Matrix Representations of Asymmetric Truncated Toeplitz Operators

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Abstract

In this paper, we describe the matrix representations of asymmetric truncated Toeplitz operators acting between two finite-dimensional model spaces K_1 and K_2 . The novelty of our approach is that here we consider matrix representations computed with respect to bases of different type in K_1 and K_2 (for example, kernel basis in K_1 and conjugate kernel basis in K_2). We thus obtain new matrix characterizations which are simpler than the ones already known for asymmetric truncated Toeplitz operators.

Keywords Model space · Truncated Toeplitz operator · Asymmetric truncated Toeplitz operator · Matrix representation

Mathematics Subject Classification 47B32 · 47B35 · 30H10

1 Introduction

Let H^2 be the classical Hardy space. The space H^2 consists of all functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ and such that $\sum_{k=0}^{\infty} |a_k|^2 < \infty$. It can also be identified (via radial limits) with the closed linear span of analytic polynomials in $L^2 = L^2(\mathbb{T})$, $\mathbb{T} = \partial\mathbb{D}$. Let P be the orthogonal projection from L^2 onto H^2 .

For $\varphi \in L^2$, the Toeplitz operator T_φ is defined on the set of all bounded analytic functions $H^\infty \subset H^2$ by

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$$T_\varphi f = P(\varphi f), \quad f \in H^\infty.$$

Since H^∞ is a dense subset of the Hardy space, the operator T_φ is densely defined. Moreover, it can be extended to a bounded linear operator $T_\varphi : H^2 \rightarrow H^2$ if and only if $\varphi \in L^\infty = L^\infty(\mathbb{T})$. Toeplitz operators have many applications and are well-studied (for more details and properties see for example [2, 19]). Two examples of these operators are the unilateral shift $S = T_z$ and the backward shift $S^* = T_{\bar{z}}$.

The Toeplitz operator T_φ can be seen as a compression to H^2 of the multiplication operator $f \mapsto \varphi f$ defined on L^2 . Recently, compressions of multiplication operators to model spaces have been intensely studied. A model space is a closed subspace of H^2 of the form $K_\alpha = H^2 \ominus \alpha H^2$, where α is an inner function ($\alpha \in H^\infty$ and $|\alpha| = 1$ a.e. on $\mathbb{T} = \partial\mathbb{D}$). Model spaces are the typical S^* -invariant subspaces of H^2 . For each $w \in \mathbb{D}$ the point evaluation functional $f \mapsto f(w)$ is bounded on K_α and so there exists a kernel function $k_w^\alpha \in K_\alpha$ with the reproducing property $f(w) = \langle f, k_w^\alpha \rangle$ for every $f \in K_\alpha$. We have

$$k_w^\alpha(z) = \frac{1 - \overline{\alpha(w)}\alpha(z)}{1 - \bar{w}z}, \quad z \in \mathbb{D}. \quad (1.1)$$

Moreover, K_α is preserved by the conjugation $C_\alpha : L^2 \rightarrow L^2$ (an antilinear, isometric involution) defined on L^2 by the formula

$$C_\alpha f(z) = \alpha(z)\overline{zf(z)}, \quad |z| = 1.$$

Note that the conjugate kernel function $\tilde{k}_w^\alpha = C_\alpha k_w^\alpha$, $w \in \mathbb{D}$, is given by

$$\tilde{k}_w^\alpha(z) = \frac{\alpha(z) - \alpha(w)}{z - w}, \quad z \in \mathbb{D}.$$

A thorough account of the subject of model spaces can be found in [11].

Let α be an inner function and let P_α denote the orthogonal projection from L^2 onto K_α . The truncated Toeplitz operator A_φ^α , $\varphi \in L^2$, is defined by

$$A_\varphi^\alpha f = P_\alpha(\varphi f), \quad f \in K_\alpha^\infty = K_\alpha \cap H^\infty.$$

More generally, for two inner functions α and β , the asymmetric truncated Toeplitz operator $A_\varphi^{\alpha, \beta}$, $\varphi \in L^2$, is defined by

$$A_\varphi^{\alpha, \beta} f = P_\beta(\varphi f), \quad f \in K_\alpha^\infty.$$

Clearly, $A_\varphi^{\alpha, \alpha} = A_\varphi^\alpha$. The operator $A_\varphi^{\alpha, \beta}$ is densely defined since K_α^∞ is a dense subset of K_α . Let

$$\mathcal{T}(\alpha, \beta) = \{A_\varphi^{\alpha, \beta} : \varphi \in L^2 \text{ and } A_\varphi^{\alpha, \beta} \text{ extends boundedly to } K_\alpha\}$$

and $\mathcal{T}(\alpha) = \mathcal{T}(\alpha, \alpha)$.

Truncated Toeplitz operators were introduced in 2007 in D. Sarason's paper [20]. Extensive study of this class of operators revealed many interesting properties and applications of truncated Toeplitz operators. For example, a truncated Toeplitz operator is not uniquely determined by its symbol; it is C_α -symmetric [20]; the fact that it can be boundedly extended to K_α does not imply that it has a symbol from L^∞ [1]. We refer the reader to the survey [10] for more results and references (see also [9]). A natural generalization of truncated Toeplitz operators, asymmetric truncated Toeplitz operators, were introduced more recently in [3,4] and [14].

Among the results on truncated Toeplitz operators are characterizations given by J.A. Cima, W.T. Ross and W.R. Wogen in [8]. The authors in [8] described truncated Toeplitz operators on finite-dimensional model spaces in terms of matrix representations with respect to some natural bases.

It is known that $\dim K_\alpha = m < \infty$ if and only if α is a finite Blaschke product with m (not necessarily distinct) zeros $a_1, \dots, a_m \in \mathbb{D}$ [11, Chapter 5]. In that case, each $f \in K_\alpha$ can be written as

$$f(z) = \frac{q(z)}{(1 - \bar{a}_1 z) \cdots (1 - \bar{a}_m z)},$$

where q is a polynomial of degree at most $m - 1$ (and so f is analytic in an open set containing the closure of $\mathbb{D}$ and here $K_\alpha = K_\alpha^\infty$). In particular, if $\alpha(z) = z^m$, then K_α is the set of all polynomials of degree at most $m - 1$.

It is also known that if $\dim K_\alpha = m < \infty$, then the set of kernel functions $\{k_{w_1}^\alpha, \dots, k_{w_m}^\alpha\}$ corresponding to any distinct points $w_1, \dots, w_m \in \mathbb{D}$ is linearly independent (for proof see [20, proof of Thm. 7.1(b)] and thus a (non-orthogonal) basis for K_α . Since C_α is an antilinear isometry, the set of conjugate kernel functions $\{\tilde{k}_{w_1}^\alpha, \dots, \tilde{k}_{w_m}^\alpha\}$ is also a basis for K_α . In particular, if the zeros $a_1, \dots, a_m$ are distinct, we have the kernel function basis $\mathcal{K}_m^\alpha = \{k_{a_1}^\alpha, \dots, k_{a_m}^\alpha\}$ and the conjugate kernel function basis $\tilde{\mathcal{K}}_m^\alpha = \{\tilde{k}_{a_1}^\alpha, \dots, \tilde{k}_{a_m}^\alpha\}$. Note that in that case, for each $j \in \{1, \dots, m\}$,

$$k_{a_j}^\alpha(z) = \frac{1}{1 - \bar{a}_j z} \quad \text{and} \quad \tilde{k}_{a_j}^\alpha(z) = \frac{\alpha(z)}{z - a_j}.$$

