m}}^{m}\left(\frac{p_l+q_l}{a_l}\right)^{j_l}\right) & \\
\qquad\times\frac{\binom{nk_1-1}{j_1}\cdots\binom{nk_{m-1}-1}{j_{m-1}}\binom{n-2}{j_m}}{\binom{nk_m+\sum_{l=1}^{m} j_l}{\sum_{l=1}^{m} j_l}} & Case m+1,
\end{cases}$$

where

$$\textit{Case I: } \frac{b}{N} < \min\left\{\frac{a_l}{(p_l+q_l)N},\ l=1,\ldots,m\right\},$$
$$\textit{Case II - Case m+1: } \frac{a_l}{(p_l+q_l)N}$$
$$< \min\left\{\frac{b}{N}, \frac{a_j}{(p_j+q_j)N},\ j\neq l,\ j=1,\ldots,m\right\},\ l=2,\ldots,m+1.$$

Proof We can see easily that Y_1^* follows an exponential distribution with mean $\frac{1}{\lambda N}$ and $X_{11}^{(l)*}, l=1,\ldots,m$ follow exponential distributions with mean $\frac{1}{\lambda_l K_l}, l=1,\ldots,m$, respectively. So,

$$\phi(X_{11}^{(1)*},\cdots,X_{11}^{(m)*},Y_1^*) = \begin{cases} 1 & K_l X_{11}^{(l)*} > N(p_l+q_l)Y_1^*,\ l=1,\ldots,m \\ 0 & \text{Otherwise}, \end{cases}$$

is an unbiased estimation of $\psi(\lambda_1,\ldots,\lambda_m,\lambda)$. Now, using the Rao–Blackwell theorem, we have

$$\begin{aligned}\widehat{\psi}_U(\lambda_1,\ldots,\lambda_m,\lambda) &= E\Big(\phi(X_{11}^{(1)*},\ldots,X_{11}^{(m)*},Y_1^*)|A_1(\alpha,\beta)\\ &= a_1,\ldots,A_m(\alpha,\beta)=a_m, B(\alpha,\beta)=b\Big)\\ &= \int\int_{\mathcal{A}}\cdots\int f_{X_{11}^{(1)*}|A_1(\alpha,\beta)=a_1}(x_1)\cdots f_{X_{11}^{(m)*}|A_m(\alpha,\beta)=a_m}(x_m) f_{Y_1^*|B(\alpha,\beta)=b}(y)\mathrm{d}x_1\cdots\mathrm{d}x_m\mathrm{d}y,\end{aligned}$$

where

$$\mathcal{A} = \left\{(x_1,\ldots,x_m,y): 0<y<\frac{b}{N}, 0<x_l<\frac{a_l}{K_l}, K_l x_l > N(p_l+q_l)y, l=1,\ldots,m\right\},$$

and the functions under integral are given in Lemma 1. We continue the proof for Case I as

$$\begin{aligned}\widehat{\psi}_U(\lambda_1,\ldots,\lambda_m,\lambda) &= \int_0^{\frac{b}{N}}\int_{\frac{(p_m+q_m)Ny}{K_m}}^{\frac{a_m}{K_m}}\cdots\int_{\frac{(p_1+q_1)Ny}{K_1}}^{\frac{a_1}{K_1}} \frac{K_1(nk_1-1)(a_1-K_1x_1)^{nk_1-2}}{a_1^{nk_1-2}}\times\cdots\\ &\times\frac{K_m(nk_m-1)(a_m-K_mx_m)^{nk_m-2}}{a_m^{nk_m-1}}\times\frac{N(n-1)(b-Ny)^{n-2}}{b^{n-1}}\mathrm{d}x_1\cdots\mathrm{d}x_m\mathrm{d}y\\ &= \int_0^{\frac{b}{N}}\Big(\int_{\frac{(p_m+q_m)Ny}{K_m}}^{\frac{a_m}{K_m}}\frac{K_m(nk_m-1)(a_m-K_mx_m)^{nk_m-2}}{a_m^{nk_m-1}}\mathrm{d}x_m\Big)\\ &\times\cdots\times\Big(\int_{\frac{(p_1+q_1)Ny}{K_1}}^{\frac{a_1}{K_1}}\frac{K_1(nk_1-1)(a_1-K_1x_1)^{nk_1-2}}{a_1^{nk_1-2}}\mathrm{d}x_1\Big)\\ &\times\frac{N(n-1)(b-Ny)^{n-2}}{b^{n-1}}\mathrm{d}y\end{aligned}$$

$$= \frac{N(n-1)}{b}\int_0^{\frac{b}{N}}\Big(1-(p_m+q_m)\frac{N}{a_m}y\Big)^{nk_m-1}\times\cdots\times\Big(1-(p_1+q_1)\frac{N}{a_1}y\Big)^{nk_1-1}$$
$$\times\Big(1-\frac{N}{b}y\Big)^{n-2}\mathrm{d}y\quad\Big\{\text{Put: } t=\frac{Ny}{b}\Big\}$$
$$=(n-1)\int_0^1(1-t)^{n-2}\Big(1-(p_m+q_m)\frac{b}{a_m}t\Big)^{nk_m-1}\Big(1-(p_1+q_1)\frac{b}{a_1}t\Big)^{nk_1-1}\mathrm{d}t$$
$$=(n-1)\int_0^1(1-t)^{n-2}\Big(\sum_{j_m=0}^{nk_m-1}(-1)^{j_m}\binom{nk_m-1}{j_m}\Big((p_m+q_m)\frac{b}{a_m}t\Big)^{j_m}\Big)\times\cdots$$
$$\times\Big(\sum_{j_1=0}^{nk_1-1}(-1)^{j_1}\binom{nk_1-1}{j_1}\Big((p_1+q_1)\frac{b}{a_1}t\Big)^{j_1}\Big)\mathrm{d}t$$
$$=\sum_{j_1=0}^{nk_1-1}\cdots\sum_{j_m=0}^{nk_m-1}(-1)^{\sum_{l=1}^{m}j_l}\Big(\prod_{l=1}^{m}\big(\frac{p_l+q_l}{a_l}\big)^{j_l}\Big)\times\frac{\binom{nk_1-1}{j_1}\cdots\binom{nk_m-1}{j_m}}{\binom{n+\sum_{l=1}^{m}j_l-1}{\sum_{l=1}^{m}j_l}}.$$

For other cases, the results can be obtained in a similar methods. □

So, the UMVUE of $R_{\mathbf{s},\mathbf{k}}$ presented by $\widehat{R}^U_{\mathbf{s},\mathbf{k}}$ can be obtained by

$$\widehat{R}^U_{\mathbf{s},\mathbf{k}}=\sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\sum_{q_1=0}^{k_1-p_1}\cdots\sum_{q_m=0}^{k_m-p_m}\Big(\prod_{l=1}^{m}\binom{k_l}{p_l}\Big)\times\Big(\prod_{l=1}^{m}\binom{k_l-p_l}{q_l}\Big)$$
$$\times(-1)^{\sum_{l=1}^{m}q_l}\,\widehat{\psi}_U(\lambda_1,\ldots,\lambda_m,\lambda). \tag{19}$$

3.3 Bayes Estimation of $R_{\mathbf{s},\mathbf{k}}$

In this section, under the squared error loss function, we infer the Bayesian estimation and corresponding credible intervals for $R_{\mathbf{s},\mathbf{k}}$, assuming that the unknown parameters are the independent gamma random variables. So, we consider the prior distributions of the parameters as

$$\lambda_l\sim\Gamma(a_l,b_l):\ \pi_l(\lambda_l)\propto\lambda_l^{a_l-1}e^{-b_l\lambda_l},\ l=1,\ldots,m,$$
$$\lambda\sim\Gamma(a_{m+1},b_{m+1}):\ \pi_{m+1}(\lambda)\propto\lambda^{a_{m+1}-1}e^{-b_{m+1}\lambda}.$$

