On a Differential Inclusion Involving Dirichlet–Laplace Operators of Fractional Orders

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Abstract

In this paper, we investigate a nonlinear differential inclusion with Dirichlet boundary conditions containing a weak Laplace operator of fractional orders (defined via the spectral decomposition of the Laplace operator $-\Delta$ under Dirichlet boundary conditions). Using variational methods, we characterize solutions of such a problem. Our approach is based on tools from convex analysis (properties of a Legendre–Fenchel transform).

Keywords Fractional Dirichlet–Laplace operator · Differential inclusion · Legendre–Fenchel transform · Variational methods

Mathematics Subject Classification 35R20 · 47F05 · 47B25

1 Introduction

In recent years, a fractional Laplacian (more precisely, the fractional powers of the Laplacian) attracted attention of many scientists due to its applications in various areas. It appears including in probability (cf. [1,6,7,13]), economics and finance (cf. [1,14]), mechanics (cf. [5,7]), optimal control theory (cf. [9,20]), fluid mechanics and hydrodynamics (cf. [8,10–12,28–30]). There exist many definitions of such operators (e.g., Fourier transform [15,17], hypersingular integral [15], Riesz potential operator [24], Bochner’s definition [27], spectral decomposition (cf. [4,14,18])).

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In this paper, we are concerned with the study of solutions to the following differential inclusion

$$\sum_{i=0}^k \alpha_i [(-\Delta)_\omega]^{\beta_i} u(x) \in \partial_u F(x, u(x)), \quad x \in \Omega \text{ a.e.}, \quad (1)$$

where $\alpha_i > 0$, $i = 0, \dots, k$ ($k \in \mathbb{N} \cup \{0\}$), $0 \leq \beta_0 < \beta_1 < \dots < \beta_k$, $\Omega \subset \mathbb{R}^N$ is an open and bounded set, $F : \Omega \times \mathbb{R} \rightarrow [0, \infty)$, $\partial_u F$ denotes a subgradient of F with respect to u , $[(-\Delta)_\omega]^\gamma$ denotes a weak fractional Laplace operator of order $\gamma > 0$ with zero Dirichlet boundary values on $\partial\Omega$ (the term “weak” is explained in Sect. 2). This work is based on the spectral definition of the mentioned operator (we shall call it a weak fractional Dirichlet–Laplace operator). More precisely, the definition of the Dirichlet Laplacian comes from the functional calculus for unbounded self-adjoint operators in a Hilbert space [16, 21] and is based on the spectral integral representation theorem for such operators [23, 25].

Let us note that if $F(t, \cdot)$ is differentiable on $\mathbb{R}$, then (1) reduces to the following boundary value problem

$$\sum_{i=0}^k \alpha_i [(-\Delta)_\omega]^{\beta_i} u(x) = D_u F(x, u(x)), \quad x \in \Omega \text{ a.e.} \quad (2)$$

Motivated by the paper [3], we characterize solutions of the problem (1). They are minimizers of a some integral functional J related to (1) due to the Legendre–Fenchel transform of the function F and the Fenchel–Young inequality (cf. Theorem 3). Using a standard method of the calculus of variations, we prove the existence and uniqueness of minimizers of J on an appropriate space of functions, being the space of solutions of (1) (cf. Theorem 4). Finally, obtained results are illustrated by some examples. In particular, we show that if F is adequately chosen, then the minimizer of J is a solution of (2). To the best of our knowledge, the problem of a type (1) was not investigated by other authors.

2 Preliminaries

In this section, we give some preliminary notions and results that will be used in the remaining part of this paper.

2.1 Weak Fractional Dirichlet–Laplace Operator

In a whole paper we assume that $\Omega \subset \mathbb{R}^N$ is an open and bounded set.

Let $T_0 = -\Delta : C_c^\infty(\Omega, \mathbb{R}) \subset L^2(\Omega, \mathbb{R}) \rightarrow L^2(\Omega, \mathbb{R})$ be the classical Dirichlet–Laplace operator.

Definition 1 (cf. [21]) We say that $u : \Omega \rightarrow \mathbb{R}$ has a weak (minus) Dirichlet–Laplacian if $u \in H_0^1(\Omega, \mathbb{R})$ and there exists a function $g \in L^2(\Omega, \mathbb{R})$ such that

$$\int_{\Omega} \nabla u(x) \nabla v(x) dx = \int_{\Omega} g(x) v(x) dx$$

for any $v \in H_0^1(\Omega, \mathbb{R})$. The function g will be called the weak Dirichlet–Laplacian and denoted by $(-\Delta)_{\omega} u$.

