On the Deformation Tensor Regularity for the Navier–Stokes Equations in Lorentz Spaces

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Abstract

This paper is concerned with the regularity criteria in terms of the middle eigenvalue of the deformation (strain) tensor $\mathcal{D}(u)$ to the 3D Navier–Stokes equations in Lorentz spaces. It is shown that a Leray–Hopf weak solution is regular on $(0, T]$ provided that the norm $\|\lambda_2^+\|_{L^{p,\infty}(0,T;L^{q,\infty}(\mathbb{R}^3))}$ with $2/p + 3/q = 2$ ($3/2 < q \leq \infty$) is small. This generalizes the corresponding works of Neustupa–Penel and Miller.

Keywords Navier–Stokes equations · Weak solutions · Regularity

Mathematics Subject Classification 35Q30 · 35B33

1 Introduction

The 3D incompressible Navier–Stokes equations describing Newtonian fluid are given by

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla \Pi = 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0. \end{cases} \quad (1.1)$$

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Here, u and Π denote the unknown velocity vector and the pressure, respectively. The initial datum u_0 is given and satisfies the divergence-free condition.

The pioneering work involving the global finite energy weak solutions is due to Leray [12] and Hopf [8]. Precisely, they showed that for any given divergence-free datum $u_0 \in L^2(\Omega)$, there exists a global weak solution u of the 3D Navier–Stokes equations such that $u \in L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; W^{1,2}(\Omega))$ where $\Omega \subseteq \mathbb{R}^3$. However, full regularity of Leray–Hopf weak solutions to the 3D Navier–Stokes system is still a fundamental open question. Starting from Serrin’s famous work, regularity criteria of Leray–Hopf weak solutions are extensively studied (see [1–3, 5, 7, 9–11, 13, 16–21, 23–26, 28, 29] and references therein). The so-called Serrin type regularity criteria is that a weak solution u is regular on $(0, T]$ if u satisfies

$$u \in L^p(0, T; L^q(\mathbb{R}^3)) \quad \text{with} \quad 2/p + 3/q = 1, \quad q \geq 3. \quad (1.2)$$

Beirao da Veiga found that condition (1.2) can be replaced by the following

$$\nabla u \in L^p(0, T; L^q(\mathbb{R}^3)) \quad \text{with} \quad 2/p + 3/q = 2, \quad q > 3/2. \quad (1.3)$$

It is clear that the gradient of the velocity field ∇u can be decomposed into a symmetric part $\mathcal{D}$ and an antisymmetric part Ω , that is,

$$\nabla u = \frac{1}{2} (\nabla u + \nabla u^T) + \frac{1}{2} (\nabla u - \nabla u^T) =: \mathcal{D}(u) + \Omega(u).$$

$\mathcal{D}(u)$ and $\Omega(u)$ are usually called the deformation tensor (or rate-of-strain tensor) and the rotation tensor, respectively (See, e.g., [14]). Denote $\lambda_1, \lambda_2, \lambda_3$ by the eigenvalue of the deformation tensor $\mathcal{D}(u)$. In a series of works [17–21], Neustupa–Penel obtained regularity criteria via only the middle eigenvalue λ_2 of the deformation tensor below

$$\lambda_2^+ = \max\{\lambda_2, 0\} \in L^p(0, T; L^q(\mathbb{R}^3)) \quad \text{with} \quad 2/p + 3/q = 2, \quad q > 3/2. \quad (1.4)$$

Very recently, a new alternative proof (1.4) was presented by Miller in [16]. Subsequently, Wu [28] generalized (1.4) to the anisotropic Lebesgue spaces in the 3D double-diffusive convection equations. Since the Lorentz spaces $L^{r,s}(\Omega)$ ($s \geq r$) are larger than the Lebesgue spaces $L^r(\Omega)$, much effort is devoted to the regularity criteria in terms of velocity, gradient, the pressure, velocity directions in Lorentz spaces (see [3, 5, 7, 9–11, 25, 26, 29] and references therein). Motivated by the references mentioned above, we extend (1.4) to Lorentz spaces. The main result of this paper reads as follows.