By this selection, we can write the joint posterior density function of $\lambda_1,\ \cdots,\lambda_m,\ \lambda$ by

$$\pi(\lambda_1,\ldots,\lambda_m,\lambda|\alpha,\beta,\text{data})=\frac{\Big(\prod_{l=1}^{m}\mu_l^{\nu_l}\Big)\mu^{\nu}}{\Big(\prod_{l=1}^{m}\Gamma(\nu_l)\Big)\Gamma(\nu)}\times\Big(\prod_{l=1}^{m}\lambda_l^{\nu_l-1}\Big)\lambda^{\nu-1}e^{-\sum_{l=1}^{m}\lambda_l\mu_l-\lambda\mu}, \tag{20}$$

where

$$\mu_l = b_l - \sum_{i=1}^{n}\sum_{j_l=1}^{k_l}\left(R_{ij_l}^{(l)}+1\right)\left(1-e^{\left(\frac{x_{ij_l}^{(l)}}{\alpha}\right)^{\beta}}\right),\ l=1,\ldots,m, \quad \mu = b_{m+1}$$
$$-\sum_{i=1}^{n}\left(S_i+1\right)\left(1-e^{\left(\frac{y_i}{\alpha}\right)^{\beta}}\right),$$
$$\nu_l = nk_l + a_l, \quad \nu = n + a_{m+1}.$$

We obtain the Bayes estimation of $R_{\mathbf{s},\mathbf{k}}$ under the squared error loss function, by solving the following multiple integral

$$\begin{aligned}\widehat{R}_{\mathbf{s},\mathbf{k}}^{B} &= \int_0^\infty\int_0^\infty\cdots\int_0^\infty R_{\mathbf{s},\mathbf{k}}\pi(\lambda_1,\ldots,\lambda_m,\lambda|\alpha,\beta,\text{data})\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_m\mathrm{d}\lambda \\ &= \sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\sum_{q_1=0}^{k_1-p_1}\cdots\sum_{q_m=0}^{k_m-p_m}\left(\prod_{l=1}^{m}\binom{k_l}{p_l}\right)\times\left(\prod_{l=1}^{m}\binom{k_l-p_l}{q_l}\right)(-1)^{\sum_{l=1}^{m}q_l} \\ &\times\int_0^\infty\int_0^\infty\cdots\int_0^\infty\frac{\lambda}{\sum_{l=1}^{m}\lambda_l(p_l+q_l)+\lambda}\pi(\lambda_1,\ldots,\lambda_m,\lambda|\alpha,\beta,\text{data})\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_m\mathrm{d}\lambda.\end{aligned} \tag{21}$$

Now, let us put the integral part in (21) by $\mathcal{B}$. So, We can simplify $\mathcal{B}$ by employing (20), as follows:

$$\mathcal{B} = \int_0^\infty\int_0^\infty\cdots\int_0^\infty \mathcal{C}_1\times\frac{\left(\prod_{l=1}^{m}\lambda_l^{\nu_l-1}\right)\lambda^{\nu}}{\sum_{l=1}^{m}\lambda_l(p_l+q_l)+\lambda}\times e^{-\sum_{l=1}^{m}\lambda_l\mu_l-\lambda\mu}\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_m\mathrm{d}\lambda,$$

where $\mathcal{C}_1 = \frac{\left(\prod_{l=1}^{m}\mu_l^{\nu_l}\right)\mu^{\nu}}{\left(\prod_{l=1}^{m}\Gamma(\nu_l)\right)\Gamma(\nu)}$. Now, define the following variables:

$$\left.\begin{aligned} \theta_1 &= \frac{\lambda_1(p_1+q_1)}{\sum_{l=1}^{m}\lambda_l(p_l+q_l)+\lambda}, \\ &\vdots \\ \theta_m &= \frac{\lambda_m(p_m+q_m)}{\sum_{l=1}^{m}\lambda_l(p_l+q_l)+\lambda}, \\ Z &= \sum_{l=1}^{m}\lambda_l(p_l+q_l)+\lambda. \end{aligned}\right\} \Rightarrow \begin{cases} \lambda_1 = \frac{\theta_1 z}{p_1+q_1}, \\ \vdots \\ \lambda_m = \frac{\theta_m z}{p_m+q_m}, \\ \lambda = z\big(1-\sum_{l=1}^{m}\theta_l\big). \end{cases}$$

In this case, $0 < \sum_{l=1}^{m}\theta_l < 1$, $z > 0$ and the Jacobian is

$$|J(\theta_1,\ldots,\theta_m,z)| = \begin{vmatrix} \frac{z}{p_1+q_1} & 0 & 0 & \cdots & 0 & \frac{\theta_1}{p_1+q_1} \\ 0 & \frac{z}{p_2+q_2} & 0 & \cdots & 0 & \frac{\theta_2}{p_2+q_2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{z}{p_m+q_m} & \frac{\theta_m}{p_m+q_m} \\ -z & -z & -z & \cdots & -z & 1-\sum_{l=1}^{m}\theta_l \end{vmatrix} = \frac{z^m}{\prod_{l=1}^{m}(p_l+q_l)}.$$

So, by this transformation, assuming that $\mathcal{D} = \{(\theta_1,\ldots,\theta_m) : 0 < \sum_{l=1}^{m}\theta_l < 1\}$, we can write

$$\begin{aligned} \mathcal{B} &= \int_{\mathcal{D}}\cdots\int\int_0^\infty \mathcal{C}_1 \frac{\prod_{l=1}^{m}\theta_l^{\nu_l-1}}{\prod_{l=1}^{m}(p_l+q_l)^{\nu_l}} \times z^{\sum_{l=1}^{m}\nu_l+\nu-1}\big(1-\sum_{l=1}^{m}\theta_l\big)^{\nu} \\ &\quad \times e^{-z\left(\sum_{l=1}^{m}\frac{\theta_l\mu_l}{p_l+q_l}+\left(1-\sum_{l=1}^{m}\theta_l\right)\mu\right)}\mathrm{d}z\mathrm{d}\theta_1\cdots\mathrm{d}\theta_m \\ &= \int_{\mathcal{D}}\cdots\int \mathcal{C}_1 \frac{\left(\prod_{l=1}^{m}\theta_l^{\nu_l-1}\right)\Gamma\left(\sum_{l=1}^{m}\nu_l+\nu\right)\left(1-\sum_{l=1}^{m}\theta_l\right)^{\nu}}{\left(\prod_{l=1}^{m}(p_l+q_l)^{\nu_l}\right)\left(\sum_{l=1}^{m}\frac{\theta_l\mu_l}{p_l+q_l}+\left(1-\sum_{l=1}^{m}\theta_l\right)\mu\right)^{\sum_{l=1}^{m}\nu_l+\nu}}\mathrm{d}\theta_1\cdots\mathrm{d}\theta_m \\ &= \int_{\mathcal{D}}\cdots\int \mathcal{C}_2\left(\prod_{l=1}^{m}\theta_l^{\nu_l-1}\right)\left(1-\sum_{l=1}^{m}\theta_l\right)^{\nu}\left(1-\sum_{l=1}^{m}\theta_l w_l\right)^{-\left(\sum_{l=1}^{m}\nu_l+\nu\right)}\mathrm{d}\theta_1\cdots\mathrm{d}\theta_m, \end{aligned} \tag{22}$$

where

$$\mathcal{C}_2=\frac{\left(\prod_{l=1}^{m}(1-w_l)^{\nu_l}\right)\Gamma\left(\sum_{l=1}^{m}\nu_l+\nu\right)}{\left(\prod_{l=1}^{m}\Gamma\left(\nu_l\right)\right)\Gamma(\nu)},\ w_l=1-\frac{\mu_l}{\mu(p_l+q_l)},l=1,\ldots,m.$$

The final integral, represented by (22), can be solved immediately using one numerical method in most of standard software programs. So, we can obtain the Bayes estimation of $R_{\mathbf{s},\mathbf{k}}$ as

$$\widehat{R}^B_{\mathbf{s},\mathbf{k}}=\sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\sum_{q_1=0}^{k_1-p_1}\cdots\sum_{q_m=0}^{k_m-p_m}\left(\prod_{l=1}^{m}\binom{k_l}{p_l}\right)\times\left(\prod_{l=1}^{m}\binom{k_l-p_l}{q_l}\right)(-1)^{\sum_{l=1}^{m}q_l}\times\mathcal{B}. \tag{23}$$

As obtaining the Bayes estimation from (23) needs to solve the numerical integrals, so, like as Sect. 2.3, when parameters α and β are known, we derive the posterior PDFs of the parameters as

$$\lambda_l|\alpha,\beta,\text{data}\sim\Gamma(nk_l+a_l,b_l-A_l(\alpha,\beta)),\ l=1,\ldots,m,$$
$$\lambda|\alpha,\beta,\text{data}\sim\Gamma(n+a_{m+1},b_{m+1}-B(\alpha,\beta)).$$

Now, we employ the Gibbs sampling method as follows to obtain the MCMC Bayes estimation and HPD credible intervals. So