If $\dim K_\alpha = m < \infty$, then α , as a finite Blaschke product, is analytic in an open set containing the closure of $\mathbb{D}$. For any $\eta \in \mathbb{T}$ one can consider k_η^α defined by (1.1) with w replaced by η . It turns out that this k_η^α also belongs to K_α and $f(\eta) = \langle f, k_\eta^\alpha \rangle$ for every $f \in K_\alpha$. Moreover,

$$\|k_\eta^\alpha\|^2 = k_\eta^\alpha(\eta) = \overline{\eta\alpha(\eta)}\alpha'(\eta) = |\alpha'(\eta)| > 0$$

and

$$\tilde{k}_\eta^\alpha = C_\alpha k_\eta^\alpha = \bar{\eta}\alpha(\eta)k_\eta^\alpha$$

(see [11, Chapter 7] for details). Now for a fixed $\lambda \in \mathbb{T}$ define

$$\alpha_\lambda = \frac{\lambda + \alpha(0)}{1 + \overline{\alpha(0)}\lambda}.$$

Then, $\alpha_\lambda \in \mathbb{T}$ and equation $\alpha(\eta) = \alpha_\lambda$ has precisely m distinct solutions $\eta_1, \dots, \eta_m$, each from $\mathbb{T}$ ([12, p. 6]). The corresponding kernel functions $\{k_{\eta_1}^\alpha, \dots, k_{\eta_m}^\alpha\}$ are pairwise orthogonal:

$$\langle k_{\eta_i}^\alpha, k_{\eta_j}^\alpha \rangle = k_{\eta_i}^\alpha(\eta_j) = \begin{cases} \|k_{\eta_i}^\alpha\|^2 > 0 & \text{for } i = j, \\ \frac{1 - |\alpha_\lambda|^2}{1 - \overline{\eta_i}\eta_j} = 0 & \text{for } i \neq j. \end{cases}$$

Hence, the set of normalized kernels

$$v_{\eta_j}^\alpha = \frac{1}{\|k_{\eta_j}^\alpha\|} k_{\eta_j}^\alpha = \frac{1}{\sqrt{|\alpha'(\eta_j)|}} k_{\eta_j}^\alpha, \quad j = 1, \dots, m,$$

is an orthonormal basis for K_α . The functions $v_{\eta_j}^\alpha$ can also be obtained as eigenvectors of the Clark operator U_λ^α - the one-dimensional unitary perturbation of the compressed shift $S_\alpha = A_z^\alpha$, given by

$$U_\lambda^\alpha = S_\alpha + \frac{\lambda + \alpha(0)}{1 - \overline{\alpha(0)}\lambda} (k_0^\alpha \otimes \tilde{k}_0^\alpha)$$

(see [7]). That is why the basis $\mathcal{V}_m^\alpha = \{v_{\eta_1}^\alpha, \dots, v_{\eta_m}^\alpha\}$ is called the Clark basis (see also [11, Chapter 11]).

It is well known that a bounded linear operator $T : H^2 \rightarrow H^2$ is a Toeplitz operator if and only if its matrix representation with respect to the standard basis is a Toeplitz matrix, that is, an infinite matrix with constant diagonals. It follows that truncated Toeplitz operators on K_α , $\alpha(z) = z^m$, can be described as those linear operators on K_α which are represented by finite Toeplitz matrices (with respect to monomial basis). In [8], the authors characterized the operators from $\mathcal{T}(\alpha)$ (for α such that $\dim K_\alpha = m < \infty$) using matrix representations of these operators with respect to the kernel basis $\mathcal{K}_m^\alpha$, with respect to the conjugate kernel basis $\tilde{\mathcal{K}}_m^\alpha$ and with respect to the Clark basis $\mathcal{V}_m^\alpha$. In each of those cases the matrix representing a truncated Toeplitz operator turns out to be determined by entries from its first (or any other fixed) row and the main diagonal. For some infinite-dimensional cases, these results were generalized in [17].

In [16], the authors generalized the characterizations from [8] to the case of asymmetric truncated Toeplitz operators acting between finite-dimensional model spaces. For example, it was proved in [16] that if $\dim K_\alpha = m < \infty$, $\dim K_\beta = n < \infty$ and the Blaschke products α, β have l common zeros, then a matrix representing an operator $A \in \mathcal{T}(\alpha, \beta)$ (with respect to $\mathcal{K}_m^\alpha$ and $\mathcal{K}_n^\beta$) is determined by its entries along the first row, first l entries along the main diagonal and last $n - l$ entries along the first column (first row and first column if $l = 0$). In this paper, we continue the study of matrix representations of operators from $\mathcal{T}(\alpha, \beta)$ (α, β - finite Blaschke products) and present some new characterizations. The novelty of our approach is that here we

consider matrix representations which are computed with respect to two different types of bases (for example the kernel basis in K_α and the conjugate kernel basis in K_β). The characterizations thus obtained turn out to be simpler than the ones from [16] as they do not depend on whether or not α and β have common zeros. Moreover, these results are new in that they were not known even for the case of truncated Toeplitz operators ($\alpha = \beta$).

In Sect. 2, we assume that α and β are two finite Blaschke products, each with distinct zeros. We then characterize operators from $\mathcal{T}(\alpha, \beta)$ using matrix representations computed with respect to the kernel basis in K_α and the conjugate kernel basis in K_β (and vice versa). In Sect. 3, we consider matrix representations computed with respect to the kernel (or conjugate kernel) basis in one space (assuming that the zeros of the corresponding finite Blaschke product are distinct) and Clark basis in the other (with no additional assumptions on the zeros of the corresponding finite Blaschke product).

2 Kernel Basis and Conjugate Kernel Basis

In what follows let α and β be two finite Blaschke products with zeros $a_1, \dots, a_m$ and $b_1, \dots, b_n$, respectively. In this section we assume that the zeros $a_1, \dots, a_m$ are distinct and that the zeros $b_1, \dots, b_n$ are also distinct.

We first consider matrix representations with respect to $\mathcal{K}_m^\alpha = \{k_{a_1}^\alpha, \dots, k_{a_m}^\alpha\}$ and $\tilde{\mathcal{K}}_n^\beta = \{\tilde{k}_{b_1}^\beta, \dots, \tilde{k}_{b_n}^\beta\}$.

Our reasoning is based on a duality relation between the kernels $\{k_{b_1}^\beta, \dots, k_{b_n}^\beta\}$ and the conjugate kernels $\{\tilde{k}_{b_1}^\beta, \dots, \tilde{k}_{b_n}^\beta\}$. Recall that for each $s \in \{1, \dots, n\}$ we have

$$\langle \tilde{k}_{b_j}^\beta, k_{b_s}^\beta \rangle = \tilde{k}_{b_j}^\beta(b_s) = \begin{cases} \beta'(b_s) & \text{for } j = s, \\ 0 & \text{for } j \neq s. \end{cases} \quad (2.1)$$

In other words, one can say that the set $\left\{ \frac{1}{\beta'(b_1)} k_{b_1}^\beta, \dots, \frac{1}{\beta'(b_n)} k_{b_n}^\beta \right\}$ is biorthogonal to $\tilde{\mathcal{K}}_n^\beta$. Therefore, if A is a linear transformation from K_α into K_β , then the element $r_{s,p}$ of its matrix representation $M_A = (r_{s,p})$ with respect to $\mathcal{K}_m^\alpha$ and $\tilde{\mathcal{K}}_n^\beta$ can be obtained by

$$r_{s,p} = \frac{1}{\beta'(b_s)} \langle A k_{a_p}^\alpha, k_{b_s}^\beta \rangle,$$

where $1 \leq p \leq m$ and $1 \leq s \leq n$. Indeed, since $A k_{a_p}^\alpha = \sum_{j=1}^n r_{j,p} \tilde{k}_{b_j}^\beta$, the above equality follows from (2.1).