The operator $(-\Delta)_{\omega}$ is called the weak Dirichlet–Laplace operator and the set

$$D((-\Delta)_{\omega}) := \left\{ u \in H_0^1(\Omega, \mathbb{R}); \quad \exists_{g \in L^2}, \quad \forall_{v \in H_0^1} \quad \int_{\Omega} \nabla u(x) \nabla v(x) dx = \int_{\Omega} g(x) v(x) dx \right\}$$

is named the domain of the operator $(-\Delta)_{\omega}$.

Using the Friedrich’s extension procedure, we obtain (cf. [21, Theorem 3.1], [23])

Theorem 1 *The operator*

$$(-\Delta)_{\omega} : D((-\Delta)_{\omega}) \subset L^2(\Omega, \mathbb{R}) \rightarrow L^2(\Omega, \mathbb{R})$$

is the self-adjoint extension of the operator T_0 and

$$T_0 \subset -\Delta \subset (-\Delta)_{\omega},$$

where $-\Delta : H_0^1(\Omega, \mathbb{R}) \cap H^2(\Omega, \mathbb{R}) \subset L^2(\Omega, \mathbb{R}) \rightarrow L^2(\Omega, \mathbb{R})$ is the (strong) Dirichlet–Laplace operator.

Remark 1 In [2, Section 8.2] $(-\Delta)_{\omega}$ is called the Laplace–Dirichlet operator (without “weak”) and denoted by $-\Delta$.

It is known (cf. [21]) that the spectrum $\sigma((-\Delta)_{\omega})$ of $(-\Delta)_{\omega}$ contains only the eigenvalues of $(-\Delta)_{\omega}$ that can be written in a non-decreasing sequence, repeating each eigenvalue according to its multiplicity $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \rightarrow \infty$. Moreover, a system $\{e_j\}$ of eigenfunctions of the operator $(-\Delta)_{\omega}$, corresponding to λ_j , is a hilbertian basis in $L^2(\Omega, \mathbb{R})$.

Remark 2 In paper [21], the author proved that if Ω is an open and bounded set of class $C^{1,1}$ or a convex polygon in $\mathbb{R}^2$, then the weak and strong Dirichlet–Laplace operators coincide.

Now, let $\beta > 0$. By the weak fractional Dirichlet–Laplace operator of order β we mean the operator $[(-\Delta)_{\omega}]^{\beta} : D([(-\Delta)_{\omega}]^{\beta}) \subset L^2(\Omega, \mathbb{R}) \rightarrow L^2(\Omega, \mathbb{R})$ defined in the following way (cf. [21, Section 3]):

$$([(-\Delta)_\omega]^\beta u)(t) = \left(\left(\int_{\sigma((-\Delta)_\omega)} \lambda^\beta E(d\lambda) \right) u \right)(t) = \sum_{j=1}^{\infty} (\lambda_j)^\beta a_j e_j(t),$$

where

$$D([(-\Delta)_\omega]^\beta) = \left\{ u \in L^2(\Omega, \mathbb{R}); \int_{\sigma((-\Delta)_\omega)} |\lambda^\beta|^2 \|E(d\lambda)u\|^2 = \sum_{j=1}^{\infty} ((\lambda_j)^\beta)^2 a_j^2 < \infty, \right.$$

where a_j -s is such that $u(t) = \left(\left(\int_{\sigma((-\Delta)_\omega)} 1 E(d\lambda) \right) u \right)(t)$

$$\left. = \sum_{j=1}^{\infty} a_j e_j(t) \right\}$$

(here E is the spectral measure given by $(-\Delta)_\omega$ and the convergence of the series is meant in $L^2(\Omega, \mathbb{R})$).

It is well known that the operator $[(-\Delta)_\omega]^\beta$ is self-adjoint, its spectrum $\sigma([(-\Delta)_\omega]^\beta)$ contains only proper values $(\lambda_j)^\beta$, $j \in \mathbb{N}$. Moreover, eigenspaces, corresponding to $(\lambda_j)^\beta$ -s and eigenspaces for $[(-\Delta)_\omega]$, corresponding to λ_j -s are the same.