1. Begin with initial values $(\lambda_{1(0)},\ldots,\lambda_{m(0)},\ \lambda_{(0)})$.
2. Set $t=1$.
3 : **m** + **2**. Generate $\lambda_{l(t)}$ from $\Gamma(nk_l+a_l,b_l-A_l(\alpha_{(t-1)},\beta_{(t-1)})),l=1,\ldots,m$.
m + **3**. Generate $\lambda_{(t)}$ from $\Gamma(n+a_{m+1},b_{m+1}-B(\alpha_{(t-1)},\beta_{(t-1)}))$.
m + **4**. Evaluate the value

$$R_{(t)\mathbf{s},\mathbf{k}}=\sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\sum_{q_1=0}^{k_1-p_1}\cdots\sum_{q_m=0}^{k_m-p_m}\left(\prod_{l=1}^{m}\binom{k_l}{p_l}\right)\times\left(\prod_{l=1}^{m}\binom{k_l-p_l}{q_l}\right)$$
$$\times(-1)^{\sum_{l=1}^{m}q_l}\frac{\lambda_{(t)}}{\sum_{l=1}^{m}\lambda_{l(t)}(p_l+q_l)+\lambda_{(t)}}.$$

m + **5**. Set $t=t+1$.
m + **6**. Repeat T times in Steps 3 : m+5.

Finally, we obtain the Bayesian estimation of $R_{\mathbf{s},\mathbf{k}}$ as follows:

$$\widehat{R}^{\text{MC}}_{\mathbf{s},\mathbf{k}}=\frac{1}{T}\sum_{t=1}^{T}R_{(t)\mathbf{s},\mathbf{k}}. \tag{24}$$

Table 1 Different censoring schemes

(k_i, K_i)		C.S.	(n, N)		C.S.
(5,10)	R_1	(0,0,0,0,5)		S_1	(0,0,0,0,5)
	R_2	(5,0,0,0,0)	(5,10)	S_2	(5,0,0,0,0)
	R_3	(1,1,1,1,1)		S_3	(1,1,1,1,1)
(10,20)	R_4	$(0^{*9},10)$		S_4	$(0^{*9},10)$
	R_5	$(10,0^{*9})$	(10,20)	S_5	$(10,0^{*9})$
	R_6	(1^{*10})		S_6	(1^{*10})

Moreover, just like as presented in Sect. 2.3, a $100(1-\eta)\%$ HPD credible interval of $R_{\mathbf{s},\mathbf{k}}$, can be provided.

4 Simulation Experiments

In this section, we employ the Monte Carlo simulation studies to compare the different estimations. In point estimates, comparing is done based on mean square errors (MSEs) and in interval estimates comparing is done based on average confidence lengths (AL) and coverage percentages (CP). We suppose the simulated system has two strength components so that, the stress random variable is $Y \sim \text{MWEx}(\alpha, \beta, \lambda)$ and strength components are $X_i \sim \text{MWEx}(\alpha, \beta, \lambda_i),\ i = 1, 2, 3$. It is noted that we implement simulation study for different censoring schemes presented in Table 1 and we generate 2000 samples to derive the simulation results.

When the common parameters are unknown, we obtain the simulation results based on $(\alpha, \beta) = (2, 3)$ and $(\lambda_1, \lambda_2, \lambda_3, \lambda) = (2, 3, 2, 4)$. Also, in this case, the repetition numbers in Gibbs sampling algorithm are $T = 3000$. Moreover, we employ two priors as

$$\text{Prior 1}: \ a_l = 0,\ b_l = 0,\ \text{Prior 2}: \ a_l = 0.2,\ b_l = 0.5,\ l = 1, \ldots, 6.$$

In this case, we obtain the MLE and Bayes estimate of $R_{\mathbf{s},\mathbf{k}}$ from (12) and (16), respectively. Also, we derive 95% asymptotic and HPD intervals for $R_{\mathbf{s},\mathbf{k}}$. The results are given in Table 2.

When the common parameters are known, we obtain the simulation results based on $(\lambda_1, \lambda_2, \lambda_3, \lambda) = (3, 1.5, 3, 2)$. Also, in this case, the repetition numbers in Gibbs sampling algorithm is $T = 3000$. Moreover, we employ two priors as

$$\text{Prior 3}: \ a_l = 0,\ b_l = 0,\ \text{Prior 4}: \ a_l = 0.25,\ b_l = 0.45,\ l = 1, \ldots, 4.$$

In this case, we obtain the MLE, UMVUE, exact and MCMC Bayes estimates of $R_{\mathbf{s},\mathbf{k}}$ from (17), (19), (23) and (24), respectively. Also, we derive 95% asymptotic and HPD intervals for $R_{\mathbf{s},\mathbf{k}}$. The results are given in Table 3.

The simulation study, from Tables 2 and 3, has the following conclusions:

Table 2 Results of the point and interval estimates of $R_{\mathbf{s},\mathbf{k}}$, when common parameters are unknown

$(k_1, k_2, k_3, n, s_1, s_2, s_3)$	C.S	MLE			Bayes					
					Prior 1			Prior 2		
		MSE	AL	CP	MSE	AL	CP	MSE	AL	CP
(5,5,5,5,2,2,2)	(R_1, R_1, R_1, S_1)	0.0510	0.5034	0.900	0.0423	0.4723	0.938	0.0356	0.4597	0.942
	(R_1, R_1, R_1, S_2)	0.0522	0.5047	0.905	0.0429	0.4752	0.937	0.0368	0.4528	0.941
	(R_1, R_1, R_1, S_3)	0.0542	0.5109	0.902	0.0428	0.4721	0.938	0.0366	0.4587	0.940
	(R_2, R_2, R_2, S_2)	0.0504	0.5095	0.900	0.0415	0.4735	0.937	0.0371	0.4532	0.942
	(R_2, R_2, R_2, S_3)	0.0515	0.5074	0.904	0.0423	0.4785	0.939	0.0360	0.4528	0.942
	(R_3, R_3, R_3, S_3)	0.0526	0.5032	0.903	0.0425	0.4790	0.937	0.0356	0.4509	0.940
	(R_1, R_1, R_1, S_4)	0.0378	0.4012	0.920	0.0307	0.3865	0.949	0.0256	0.3521	0.951
(5,5,5,10,2,2,2)	(R_1, R_1, R_1, S_5)	0.0354	0.4051	0.922	0.0300	0.3819	0.949	0.0259	0.3582	0.951
	(R_1, R_1, R_1, S_6)	0.0368	0.4068	0.923	0.0304	0.3879	0.948	0.0260	0.3560	0.950
	(R_2, R_2, R_2, S_5)	0.0372	0.4097	0.925	0.0310	0.3896	0.949	0.0251	0.3500	0.952
	(R_2, R_2, R_2, S_6)	0.0395	0.4023	0.920	0.3030	0.3840	0.948	0.0262	0.3574	0.951
	(R_3, R_3, R_3, S_6)	0.0347	0.4018	0.923	0.0300	0.3814	0.949	0.0277	0.3596	0.952
	(R_4, R_4, R_4, S_1)	0.0423	0.4412	0.915	0.0395	0.4190	0.943	0.0311	0.3976	0.948
(10,10,10,5,2,2,2)	(R_4, R_4, R_4, S_2)	0.0412	0.4425	0.914	0.0385	0.4112	0.942	0.0315	0.3925	0.949
	(R_4, R_4, R_4, S_3)	0.0425	0.4475	0.915	0.0390	0.4182	0.943	0.0323	0.3924	0.948
	(R_5, R_5, R_5, S_2)	0.0433	0.4444	0.916	0.0375	0.4196	0.940	0.0319	0.3975	0.949
	(R_5, R_5, R_5, S_3)	0.0410	0.4468	0.915	0.0384	0.4178	0.942	0.0309	0.3964	0.947
	(R_6, R_6, R_6, S_3)	0.0444	0.4437	0.914	0.0369	0.4108	0.943	0.0327	0.3974	0.949
	(R_4, R_4, R_4, S_4)	0.0267	0.3745	0.929	0.0205	0.3542	0.952	0.0195	0.3312	0.955