Theorem 1.1 *Suppose that u is a weak solution to (1.1) with the divergence-free initial datum $u_0(x) \in W^{1,2}(\mathbb{R}^3)$. Then, there exists a positive constant ε_1 such that $u(x, t)$ is a regular solution on $(0, T]$ provided holds $\lambda_2^+ \in L^{p,\infty}(0, T; L^{q,\infty}(\mathbb{R}^3))$ and*

$$\|\lambda_2^+\|_{L^{p,\infty}(0,T;L^{q,\infty}(\mathbb{R}^3))} \leq \varepsilon_1, \quad \text{with} \quad 2/p + 3/q = 2, \quad 3/2 < q \leq \infty;$$

Remark 1.1 Theorem 1.1 is an improvement of regularity criteria based on only the middle eigenvalue of the deformation tensor (1.4). It seems that a slightly modified the technique in Theorem 1.1 can be applied to other incompressible fluid equations such as Navier–Stokes equations with fractional dissipation and the double-diffusive convection equations in [28].

As a by-production of Theorem 1.1, by the absolute continuity of norm in Lorentz space $L^{p,r}(0, T)$ with $r < \infty$, we have

Corollary 1.2 Suppose that u is a weak solution to (1.1) with the divergence-free initial datum $u_0(x) \in W^{1,2}(\mathbb{R}^3)$. Then, $u(x, t)$ is a regular solution on $(0, T]$ provided that, for $p \leq r < \infty$,

$$\lambda_2^+ \in L^{p,r}(0, T; L^{q,\infty}(\mathbb{R}^3)) \quad \text{with} \quad 2/p + 3/q = 2, \quad q > 3/2.$$

Remark 1.2 Corollary 1.2 is an extension of regularity criteria (1.4).

2 Notations and auxiliary lemmas

In this section, we give some preliminaries on functional settings and some useful lemmas, which will be used in proof of our main results.

For $p \in [1, \infty]$, the notation $L^p(0, T; X)$ stands for the set of measurable functions $f(x, t)$ on the interval $(0, T)$ with values in X and $\|f(\cdot, t)\|_X$ belonging to $L^p(0, T)$. The classical Sobolev space $W^{k,2}(\mathbb{R}^3)$ is equipped with the norm $\|f\|_{W^{k,2}(\mathbb{R}^3)} = \sum_{\alpha=0}^k \|D^\alpha f\|_{L^2(\mathbb{R}^3)}$. $|\Omega|$ represents the n -dimensional Lebesgue measure of a set $\Omega \subset \mathbb{R}^n$. We will use the summation convention on repeated indices.

We now present some basic facts on Lorentz spaces. For $p, q \in [1, \infty]$, we define

$$\|f\|_{L^{p,q}(\Omega)} = \begin{cases} \left(p \int_0^\infty \alpha^q |\{x \in \Omega : |f(x)| > \alpha\}|^{\frac{q}{p}} \frac{d\alpha}{\alpha} \right)^{\frac{1}{q}}, & q < \infty, \\ \sup_{\alpha > 0} \alpha |\{x \in \Omega : |f(x)| > \alpha\}|^{\frac{1}{p}}, & q = \infty. \end{cases}$$

Moreover,

$$L^{p,q}(\Omega) = \{f : f \text{ is a measurable function on } \Omega \text{ and } \|f\|_{L^{p,q}(\Omega)} < \infty\}.$$

Similarly, one can define Lorentz spaces $L^{p,q}(0, T; X)$ in time for $p \leq q \leq \infty$. $f \in L^{p,q}(0, T; X)$ means that $\|f\|_{L^{p,q}(0,T;X)} < \infty$, where

$$\|f\|_{L^{p,q}(0,T;X)} = \begin{cases} \left(p \int_0^\infty \alpha^q |\{t \in (0, T) : \|f(t)\|_X > \alpha\}|^{\frac{q}{p}} \frac{d\alpha}{\alpha} \right)^{\frac{1}{q}}, & q < \infty, \\ \sup_{\alpha > 0} \alpha |\{t \in (0, T) : \|f(t)\|_X > \alpha\}|^{\frac{1}{p}}, & q = \infty. \end{cases}$$

Now, we list some properties of Lorentz spaces.

