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Second Hankel Determinant of Logarithmic Coefficients of Convex and Starlike Functions of Order Alpha

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Abstract
In the present paper, we found sharp bounds of the second Hankel determinant of logarithmic coefficients of starlike and convex functions of order α.

Keywords Starlike function of order α · Convex function of order α · Carathéodory function · Hankel determinant · Logarithmic coefficient

Mathematics Subject Classification 30C45 · 30C50

1 Introduction

Let $\mathcal{H}$ be the class of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad z \in \mathbb{D}, \tag{1.1}$$

and $\mathcal{A}$ be its subclass of all f normalized by $f'(0) = 1$. By $\mathcal{S}$ we denote the subclass of $\mathcal{A}$ of univalent functions.

For $f \in \mathcal{S}$ let

$$F_f(z) := \log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n(f) z^n, \quad z \in \mathbb{D}, \ \log 1 := 0. \tag{1.2}$$

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The numbers $\gamma_n := \gamma_n(f)$ are called logarithmic coefficients of f. It is well known that the logarithmic coefficients play a crucial role in Milin conjecture ([16], see also [8, p. 155]). It is surprising that for the whole class $\mathcal{S}$ the sharp estimates of single logarithmic coefficients are known only for γ_1 and γ_2, namely

$$|\gamma_1| \le 1, \quad |\gamma_2| \le \frac{1}{2} + \frac{1}{e^2} = 0.635\ldots$$

and are unknown for $n \ge 3$. Logarithmic coefficients is one of the topic recently being of the research interest by various authors (e.g., [1,2,6,9,13,20]).

For $q, n \in \mathbb{N}$, the Hankel determinant $H_{q,n}(f)$ of $f \in \mathcal{A}$ of form (1.1) is defined as

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

Recently, many authors examined the second and the third Hankel determinants $H_{2,2}(f)$ and $H_{3,1}(f)$ over selected subclasses of $\mathcal{A}$, particularly of $\mathcal{S}$ (see e.g., [4], [11] for further references).

Based on the ideas mentioned above, in [12] was begun the research study of the Hankel determinant $H_{q,n}(F_f/2)$ which entries are logarithmic coefficients of f, i.e.,

$$H_{q,n}(F_f/2) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2(q-1)} \end{vmatrix}.$$

Due to the great importance of logarithmic coefficients, the proposed topic seems reasonable and interesting. The results which can be obtained on Hankel determinants $H_{q,n}(F_f/2)$ broaden the knowledge of logarithmic coefficients. In this paper, we continue this research dealing with $H_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2$ for the classes of starlike and convex functions of order α. Recall that $H_{2,1}(F_f/2)$ corresponds to the well-known functional $H_{2,1}(f) = a_3 - a_2^2$ over the class $\mathcal{S}$ or its subclasses. For the class $\mathcal{S}$ the functional $H_{2,1}(f)$ was estimated in 1916 by Bieberbach (see e.g., [10, Vol. I, p. 35]).

Differentiating (1.2) and using (1.1) we get

$$\gamma_1 = \frac{1}{2}a_2, \quad \gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \quad \gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right). \tag{1.3}$$

Therefore,

$$H_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2 = \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right). \tag{1.4}$$

Observe that when $f \in \mathcal{S}$, then for $f_\theta(z) := \mathrm{e}^{-\mathrm{i}\theta} f(\mathrm{e}^{\mathrm{i}\theta} z),\ \theta \in \mathbb{R}$,

$$H_{2,1}(F_{f_\theta}/2) = \frac{\mathrm{e}^{4\mathrm{i}\theta}}{4}\left(a_2 a_4 - a_3^2 + \frac{1}{12} a_2^4\right) = \mathrm{e}^{4\mathrm{i}\theta} H_{2,1}(F_f/2). \tag{1.5}$$

The main goal of this paper is to find sharp upper bounds for $H_{2,1}(F_f/2)$ in case when f is a starlike or convex function of order α. Given $\alpha \in [0, 1)$, a function $f \in \mathcal{A}$ is called starlike of order α if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathbb{D}. \tag{1.6}$$

Further, a function $f \in \mathcal{A}$ is called convex of order α if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad z \in \mathbb{D}. \tag{1.7}$$