As in the proof of [16, Thm. 3.1], for $A \in \mathcal{T}(\alpha, \beta)$ we will write $A k_{a_p}^\alpha$ as a linear combination of $k_{a_p}^\beta$ and $\{\tilde{k}_{b_1}^\beta, \dots, \tilde{k}_{b_n}^\beta\}$. It will then follow from (2.1) that in this case $r_{s,p}$ (for fixed s and p) is essentially determined by two coefficients of that linear combination, namely the ones corresponding to $k_{a_p}^\beta$ and $\tilde{k}_{b_s}^\beta$ (in contrast to the proof of [16, Thm. 3.1], where more of the coefficients were important). This will lead to simpler calculations.

Theorem 2.1 *Let α be a finite Blaschke product with $m > 0$ distinct zeros $a_1, \dots, a_m$, let β be a finite Blaschke product with $n > 0$ distinct zeros $b_1, \dots, b_n$ and let A be any linear transformation from K_α into K_β . If $M_A = (r_{s,p})$ is the matrix representation of A with respect to the bases $\mathcal{K}_m^\alpha = \{k_{a_1}^\alpha, \dots, k_{a_m}^\alpha\}$ and $\tilde{\mathcal{K}}_n^\beta = \{\tilde{k}_{b_1}^\beta, \dots, \tilde{k}_{b_n}^\beta\}$, then $A \in \mathcal{T}(\alpha, \beta)$ if and only if*

$$r_{s,p} = \frac{\beta'(b_s)(1 - \bar{a}_1 b_s) r_{s,1} + \beta'(b_1)(\bar{a}_1 b_1 - 1) r_{1,1} + \beta'(b_1)(1 - \bar{a}_p b_1) r_{1,p}}{\beta'(b_s)(1 - \bar{a}_p b_s)} \quad (2.2)$$

for all $1 \leq p \leq m$ and $1 \leq s \leq n$.

Proof The proof is similar to the proofs of [8, Thm. 1.4] and [16, Thm. 3.1].

Assume first that $A \in \mathcal{T}(\alpha, \beta)$ and let

$$A = A_{\bar{\chi} + \psi}^{\alpha, \beta} \text{ with } \chi \in K_\alpha, \psi \in K_\beta$$

(this is possible by [15, Cor. 2.6]). Moreover, write

$$\chi(z) = \sum_{i=1}^m c_i \tilde{k}_{a_i}^\alpha(z) = \sum_{i=1}^m c_i \frac{\alpha(z)}{z - a_i} \quad \text{and} \quad \psi(z) = \sum_{j=1}^n d_j \tilde{k}_{b_j}^\beta(z) = \sum_{j=1}^n d_j \frac{\beta(z)}{z - b_j}.$$

Then

$$A = A_{\bar{\chi} + \psi}^{\alpha, \beta} = \sum_{i=1}^m \bar{c}_i A_{\frac{\alpha(z)}{\bar{z} - \bar{a}_i}}^{\alpha, \beta} + \sum_{j=1}^n d_j A_{\frac{\beta(z)}{\bar{z} - \bar{b}_j}}^{\alpha, \beta}$$

and, since by [15, Prop. 3.1(a)],

$$A_{\frac{\alpha(z)}{\bar{z} - \bar{a}_i}}^{\alpha, \beta} = k_{a_i}^\beta \otimes \tilde{k}_{a_i}^\alpha \quad \text{and} \quad A_{\frac{\beta(z)}{\bar{z} - \bar{b}_j}}^{\alpha, \beta} = \tilde{k}_{b_j}^\beta \otimes k_{b_j}^\alpha$$

(here $f \otimes g$ denotes the standard rank-one operator given by $f \otimes g(h) = \langle h, g \rangle f$), we get

$$\begin{aligned} A k_{a_p}^\alpha &= \sum_{i=1}^m \bar{c}_i \langle k_{a_p}^\alpha, \tilde{k}_{a_i}^\alpha \rangle k_{a_i}^\beta + \sum_{j=1}^n d_j \langle k_{a_p}^\alpha, k_{b_j}^\alpha \rangle \tilde{k}_{b_j}^\beta \\ &= \bar{c}_p \overline{\alpha'(a_p)} k_{a_p}^\beta + \sum_{j=1}^n \frac{d_j}{1 - \bar{a}_p b_j} \tilde{k}_{b_j}^\beta \end{aligned}$$

for each $1 \leq p \leq m$. Therefore, for each $1 \leq p \leq m$ and $1 \leq s \leq n$ we have

$$\begin{aligned} r_{s,p} &= \frac{1}{\beta'(b_s)} \langle A k_{a_p}^\alpha, k_{b_s}^\beta \rangle = \frac{1}{\beta'(b_s)} \left\langle \bar{c}_p \overline{\alpha'(a_p)} k_{a_p}^\beta + \sum_{j=1}^n \frac{d_j}{1 - \bar{a}_p b_j} \tilde{k}_{b_j}^\beta, k_{b_s}^\beta \right\rangle \\ &= \frac{1}{\beta'(b_s)} \bar{c}_p \overline{\alpha'(a_p)} k_{a_p}^\beta(b_s) + \frac{1}{\beta'(b_s)} \sum_{j=1}^n \frac{d_j}{1 - \bar{a}_p b_j} \tilde{k}_{b_j}^\beta(b_s) \\ &= \frac{1}{\beta'(b_s)} \bar{c}_p \overline{\alpha'(a_p)} \frac{1}{1 - \bar{a}_p b_s} + \frac{d_s}{1 - \bar{a}_p b_s} = \frac{d_s \beta'(b_s) + \bar{c}_p \overline{\alpha'(a_p)}}{\beta'(b_s)(1 - \bar{a}_p b_s)}. \end{aligned}$$

It is now easy to verify that (2.2) holds.

To complete the proof recall that $\mathcal{T}(\alpha, \beta)$, and so the linear space V_0 of all $n \times m$ matrices representing operators from $\mathcal{T}(\alpha, \beta)$ (with respect to the bases in question), has dimension $m + n - 1$ [16, Prop.2.1]. Since $V_0 \subset V$, where V is the linear space of all $n \times m$ matrices satisfying (2.2), and clearly the dimension of V is also $m + n - 1$, we have $V = V_0$. $\square$

Remark 2.2 (a) By Theorem 2.1 a matrix representing an operator from $\mathcal{T}(\alpha, \beta)$ is completely determined by $m + n - 1$ of its entries: the entries along the first row and the first column. The same reasoning can be used to show that the first row and the first column can be replaced by any other row and any other column, respectively. More precisely, for any fixed $p_0 \in \{1, \dots, m\}$ and $s_0 \in \{1, \dots, n\}$, condition (2.2) can be replaced by

$$r_{s,p} = \frac{\beta'(b_s)(1 - \bar{a}_{p_0} b_s) r_{s,p_0} + \beta'(b_{s_0})(\bar{a}_{p_0} b_{s_0} - 1) r_{s_0,p_0} + \beta'(b_{s_0})(1 - \bar{a}_p b_{s_0}) r_{s_0,p}}{\beta'(b_s)(1 - \bar{a}_p b_s)}$$

for all $1 \leq p \leq m$ and $1 \leq s \leq n$.