Table 2 continued

$(k_1, k_2, k_3, n, s_1, s_2, s_3)$	C.S	MLE			Bayes					
					Prior 1			Prior 2		
		MSE	AL	CP	MSE	AL	CP	MSE	AL	CP
(10,10,10,10,2,2,2)	(R_4, R_4, R_4, S_5)	0.0250	0.3758	0.930	0.0209	0.3522	0.950	0.0180	0.3333	0.955
	(R_4, R_4, R_4, S_6)	0.0278	0.3760	0.929	0.0210	0.3560	0.952	0.0186	0.3315	0.954
	(R_5, R_5, R_5, S_5)	0.0259	0.3740	0.931	0.0208	0.3514	0.951	0.0179	0.3374	0.954
	(R_5, R_5, R_5, S_6)	0.0275	0.3705	0.929	0.0200	0.3581	0.950	0.0182	0.3394	0.955
	(R_6, R_6, R_6, S_6)	0.0280	0.3714	0.930	0.0204	0.3509	0.952	0.0179	0.3360	0.954
	(R_1, R_1, R_1, S_1)	0.0585	0.5143	0.902	0.0562	0.4832	0.939	0.0546	0.4412	0.943
(5,5,5,5,4,4,4)	(R_1, R_1, R_1, S_2)	0.0596	0.5185	0.900	0.0550	0.4825	0.937	0.0540	0.4428	0.942
	(R_1, R_1, R_1, S_3)	0.0575	0.5126	0.902	0.0565	0.4896	0.939	0.0535	0.4475	0.943
	(R_2, R_2, R_2, S_2)	0.0578	0.5196	0.901	0.0561	0.4852	0.938	0.0538	0.4462	0.942
	(R_2, R_2, R_2, S_3)	0.0584	0.5108	0.900	0.0557	0.4806	0.937	0.0534	0.4485	0.943
	(R_3, R_3, R_3, S_3)	0.0570	0.5113	0.902	0.0569	0.4862	0.937	0.0549	0.4426	0.942
	(R_1, R_1, R_1, S_4)	0.0290	0.3712	0.924	0.0236	0.3510	0.949	0.0225	0.3317	0.951

Table 2 continued

$(k_1, k_2, k_3, n, s_1, s_2, s_3)$	C.S	MLE			Bayes					
					Prior 1			Prior 2		
		MSE	AL	CP	MSE	AL	CP	MSE	AL	CP
(5,5,5,10,4,4,4)	(R_1, R_1, R_1, S_5)	0.0285	0.3708	0.925	0.0240	0.3562	0.948	0.0220	0.3315	0.950
	(R_1, R_1, R_1, S_6)	0.0298	0.3725	0.926	0.0239	0.3520	0.949	0.0227	0.3362	0.950
	(R_2, R_2, R_2, S_5)	0.0281	0.3725	0.924	0.0248	0.3574	0.948	0.0229	0.3304	0.951
	(R_2, R_2, R_2, S_6)	0.0296	0.3752	0.925	0.0230	0.3590	0.949	0.0220	0.3389	0.951
	(R_3, R_3, R_3, S_6)	0.0287	0.3766	0.925	0.0244	0.3584	0.949	0.0221	0.3379	0.950
(10,10,10,5,4,4,4)	(R_4, R_4, R_4, S_1)	0.0386	0.4498	0.916	0.0351	0.4135	0.942	0.0322	0.4081	0.948
	(R_4, R_4, R_4, S_2)	0.0395	0.4452	0.917	0.0348	0.4152	0.943	0.0320	0.4025	0.949
	(R_4, R_4, R_4, S_3)	0.0380	0.493	0.916	0.0356	0.4168	0.942	0.0329	0.4068	0.948
	(R_5, R_5, R_5, S_2)	0.0388	0.4470	0.915	0.0340	0.4179	0.943	0.0330	0.4096	0.949
	(R_5, R_5, R_5, S_3)	0.0394	0.4436	0.915	0.0350	0.4127	0.942	0.0327	0.4051	0.949
	(R_6, R_6, R_6, S_3)	0.0380	0.4412	0.915	0.0359	0.4169	0.942	0.0325	0.4116	0.948
(10,10,10,10,4,4,4)	(R_4, R_4, R_4, S_4)	0.0168	0.3043	0.930	0.0110	0.2801	0.952	0.0084	0.2512	0.954
	(R_4, R_4, R_4, S_5)	0.0160	0.3014	0.932	0.0100	0.2851	0.950	0.0080	0.2510	0.955
	(R_4, R_4, R_4, S_6)	0.0175	0.3068	0.935	0.0109	0.2894	0.951	0.0089	0.2561	0.954
	(R_5, R_5, R_5, S_5)	0.162	0.3072	0.934	0.0115	0.2864	0.952	0.0078	0.2532	0.954
	(R_5, R_5, R_5, S_6)	0.0165	0.3095	0.930	0.0117	0.2847	0.952	0.0089	0.2519	0.955
	(R_6, R_6, R_6, S_6)	0.0170	0.3045	0.932	0.0107	0.2815	0.952	0.0091	0.2534	0.954

Table 3 Results of the point and interval estimates of $R_{\mathbf{s},\mathbf{k}}$, when common parameters are known

$(k_1, k_2, k_3, n, s_1, s_2, s_3)$					Bayes								
					Prior 1				Prior 2				
		MLE			MCMC			Exact	MCMC			Exact	UMVUE
	C.S	MSE	AL	CP	MSE	AL	CP	MSE	MSE	AL	CP	MSE	MSE
(5,5,5,5,2,2,2)	(R_1, R_1, R_1, S_1)	0.0423	0.5534	0.904	0.0394	0.5032	0.930	0.0401	0.0380	0.4794	0.942	0.0390	0.0498
	(R_1, R_1, R_1, S_2)	0.0425	0.5521	0.905	0.0399	0.5026	0.932	0.0402	0.0385	0.4752	0.944	0.0392	0.0488
	(R_1, R_1, R_1, S_3)	0.0430	0.5565	0.904	0.0390	0.5047	0.930	0.0405	0.0379	0.4725	0.942	0.0395	0.0492
	(R_2, R_2, R_2, S_2)	0.0452	0.5541	0.904	0.0395	0.5068	0.931	0.0400	0.0375	0.4735	0.940	0.0390	0.0483
	(R_2, R_2, R_2, S_3)	0.0453	0.5585	0.906	0.0399	0.5098	0.931	0.0408	0.0385	0.4722	0.942	0.0396	0.0480
	(R_3, R_3, R_3, S_3)	0.0432	0.5534	0.904	0.0394	0.5067	0.930	0.0402	0.0374	0.4761	0.943	0.0394	0.0493
	(R_1, R_1, R_1, S_4)	0.0275	0.4598	0.919	0.0244	0.4201	0.945	0.0262	0.0210	0.4036	0.948	0.0255	0.0301
(5,5,5,10,2,2,2)	(R_1, R_1, R_1, S_5)	0.0273	0.4582	0.920	0.0240	0.4254	0.944	0.0265	0.0209	0.4020	0.949	0.0250	0.0309
	(R_1, R_1, R_1, S_6)	0.0276	0.4562	0.918	0.0245	0.4284	0.945	0.0260	0.0211	0.4036	0.948	0.0249	0.0300
	(R_2, R_2, R_2, S_5)	0.0275	0.4521	0.919	0.0243	0.4261	0.944	0.0263	0.0205	0.4068	0.949	0.0256	0.0304
	(R_2, R_2, R_2, S_6)	0.0278	0.4565	0.918	0.0242	0.4280	0.945	0.0267	0.0208	0.4025	0.948	0.0250	0.0305
	(R_3, R_3, R_3, S_6)	0.0270	0.4532	0.920	0.0246	0.4267	0.945	0.0260	0.0210	0.4085	0.948	0.0253	0.0309
	(R_4, R_4, R_4, S_1)	0.0364	0.5032	0.913	0.0300	0.4777	0.940	0.0333	0.0282	0.4432	0.945	0.0311	0.0423
(10,10,10,5,2,2,2)	(R_4, R_4, R_4, S_2)	0.0360	0.5065	0.915	0.0309	0.4720	0.939	0.0335	0.0283	0.4485	0.946	0.0315	0.0432
	(R_4, R_4, R_4, S_3)	0.0367	0.5087	0.913	0.0295	0.4761	0.942	0.0339	0.0285	0.4401	0.945	0.0310	0.0435
	(R_5, R_5, R_5, S_2)	0.0369	0.5037	0.914	0.0299	0.4715	0.940	0.0337	0.0281	0.4462	0.947	0.0314	0.0430
	(R_5, R_5, R_5, S_3)	0.0360	0.5068	0.915	0.0300	0.4763	0.940	0.0340	0.0289	0.4485	0.945	0.0316	0.0425
	(R_6, R_6, R_6, S_3)	0.0364	0.5094	0.914	0.0305	0.4765	0.942	0.0328	0.0286	0.4430	0.946	0.0319	0.0429
	(R_4, R_4, R_4, S_4)	0.0128	0.4030	0.927	0.0110	0.3662	0.947	0.0120	0.0100	0.3008	0.951	0.0114	0.0157