Both classes usually denoted as $\mathcal{S}^*(\alpha)$ and $\mathcal{S}^c(\alpha)$, respectively, were introduced by Robertson [19] (e.g., [10, Vol. I, p. 138]). The classes $\mathcal{S}^*(0) =: \mathcal{S}^*$ and $\mathcal{S}^c(0) =: \mathcal{S}^c$ consist of starlike and convex functions, respectively. Let us mention that in the class $\mathcal{S}^*(\alpha)$, $|\gamma_n| \le (1-\alpha)/n$ and in the class $\mathcal{S}^c(\alpha)$, $|\gamma_n| \le (1-\eta(\alpha))/n$ for $n \in \mathbb{N}$, with sharpness (cf. [21, p. 263]), where

$$\eta(\alpha) := \begin{cases} \dfrac{1-2\alpha}{4^{1-\alpha}(1-2^{2\alpha-1})}, & \alpha \ne 1/2, \\ \dfrac{1}{2\log 2}, & \alpha = 1/2. \end{cases}$$

In particular, for $n \in \mathbb{N}$, $|\gamma_n| \le 1/n$ in the class $\mathcal{S}^*$ of starlike functions and $|\gamma_n| \le 1/(2n)$ in the class of convex functions.

In view of (1.6) and (1.7) both classes $\mathcal{S}^*(\alpha)$ and $\mathcal{S}^c(\alpha)$ have a representation with using the Carathéodory class $\mathcal{P}$, i.e., the class of analytic functions p in $\mathbb{D}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{1.8}$$

having a positive real part in $\mathbb{D}$. Therefore, the coefficients of functions in $\mathcal{S}^*(\alpha)$ and $\mathcal{S}^c(\alpha)$ have a suitable representation expressed by coefficients of functions in $\mathcal{P}$. Thus, to get the upper bound of $H_{2,1}(F_f/2)$, we based our computing on the well-known formulas on coefficient c_2 (e.g., [18, p. 166]) and the formula c_3 due to Libera and Zlotkiewicz [14,15]; cf. [17, Proposition 6]. Further remarks related to extremal functions see [5].

Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

Lemma 1.1 *If $p \in \mathcal{P}$ is of form (1.6) with $c_1 \geq 0$, then*

$$c_1 = 2\zeta_1, \tag{1.9}$$
$$c_2 = 2\zeta_1^2 + 2(1-\zeta_1^2)\zeta_2 \tag{1.10}$$

and

$$c_3 = 2\zeta_1^3 + 4(1-\zeta_1^2)\zeta_1\zeta_2 - 2(1-\zeta_1^2)\zeta_1\zeta_2^2 + 2(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_3 \tag{1.11}$$

for some $\zeta_1 \in [0, 1]$ and $\zeta_2, \zeta_3 \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| < 1\}$.

For $\zeta_1 \in \mathbb{D}$ and $\zeta_2 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with c_1 and c_2 as in (1.9)–(1.10), namely

$$p(z) = \frac{1 + \left(\overline{\zeta_1}\zeta_2 + \zeta_1\right)z + \zeta_2 z^2}{1 + \left(\overline{\zeta_1}\zeta_2 - \zeta_1\right)z - \zeta_2 z^2}, \quad z \in \mathbb{D}. \tag{1.12}$$

We will also apply the following lemma.

Lemma 1.2 (Choi et al. [7]) *Given real numbers A, B, C, let*

$$Y(A, B, C) := \max\left\{|A + Bz + Cz^2| + 1 - |z|^2 : z \in \overline{\mathbb{D}}\right\}.$$

I. *If $AC \geq 0$, then*

$$Y(A, B, C) = \begin{cases} |A| + |B| + |C|, & |B| \geq 2(1-|C|), \\ 1 + |A| + \dfrac{B^2}{4(1-|C|)}, & |B| < 2(1-|C|). \end{cases}$$

II. *If $AC < 0$, then*

$$Y(A, B, C) = \begin{cases} 1 - |A| + \dfrac{B^2}{4(1-|C|)}, & -4AC(C^{-2}-1) \leq B^2 \wedge |B| < 2(1-|C|), \\ 1 + |A| + \dfrac{B^2}{4(1+|C|)}, & B^2 < \min\left\{4(1+|C|)^2, -4AC(C^{-2}-1)\right\}, \\ R(A, B, C), & \text{otherwise}, \end{cases}$$

where

$$R(A, B, C) := \begin{cases} |A| + |B| - |C|, & |C|(|B| + 4|A|) \leq |AB|, \\ -|A| + |B| + |C|, & |AB| \leq |C|(|B| - 4|A|), \\ (|C| + |A|)\sqrt{1 - \dfrac{B^2}{4AC}}, & \text{otherwise}. \end{cases}$$

2 Starlike functions of order α

We now discuss $H_{2,1}(F_f/2)$ for the class $\mathcal{S}^*(\alpha)$.