(b) Note that if A is a linear transformation represented by the matrix $M_A = (r_{s,p})$, then $A = A_\varphi^{\alpha,\beta}$, where (by the proof of Theorem 2.1) the symbol φ of the form

$$\varphi = \overline{\chi} + \psi = \sum_{i=1}^m c_i \tilde{k}_{a_i}^\alpha + \sum_{j=1}^n d_j \tilde{k}_{b_j}^\beta$$

can be obtained from M_A using the following system of $m + n - 1$ equations:

$$\begin{cases} r_{1,p} = \frac{\beta'(b_1)}{\beta'(b_1)(1 - \bar{a}_p b_1)} d_1 + \frac{\overline{\alpha'(a_p)}}{\beta'(b_1)(1 - \bar{a}_p b_1)} \bar{c}_p, & 1 \leq p \leq m, \\ r_{s,1} = \frac{\beta'(b_s)}{\beta'(b_s)(1 - \bar{a}_1 b_s)} d_s + \frac{\overline{\alpha'(a_1)}}{\beta'(b_s)(1 - \bar{a}_1 b_s)} \bar{c}_1, & 1 < s \leq n. \end{cases}$$

Using Theorem 2.1 we can now describe operators from $\mathcal{T}(\alpha, \beta)$ in terms of their matrix representations with respect to $\tilde{\mathcal{K}}_m^\alpha = \{\tilde{k}_{a_1}^\alpha, \dots, \tilde{k}_{a_m}^\alpha\}$ and $\mathcal{K}_n^\beta = \{k_{b_1}^\beta, \dots, k_{b_n}^\beta\}$.

If A is a linear transformation from K_α into K_β , then its matrix representation $\tilde{M}_A = (t_{s,p})$ with respect to $\tilde{\mathcal{K}}_m^\alpha$ and $\mathcal{K}_n^\beta$ is given by

$$t_{s,p} = \frac{1}{\overline{\beta'(b_s)}} \langle A\tilde{k}_{a_p}^\alpha, \tilde{k}_{b_s}^\beta \rangle$$

for all $1 \leq p \leq m$ and $1 \leq s \leq n$ (as before, this follows from (2.1)).

Consider $B = C_\beta A C_\alpha$. Then

$$\begin{aligned} t_{s,p} &= \frac{1}{\overline{\beta'(b_s)}} \langle A\tilde{k}_{a_p}^\alpha, \tilde{k}_{b_s}^\beta \rangle = \overline{\frac{1}{\beta'(b_s)} \langle C_\beta k_{b_s}^\beta, A C_\alpha k_{a_p}^\alpha \rangle} \\ &= \overline{\frac{1}{\beta'(b_s)} \langle C_\beta A C_\alpha k_{a_p}^\alpha, k_{b_s}^\beta \rangle} = \bar{r}_{s,p}, \end{aligned} \quad (2.3)$$

where $M_B = (r_{s,p})$ is the matrix representation of B with respect to the bases $\mathcal{K}_m^\alpha$ and $\tilde{\mathcal{K}}_n^\beta$. Since $A \in \mathcal{T}(\alpha, \beta)$ if and only if $B \in \mathcal{T}(\alpha, \beta)$ ([14, Prop. 3.2(a), Rem. 3.3(a)]), we get the following characterization.

Theorem 2.3 *Let α be a finite Blaschke product with $m > 0$ distinct zeros $a_1, \dots, a_m$, let β be a finite Blaschke product with $n > 0$ distinct zeros $b_1, \dots, b_n$ and let A be any linear transformation from K_α into K_β . If $\tilde{M}_A = (t_{s,p})$ is the matrix representation of A with respect to the bases $\tilde{\mathcal{K}}_m^\alpha = \{\tilde{k}_{a_1}^\alpha, \dots, \tilde{k}_{a_m}^\alpha\}$ and $\mathcal{K}_n^\beta = \{k_{b_1}^\beta, \dots, k_{b_n}^\beta\}$, then $A \in \mathcal{T}(\alpha, \beta)$ if and only if*

$$t_{s,p} = \frac{\overline{\beta'(b_s)}(1 - a_1 \bar{b}_s)t_{s,1} + \overline{\beta'(b_1)}(a_1 \bar{b}_1 - 1)t_{1,1} + \overline{\beta'(b_1)}(1 - a_p \bar{b}_1)t_{1,p}}{\overline{\beta'(b_s)}(1 - a_p \bar{b}_s)} \quad (2.4)$$

for all $1 \leq p \leq m$ and $1 \leq s \leq n$.

Proof As above, let $\tilde{M}_A = (t_{s,p})$ be the matrix representation of A with respect to $\tilde{\mathcal{K}}_m^\alpha$ and $\mathcal{K}_n^\beta$, and let $M_B = (r_{s,p})$ be the matrix representation of $B = C_\beta A C_\alpha$ with respect to $\mathcal{K}_m^\alpha$ and $\tilde{\mathcal{K}}_n^\beta$. Then $A \in \mathcal{T}(\alpha, \beta)$ if and only if $B \in \mathcal{T}(\alpha, \beta)$, which by Theorem 2.1 happens if and only if M_B satisfies (2.2). Using (2.3) it is now easy to see that the latter happens if and only if M_A satisfies (2.4). $\square$

Remark 2.4 (a) In this case, a matrix representing an operator from $\mathcal{T}(\alpha, \beta)$ is also completely determined by its entries along the first row and the first column. Again, the first row and column can be replaced by any other row and column: for any fixed $p_0 \in \{1, \dots, m\}$ and $s_0 \in \{1, \dots, n\}$, condition (2.4) can be replaced by

$$t_{s,p} = \frac{\overline{\beta'(b_s)}(1 - a_{p_0} \bar{b}_s)t_{s,p_0} + \overline{\beta'(b_{s_0})}(a_{p_0} \bar{b}_{s_0} - 1)t_{s_0,p_0} + \overline{\beta'(b_{s_0})}(1 - a_p \bar{b}_{s_0})t_{s_0,p}}{\overline{\beta'(b_s)}(1 - a_p \bar{b}_s)}$$

for all $1 \leq p \leq m$ and $1 \leq s \leq n$.

(b) If $\tilde{M}_A = (t_{s,p})$ is the matrix representation of A with respect to $\tilde{\mathcal{K}}_m^\alpha$ and $\mathcal{K}_n^\beta$, and $B = C_\beta A C_\alpha$, then $M_B = (\bar{t}_{s,p})$ is the matrix representation of B with respect to $\mathcal{K}_m^\alpha$ and $\tilde{\mathcal{K}}_n^\beta$ (by (2.3)). If $\tilde{M}_A$ satisfies (2.4), then M_B satisfies (2.2) and, as in

Remark 2.2(b), we can use $(\bar{t}_{s,p})$ to find $\varphi \in L^2$ such that $B = A_{\varphi}^{\alpha,\beta}$. Then $\overline{\alpha\varphi}\beta$ is a symbol for A since $A = C_{\beta}A_{\varphi}^{\alpha,\beta}C_{\alpha} = A_{\overline{\alpha\varphi}\beta}^{\alpha,\beta}$ (see [14, Prop. 3.2(a)]).

- (c) Note that characterizations given in Theorem 2.1 and Theorem 2.3 do not depend on whether α and β have common zeros or not (in contrast to the characterizations given in [16, Thm. 3.1, Thm. 3.3]).