(1) Interpolation characteristic of Lorentz spaces [4]:

$$(L^{p_0, q_0}(\mathbb{R}^n), L^{p_1, q_1}(\mathbb{R}^n))_{\theta, q} = L^{p, q}(\mathbb{R}^n) \quad \text{with} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad 0 < \theta < 1. \quad (2.1)$$

(2) Hölder's inequality in Lorentz spaces [22]:

$$\begin{aligned} \|fg\|_{L^{r, s}(\mathbb{R}^n)} &\leq \|f\|_{L^{r_1, s_1}(\mathbb{R}^n)} \|g\|_{L^{r_2, s_2}(\mathbb{R}^n)}, \\ \frac{1}{r} &= \frac{1}{r_1} + \frac{1}{r_2}, \quad \frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}. \end{aligned} \quad (2.2)$$

(3) The Lorentz spaces increase as the exponent q increases [6, 15]. For $1 \leq p \leq \infty$ and $1 \leq q_1 < q_2 \leq \infty$,

$$\|f\|_{L^{p, q_2}(\mathbb{R}^n)} \leq \left(\frac{q_1}{p}\right)^{\frac{1}{q_1} - \frac{1}{q_2}} \|f\|_{L^{p, q_1}(\mathbb{R}^n)}. \quad (2.3)$$

(4) Sobolev inequality in Lorentz spaces [22, 27]:

$$\|f\|_{L^{\frac{np}{n-p}, p}(\mathbb{R}^n)} \leq \|\nabla f\|_{L^p(\mathbb{R}^n)} \quad \text{with} \quad 1 \leq p < n. \quad (2.4)$$

We recall the following useful Gronwall lemma first shown by Bosia, Pata and Robinson in [5].

Lemma 2.1 ([5]). *Let ϕ be a measurable positive function defined on the interval $[0, T]$. Suppose that there exists $\kappa_0 > 0$ such that for all $0 < \kappa < \kappa_0$ and a.e. $t \in [0, T]$, ϕ satisfies the inequality*

$$\frac{d}{dt} \phi \leq \mu \lambda^{1-\kappa} \phi^{1+2\kappa},$$

where $0 < \lambda \in L^{1, \infty}(0, T)$ and $\mu > 0$ with

$$\mu \|\lambda\|_{L^{1, \infty}(0, T)} < \frac{1}{2}.$$

Then, ϕ is bounded on $[0, T]$.

To apply the above lemma, we need the following fact obtained in [9].

Lemma 2.2 ([9]). *Assume that the pair (p, q) satisfies $\frac{2}{p} + \frac{3}{q} = a$ with $a, q \geq 1$ and $p > 0$. Then, for every $\kappa \in [0, 1]$ and given $b, c_0 \geq 1$, there exist $p_\kappa > 0$ and $\min\{q, b\} \leq q_\kappa \leq \max\{q, b\}$ such that*

$$\begin{cases} \frac{2}{p_\kappa} + \frac{3}{q_\kappa} = a, \\ \frac{p_\kappa}{q_\kappa} = \frac{p(1-\kappa)}{q} + \frac{c_0\kappa}{b}. \end{cases} \quad (2.5)$$

To proceed further, we recall the Kronecker symbol δ_{ij} and the three-dimensional Levi-Civita symbol ε_{ijk} , respectively,

$$\delta_{ij} = \begin{cases} 0, & \text{for } i \neq j, \\ 1, & \text{for } i = j, \end{cases}$$

and

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } (i, j, k) \text{ is } (1, 2, 3), (2, 3, 1) \text{ or } (3, 1, 2), \\ -1, & \text{if } (i, j, k) \text{ is } (3, 2, 1), (1, 3, 2) \text{ or } (2, 1, 3), \\ 0, & \text{else.} \end{cases}$$

Then, a useful identity (Levi-Civita and Kronecker Delta identity) holds

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}.$$

We denote $a_{ij} = \frac{1}{2}(\partial_j u_i - \partial_i u_j)$ and $d_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. Next, we recall the following result stated in [17]. Since this result plays an important role in the derivation of regularity criteria via only the middle eigenvalue of the deformation tensor, we present its proof here.

Lemma 2.3 *Let $\omega = \text{curl} u = (\omega_1, \omega_2, \omega_3)$ be the vorticity field. Then,*

$$d_{kj}a_{ij}a_{ik} = -\frac{1}{4}d_{kj}w_jw_k. \quad (2.6)$$

Proof Straightforward calculations show that

$$w_i = \varepsilon_{ijk}\partial_j u_k = \varepsilon_{ijk}a_{kj}. \quad (2.7)$$

Multiplying both side of (2.7) by ε_{imn} , we obtain

$$\varepsilon_{imn}w_i = \varepsilon_{imn} \cdot \varepsilon_{ijk}a_{kj}.$$

This together with the aforementioned Levi-Civita and Kronecker Delta identity yields that

$$\varepsilon_{imn}w_i = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) \cdot a_{kj} = \delta_{kn}a_{km} - \delta_{km}a_{kn} = -2a_{mn},$$

where the definition of a_{ij} was used.