Theorem 2.1 *Let* $\alpha \in [0, 1)$. *If* $f \in \mathcal{S}^*(\alpha)$, *then*

$$|\gamma_1\gamma_3 - \gamma_2^2| \le \frac{1}{4}(1-\alpha)^2. \tag{2.1}$$

The inequality is sharp.

Proof Fix $\alpha \in [0, 1)$ and let $f \in \mathcal{S}^*(\alpha)$ be of form (1.1). Then, by (1.6),

$$zf'(z) = ((1-\alpha)p(z) + \alpha)f(z), \quad z \in \mathbb{D}, \tag{2.2}$$

for some $p \in \mathcal{P}$ of form (1.8). Since the class $\mathcal{P}$ is invariant under the rotations and (1.5) holds, we may assume that $c_1 \in [0, 2]$ ([3], see also [10, Vol. I, p. 80, Theorem 3]), i.e., in view of (1.9) that $\zeta_1 \in [0, 1]$. Substituting series (1.1) and (1.8) into (2.2) and equating coefficients we get

$$\begin{aligned} a_2 &= (1-\alpha)c_1, \quad a_3 = \frac{1}{2}(1-\alpha)\left(c_2 + (1-\alpha)c_1^2\right), \\ a_4 &= \frac{1}{6}(1-\alpha)\left[2c_3 + 3(1-\alpha)c_1c_2 + (1-\alpha)^2c_1^3\right]. \end{aligned}$$

Hence by using (1.4) and (1.9)–(1.11) we obtain

$$\begin{aligned} \gamma_1\gamma_3 - \gamma_2^2 &= \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right) = \frac{1}{48}(1-\alpha)^2\left(4c_1c_3 - 3c_2^2\right) \\ &= \frac{(1-\alpha)^2}{12}\Big[\zeta_1^4 + 2(1-\zeta_1^2)\zeta_1^2\zeta_2 - (1-\zeta_1^2)(3+\zeta_1^2)\zeta_2^2 \\ &\quad + 4(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_1\zeta_3\Big]. \end{aligned} \tag{2.3}$$

A. Suppose that $\zeta_1 = 1$. Then, by (2.3),

$$|\gamma_1\gamma_3 - \gamma_2^2| = \frac{1}{12}(1-\alpha)^2.$$

B. Suppose that $\zeta_1 = 0$. Then, by (2.3),

$$|\gamma_1\gamma_3 - \gamma_2^2| = \frac{1}{4}(1-\alpha)^2|\zeta_2|^2 \le \frac{1}{4}(1-\alpha)^2.$$

C. Suppose that $\zeta_1 \in (0,1)$. By the fact that $|\zeta_3| \le 1$ from (2.3) we obtain

$$\begin{aligned} |\gamma_1\gamma_3 - \gamma_2^2| &\le \frac{(1-\alpha)^2}{12}\Big[\Big|\zeta_1^4 + 2(1-\zeta_1^2)\zeta_1^2\zeta_2 - (1-\zeta_1^2)(3+\zeta_1^2)\zeta_2^2\Big| \\ &\quad + 4(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_1\Big] \\ &= \frac{1}{3}(1-\alpha)^2\zeta_1(1-\zeta_1^2)\Big[|A + B\zeta_2 + C\zeta_2^2| + 1 - |\zeta_2|^2\Big], \end{aligned} \tag{2.4}$$

where

$$A := \frac{\zeta_1^3}{4(1-\zeta_1^2)}, \quad B := \frac{1}{2}\zeta_1, \quad C := -\frac{3+\zeta_1^2}{4\zeta_1}.$$

Since $AC < 0$, we apply Lemma 1.2 only for the case II.