Remark 2.5 In [18] the authors consider matrix representations of the so-called truncated Hankel operators. These operators may be defined in several ways. The authors of [18] follow the definition proposed by C. Gu [13]: the truncated Hankel operator B_{φ}^{α} , $\varphi \in L^2$, is defined on a dense subset of K_{α} by

$$B_{\varphi}^{\alpha} f = P_{\alpha} J(I - P)(\varphi f), \quad f \in K_{\alpha}^{\infty},$$

where $J : L^2 \rightarrow L^2$, $Jf(z) = \bar{z}f(\bar{z})$ for $|z| = 1$. It is shown in [18, Thm. 3.1] that, for a finite Blaschke product α with distinct zeros $a_1, \dots, a_n$, a linear transformation B on K_{α} is a truncated Hankel operator if and only if its matrix representation $M_A = (r_{s,p})$ with respect to the kernel basis $\mathcal{K}_n^{\alpha} = \{k_{a_1}^{\alpha}, \dots, k_{a_n}^{\alpha}\}$ satisfies

$$r_{s,p} = \frac{\overline{\alpha'(a_s)}(1 - a_s a_1) r_{s,1} - \overline{\alpha'(a_1)}(1 - a_1^2) r_{1,1} + \overline{\alpha'(a_p)}(1 - a_1 a_p) r_{1,p}}{\overline{\alpha'(a_s)}(1 - a_s a_p)} \quad (2.5)$$

for all $1 \leq s, p \leq n$.

Recall from [14] that B is a truncated Hankel operator if and only if $A = J^{\#} B C_{\alpha}$ (here $J^{\#} : L^2 \rightarrow L^2$, $J^{\#} f(z) = \overline{f(\bar{z})}$) belongs to $\mathcal{T}(\alpha, \beta)$ with $\beta = \alpha^{\#} = J^{\#} \alpha$. Note that if α is a finite Blaschke product with distinct zeros $a_1, \dots, a_n$, then $\beta = \alpha^{\#}$ is a finite Blaschke product with distinct zeros $b_1, \dots, b_n$, where $b_s = \bar{a}_s$, $s = 1, \dots, n$. So, by Theorem 2.3, $A \in \mathcal{T}(\alpha, \beta)$ if and only if its matrix representation $\tilde{M}_A = (t_{s,p})$ with respect to $\tilde{\mathcal{K}}_n^{\alpha}$ and $\mathcal{K}_n^{\beta}$ satisfies

$$t_{s,p} = \frac{\overline{\beta'(\bar{a}_s)}(1 - a_1 a_s) t_{s,1} + \overline{\beta'(\bar{a}_1)}(a_1 a_1 - 1) t_{1,1} + \overline{\beta'(\bar{a}_1)}(1 - a_p a_1) t_{1,p}}{\overline{\beta'(\bar{a}_s)}(1 - a_p a_s)} \quad (2.6)$$

for all $1 \leq p, s \leq n$. Since here $\beta'(b_s) = \beta'(\bar{a}_s) = \overline{\alpha'(a_s)}$ and

$$J^{\#} \tilde{k}_{b_s}^{\beta} = J^{\#} C_{\alpha^{\#}} k_{a_s}^{\alpha^{\#}} = C_{\alpha} J^{\#} k_{a_s}^{\alpha^{\#}} = C_{\alpha} k_{a_s}^{\alpha} = \tilde{k}_{a_s}^{\alpha}$$

(see [6]), we get

$$\begin{aligned} t_{s,p} &= \frac{1}{\beta'(b_s)} \langle A \tilde{k}_{a_p}^{\alpha}, \tilde{k}_{b_s}^{\beta} \rangle = \frac{1}{\alpha'(a_s)} \langle J^{\#} B C_{\alpha} \tilde{k}_{a_p}^{\alpha}, \tilde{k}_{b_s}^{\beta} \rangle \\ &= \frac{1}{\alpha'(a_s)} \langle B k_{a_p}^{\alpha}, J^{\#} \tilde{k}_{b_s}^{\beta} \rangle = \frac{1}{\alpha'(a_s)} \langle B k_{a_p}^{\alpha}, \tilde{k}_{a_s}^{\alpha} \rangle = \bar{r}_{s,p}. \end{aligned}$$

It follows that (2.5) is equivalent to (2.6) and [18, Thm. 3.1] can be obtained from Theorem 2.3. Similarly, [18, Thm. 3.2] can be obtained from Theorem 2.1.

3 Kernel Basis and Clark Basis, Conjugate Kernel Basis and Clark Basis

Let α and β be two finite Blaschke products with (not necessarily distinct) zeros $a_1, \dots, a_m$ and $b_1, \dots, b_n$, respectively.

For a fixed $\lambda_1 \in \mathbb{T}$ let $\eta_1, \dots, \eta_m \in \mathbb{T}$ be the distinct solutions of the equation

$$\alpha(\eta) = \alpha_{\lambda_1} = \frac{\lambda_1 + \alpha(0)}{1 + \overline{\alpha(0)}\lambda_1} \quad (3.1)$$

and let $\mathcal{V}_m^\alpha = \{v_{\eta_1}^\alpha, \dots, v_{\eta_m}^\alpha\}$, $v_{\eta_j}^\alpha = k_{\eta_j}^\alpha / \sqrt{|\alpha'(\eta_j)|}$ for $j = 1, \dots, m$, be the corresponding Clark basis. Similarly, for a fixed $\lambda_2 \in \mathbb{T}$ let $\zeta_1, \dots, \zeta_n \in \mathbb{T}$ be the solutions of

$$\beta(\zeta) = \beta_{\lambda_2} = \frac{\lambda_2 + \beta(0)}{1 + \overline{\beta(0)}\lambda_2}$$

and let $\mathcal{V}_n^\beta = \{v_{\zeta_1}^\beta, \dots, v_{\zeta_n}^\beta\}$, where $v_{\zeta_j}^\beta = k_{\zeta_j}^\beta / \sqrt{|\beta'(\zeta_j)|}$ for $j = 1, \dots, n$.

3.1 $\mathcal{V}_m^\alpha$ and $\mathcal{K}_n^\beta$, $\mathcal{V}_m^\alpha$ and $\widetilde{\mathcal{K}}_n^\beta$

We first assume that the zeros $b_1, \dots, b_n$ are distinct and we consider matrix representations with respect to $\mathcal{V}_m^\alpha$ and $\mathcal{K}_n^\beta = \{k_{b_1}^\beta, \dots, k_{b_n}^\beta\}$.

As in Sect. 2, (2.1) implies that the elements of the matrix representation $M_A = (r_{s,p})$ of a linear transformation $A : K_\alpha \rightarrow K_\beta$ (with respect to $\mathcal{V}_m^\alpha$ and $\mathcal{K}_n^\beta$) are given by

$$r_{s,p} = \frac{1}{\beta'(b_s)} \langle A v_{\eta_p}^\alpha, \widetilde{k}_{b_s}^\beta \rangle. \quad (3.2)$$

Theorem 3.1 *Let α be a finite Blaschke product with $m > 0$ (not necessarily distinct) zeros, let $\mathcal{V}_m^\alpha = \{v_{\eta_1}^\alpha, \dots, v_{\eta_m}^\alpha\}$ be the Clark basis for K_α corresponding to $\lambda_1 \in \mathbb{T}$ and assume that β is a finite Blaschke product with $n > 0$ distinct zeros $b_1, \dots, b_n$. Moreover, let A be any linear transformation from K_α into K_β . If $M_A = (r_{s,p})$ is the matrix representation of A with respect to the bases $\mathcal{V}_m^\alpha$ and $\mathcal{K}_n^\beta$, then $A \in \mathcal{T}(\alpha, \beta)$ if and only if*

$$\begin{aligned} r_{s,p} = & \frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_p)|}} \frac{\bar{\eta}_1 - \bar{b}_s}{\bar{\eta}_p - \bar{b}_s} r_{s,1} + \frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_p)|}} \frac{\overline{\beta'(b_1)}}{\overline{\beta'(b_s)}} \frac{\bar{b}_1 - \bar{\eta}_1}{\bar{\eta}_p - \bar{b}_s} r_{1,1} \\ & + \frac{\overline{\beta'(b_1)}}{\overline{\beta'(b_s)}} \frac{\bar{\eta}_p - \bar{b}_1}{\bar{\eta}_p - \bar{b}_s} r_{1,p} \end{aligned} \quad (3.3)$$

for all $1 \leq p \leq m$ and $1 \leq s \leq n$.