In the space $D([(-\Delta)_\omega]^\beta)$, we define two scalar products:

$$\begin{aligned} \langle u, v \rangle_\beta &:= \langle u, v \rangle_{L^2} + \langle [(-\Delta)_\omega]^\beta u, [(-\Delta)_\omega]^\beta v \rangle_{L^2}, \\ \langle u, v \rangle_{\sim\beta} &:= \langle [(-\Delta)_\omega]^\beta u, [(-\Delta)_\omega]^\beta v \rangle_{L^2} \end{aligned}$$

which generate norms:

$$\|u\|_\beta = \left(\|u\|_{L^2}^2 + \|[(-\Delta)_\omega]^\beta u\|_{L^2}^2 \right)^{\frac{1}{2}}, \quad (3)$$

$$\|u\|_{\sim\beta} = \|[(-\Delta)_\omega]^\beta u\|_{L^2}, \quad (4)$$

respectively. Norms (3) and (4) are equivalent due to the following Poincaré inequality in $D([(-\Delta)_\omega]^\beta)$ (cf. [21, inequality (3.2)]):

$$\|u\|_{L^2}^2 \leq M_\beta \|u\|_{\sim\beta}^2, \quad (5)$$

where

$$M_\beta = \begin{cases} 1 & \text{if } \lambda_1 \geq 1 \\ \frac{1}{((\lambda_1)^\beta)^2} & \text{if } \lambda_1 < 1 \end{cases}$$

(here $\lambda_1 > 0$ is the first (the smallest) eigenvalue of the operator $(-\Delta)_\omega$).

Now, let us consider the function

$$w : \mathbb{R} \ni \lambda \rightarrow \begin{cases} 0 & \lambda < 0 \\ \alpha_k \lambda^{\beta_k} + \dots + \alpha_1 \lambda^{\beta_1} + \alpha_0 \lambda^{\beta_0} & \lambda \geq 0, \end{cases} \quad (6)$$

where $0 \leq \beta_0 < \beta_1 < \dots < \beta_k$, $k \in \mathbb{N} \cup \{0\}$, $\alpha_i > 0$, $i = 0, \dots, k$. Then, we can define the operator $w((-\Delta)_\omega) : D(w((-\Delta)_\omega)) \subset L^2(\mathcal{Q}, \mathbb{R}) \rightarrow L^2(\mathcal{Q}, \mathbb{R})$ as follows:

$$w((-\Delta)_\omega)u := \sum_{i=0}^k \alpha_i ((-\Delta)_\omega)^{\beta_i} u.$$

The operator $w((-\Delta)_\omega)$ is self-adjoint and $D(w((-\Delta)_\omega)) = D(((\Delta)_\omega)^{\beta_k})$ (cf. [21]). Moreover, in $D([(-\Delta)_\omega]^{\beta_k})$ one can introduce a new scalar product of the form

$$\langle u, v \rangle_{w((-\Delta)_\omega)} := \langle w((-\Delta)_\omega)u, w((-\Delta)_\omega)v \rangle_{L^2} \quad (7)$$

which generates a norm

$$\|u\|_{w((-\Delta)_\omega)} = \|w((-\Delta)_\omega)u\|_{L^2}. \quad (8)$$

Norms (4) and (8) (the norm (8) will be used in the main part of this article) are equivalent in $D([(-\Delta)_\omega]^{\beta_k})$ (cf. [21, Lemma 3.6]) and

$$\|u\|_{\sim_{\beta_k}}^2 \leq \frac{1}{\alpha_k^2} \|u\|_{w((-\Delta)_\omega)}^2 \leq \frac{C_1}{\alpha_k^2} \left(1 + kC_2^2\right) \|u\|_{\sim_{\beta_k}}^2, \quad (9)$$

where $C_1, C_2 > 0$ are constants from [21, Lemma 3.6]. Hence and from (5) we obtain the following Poincaré inequality in $D([(-\Delta)_\omega]^{\beta_k})$:

$$\|u\|_{L^2}^2 \leq \frac{1}{M\beta\alpha_k^2} \|u\|_{w((-\Delta)_\omega)}^2. \quad (10)$$

Moreover, the space $D(w((-\Delta)_\omega)) = D(((\Delta)_\omega)^{\beta_k})$ with the scalar product (7) is complete (cf. [21, Lemma 3.6]).