Table 3 continued

$(k_1, k_2, k_3, n, s_1, s_2, s_3)$		Bayes											
					Prior 1				Prior 2				
		MLE			MCMC			Exact	MCMC			Exact	UMVUE
	C.S	MSE	AL	CP	MSE	AL	CP	MSE	MSE	AL	CP	MSE	MSE
(10,10,10,10,2,2,2)	(R_4, R_4, R_4, S_5)	0.0129	0.4098	0.929	0.0115	0.3649	0.948	0.0122	0.0095	0.3001	0.952	0.0115	0.0160
	(R_4, R_4, R_4, S_6)	0.0130	0.4085	0.927	0.0119	0.3625	0.947	0.0125	0.0100	0.3065	0.95	0.0118	0.0163
	(R_5, R_5, R_5, S_5)	0.0128	0.4068	0.928	0.0110	0.3615	0.948	0.0123	0.0099	0.3042	0.951	0.0114	0.0157
	(R_5, R_5, R_5, S_6)	0.0129	0.4032	0.928	0.0114	0.3691	0.948	0.0120	0.0105	0.3068	0.952	0.0116	0.0153
	(R_6, R_6, R_6, S_6)	0.0128	0.4028	0.929	0.0110	0.3684	0.948	0.0124	0.0103	0.3074	0.951	0.0115	0.0150
	(R_1, R_1, R_1, S_1)	0.0410	0.5210	0.905	0.0395	0.4983	0.933	0.0401	0.0351	0.4592	0.942	0.0378	0.0511
(5,5,5,5,4,4,4)	(R_1, R_1, R_1, S_2)	0.0415	0.5252	0904	0.0390	0.4965	0.935	0.0400	0.0352	0.4516	0.942	0.0380	0.0510
	(R_1, R_1, R_1, S_3)	0.0412	0.5064	0.905	00398	0.4949	0.935	0.0403	0.0356	0.4597	0.943	0.0379	0.0515
	(R_2, R_2, R_2, S_2)	0.0410	0.5097	0.906	0.0396	0.4974	0.933	0.0409	0.0350	0.4572	0.942	0.0375	0.0513
	(R_2, R_2, R_2, S_3)	0.0418	0.5238	0.905	0.0398	0.4925	0.934	0.0407	0.0357	0.4535	0.943	0.0376	0.0510
	(R_3, R_3, R_3, S_3)	0.0416	0.5274	0.906	0.0395	0.4953	0.933	0.0406	0.0354	0.4502	0.942	0.0378	0.0514
	(R_1, R_1, R_1, S_4)	0.0202	0.4439	0.923	0.0156	0.4105	0.944	0.0184	0.0149	0.3759	0.949	0.0170	0.0296

Table 3 continued

$(k_1, k_2, k_3, n, s_1, s_2, s_3)$		Bayes											
					Prior 1				Prior 2				
		MLE			MCMC			Exact	MCMC			Exact	UMVUE
	C.S	MSE	AL	CP	MSE	AL	CP	MSE	MSE	AL	CP	MSE	MSE
(5,5,5,10,4,4,4)	(R_1, R_1, R_1, S_5)	0.0200	0.4496	0.925	0.0153	0.4111	0.945	0.0189	0.0140	0.3795	0.948	0.0172	0.0300
	(R_1, R_1, R_1, S_6)	0.0205	0.4435	0.923	0.0155	0.4168	0.944	0.0190	0.0145	0.3715	0.948	0.0170	0.0305
	(R_2, R_2, R_2, S_5)	0.0207	0.4408	0.925	0.0150	0.4128	0.946	0.0183	0.0146	0.3762	0.948	0.0179	0.0295
	(R_2, R_2, R_2, S_6)	0.0208	0.4415	0.924	0.0157	0.4109	0.944	0.0180	0.0140	0.3750	0.949	0.0176	0.0294
	(R_3, R_3, R_3, S_6)	0.0206	0.4496	0.923	0.0156	0.4137	0.945	0.0186	0.0146	0.3749	0.948	0.0173	0.0301
	(R_4, R_4, R_4, S_1)	0.0298	0.4756	0.915	0.0255	0.4365	0.941	0.0279	0.0200	0.4022	0.945	0.0260	0.0377
(10,10,10,5,4,4,4)	(R_4, R_4, R_4, S_2)	0.0290	0.4723	0.917	0.0253	0.4312	0.940	0.0280	0.0195	0.4052	0.946	0.0262	0.0380
	(R_4, R_4, R_4, S_3)	0.0294	0.4760	0.916	0.0250	0.4396	0.943	0.0275	0.0199	0.4067	0.945	0.0265	0.0370
	(R_5, R_5, R_5, S_2)	0.0296	0.4751	0.916	0.0255	0.4385	0.942	0.0270	0.0205	0.4062	0.946	0.0260	0.0379
	(R_5, R_5, R_5, S_3)	0.0296	0.4725	0.917	0.0257	0.4394	0.942	0.0276	0.0195	0.4069	0.944	0.0268	0.0372
	(R_6, R_6, R_6, S_3)	0.0298	0.4763	0.916	0.0260	0.4319	0.940	0.0278	0.0190	0.4038	0.946	0.0260	0.0382
	(R_4, R_4, R_4, S_4)	0.0103	0.3888	0.931	0.0083	0.3376	0.949	0.0094	0.0067	0.2986	0.951	0.0080	0.0123
(10,10,10,10,4,4,4)	(R_4, R_4, R_4, S_5)	0.0100	0.3855	0.930	0.0085	0.3364	0.949	0.0095	0.0070	0.2929	0.952	0.0075	0.0125
	(R_4, R_4, R_4, S_6)	0.0105	0.3880	0.932	0.0080	0.3375	0.950	0.0096	0.0065	0.2915	0.952	0.0070	0.0130
	(R_5, R_5, R_5, S_5)	0.0107	0.3896	0.932	0.0086	0.3315	0.949	0.0097	0.0063	0.2947	0.952	0.0085	0.0129
	(R_5, R_5, R_5, S_6)	0.0100	0.3842	0.930	0.0080	0.3392	0.948	0.0097	0.0060	0.2968	0.952	0.0086	0.0128
	(R_6, R_6, R_6, S_6)	0.0105	0.3864	0.930	0.0084	0.3308	0.950	0.0095	0.0068	0.2943	0.951	0.0089	0.0122

- In comparison with point estimations, Bayes estimations perform better than the others and in the Bayes estimates, informative priors perform than the non-informative ones, based on MSEs.
- In comparison with interval estimations, Bayes estimations have better performance than the others and in the Bayes estimates, informative priors have better performance than the non-informative ones, based on ALs and CPs.
- By increasing n, for fixed **s** and **k**, MSEs and ALs decrease and CPs increase.
- By increasing in **k**, for fixed **s** and n, MSEs and ALs decrease and CPs increase.

The two last conclusions may occur due to the fact that by increasing the sample sizes, more information is gathered.

5 General Case

In analyzing of the real data set, researcher usually faces the general case, when the common parameters are different. So, studying this case is very important. On the other hand, some formulas in Sect. 2 can be obtained from this case.