Proof Assume that $A \in \mathcal{T}(\alpha, \beta)$ and let $M_A = (r_{s,p})$ be its matrix representation with respect to $\mathcal{V}_m^\alpha$ and $\mathcal{K}_n^\beta$. To show that $r_{s,p}$ satisfies (3.3), we follow the reasoning used in the proof of [16, Thm. 3.4] (compare [8, Proof of Thm. 1.11]).

Pick $m + n - 1$ distinct points $\xi_1, \dots, \xi_{m+n-1}$ from $\mathbb{T} \setminus \{\eta_1, \dots, \eta_m\}$. By [16, Cor. 2.4], the set $\{k_{\xi_i}^\beta \otimes k_{\xi_i}^\alpha : i = 1, \dots, m + n - 1\}$ is a basis for $\mathcal{T}(\alpha, \beta)$. Hence,

$$A = \sum_{i=1}^{m+n-1} c_i (k_{\xi_i}^{\beta} \otimes k_{\xi_i}^{\alpha})$$

for some $c_1, \dots, c_{m+n-1} \in \mathbb{C}$. From this and (3.2) it follows that

$$\begin{aligned} r_{s,p} &= \frac{1}{\beta'(b_s)} \left\langle \sum_{i=1}^{m+n-1} c_i (k_{\xi_i}^{\beta} \otimes k_{\xi_i}^{\alpha}) v_{\eta_p}^{\alpha}, \tilde{k}_{b_s}^{\beta} \right\rangle \\ &= \frac{1}{\beta'(b_s)} \frac{1}{\sqrt{|\alpha'(\eta_p)|}} \left\langle \sum_{i=1}^{m+n-1} c_i \langle k_{\eta_p}^{\alpha}, k_{\xi_i}^{\alpha} \rangle k_{\xi_i}^{\beta}, \tilde{k}_{b_s}^{\beta} \right\rangle \\ &= \frac{1}{\beta'(b_s)} \frac{1}{\sqrt{|\alpha'(\eta_p)|}} \sum_{i=1}^{m+n-1} c_i k_{\eta_p}^{\alpha}(\xi_i) \overline{\tilde{k}_{b_s}^{\beta}(\xi_i)} \\ &= \frac{1}{\beta'(b_s)} \frac{1}{\sqrt{|\alpha'(\eta_p)|}} \sum_{i=1}^{m+n-1} c_i \frac{1 - \overline{\alpha(\eta_p)} \alpha(\xi_i)}{1 - \overline{\eta_p} \xi_i} \frac{\overline{\beta(\xi_i)}}{\xi_i - \overline{b_s}}. \end{aligned}$$

Since $\alpha(\eta_p) = \alpha_{\lambda_1}$ for all $1 \leq p \leq m$ (see (3.1)), we have

$$r_{s,p} = \frac{1}{\beta'(b_s)} \frac{1}{\sqrt{|\alpha'(\eta_p)|}} \sum_{i=1}^{m+n-1} d_i \bar{\xi}_i \frac{1}{(\overline{\eta_p} - \bar{\xi}_i)(\overline{b_s} - \bar{\xi}_i)},$$

where $d_i = (1 - \overline{\alpha_{\lambda_1}} \alpha(\xi_i)) \overline{\beta(\xi_i)}$, $1 \leq i \leq m+n-1$, are constants which do not depend on p and s . The condition (3.3) can now be verified using the equality

$$\frac{\overline{\eta_p} - \overline{b_s}}{(\overline{\eta_p} - \bar{\xi}_i)(\overline{b_s} - \bar{\xi}_i)} = \frac{1}{\overline{b_s} - \bar{\xi}_i} - \frac{1}{\overline{\eta_p} - \bar{\xi}_i}.$$

Indeed, for $1 \leq p \leq m$ and $1 \leq s \leq n$ we have

$$\begin{aligned} r_{s,p} &= \frac{1}{\beta'(b_s)} \frac{1}{\sqrt{|\alpha'(\eta_p)|}} \frac{1}{\overline{\eta_p} - \overline{b_s}} \sum_{i=1}^{m+n-1} d_i \bar{\xi}_i \frac{\overline{\eta_p} - \overline{b_s}}{(\overline{\eta_p} - \bar{\xi}_i)(\overline{b_s} - \bar{\xi}_i)} \\ &= \frac{1}{\beta'(b_s)} \frac{(\overline{\eta_p} - \overline{b_s})^{-1}}{\sqrt{|\alpha'(\eta_p)|}} \sum_{i=1}^{m+n-1} d_i \bar{\xi}_i \left(\frac{\overline{\eta_p} - \overline{b_1}}{(\overline{\eta_p} - \bar{\xi}_i)(\overline{b_1} - \bar{\xi}_i)} + \frac{\overline{b_1} - \overline{\eta_1}}{(\overline{b_1} - \bar{\xi}_i)(\overline{\eta_1} - \bar{\xi}_i)} + \frac{\overline{\eta_1} - \overline{b_s}}{(\overline{\eta_1} - \bar{\xi}_i)(\overline{b_s} - \bar{\xi}_i)} \right) \\ &= \frac{1}{\beta'(b_s)} \frac{1}{\sqrt{|\alpha'(\eta_p)|}} \frac{1}{\overline{\eta_p} - \overline{b_s}} \left(\overline{\beta'(b_1)} \sqrt{|\alpha'(\eta_p)|} (\overline{\eta_p} - \overline{b_1}) r_{1,p} \right. \\ &\quad \left. + \overline{\beta'(b_1)} \sqrt{|\alpha'(\eta_1)|} (\overline{b_1} - \overline{\eta_1}) r_{1,1} + \overline{\beta'(b_s)} \sqrt{|\alpha'(\eta_1)|} (\overline{\eta_1} - \overline{b_s}) r_{s,1} \right) \\ &= \frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_p)|}} \frac{\overline{\eta_1} - \overline{b_s}}{\overline{\eta_p} - \overline{b_s}} r_{s,1} + \frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_p)|}} \frac{\overline{\beta'(b_1)}}{\beta'(b_s)} \frac{\overline{b_1} - \overline{\eta_1}}{\overline{\eta_p} - \overline{b_s}} r_{1,1} + \frac{\overline{\beta'(b_1)}}{\beta'(b_s)} \frac{\overline{\eta_p} - \overline{b_1}}{\overline{\eta_p} - \overline{b_s}} r_{1,p}. \end{aligned}$$

The reminder of the proof is analogous to that of Theorem 2.1 (use the dimension argument). $\square$

To describe the operators from $\mathcal{T}(\alpha, \beta)$ in terms of their matrix representations with respect to $\mathcal{V}_m^\alpha$ and $\tilde{\mathcal{K}}_n^\beta = \{\tilde{k}_{b_1}^\beta, \dots, \tilde{k}_{b_n}^\beta\}$ we again use the fact that $A \in \mathcal{T}(\alpha, \beta)$ if and only if $B = C_\beta A C_\alpha \in \mathcal{T}(\alpha, \beta)$.