C1. Note that the inequality

$$-4AC\left(\frac{1}{C^2} - 1\right) - B^2 = \frac{(3+\zeta_1^2)\zeta_1^2}{4(1-\zeta_1^2)}\left(\frac{16\zeta_1^2}{(3+\zeta_1^2)^2} - 1\right) - \frac{1}{4}\zeta_1^2 \le 0$$

is equivalent to $-9(1-\zeta_1^2) \le 3(1-\zeta_1^2)$, which evidently holds for $\zeta_1 \in (0,1)$. Moreover, the inequality $|B| < 2(1-|C|)$ is equivalent to $2\zeta_1^2 - 4\zeta_1 + 3 < 0$, which is false for $\zeta_1 \in (0,1)$.

C2. Since

$$4(1+|C|)^2 = 4\frac{(\zeta_1^2 + 4\zeta_1 + 3)^2}{16\zeta_1^2} > 0, \quad -4AC\left(\frac{1}{C^2} - 1\right) = -\frac{\zeta_1^2(9-\zeta_1^2)}{4(3+\zeta_1^2)} < 0,$$

we see that the inequality

$$\frac{\zeta_1^2}{4} = B^2 < \min\left\{4(1+|C|)^2, -4AC\left(\frac{1}{C^2} - 1\right)\right\} = -\frac{\zeta_1^2(9-\zeta_1^2)}{4(3+\zeta_1^2)}$$

is false for $\zeta_1 \in (0,1)$.

C3. Observe that the inequality

$$|C|(|B| + 4|A|) - |AB| = \frac{3+\zeta_1^2}{4\zeta_1}\left(\frac{1}{2}\zeta_1 + \frac{\zeta_1^3}{1-\zeta_1^2}\right) - \frac{\zeta_1^4}{8(1-\zeta_1^2)} \le 0$$

is equivalent to $3 + 4\zeta_1^2 \le 0$, which is false for $\zeta_1 \in (0,1)$.

C4. Note that the inequality

$$|AB| - |C|(|B| - 4|A|) = \frac{\zeta_1^4}{8(1-\zeta_1^2)} - \frac{3+\zeta_1^2}{4\zeta_1}\left(\frac{1}{2}\zeta_1 - \frac{\zeta_1^3}{1-\zeta_1^2}\right) \le 0$$

is equivalent to $4\zeta_1^4 + 8\zeta_1^2 - 3 \leq 0$, which is true for $0 < \zeta_1 \leq \zeta' := \sqrt{\sqrt{7}/2 - 1} \approx 0.5682$. Then, by (2.3) and Lemma 1.2,

$$\begin{aligned} |\gamma_1\gamma_3 - \gamma_2^2| &\leq \frac{1}{3}(1-\alpha)^2\zeta_1(1-\zeta_1^2)(-|A|+|B|+|C|) \\ &= \frac{1}{12}(1-\alpha)^2(3-4\zeta_1^4) \leq \frac{1}{4}(1-\alpha)^2, \end{aligned} \tag{2.5}$$

for $0 < \zeta_1 \leq \zeta'$.

C5. It remains to consider the last case in Lemma 1.2, which taking into account C4 holds for $\zeta' < \zeta_1 < 1$. Then, by (2.3),

$$\begin{aligned} |\gamma_1\gamma_3 - \gamma_2^2| &\leq \tfrac{1}{3}(1-\alpha)^2\zeta_1(1-\zeta_1^2)(|C|+|A|)\sqrt{1-\frac{B^2}{4AC}} \\ &= (1-\alpha)^2\psi(\zeta_1) \leq (1-\alpha)^2\psi(\zeta') = (1-\alpha)^2\frac{5-\sqrt{7}}{3\sqrt{8+2\sqrt{7}}}, \end{aligned} \tag{2.6}$$

where

$$\psi(t) := \frac{3-2t^2}{6\sqrt{3+t^2}}, \quad \zeta' \leq t \leq 1.$$

Indeed, to see that the last inequality in (2.6) is true, observe that since

$$\psi'(t) = -\frac{15t+2t^3}{6(3+t^2)^{3/2}} < 0, \quad \zeta' < t < 1,$$

the function ψ decrease which yields $\psi(t) \leq \psi(\zeta') \approx 0.21525$ for $\zeta' < t < 1$.