As a consequence, we infer that

$$a_{mn} = -\frac{1}{2}\varepsilon_{imn}w_i. \quad (2.8)$$

Using (2.8) twice and (2.7), we see that

$$d_{kj}a_{ij}a_{ik} = \frac{1}{4}d_{kj}\varepsilon_{kij}w_k\varepsilon_{jik}w_j = -\frac{1}{4}d_{jk}w_kw_j.$$

This completes the proof of Lemma 2.3. $\square$

3 Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1.

Proof of Theorem 1.1 Multiplying both side of the Navier–Stokes equations (1.1) by Δu , integrating by parts and using the divergence-free condition $\operatorname{div} u = 0$, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} |\Delta u|^2 dx = \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta u dx. \quad (3.1)$$

Using the divergence-free condition $\operatorname{div} u = 0$, we write the right-hand side of (3.1) as

$$\begin{aligned} \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta u dx &= \int_{\mathbb{R}^3} u_j \partial_j u_i \partial_k \partial_k u_i dx \\ &= - \int_{\mathbb{R}^3} \partial_k u_j \partial_j u_i \partial_k u_i dx \\ &= - \int_{\mathbb{R}^3} d_{kj} d_{ij} d_{ik} - \int_{\mathbb{R}^3} d_{kj} d_{ij} a_{ik} \\ &\quad - \int_{\mathbb{R}^n} d_{kj} a_{ij} d_{ik} - \int_{\mathbb{R}^3} d_{kj} a_{ij} a_{ik} dx. \end{aligned} \quad (3.2)$$

Interchanging i with k , we have

$$d_{kj} d_{ij} a_{ik} = -d_{kj} d_{ij} a_{ki} = -d_{ij} d_{kj} a_{ik} = -d_{kj} d_{ij} a_{ik}.$$

Hence, we see that

$$d_{kj} d_{ij} a_{ik} = 0. \quad (3.3)$$

Arguing in the same manner as the above, we observe that

$$a_{ij} d_{ik} d_{kj} = 0. \quad (3.4)$$

Substituting (3.3) and (3.4) into (3.2),

$$- \int_{\mathbb{R}^3} \partial_k u_j \partial_j u_i \partial_k u_i dx = - \int_{\mathbb{R}^3} d_{kj} d_{ij} d_{ik} dx - \int_{\mathbb{R}^3} d_{kj} a_{ij} a_{ik} dx. \quad (3.5)$$

We derive from Lemma 2.3 that

$$-\int_{\mathbb{R}^3} d_{kj} a_{ij} a_{ik} dx = \frac{1}{4} \int_{\mathbb{R}^3} d_{kj} w_j w_k dx.$$

Collecting the above estimates, we see that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} |\Delta u|^2 dx = - \int_{\mathbb{R}^3} d_{kj} d_{ij} d_{ik} dx + \frac{1}{4} \int_{\mathbb{R}^3} d_{kj} w_j w_k dx. \quad (3.6)$$

As in [17], we invoke the vorticity equation to control the first term in the right hand side of (3.6). Recall that the vorticity equation of the Navier–Stokes system is given by

$$w_t - \Delta w + (u \cdot \nabla) w = (w \cdot \nabla) u. \quad (3.7)$$

Multiplying Eq. (3.7) by ω , we infer that

$$\frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} \omega \cdot \nabla u \cdot \omega dx = \int_{\mathbb{R}^3} d_{kj} w_j w_k dx. \quad (3.8)$$

Notice that,

$$\int_{\mathbb{R}^3} |\omega|^2 dx = \int_{\mathbb{R}^3} |\nabla u|^2 dx, \quad \int_{\mathbb{R}^3} |\nabla \omega|^2 dx = \int_{\mathbb{R}^3} |\Delta u|^2 dx.$$

Thus, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 + \|\Delta u\|_{L^2(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} d_{kj} w_j w_k dx. \quad (3.9)$$

Multiplying Eq. (3.9) by $-\frac{1}{4}$ and summing with (3.6), it immediately follows that

$$\frac{3}{8} \frac{d}{dt} \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 + \frac{3}{4} \|\Delta u\|_{L^2(\mathbb{R}^3)}^2 = - \int_{\mathbb{R}^3} d_{kj} d_{ij} d_{ik} dx. \quad (3.10)$$