Here, the matrix representation $\tilde{M}_A = (t_{s,p})$ of a linear transformation $A : K_\alpha \rightarrow K_\beta$ with respect to the bases $\mathcal{V}_m^\alpha$ and $\tilde{\mathcal{K}}_n^\beta$ is given by

$$t_{s,p} = \frac{1}{\beta'(b_s)} \langle A v_{\eta_p}^\alpha, k_{b_s}^\beta \rangle.$$

Since

$$C_\alpha v_{\eta_p}^\alpha = \bar{\eta}_p \alpha(\eta_p) v_{\eta_p}^\alpha = \bar{\eta}_p \alpha_{\lambda_1} v_{\eta_p}^\alpha,$$

we get that

$$\begin{aligned} t_{s,p} &= \frac{1}{\beta'(b_s)} \langle A v_{\eta_p}^\alpha, k_{b_s}^\beta \rangle = \frac{1}{\beta'(b_s)} \langle C_\beta k_{b_s}^\beta, C_\beta A v_{\eta_p}^\alpha \rangle \\ &= \frac{1}{\beta'(b_s)} \overline{\langle B C_\alpha v_{\eta_p}^\alpha, \tilde{k}_{b_s}^\beta \rangle} = \eta_p \bar{\alpha}_{\lambda_1} \overline{\left(\frac{1}{\beta'(b_s)} \langle B v_{\eta_p}^\alpha, \tilde{k}_{b_s}^\beta \rangle \right)} = \eta_p \bar{\alpha}_{\lambda_1} \bar{r}_{s,p}, \end{aligned} \quad (3.4)$$

where $M_B = (r_{s,p})$ is the matrix representation of B with respect to $\mathcal{V}_m^\alpha$ and $\mathcal{K}_n^\beta$.

Theorem 3.2 *Let α be a finite Blaschke product with $m > 0$ (not necessarily distinct) zeros, let $\mathcal{V}_m^\alpha = \{v_{\eta_1}^\alpha, \dots, v_{\eta_m}^\alpha\}$ be the Clark basis for K_α corresponding to $\lambda_1 \in \mathbb{T}$ and assume that β is a finite Blaschke product with $n > 0$ distinct zeros $b_1, \dots, b_n$. Moreover, let A be any linear transformation from K_α into K_β . If $\tilde{M}_A = (t_{s,p})$ is the matrix representation of A with respect to the bases $\mathcal{V}_m^\alpha$ and $\tilde{\mathcal{K}}_n^\beta$, then $A \in \mathcal{T}(\alpha, \beta)$ if and only if*

$$\begin{aligned} t_{s,p} &= \frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_p)|}} \frac{1 - \bar{\eta}_1 b_s}{1 - \bar{\eta}_p b_s} t_{s,1} + \frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_p)|}} \frac{\beta'(b_1)}{\beta'(b_s)} \frac{\bar{\eta}_1 b_1 - 1}{1 - \bar{\eta}_p b_s} t_{1,1} \\ &\quad + \frac{\beta'(b_1)}{\beta'(b_s)} \frac{1 - \bar{\eta}_p b_1}{1 - \bar{\eta}_p b_s} t_{1,p} \end{aligned} \quad (3.5)$$

for all $1 \leq p \leq m$ and $1 \leq s \leq n$.

Proof Let $\tilde{M}_A = (t_{s,p})$ be the matrix representation of A with respect to $\mathcal{V}_m^\alpha$ and $\tilde{\mathcal{K}}_n^\beta$, and let $M_B = (r_{s,p})$ be the matrix representation of $B = C_\beta A C_\alpha$ with respect to $\mathcal{V}_m^\alpha$ and $\mathcal{K}_n^\beta$. As in the proof of Theorem 2.3, $A \in \mathcal{T}(\alpha, \beta)$ if and only if $B \in \mathcal{T}(\alpha, \beta)$. By Theorem 3.1 the latter happens if and only if M_B satisfies (3.3). It is not difficult to verify using (3.4) that M_B satisfies (3.3) if and only if $\tilde{M}_A$ satisfies (3.5). $\square$

Remark 3.3 (a) Here, a matrix representing an asymmetric truncated Toeplitz operator is also determined by its entries along the first row and the first column. Again, the first row and column can be replaced by any other row and column.

(b) To determine a symbol for A from $M_A = (r_{s,p})$ note that by the proof of Theorem 3.1,

$$A = \sum_{i=1}^{m+n-1} c_i (k_{\xi_i}^\beta \otimes k_{\xi_i}^\alpha) = A_\psi^{\alpha,\beta} \quad \text{with} \quad \psi = \sum_{i=1}^{m+n-1} c_i (k_{\xi_i}^\beta + \bar{k}_{\xi_i}^\alpha - 1).$$

(see [16, Prop. 2.3(b)]). Now $c_1, \dots, c_{m+n-1}$ are determined by

$$\begin{cases} r_{1,p} = \frac{1}{\beta'(b_1)} \frac{1}{\sqrt{|\alpha'(\eta_p)|}} \sum_{i=1}^{m+n-1} c_i \frac{1-\bar{\alpha}_{\lambda_1} \alpha(\xi_i)}{1-\bar{\eta}_p \xi_i} \frac{\bar{\beta}(\xi_i)}{\xi_i - b_1}, & 1 \leq p \leq m, \\ r_{s,1} = \frac{1}{\beta'(b_s)} \frac{1}{\sqrt{|\alpha'(\eta_1)|}} \sum_{i=1}^{m+n-1} c_i \frac{1-\bar{\alpha}_{\lambda_1} \alpha(\xi_i)}{1-\bar{\eta}_1 \xi_i} \frac{\bar{\beta}(\xi_i)}{\xi_i - b_s}, & 1 < s \leq n. \end{cases}$$

To determine a symbol for A from $\tilde{M}_A = (t_{s,p})$ use (3.4) and [14, Prop. 3.2(a)] as in Remark 2.4(b).

3.2 $\tilde{\mathcal{K}}_m^\alpha$ and $\mathcal{V}_n^\beta, \mathcal{K}_m^\alpha$ and $\mathcal{V}_n^\beta$

We now assume that the zeros $a_1, \dots, a_m$ are distinct and we first consider matrix representations with respect to $\tilde{\mathcal{K}}_m^\alpha$ and $\mathcal{V}_n^\beta$.