D. Summarizing from parts A–C it follows inequality (2.1). Equality holds for the function $f \in \mathcal{A}$ given by

$$\frac{zf'(z)}{f(z)} = \frac{1+(1-2\alpha)z^2}{1-z^2}, \quad z \in \mathbb{D},$$

for which $a_2 = a_4 = 0$ and $a_3 = 1-\alpha$. □

For $\alpha = 0$ we get the estimate for the class $\mathcal{S}^*$ of starlike functions [12].

Corollary 2.2 *If $f \in \mathcal{S}^*$, then*

$$|\gamma_1\gamma_3 - \gamma_2^2| \leq \frac{1}{4}.$$

The inequality is sharp.

3 Convex functions of order α

Now we deal with $H_{2,1}(F_f/2)$ for the class $\mathcal{S}^c(\alpha)$.

Theorem 3.1 *Let* $\alpha \in [0, 1)$. *If* $f \in \mathcal{S}^c(\alpha)$, *then*

$$|\gamma_1\gamma_3 - \gamma_2^2| \le \frac{(1-\alpha)^2(\alpha^2+4\alpha-16)}{48(\alpha^2+2\alpha-11)}. \tag{3.1}$$

The inequality is sharp.

Proof Fix $\alpha \in [0, 1)$ and let $f \in \mathcal{S}^c(\alpha)$ be of form (1.1). Then, by (1.7),

$$f'(z) + zf''(z) = ((1-\alpha)p(z)+\alpha)f'(z), \quad z \in \mathbb{D}, \tag{3.2}$$

for some $p \in \mathcal{P}$ of form (1.8). As in the proof of Theorem 2.1 we may assume that $c_1 \in [0, 2]$, i.e., in view of (1.9) that $\zeta_1 \in [0, 1]$. Substituting series (1.1) and (1.8) into (3.2) and equating coefficients we get

$$\begin{aligned} a_2 &= \frac{1}{2}(1-\alpha)c_1, \quad a_3 = \frac{1}{6}(1-\alpha)\left(c_2+(1-\alpha)c_1^2\right), \\ a_4 &= \frac{1}{24}(1-\alpha)\left[2c_3+3(1-\alpha)c_1c_2+(1-\alpha)^2c_1^3\right]. \end{aligned}$$

Hence by using (1.4) and (1.9)–(1.11) we obtain

$$\begin{aligned} \gamma_1\gamma_3 - \gamma_2^2 &= \frac{(1-\alpha)^2}{2304}\left[-(1-\alpha)^2c_1^4+4(1-\alpha)c_1^2c_2+24c_1c_3-16c_2^2\right] \\ &= \frac{(1-\alpha)^2}{144}\Big[(3-\alpha^2)\zeta_1^4+(6-2\alpha)(1-\zeta_1^2)\zeta_1^2\zeta_2 \\ &\quad - 2(1-\zeta_1^2)(2+\zeta_1^2)\zeta_2^2+6(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_1\zeta_3\Big]. \end{aligned} \tag{3.3}$$

A. Suppose that $\zeta_1 = 1$. Then, by (3.3),

$$|\gamma_1\gamma_3 - \gamma_2^2| = \frac{1}{144}(1-\alpha)^2(3-\alpha^2).$$

B. Suppose that $\zeta_1 = 0$. Then, by (3.3),

$$|\gamma_1\gamma_3 - \gamma_2^2| = \frac{1}{36}(1-\alpha)^2|\zeta_2|^2 \le \frac{1}{36}(1-\alpha)^2.$$

C. Suppose that $\zeta_1 \in (0,1)$. By the fact that $|\zeta_3| \le 1$ from (3.3) we obtain

$$
\begin{aligned}
&|\gamma_1\gamma_3 - \gamma_2^2| \\
&\le \frac{(1-\alpha)^2}{144}\Big[\Big|(3-\alpha^2)\zeta_1^4 + (6-2\alpha)(1-\zeta_1^2)\zeta_1^2\zeta_2 - 2(1-\zeta_1^2)(2+\zeta_1^2)\zeta_2^2\Big| \\
&\quad + 6(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_1\Big] \\
&= \frac{1}{24}(1-\alpha)^2\zeta_1(1-\zeta_1^2)\Big[|A + B\zeta_2 + C\zeta_2^2| + 1 - |\zeta_2|^2\Big],
\end{aligned}
\tag{3.4}
$$

where

$$A := \frac{(3-\alpha^2)\zeta_1^3}{6(1-\zeta_1^2)}, \quad B := \frac{3-\alpha}{3}\zeta_1, \quad C := -\frac{2+\zeta_1^2}{3\zeta_1}.$$

Since $AC < 0$, we apply Lemma 1.2 only for the case II.