2.2 Basic Facts from Convex Analysis

In this part we recall some basic definitions and facts concerning the convex analysis. More details can be found in [19].

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$. We shall say that a vector $y \in \mathbb{R}^n$ is a subgradient of f at $x \in \mathbb{R}^n$ if

$$f(z) - f(x) \geq \langle y, z - x \rangle_{\mathbb{R}^n}, \quad \text{for every } z \in \mathbb{R}^n.$$

The set of all subgradients of f at x is called a subdifferential and is denoted by $\partial f(x)$. If $\partial f(x) \neq \emptyset$ then the function f is called subdifferentiable at x . If f is a convex function, then the subdifferential is a nonempty, convex and compact set. Moreover, if f is Gateaux differentiable at x , then $\partial f(x) = \{\nabla f(x)\}$. The function $f^* : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ defined by

$$f^*(y) := \sup_{x \in \mathbb{R}^n} \{\langle y, x \rangle_{\mathbb{R}^n} - f(x)\}$$

is called the Legendre–Fenchel transform of the function f . In the main part of this paper we use the following facts:

$$f(x) \leq g(x), \quad x \in \mathbb{R}^n \implies f^*(y) \geq g^*(y), \quad y \in \mathbb{R}^n. \quad (11)$$

Theorem 2 (Fenchel–Young inequality) *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$. Any points $x, y \in \mathbb{R}^n$ satisfy the inequality*

$$f(x) + f^*(y) \geq \langle y, x \rangle.$$

Equality holds if and only if $y \in \partial f(x)$.

In conclusion of this section we give another useful fact (cf. [22, Part II, Lemma 4.3.1]).

Lemma 1 *Let $\Omega \subset \mathbb{R}^d$ be open, $f : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$, with $f(\cdot, v)$ measurable for all $v \in \mathbb{R}^d$, $f(x, \cdot)$ continuous for almost all $x \in \Omega$, and*

$$f(x, v) \geq -a(x) + b|v|^p, \quad x \in \Omega \text{ a.e.}, \quad v \in \mathbb{R}^d,$$

where $a \in L^1(\Omega, \mathbb{R})$, $b \in \mathbb{R}$, $p \geq 1$. Then, the functional $\Phi : L^p(\Omega, \mathbb{R}^d) \rightarrow \mathbb{R} \cup \{\infty\}$,

$$\Phi(v) := \int_{\Omega} f(x, v(x)) dx$$

is a sequentially lower semicontinuous on $L^p(\Omega, \mathbb{R}^d)$.

Remark 3 The above lemma can be also proved in case of the function f defined on $\Omega \times \mathbb{R}^n$, where $n \neq d$.

3 Differential Inclusion with Fractional Dirichlet–Laplace Operators

In this section we prove the main result of this paper, namely a theorem on the existence of minimizers of a some functional associated with the problem (1).

By a solution of inclusion (1) we mean a function $u \in D(w((-\Delta)_{\omega}))$ satisfying (1) a.e. on Ω .

Let us assume that F satisfies the following conditions:

(F1): $F(\cdot, u)$ is measurable on Ω for all $u \in \mathbb{R}$ and $F(x, \cdot)$ is convex on $\mathbb{R}$ for a.e. $x \in \Omega$.

(F2): $F(x, \cdot)$ is coercive on $\mathbb{R}$ for a.e. $x \in \Omega$, i.e.

$$\lim_{|u| \rightarrow \infty} \frac{F(x, u)}{|u|} = \infty, \quad x \in \Omega \text{ a.e.} \quad (12)$$

(F3): there exist a function $v \in L^1(\Omega, \mathbb{R})$ and a constant $c > 0$ such that

$$F(x, u) \leq v(x) + c|u|^2, \quad x \in \Omega \text{ a.e., } u \in \mathbb{R}. \quad (13)$$

Remark 4 Of course, since $F(x, \cdot)$ is convex, therefore it is continuous on $\mathbb{R}$ for a.e. $x \in \Omega$ (cf. [26, Corollary 10.1.1]).

Let us consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(u) := v + c|u|^2, \quad (14)$$

where $c > 0$ and $v \in \mathbb{R}$ are fixed. Then, it easy to check that

$$f^*(z) = -v + \frac{1}{4c}|z|^2, \quad z \in \mathbb{R}. \quad (15)$$

In the next result we shall use the following

Lemma 2 Let $G : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function and the function $H : \Omega \rightarrow \mathbb{R} \cup \{\infty\}$ is defined by

$$H(x) := \sup_{w \in \mathbb{R}} \{G(x, w)\}, \quad x \in \Omega \text{ a.e.}$$

Then H is measurable on Ω .