5.1 MLE of $R_{\mathbf{s},\mathbf{k}}$

We suppose that $X_1 \sim \text{MWEx}(\alpha_1, \beta_1, \lambda_1)$, $X_2 \sim \text{MWEx}(\alpha_2, \beta_2, \lambda_2)$, $\cdots$, $X_m \sim \text{MWEx}(\alpha_m, \beta_m, \lambda_m)$ and $Y \sim \text{MWEx}(\alpha, \beta, \lambda)$ are independent random variables. Using the equation (1) and (2), we can obtain the multi-component reliability with nonidentical-component strengths in (4) as follows:

$$R_{\mathbf{s},\mathbf{k}} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \left(\prod_{l=1}^{m} \binom{k_l}{p_l} \right) \int_0^\infty e^{\sum_{l=1}^{m} \lambda_l p_l \alpha_l (1-e^{(\frac{y}{\alpha_l})^{\beta_l}})} \times \prod_{l=1}^{m} \left(1 - e^{\lambda_l \alpha_l (1-e^{(\frac{y}{\alpha_l})^{\beta_l}})} \right)^{k_l - p_l} \lambda\beta(\frac{y}{\alpha})^{\beta-1} e^{\lambda\alpha(1-e^{(\frac{y}{\alpha})^{\beta}})+(\frac{y}{\alpha})^{\beta}} dy.$$

Now, similar to Sect. 2, we can obtain the likelihood function, based on observed data, as

$$L(\lambda_1, \ldots, \lambda_m, \lambda, \alpha_1, \ldots, \alpha_m, \alpha, \beta_1, \ldots, \beta_m, \beta|\text{data}) \propto \left(\prod_{l=1}^{m} (\lambda_l \beta_l)^{nk_l} \right) (\beta\lambda)^n \times \left(\prod_{i=1}^{n} \prod_{l=1}^{m} \prod_{j_l=1}^{k_l} (\frac{x_{ij_l}^{(l)}}{\alpha_l})^{\beta_l - 1} \right) \times \left(\prod_{i=1}^{n} (\frac{y_i}{\alpha})^{\beta-1} \right) \times e^{\sum_{i=1}^{n} \sum_{l=1}^{m} \sum_{j_l=1}^{k_l} \left(\frac{x_{ij_l}^{(l)}}{\alpha_l} \right)^{\beta_l} + \sum_{i=1}^{n} \left(\frac{y_i}{\alpha} \right)^{\beta}} \times e^{\sum_{l=1}^{m} \lambda_l \alpha_l A_l(\alpha_l, \beta_l) + \alpha\lambda B(\alpha, \lambda)},$$

where $A_l(\cdot,\cdot), l = 1,\dots,m$ and $B(\cdot,\cdot)$ are given in (7) and (8), respectively. To obtain the MLEs of unknown parameters, after deriving the log-likelihood function from the above function, we should solve together the following equations:

$$\frac{\partial \ell}{\partial \lambda_l} = \frac{nk_l}{\lambda_l} + \alpha_l A_l(\alpha_l,\beta_l),\ l = 1,\dots,m, \quad \frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + \alpha B(\alpha,\beta),$$

$$\frac{\partial \ell}{\partial \beta_l} = \frac{nk_l}{\beta_l} + \sum_{i=1}^{n}\sum_{j_l=1}^{k_l} \log\left(\frac{x_{ij_l}^{(l)}}{\alpha_l}\right) + \sum_{i=1}^{n}\sum_{j_l=1}^{k_l} \left(\frac{x_{ij_l}^{(l)}}{\alpha_l}\right)^{\beta} \log\left(\frac{x_{ij_l}^{(l)}}{\alpha_l}\right)$$
$$\times\left(1-\alpha_l\lambda_l\Big(R_{ij_l}^{(l)}+1\Big)e^{\left(\frac{x_{ij_l}^{(l)}}{\alpha_l}\right)^{\beta_l}}\right),\ l=1,\dots,m,$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log\left(\frac{y_i}{\alpha}\right) + \sum_{i=1}^{n}\left(\frac{y_i}{\alpha}\right)^{\beta}\log\left(\frac{y_i}{\alpha}\right)\left(1-\alpha\lambda\Big(S_i+1\Big)e^{\left(\frac{y_i}{\alpha}\right)^{\beta}}\right)$$

$$\frac{\partial \ell}{\partial \alpha_l} = -\frac{nk_l(\beta_l-1)}{\alpha_l} - \frac{\beta_l}{\alpha_l}\sum_{i=1}^{n}\sum_{j_l=1}^{k_l}\left(\frac{x_{ij_l}^{(l)}}{\alpha_l}\right)^{\beta_l}$$
$$+\sum_{i=1}^{n}\sum_{j_l=1}^{k_l}\lambda_l\Big(R_{ij_l}^{(l)}+1\Big)\left(1-e^{\left(\frac{x_{ij_l}^{(l)}}{\alpha_l}\right)^{\beta_l}} + \beta_l\left(\frac{x_{ij_l}^{(l)}}{\alpha_l}\right)^{\beta_l}e^{\left(\frac{x_{ij_l}^{(l)}}{\alpha_l}\right)^{\beta_l}}\right),\ l=1,\dots,m,$$

$$\frac{\partial \ell}{\partial \alpha} = -\frac{n(\beta-1)}{\alpha} - \frac{\beta}{\alpha}\sum_{i=1}^{n}\left(\frac{y_i}{\alpha}\right)^{\beta}\lambda\sum_{i=1}^{n}\Big(S_i+1\Big)\left(1-e^{\left(\frac{y_i}{\alpha_l}\right)^{\beta}} + \beta\left(\frac{y_i}{\alpha}\right)^{\beta}e^{\left(\frac{y_i}{\alpha_l}\right)^{\beta}}\right)$$

The MLEs of the unknown parameters can be obtained from simultaneous solution of the above equations, using one numerical method such as Newton–Raphson algorithm. Finally, the invariance property of MLE conclude that the MLE of $R_{\mathbf{s},\mathbf{k}}$, presented by $\widehat{R}_{\mathbf{s},\mathbf{k}}^{\mathrm{MLE}}$, can be obtained as

$$\widehat{R}_{\mathbf{s},\mathbf{k}}^{\mathrm{MLE}} = \sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\left(\prod_{l=1}^{m}\binom{k_l}{p_l}\right)\int_0^\infty e^{\sum_{l=1}^{m}\widehat{\lambda}_l p_l\widehat{\alpha}_l(1-e^{(\frac{y}{\widehat{\alpha}_l})^{\widehat{\beta}_l}})}$$
$$\times\prod_{l=1}^{m}\left(1-e^{\widehat{\lambda}_l\widehat{\alpha}_l(1-e^{(\frac{y}{\widehat{\alpha}_l})^{\widehat{\beta}_l}})}\right)^{k_l-p_l}\widehat{\lambda}\widehat{\beta}\left(\frac{y}{\widehat{\alpha}}\right)^{\widehat{\beta}-1}e^{\widehat{\lambda}\widehat{\alpha}(1-e^{(\frac{y}{\widehat{\alpha}})^{\widehat{\beta}}})+(\frac{y}{\widehat{\alpha}})^{\widehat{\beta}}}\mathrm{d}y. \qquad (25)$$

5.2 Bayes Estimation of $R_{\mathbf{s},\mathbf{k}}$

In this section, under the squared error loss function, we infer the Bayesian estimation and corresponding credible intervals for $R_{\mathbf{s},\mathbf{k}}$, assuming that the unknown parameters are the independent gamma random variables. So, we consider the prior distributions of the parameters as

$$\lambda_l \sim \Gamma(a_l, b_l),\ l = 1,\dots,m, \quad \lambda \sim \Gamma(a_{m+1}, b_{m+1}),$$

$$\alpha_l \sim \Gamma(c_l, d_l),\ l = 1, \ldots, m, \quad \alpha \sim \Gamma(a_{m+2}, b_{m+2}),$$
$$\beta_l \sim \Gamma(e_l, f_l),\ l = 1, \ldots, m, \quad \beta \sim \Gamma(a_{m+3}, b_{m+3}).$$

Similar to Sect. 2.3, we can obtain the posterior PDFs of the parameters as

$$\lambda_l|\alpha_l, \beta_l, \text{data} \sim \Gamma(nk_l + a_l, b_l - \alpha_l A_l(\alpha_l, \beta_l)),\ l = 1, \ldots, m,$$
$$\lambda|\alpha, \beta, \text{data} \sim \Gamma(n + a_{m+1}, b_{m+1} - \alpha B(\alpha, \beta)),$$
$$\pi(\alpha_l|\lambda_l, \beta_l, \text{data}) \propto \alpha_l^{c_l-1}$$
$$\times \Big(\prod_{i=1}^{n}\prod_{j_l=1}^{k_l}(\frac{x_{ij_l}^{(l)}}{\alpha_l})^{\beta_l-1}\Big)e^{-d_l\alpha_l+\sum_{i=1}^{n}\sum_{j_l=1}^{k_l}\left(\frac{x_{ij_l}^{(l)}}{\alpha_l}\right)^{\beta_l}+\lambda_l\alpha_l A_l(\alpha_l,\beta_l)},\ l = 1, \ldots, m,$$
$$\pi(\alpha|\lambda, \beta, \text{data}) \propto \alpha^{a_{m+2}-1} \times \Big(\prod_{i=1}^{n}(\frac{y_i}{\alpha})^{\beta-1}\Big)e^{-b_{m+2}\alpha+\sum_{i=1}^{n}\left(\frac{y_i}{\alpha}\right)^{\beta}+\lambda\alpha B(\alpha,\beta)},$$
$$\pi(\beta_l|\lambda_l, \alpha_l, \text{data}) \propto \beta_l^{nk_l+e_l-1}$$
$$\times \Big(\prod_{i=1}^{n}\prod_{j_l=1}^{k_l}(\frac{x_{ij_l}^{(l)}}{\alpha_l})^{\beta_l-1}\Big)e^{-f_l\beta_l+\sum_{i=1}^{n}\sum_{j_l=1}^{k_l}\left(\frac{x_{ij_l}^{(l)}}{\alpha_l}\right)^{\beta_l}+\lambda_l\alpha_l A_l(\alpha_l,\beta_l)},\ l = 1, \ldots, m,$$
$$\pi(\beta|\lambda, \alpha, \text{data}) \propto \beta^{n+a_{m+3}-1} \times \Big(\prod_{i=1}^{n}(\frac{y_i}{\alpha})^{\beta-1}\Big)e^{-b_{m+3}\beta+\sum_{i=1}^{n}\left(\frac{y_i}{\alpha}\right)^{\beta}+\lambda\alpha B(\alpha,\beta)}.$$