Without loss of generality, we assume $\lambda_1 \leq \lambda_2 \leq \lambda_3$. The tensor (d_{ij}) is symmetric so that the corresponding eigenvectors are orthogonal. Due to the fact $\operatorname{div} u = 0$, we deduce that

$$\lambda_1 + \lambda_2 + \lambda_3 = 0, \quad (3.11)$$

and

$$(d_{kj}d_{ij}d_{ik})(x, t) = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = \lambda_1^3 + \lambda_2^3 - (\lambda_1 + \lambda_2)^3 = 3\lambda_1\lambda_2\lambda_3 = 3|d_{ij}|. \quad (3.12)$$

The quantity $\lambda_1\lambda_2\lambda_3 = |d_{ij}|$ is independent of choice of the system of coordinates. Hence, we obtain

$$\frac{3}{8} \frac{d}{dt} \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 + \|\Delta u\|_{L^2(\mathbb{R}^3)}^2 = -3 \int_{\mathbb{R}^3} \lambda_1\lambda_2\lambda_3 dx. \quad (3.13)$$

Using the Hölder inequality (2.2) or interpolation characteristic (2.1), Sobolev embedding (2.4) in Lorentz spaces and the Young inequality, we have

$$\begin{aligned} \int_{\mathbb{R}^3} \lambda_1\lambda_2^+\lambda_3 dx &\leq C \cdot \|\lambda_2^+\|_{L^{q,\infty}(\mathbb{R}^3)} \|\lambda_1\|_{L^{\frac{2q}{q-1},2}(\mathbb{R}^3)} \|\lambda_3\|_{L^{\frac{2q}{q-1},2}(\mathbb{R}^3)} \\ &\leq C \cdot \|\lambda_2^+\|_{L^{q,\infty}(\mathbb{R}^3)} \|\nabla u\|_{L^{\frac{2q}{q-1},2}(\mathbb{R}^3)}^2 \\ &\leq C \cdot \|\lambda_2^+\|_{L^{q,\infty}(\mathbb{R}^3)} \|\nabla u\|_{L^2(\mathbb{R}^3)}^{2-\frac{3}{q}} \|\nabla^2 u\|_{L^2(\mathbb{R}^3)}^{\frac{3}{q}} \\ &\leq C \cdot \|\lambda_2^+\|_{L^{q,\infty}(\mathbb{R}^3)}^{\frac{2q}{q-3}} \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{8} \|\nabla^2 u\|_{L^2(\mathbb{R}^3)}^2. \end{aligned} \quad (3.14)$$

Inserting (3.14) into (3.13), we get

$$\frac{d}{dt} \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 dx \leq C \cdot \|\lambda_2^+\|_{L^{q,\infty}(\mathbb{R}^3)}^{\frac{2q}{q-3}} \|\nabla u\|_{L^2(\mathbb{R}^3)}^2. \quad (3.15)$$

With the help of interpolation characteristic of Lorentz spaces (2.1) or the Hölder inequality (2.2), applying Lemma 2.2 with $a = b = 2$, $c_0 = 4$ and (2.3), we see that

$$\begin{aligned} \|\lambda_2^+\|_{L^{q_\kappa,\infty}(\mathbb{R}^3)}^{p_\kappa} &\leq \|\lambda_2^+\|_{L^{q,\infty}(\mathbb{R}^3)}^{p(1-\kappa)} \|\lambda_2^+\|_{L^{2,\infty}(\mathbb{R}^3)}^{4\kappa} \\ &\leq \|\lambda_2^+\|_{L^{q,\infty}(\mathbb{R}^3)}^{p(1-\kappa)} \|\nabla u\|_{L^{2,\infty}(\mathbb{R}^3)}^{4\kappa}. \end{aligned} \quad (3.16)$$

Since the pair (p_κ, q_κ) also meets $2/p_\kappa + 3/q_\kappa = 2$, we insert (3.16) into (3.15) to obtain

$$\frac{d}{dt} \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 dx \leq C \cdot \|\lambda_2^+\|_{L^{q,\infty}(\mathbb{R}^3)}^{p(1-\kappa)} \|\nabla u\|_{L^{2,\infty}(\mathbb{R}^3)}^{2(1+2\kappa)}. \quad (3.17)$$

Applying Lemma 2.1 and (1.2) completes the proof of Theorem 1.1. $\square$

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