Since the Clark basis $\mathcal{V}_n^\beta$ is orthogonal, the elements of the matrix representation $M_A = (r_{s,p})$ of a linear transformation $A : K_\alpha \rightarrow K_\beta$ with respect to $\tilde{\mathcal{K}}_m^\alpha$ and $\mathcal{V}_n^\beta$ are given by

$$r_{s,p} = \langle A \tilde{k}_{a_p}^\alpha, v_{\xi_s}^\beta \rangle.$$

Observe that here, for $1 \leq p \leq m$ and $1 \leq s \leq n$,

$$r_{s,p} = \langle \tilde{k}_{a_p}^\alpha, A^* v_{\xi_s}^\beta \rangle = \alpha'(a_p) \overline{\left(\frac{1}{\alpha'(a_p)} \langle A^* v_{\xi_s}^\beta, \tilde{k}_{a_p}^\alpha \rangle \right)} = \alpha'(a_p) \overline{r_{p,s}^*}, \quad (3.6)$$

where $M_{A^*} = (r_{p,s}^*)$ is the matrix representation of the adjoint $A^* : K_\beta \rightarrow K_\alpha$ with respect to $\mathcal{V}_n^\beta$ and $\mathcal{K}_m^\alpha$ (compare with (3.2)). By Theorem 3.1 (with α in place of β and β in place of α), $A^* \in \mathcal{T}(\beta, \alpha)$ if and only if

$$\begin{aligned} r_{p,s}^* &= \frac{\sqrt{|\beta'(\xi_1)|}}{\sqrt{|\beta'(\xi_s)|}} \frac{\bar{\xi}_1 - \bar{a}_p}{\bar{\xi}_s - \bar{a}_p} r_{p,1}^* + \frac{\sqrt{|\beta'(\xi_1)|}}{\sqrt{|\beta'(\xi_s)|}} \frac{\overline{\alpha'(a_1)}}{\overline{\alpha'(a_p)}} \frac{\bar{a}_1 - \bar{\xi}_1}{\bar{\xi}_s - \bar{a}_p} r_{1,1}^* \\ &\quad + \frac{\overline{\alpha'(a_1)}}{\overline{\alpha'(a_p)}} \frac{\bar{\xi}_s - \bar{a}_1}{\bar{\xi}_s - \bar{a}_p} r_{1,s}^* \end{aligned} \quad (3.7)$$

for all $1 \leq p \leq m$ and $1 \leq s \leq n$. Since $A \in \mathcal{T}(\alpha, \beta)$ if and only if $A^* \in \mathcal{T}(\beta, \alpha)$ [14], the next theorem follows easily from (3.6) and (3.7).

Theorem 3.4 *Let α be a finite Blaschke product with $m > 0$ distinct zeros $a_1, \dots, a_m$, let β be a finite Blaschke product with $n > 0$ (not necessarily distinct) zeros and let $\mathcal{V}_n^\beta = \{v_{\zeta_1}^\beta, \dots, v_{\zeta_n}^\beta\}$ be the Clark basis for K_β corresponding to $\lambda_2 \in \mathbb{T}$. Moreover, let A be any linear transformation from K_α into K_β . If $M_A = (r_{s,p})$ is the matrix representation of A with respect to the bases $\tilde{\mathcal{K}}_m^\alpha$ and $\mathcal{V}_n^\beta$, then $A \in \mathcal{T}(\alpha, \beta)$ if and only if*

$$r_{s,p} = \frac{a_1 - \zeta_s}{a_p - \zeta_s} r_{s,1} + \frac{\sqrt{|\beta'(\zeta_1)|}}{\sqrt{|\beta'(\zeta_s)|}} \frac{\zeta_1 - a_1}{a_p - \zeta_s} r_{1,1} + \frac{\sqrt{|\beta'(\zeta_1)|}}{\sqrt{|\beta'(\zeta_s)|}} \frac{a_p - \zeta_1}{a_p - \zeta_s} r_{1,p} \quad (3.8)$$

for all $1 \leq p \leq m$ and $1 \leq s \leq n$.

Similarly, the matrix representation $\tilde{M}_A = (t_{s,p})$ of a linear transformation $A : K_\alpha \rightarrow K_\beta$ with respect to $\mathcal{K}_m^\alpha$ and $\mathcal{V}_n^\beta$ is given by

$$t_{s,p} = \langle Ak_{a_p}^\alpha, v_{\zeta_s}^\beta \rangle$$

and

$$t_{s,p} = \overline{\alpha'(a_p) t_{p,s}^*},$$

where $\tilde{M}_{A^*} = (t_{p,s}^*)$ is the matrix representation of $A^* : K_\beta \rightarrow K_\alpha$ with respect to $\mathcal{V}_n^\beta$ and $\tilde{\mathcal{K}}_m^\alpha$. By Theorem 3.2, $A^* \in \mathcal{T}(\beta, \alpha)$ if and only if

$$t_{p,s}^* = \frac{\sqrt{|\beta'(\zeta_1)|}}{\sqrt{|\beta'(\zeta_s)|}} \frac{1 - \bar{\zeta}_1 a_p}{1 - \bar{\zeta}_s a_p} t_{p,1}^* + \frac{\sqrt{|\beta'(\zeta_1)|}}{\sqrt{|\beta'(\zeta_s)|}} \frac{\alpha'(a_1)}{\alpha'(a_p)} \frac{\bar{\zeta}_1 a_1 - 1}{1 - \bar{\zeta}_s a_p} t_{1,1}^* + \frac{\alpha'(a_1)}{\alpha'(a_p)} \frac{1 - \bar{\zeta}_s a_1}{1 - \bar{\zeta}_s a_p} t_{1,s}^*$$

for all $1 \leq p \leq m$, $1 \leq s \leq n$, and we get the following.

Theorem 3.5 *Let α be a finite Blaschke product with $m > 0$ distinct zeros $a_1, \dots, a_m$, let β be a finite Blaschke product with $n > 0$ (not necessarily distinct) zeros and let $\mathcal{V}_n^\beta = \{v_{\zeta_1}^\beta, \dots, v_{\zeta_n}^\beta\}$ be the Clark basis for K_β corresponding to $\lambda_2 \in \mathbb{T}$. Moreover, let A be any linear transformation from K_α into K_β . If $\tilde{M}_A = (t_{s,p})$ is the matrix representation of A with respect to the bases $\mathcal{K}_m^\alpha$ and $\mathcal{V}_n^\beta$, then $A \in \mathcal{T}(\alpha, \beta)$ if and only if*

$$t_{s,p} = \frac{1 - \bar{a}_1 \zeta_s}{1 - \bar{a}_p \zeta_s} t_{s,1} + \frac{\sqrt{|\beta'(\zeta_1)|}}{\sqrt{|\beta'(\zeta_s)|}} \frac{\bar{a}_1 \zeta_1 - 1}{1 - \bar{a}_p \zeta_s} t_{1,1} + \frac{\sqrt{|\beta'(\zeta_1)|}}{\sqrt{|\beta'(\zeta_s)|}} \frac{1 - \bar{a}_p \zeta_1}{1 - \bar{a}_p \zeta_s} t_{1,p} \quad (3.9)$$

for all $1 \leq p \leq m$ and $1 \leq s \leq n$.

Remark 3.6 Recall that $(A_{\varphi}^{\alpha,\beta})^* = A_{\varphi}^{\beta,\alpha}$ [14]. To find a symbol for $A \in \mathcal{T}(\alpha, \beta)$ it is therefore enough to find a symbol for $A^* \in \mathcal{T}(\beta, \alpha)$. Now if the matrix representation $M_A = (r_{s,p})$ with respect to $\tilde{\mathcal{K}}_m^{\alpha}$ and $\mathcal{V}_n^{\beta}$ satisfies (3.8), then a symbol for A^* can be obtained from $M_{A^*} = (r_{p,s}^*) = \left(\frac{1}{\alpha'(a_p)} \bar{r}_{s,p} \right)$ as in Remark 3.3. If the matrix representation $\tilde{M}_A = (t_{s,p})$ with respect to $\mathcal{K}_m^{\alpha}$ and $\mathcal{V}_n^{\beta}$ satisfies (3.9), then a symbol for A^* can be obtained from $\tilde{M}_{A^*} = (t_{p,s}^*) = \left(\frac{1}{\alpha'(a_p)} \bar{t}_{s,p} \right)$.

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