Because the posterior PDFs of $\lambda_l,\ l = 1, \ldots, m$ and λ are gamma distributions, we generate random samples from them, easily. But, the posterior PDFs of $\alpha_l,\ \beta_l,\ l = 1, \ldots, m, \alpha$ and β are not well-known distributions. So, we use Metropolis–Hastings method, to generate random samples from them. Therefore, the Gibbs sampling algorithm can be implemented by the following:

1. Begin with initial values $(\lambda_{1(0)}, \ldots, \lambda_{m(0)}, \lambda_{(0)}, \alpha_{1(0)}, \ldots, \alpha_{m(0)}, \alpha_{(0)}, \beta_{1(0)}, \ldots, \beta_{m(0)}, \beta_{(0)})$.
2. Set $t = 1$.
$\mathbf{3 : m + 2}$. Generate $\alpha_{l(t)}$ from $\pi(\alpha_l|\lambda_{l(t-1)}, \beta_{l(t-1)}, \text{data})$ using Metropolis–Hastings method, with $N(\alpha_{l(t-1)}, 1), l = 1, \ldots, m$ as proposal distribution.
$\mathbf{m + 3}$. Generate $\alpha_{(t)}$ from $\pi(\alpha|\lambda_{(t-1)}, \beta_{(t-1)}, \text{data})$ using Metropolis–Hastings method, with $N(\alpha_{(t-1)}, 1)$ as proposal distribution.
$\mathbf{m + 4 : 2m + 3}$. Generate $\beta_{l(t)}$ from $\pi(\beta_l|\lambda_{l(t-1)}, \alpha_{l(t-1)}, \text{data})$ using Metropolis–Hastings method, with $N(\beta_{l(t-1)}, 1), l = 1, \ldots, m$ as proposal distribution.
$\mathbf{2m + 4}$. Generate $\beta_{(t)}$ from $\pi(\beta|\lambda_{(t-1)}, \alpha_{(t-1)}, \text{data})$ using Metropolis–Hastings method, with $N(\beta_{(t-1)}, 1)$ as proposal distribution.
$\mathbf{2m + 5 : 3m + 4}$. Generate $\lambda_{l(t)}$ from $\Gamma(nk_l + a_l, b_l - \alpha_{l(t-1)}A_l(\alpha_{l(t-1)}, \beta_{l(t-1)}))$, $l = 1, \ldots, m$.
$\mathbf{3m + 5}$. Generate $\lambda_{(t)}$ from $\Gamma(n + a_{m+1}, b_{m+1} - \alpha_{(t-1)}B(\alpha_{(t-1)}, \beta_{(t-1)}))$.

3m + 6. Evaluate the value

$$R_{(t)\mathbf{s},\mathbf{k}} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \left(\prod_{l=1}^{m} \binom{k_l}{p_l} \right) \int_0^\infty e^{\sum_{l=1}^{m} \lambda_{l(t)} p_l \alpha_{l(t)} \left(1-e^{\left(\frac{y}{\alpha_{l(t)}}\right)^{\beta_{l(t)}}}\right)}$$

$$\times \prod_{l=1}^{m} \left(1 - e^{\lambda_{l(t)}\alpha_{l(t)}\left(1-e^{\left(\frac{y}{\alpha_{l(t)}}\right)^{\beta_{l(t)}}}\right)}\right)^{k_l - p_l}$$

$$\lambda_{l(t)}\beta_{l(t)}\left(\frac{y}{\alpha_{l(t)}}\right)^{\beta_{l(t)}-1} e^{\lambda_{l(t)}\alpha_{l(t)}\left(1-e^{\left(\frac{y}{\alpha_{l(t)}}\right)^{\beta_{l(t)}}}\right)+\left(\frac{y}{\alpha_{l(t)}}\right)^{\beta_{l(t)}}} \mathrm{d}y.$$

3m + 7. Set $t = t + 1$.
3m + 8. Repeat T times in Steps 3 : 3m+7.

Finally, we obtain the Bayesian estimation of $R_{\mathbf{s},\mathbf{k}}$ as follows:

$$\widehat{R}_{\mathbf{s},\mathbf{k}}^{\mathrm{MC}} = \frac{1}{T}\sum_{t=1}^{T} R_{(t)\mathbf{s},\mathbf{k}}. \tag{26}$$

Moreover, just like as presented in Sect. 2.3, a $100(1-\eta)\%$ HPD credible interval of $R_{\mathbf{s},\mathbf{k}}$, can be provided.

6 Real Data Analysis

In this section, we analyze one real data set, for illustrative aims. The data which is considered in this section demonstrates the breaking strengths of jute fiber at 10 mm, 15 mm and 20 mm gauge lengths and can be founded in [27]. Recently, this data is investigated by Kang et al. [13] as a stress–strength model for exponential distribution. Now, we suppose that a system contains two different gauge lengths of jute fiber, so that the jute fiber at 10 mm and 15 mm gauge lengths is considered as the strength and the jute fiber at 10 mm gauge lengths is the stress of the system. Therefore, we set that X_1, X_2 and Y denote the jute fiber with length 10 mm, 15 mm and 20 mm, respectively. So, the observations of X_1, X_2 and Y can be considered, respectively, as follows:

$$\begin{bmatrix} 693.73 & 704.66 & 232.83 & 778.17 & 126.06 \\ 637.66 & 383.43 & 151.48 & 108.94 & 50.16 \\ 671.49 & 183.16 & 257.44 & 727.23 & 291.27 \\ 101.15 & 376.42 & 163.40 & 141.38 & 700.74 \\ 262.90 & 353.24 & 422.11 & 43.93 & 590.48 \\ 212.13 & 303.90 & 506.60 & 530.55 & 177.25 \end{bmatrix},$$

$$\begin{bmatrix} 594.40 & 202.75 & 168.37 & 574.86 & 225.65 \\ 76.38 & 156.67 & 127.81 & 813.87 & 562.39 \\ 468.47 & 135.09 & 72.24 & 497.94 & 355.56 \\ 569.07 & 640.48 & 200.76 & 550.42 & 748.75 \\ 489.66 & 678.06 & 457.71 & 106.73 & 716.30 \\ 42.66 & 80.40 & 339.22 & 70.09 & 193.42 \end{bmatrix}, \begin{bmatrix} 71.46 \\ 113.85 \\ 578.62 \\ 707.36 \\ 547.44 \\ 48.01 \end{bmatrix}.$$

To simplify calculations, we re-normalized the data on a scale of 0 to 1. We noted that this work has no effect on statistical inference. Now, first, we have fitted the MWEx distribution on the three data sets, separately and obtained the results as follows. For X_1, $\widehat{\alpha}_1 = 1.4390$, $\widehat{\beta}_1 = 1.5830$, $\widehat{\lambda}_1 = 2.8930$ and the p-value$= 0.7635$. For X_2, $\widehat{\alpha}_2 = 1.0670$, $\widehat{\beta}_2 = 1.4710$, $\widehat{\lambda}_2 = 2.1701$ and the p-value$= 0.2410$. For Y, $\widehat{\alpha} = 0.9700$, $\widehat{\beta} = 1.0810$, $\widehat{\lambda} = 1.3447$ and the p-value$= 0.4819$. From the p-values, we conclude that the MWEx distribution gives suitable fits for X_1, X_2 and Y data sets. The estimated parameters for different data sets show that only the general case can be considered for analyzing of them. For these three data sets, we provide the empirical distribution functions and PP-plots in Fig. 2.