C1. Note that the inequality

$$
\begin{aligned}
&-4AC\left(\frac{1}{C^2}-1\right) - B^2 \\
&\quad = \frac{2(3-\alpha^2)(2+\zeta_1^2)\zeta_1^2}{9(1-\zeta_1^2)}\left(\frac{9\zeta_1^2}{(2+\zeta_1^2)^2} - 1\right) - \frac{(3-\alpha)^2}{9}\zeta_1^2 \le 0
\end{aligned}
$$

is equivalent to $-2(3-\alpha^2)(4-\zeta_1^2) \le (3-\alpha)^2(2+\zeta_1^2)$, which holds for $\zeta_1 \in (0,1)$. Moreover, the inequality $|B| < 2(1-|C|)$ is equivalent to $(3-\alpha)\zeta_1^2 \le -2(1-\zeta_1)(2-\zeta_1)$, which is false for $\zeta_1 \in (0,1)$.

C2. Since

$$4(1+|C|)^2 = 4\frac{(\zeta_1^2+3\zeta_1+2)^2}{9\zeta_1^2} > 0$$

and

$$-4AC\left(\frac{1}{C^2}-1\right) = -\frac{2(3-\alpha^2)\zeta_1^2(4-\zeta_1^2)}{3(2+\zeta_1^2)} < 0,$$

we see that the inequality

$$
\begin{aligned}
\frac{1}{9}(3-\alpha)^2\zeta_1^2 = B^2 &< \min\left\{4(1+|C|)^2, -4AC\left(\frac{1}{C^2}-1\right)\right\} \\
&= -\frac{2(3-\alpha^2)\zeta_1^2(4-\zeta_1^2)}{9(2+\zeta_1^2)}
\end{aligned}
$$

is false for $\zeta_1 \in (0,1)$.

C3. Observe that the inequality

$$
\begin{aligned}
&|C|(|B|+4|A|)-|AB|\\
&\quad=\frac{2+\zeta_1^2}{3\zeta_1}\left(\frac{3-\alpha}{3}\zeta_1+\frac{2(3-\alpha^2)\zeta_1^3}{3(1-\zeta_1^2)}\right)-\frac{(3-\alpha^2)(3-\alpha)\zeta_1^4}{18(1-\zeta_1^2)}\le 0
\end{aligned}
$$

is equivalent to

$$
(1-\alpha)^2(\alpha+3)t^2+2(4\alpha^2-\alpha-9)t-4(3-\alpha)\ge 0, \tag{3.5}
$$

where $t:=\zeta_1^2\in(0,1)$. Note that $\Delta:=4\left(12\alpha^4-39\alpha^2-54\alpha+117\right)>0$ for $\alpha\in[0,1)$. Consider now

$$
t_{1,2}:=\frac{-4\alpha^2+\alpha+9\mp\sqrt{12\alpha^4-39\alpha^2-54\alpha+117}}{(1-\alpha)^2(3+\alpha)}.
$$

Observe first that $t_1<0$. Indeed, since $-4\alpha^2+\alpha+9>0$ for $\alpha\in[0,1)$, the inequality $t_1<0$ is equivalent to

$$
\sqrt{12\alpha^4-39\alpha^2-54\alpha+117}>-4\alpha^2+\alpha+9,
$$

which is equivalent to the true inequality

$$
4\alpha^4-8\alpha^3-32\alpha^2+72\alpha-36<0,\quad \alpha\in[0,1).
$$

Further, the inequality $t_2>1$ is equivalent to the true inequality

$$
12\alpha^4-\alpha^3-44\alpha^2-48\alpha+123>0,\quad \alpha\in[0,1).
$$

Thus, we state that inequality (3.5) is false.