Proof From the fact that $G(\cdot, w)$ is continuous on $\mathbb{R}$ it follows that

$$H(x) = \sup_{w \in \mathbb{Q}} \{G(x, w)\}, \quad x \in \Omega \text{ a.e.,}$$

where $\mathbb{Q}$ denotes the set of rational numbers. Since the supremum of the countable set of measurable functions is a measurable set, therefore H is measurable on Ω . $\square$

Now, we formulate and prove the following result.

Proposition 1 Let us assume that conditions (F1)–(F3) are satisfied. Then, the Legendre–Fenchel transform of the function F with respect to u $F^* : \Omega \times \mathbb{R} \rightarrow (-\infty, \infty]$ given by

$$F^*(x, z) := \sup_{u \in \mathbb{R}} [\langle u, z \rangle_{\mathbb{R}} - F(x, u)], \quad x \in \Omega \text{ a.e., } z \in \mathbb{R}$$

has the following properties:

- (a) $F^*(x, z)$ is finite for a.e. $x \in \Omega$ and all $z \in \mathbb{R}$,
- (b) $F^*(\cdot, z)$ is measurable on Ω for all $z \in \mathbb{R}$ and $F^*(x, \cdot)$ is continuous on $\mathbb{R}$ for a.e. $x \in \Omega$,
- (c) for a.e. $x \in \Omega$ and all $z \in \mathbb{R}$

$$F^*(x, z) \geq -v(x) + \frac{1}{4c}|z|^2, \quad (16)$$

where the constant c and the function v are from assumption (F3).

Proof proof of the property (a): Let us fix any $z \in \mathbb{R}$ and $x \in S \subset \Omega$, where $\mu(\Omega \setminus S) = 0$ (μ denotes the Lebesgue measure on $\mathbb{R}^N$). From (F2) it follows that for any $R > 0$ there exists $r > 0$ such that for $|u| > r$ we have $F(x, u) \geq R|u|$. In particular, putting $R = |z|$, we get

$$F(x, u) \geq |z||u|, \quad |u| > r.$$

Consequently,

$$\langle u, z \rangle_{\mathbb{R}} - F(x, u) \leq |u||z| - R|u| = 0, \quad |u| > r.$$

Hence and from continuity of the function $F(\cdot, u)$ we obtain

$$\begin{aligned} F^*(x, z) &= \sup_{u \in \mathbb{R}} [\langle u, z \rangle_{\mathbb{R}} - F(x, u)] \\ &= \max \left\{ \sup_{|u| \leq r} [\langle u, z \rangle_{\mathbb{R}} - F(x, u)], \sup_{|u| > r} [\langle u, z \rangle_{\mathbb{R}} - F(x, u)] \right\} \\ &\leq \max\{\text{const}, 0\} < \infty. \end{aligned}$$

This means that F^* is finite for a.e. $x \in \Omega$ and all $z \in \mathbb{R}$.

proof of the property (b): Measurability of $F^*(\cdot, z)$ on Ω for any fixed $z \in \mathbb{R}$ follows from Lemma 2. From the definition of the Legendre–Fenchel transform it follows that $F^*(x, \cdot)$ is convex on $\mathbb{R}$ for a.e. $x \in \Omega$. Thus and from the proved condition (a) of this proposition we obtain continuity of F^* with respect to z for a.e. $x \in \Omega$ (cf. [26, Corollary 10.1.1]).

proof of the property (c): The condition (16) follows from assumption (F3), property (11) and equality (15). $\square$

Now, let us consider the functional $J : D(w((-\Delta)_{\omega})) \rightarrow \mathbb{R} \cup \{\infty\}$ given by:

$$J(u) := \int_{\Omega} (F(x, u(x)) + F^*(x, w((-\Delta)_{\omega})u(x)) - \langle u(x), w((-\Delta)_{\omega})u(x) \rangle_{\mathbb{R}}) dx. \quad (17)$$

Let us note that since all terms in the above integral are measurable on Ω (the first and third terms are even summable on Ω), therefore J is well defined.