For complete data set, putting $\mathbf{s} = (2, 2)$ and $\mathbf{k} = (5, 5)$, we obtain the MLEs of α_1, β_1, λ_1, α_2, β_2, λ_2, α, β, λ_1 and $\widehat{R}^{\text{MLE}}_{\mathbf{s},\mathbf{k}}$ by 1.4930, 1.5830, 2.8930, 1.0670, 1.4710, 2.1701, 0.9700, 1.0810, 1.3447 and 0.5463, respectively. Also, with non-informative priors, we obtain $\widehat{R}^{\text{MC}}_{\mathbf{s},\mathbf{k}}$ and the corresponding 95% HPD interval by 0.5449 and (0.2658, 0.8290), respectively.

Now, we generate two different censoring progressive scheme as follows:

$$\text{Scheme 1: } R^{(1)} = R^{(2)} = [0, 0, 1, 0], \quad S = [0, 0, 1, 0, 0], \quad (\mathbf{k} = (4, 4), \mathbf{s} = (2, 2)).$$
$$\text{Scheme 2: } R^{(1)} = R^{(2)} = [0, 1, 1], \quad S = [1, 0, 0, 1], \quad (\mathbf{k} = (3, 3), \mathbf{s} = (1, 1)).$$

For Scheme 1, we obtain the MLEs of α_1, β_1, λ_1, α_2, β_2, λ_2, α, β, λ_1 and $\widehat{R}^{\text{MLE}}_{\mathbf{s},\mathbf{k}}$ by 1.2500, 1.8520, 2.4146, 1.2600, 1.3930, 2.5659, 1.8410, 0.9270, 1.2359 and 0.4891, respectively. Also, with non-informative priors, we obtain $\widehat{R}^{\text{MC}}_{\mathbf{s},\mathbf{k}}$ and the corresponding 95% HPD interval by 0.5119 and (0.1959, 0.7893), respectively. For Scheme 2, we obtain the MLEs of $\alpha_1, \beta_1, \lambda_1, \alpha_2, \beta_2, \lambda_2, \alpha, \beta, \lambda_1$ and $\widehat{R}^{\text{MLE}}_{\mathbf{s},\mathbf{k}}$ by 0.8430, 2.6120, 0.5292, 0.8870, 1.8510, 0.7164, 0.9220, 1.1840, 1.1563 and 0.8039, respectively. Also, with non-informative priors, we obtain $\widehat{R}^{\text{MC}}_{\mathbf{s},\mathbf{k}}$ and the corresponding 95% HPD interval by 0.7417 and (0.3315, 0.9707), respectively.

To see the effect of hyper-parameters, we obtain the Bayes estimates and HPD intervals of $\widehat{R}^{\text{MC}}_{\mathbf{s},\mathbf{k}}$ with informative priors. The hyper-parameters can be obtained using re-sampling method by $a_1 = 3.48$, $b_1 = 0.91$, $a_2 = 5.04$, $b_2 = 2.01$, $a_3 = 1.33$, $b_3 = 0.83$, $c_1 = 6.94$, $d_1 = 4.21$, $c_2 = 4.37$, $d_2 = 2.34$, $c_3 = 4.42$, $d_3 = 2.46$, $e_1 = 75.41$, $f_1 = 45.92$, $e_2 = 53.35$, $f_2 = 35.19$, $e_3 = 6.33$ and $f_3 = 4.75$. So, for complete data, we obtain $\widehat{R}^{\text{MC}}_{\mathbf{s},\mathbf{k}}$ and the corresponding 95% HPD interval is equal to 0.5367 and (0.2985, 0.7402), respectively. Also, for Scheme 1, we obtain $\widehat{R}^{\text{MC}}_{\mathbf{s},\mathbf{k}}$ and the corresponding 95% HPD interval is equal to 0.5167 and (0.2286, 0.7275), respectively. Moreover, for Scheme 2, we obtain $\widehat{R}^{\text{MC}}_{\mathbf{s},\mathbf{k}}$ and the corresponding 95% HPD interval is equal to 0.7731 and (0.3791, 0.9341), respectively.

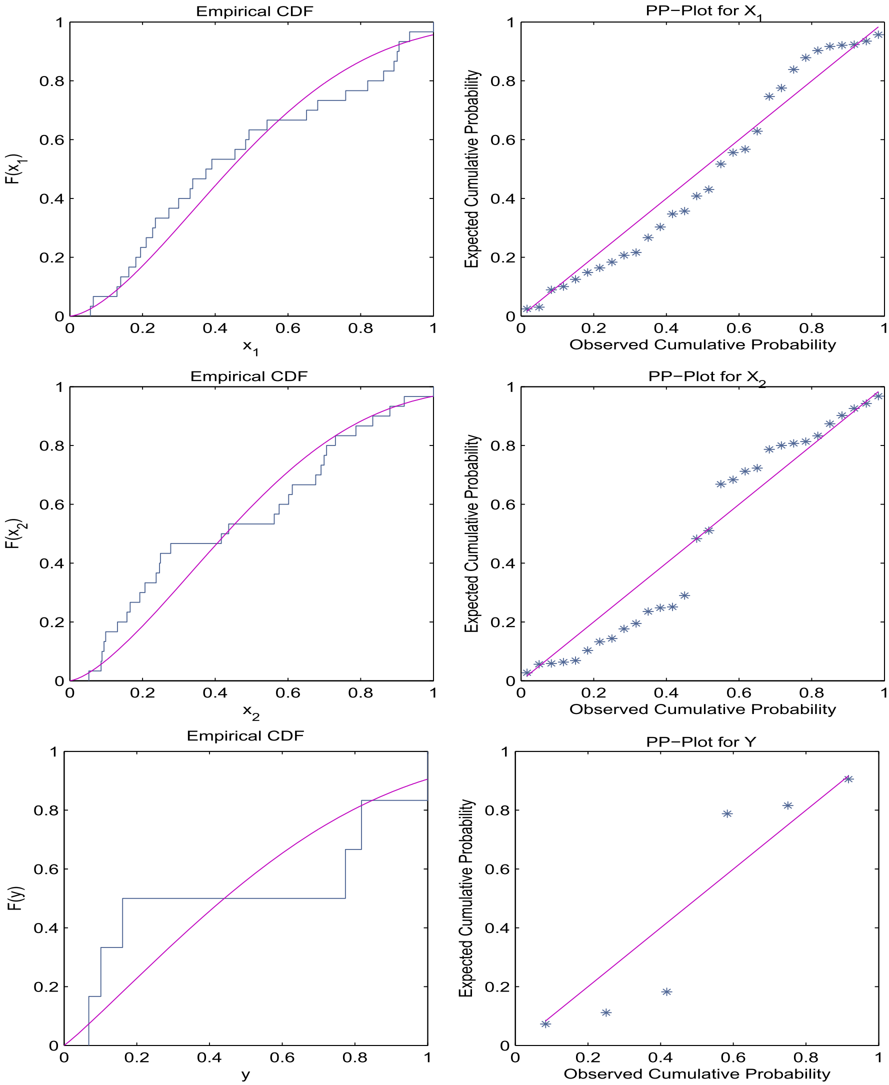

Fig. 2 Empirical distribution function (left) and the PP-plot (right) for X_1 (first row), for X_2 (middle row) and for Y (third row)

With comparing the different point and interval estimates, it seems that the estimates in Scheme 1 perform better than in Scheme 2. Moreover, we observe that HPD intervals with informative priors are smaller than the non-informative ones. So, it is reasonable that we should use the informative priors, if they are available.

7 Conclusion

In this paper, the statistical inference of multi-component stress–strength system with nonidentical-component strengths is studied, for the MWEx distribution, in the presence of the progressive censoring scheme. For this aim, we derived some point and interval estimations in classical and Bayesian inference, such as MLE, UMVUE, asymptotic and HPD intervals. Also, we considered these estimations in some cases, when the common parameters are unknown, known and in general case.

The theoretical methods are compared with Monte Carlo simulation study. The important results can be described as follows. The Bayes estimates performed better than the classical ones. Also, in Bayesian estimates, the performance of informative priors is better than the non-informative ones, in terms of point and interval estimates.

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Declarations

Conflict of interest The authors have no conflicts of interest to declare. All co-authors have seen and agreed with the contents of the manuscript. We certify that the submission is original work and is not under review at any other publication.

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