C4. Note that the inequality

$$
\begin{aligned}
&|AB|-|C|(|B|-4|A|)\\
&\quad=\frac{(3-\alpha^2)(3-\alpha)\zeta_1^4}{18(1-\zeta_1^2)}-\frac{2+\zeta_1^2}{3\zeta_1}\left(\frac{3-\alpha}{3}\zeta_1-\frac{2(3-\alpha^2)\zeta_1^3}{3(1-\zeta_1^2)}\right)\le 0
\end{aligned}
$$

is equivalent to

$$
\begin{aligned}
&(\alpha^3-7\alpha^2-5\alpha+27)t^2+(-8\alpha^2-2\alpha+30)t\\
&\quad+4\alpha-12\le 0,\quad \alpha\in[0,1),
\end{aligned} \tag{3.6}
$$

where $t:=\zeta_1^2\in(0,1)$. We have $\Delta:=4\left(12\alpha^4+48\alpha^3-183\alpha^2-198\alpha+549\right)>0$ for $\alpha\in[0,1)$. Since $4\alpha^2+\alpha-15<0$ and $\alpha^3-7\alpha^2-5\alpha+27>0$ for

$\alpha \in [0, 1)$, so $s_1 < 0$, where

$$s_{1,2} := \frac{4\alpha^2 + \alpha - 15 \mp \sqrt{12\alpha^4 + 48\alpha^3 - 183\alpha^2 - 198\alpha + 549}}{\alpha^3 - 7\alpha^2 - 5\alpha + 27}.$$

Further, the condition $s_2 > 0$ is equivalent to

$$12\alpha^4 + 48\alpha^3 - 183\alpha^2 - 198\alpha + 549 > (-4\alpha^2 - \alpha + 15)^2,$$

which is equivalent to the true inequality

$$\alpha^4 - 10\alpha^3 + 16\alpha^2 + 42\alpha - 81 < 0, \quad \alpha \in [0, 1).$$

Moreover, the inequality $s_2 < 1$ is equivalent to the inequality

$$-\alpha^6 + 22\alpha^5 - 97\alpha^4 - 168\alpha^3 + 705\alpha^2 + 306\alpha - 1215 < 0$$

which is valid for $\alpha \in [0, 1)$. Thus, inequality (3.6) holds only when

$$0 < \zeta_1 \leq \sqrt{s_2} =: \zeta'.$$

Then, by (3.4) and Lemma 1.2,

$$\begin{aligned} |\gamma_1\gamma_3 - \gamma_2^2| &\leq \frac{(1-\alpha)^2}{24}\zeta_1(1-\zeta_1^2)(-|A| + |B| + |C|) = \varphi(\zeta_1) \leq \varphi(u_0) \\ &= \frac{(1-\alpha)^2(\alpha^2 + 4\alpha - 16)}{48(\alpha^2 + 2\alpha - 11)}, \end{aligned} \tag{3.7}$$

where

$$\varphi(u) := \frac{(1-\alpha)^2}{144}\left[(\alpha^2 + 2\alpha - 11)u^4 + (4 - 2\alpha)u^2 + 4\right], \quad 0 \leq u \leq \zeta_1',$$

and

$$0 < u_0 := \sqrt{\frac{\alpha - 2}{\alpha^2 + 2\alpha - 11}} < \zeta', \quad \alpha \in [0, 1), \tag{3.8}$$

is a unique critical point, namely the maximum of φ. Observe here that $u_0 < \zeta'$ leads to

$$\begin{aligned} &3\alpha^8 - 12\alpha^7 - 135\alpha^6 + 522\alpha^5 - 471\alpha^4 + 1440\alpha^3 - 489\alpha^2 - 13950\alpha + 18468 \\ &\quad > 3\alpha^8 + 522\alpha^5 + 1440\alpha^3 + 3411 > 0, \quad \alpha \in [0, 1), \end{aligned}$$

and the last inequality is true.

C5. It remains to consider the last case in Lemma 1.2, which taking into account C4 holds for $\zeta' < \zeta_1 < 1$. Then, by (3.4),

$$\begin{aligned} |\gamma_1\gamma_3 - \gamma_2^2| &\le \frac{1}{24}(1-\alpha)^2\zeta_1(1-\zeta_1^2)(|C|+|A|)\sqrt{1-\frac{B^2}{4AC}} = \psi(\zeta_1) \\ &\le \psi(\zeta') = \frac{(1-\alpha)^2(a_1 - a_2\sqrt{b})}{144d^2}\sqrt{\frac{a_3 - 3(\alpha-1)^2\sqrt{b}}{a_4 + 2(3-\alpha^2)\sqrt{b}}}, \end{aligned} \tag{3.9}$$

where

$$\psi(u) := \frac{(1-\alpha)^2}{144}\left[(1-\alpha^2)u^4 - 2u^2 + 4\right]\sqrt{\frac{3(-\alpha^2-2\alpha+7) - 3(1-\alpha)^2u^2}{2(3-\alpha^2)(2+u^2)}}$$

for $\zeta' \le u \le 1$, and

$$\begin{aligned} a_1 &:= -24\alpha^6 - 120\alpha^5 + 540\alpha^4 + 924\alpha^3 - 2904\alpha^2 - 1572\alpha + 4500, \\ a_2 &:= 8\alpha^4 + 4\alpha^3 - 52\alpha^2 - 12\alpha + 84, \\ a_3 &:= -3\alpha^5 + 3\alpha^4 + 99\alpha^3 - 159\alpha^2 - 360\alpha + 612, \\ a_4 &:= -4\alpha^5 + 20\alpha^4 + 30\alpha^3 - 138\alpha^2 - 54\alpha + 234, \\ b &:= 12\alpha^4 + 48\alpha^3 - 183\alpha^2 - 198\alpha + 549, \\ d &:= \alpha^3 - 7\alpha^2 - 5\alpha + 27 \end{aligned}$$

for $\alpha \in [0, 1)$. To see that the last inequality in (3.9) holds observe that ψ is decreasing. Indeed, we have

$$\begin{aligned} \psi'(u) = &\frac{-3(1-\alpha)^2u}{288(3-\alpha^2)(2+u^2)^2}\sqrt{\frac{2(3-\alpha^2)(2+u^2)}{3(-\alpha^2-2\alpha+7) - 3(1-\alpha)^2u^2}} \\ &\times\Big[4(2+u^2)\left(1-(1-\alpha^2)u^2\right)\left(7-2\alpha-\alpha^2-(1-\alpha)^2u^2\right) \\ &+ \left((1-\alpha^2)u^4 - 2u^2 + 4\right)(\alpha-3)^2\Big], \quad \zeta' < u < 1. \end{aligned}$$

Since for $\zeta' < u < 1$,

$$7 - 2\alpha - \alpha^2 - (1-\alpha)^2u^2 \ge 7 - 2\alpha - \alpha^2 - (1-\alpha)^2 = 6 - 2\alpha^2 > 0$$

and

$$\begin{aligned} (1-\alpha^2)u^4 - 2u^2 + 4 &= 4 - u^2\left(2-(1-\alpha)^2u^2\right) \ge 4 - \left(2-(1-\alpha)^2u^2\right) \\ &= 2 + (1-\alpha)^2u^2 > 0, \end{aligned}$$

we deduce that $\psi' \leq 0$ for $\zeta' \leq u < 1$, which confirm that ψ decreases. Simple however tedious computations which we omit show that

$$\varphi(\zeta') = \psi(\zeta')$$

for each $\alpha \in [0, 1)$. Hence taking into account (3.7) and (3.9) we see that

$$\psi(\zeta') \leq \varphi(u_0) = \frac{(1-\alpha)^2(\alpha^2+4\alpha-16)}{48(\alpha^2+2\alpha-11)}.$$

D. Summarizing from parts A–C it follows inequality (3.1). Equality holds for the function $f \in \mathcal{A}$ given by (3.2), where the function $p \in \mathcal{P}$ is of form (1.12) with $\zeta_1 = u_0 =: \tau$, with u_0 given by (3.8) and $\zeta_2 = -1$, i.e.,

$$p(z) = \frac{1-z^2}{1-2\tau z+z^2}, \quad z \in \mathbb{D}.$$

□

For $\alpha = 0$ we get the estimate for the class $\mathcal{S}^c$ of convex functions [12].

Corollary 3.2 *If $f \in \mathcal{S}^c$, then*

$$|\gamma_1\gamma_3 - \gamma_2^2| \leq \frac{1}{33}.$$

The inequality is sharp.

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