In the proof of the main result we use the following fact.

Lemma 3 *If assumptions (F1)–(F3) are satisfied, then the functional $I : D(w((-\Delta)_\omega)) \rightarrow \mathbb{R} \cup \{\infty\}$ given by*

$$I(u) = \int_{\Omega} F^*(x, w((-\Delta)_\omega)u(x)) dx \quad (18)$$

is sequentially weakly lower semicontinuous on $D(w((-\Delta)_\omega))$.

Proof Let us consider the functional $I_1 : L^2(\Omega, \mathbb{R}) \rightarrow \mathbb{R} \cup \{\infty\}$ given by

$$I_1(z) = \int_{\Omega} F^*(x, z(x)) dx.$$

From Proposition 1, Lemma 1 and Remark 3 it follows that I_1 is a sequentially lower semicontinuous functional on $L^2(\Omega, \mathbb{R})$.

Let $(u_n)_{n \in \mathbb{N}} \subset D(w((-\Delta)_\omega))$ be a sequence convergent to a some element u in the space $D(w((-\Delta)_\omega))$. Then, the sequence $(w((-\Delta)_\omega)u_n)_{n \in \mathbb{N}} \subset L^2(\Omega, \mathbb{R})$ is convergent to $w((-\Delta)_\omega)u$ in $L^2(\Omega, \mathbb{R})$. From the proved part of this Lemma it follows that the functional I is sequentially lower semicontinuous on $D(w((-\Delta)_\omega))$. Since I is convex on $D(w((-\Delta)_\omega))$ ($F^*(x, \cdot)$ is convex on $\mathbb{R}$), therefore it is sequential weak lower semicontinuous on $D(w((-\Delta)_\omega))$.

The proof is completed. $\square$

The next theorem plays a key role in the study of the existence of solutions to problem (1).

Theorem 3 *Assume that conditions (F1)–(F3) are satisfied. A function $u_* \in D(w((-\Delta)_\omega))$ is a solution of the inclusion (1) if and only if u_* is a minimizer of J on $D(w((-\Delta)_\omega))$ and $J(u_*) = 0$.*

Proof Let $u_* \in D(w((-\Delta)_\omega))$ be a solution of the problem (1). From Theorem 2 it follows that

$$F(x, u_*(x)) + F^*(x, w((-\Delta)_\omega)u_*(x)) - \langle u_*(x), w((-\Delta)_\omega)u_*(x) \rangle_{\mathbb{R}} = 0, \\ x \in \Omega \text{ a.e.}$$

Consequently, $J(u_*) = 0$. On the other hand, for every $u \in D(w((-\Delta)_\omega))$, using once again Theorem 2, we have

$$F(x, u(x)) + F^*(x, w((-\Delta)_\omega)u(x)) - \langle u(x), w((-\Delta)_\omega)u(x) \rangle_{\mathbb{R}} \geq 0, \quad x \in \Omega \text{ a.e.} \quad (19)$$

This means that $J(u) \geq 0$ for every $u \in D(w((-\Delta)_\omega))$, so u_* is a minimizer of J on $D(w((-\Delta)_\omega))$.

Now, we assume that u_* is a minimizer of J on $D(w((-\Delta)_\omega))$ and $J(u_*) = 0$. This means in particular that

$$\begin{aligned} 0 &= J(u_*) \\ &= \int_{\Omega} (F(x, u_*(x)) + F^*(x, w((-\Delta)_\omega)u_*(x)) - \langle u_*(x), w((-\Delta)_\omega)u_*(x) \rangle_{\mathbb{R}}) dx. \end{aligned}$$

On the other hand, from Theorem 2 it follows that

$$\begin{aligned} F(x, u_*(x)) + F^*(x, w((-\Delta)_\omega)u_*(x)) - \langle u_*(x), w((-\Delta)_\omega)u_*(x) \rangle_{\mathbb{R}} &\geq 0, \\ x \in \Omega \text{ a.e.} \end{aligned}$$

Consequently,

$$\begin{aligned} F(x, u_*(x)) + F^*(x, w((-\Delta)_\omega)u_*(x)) - \langle u_*(x), w((-\Delta)_\omega)u_*(x) \rangle_{\mathbb{R}} &= 0, \\ x \in \Omega \text{ a.e.} \end{aligned}$$

The second part of Theorem 2 guaranties that $u_* \in D(w((-\Delta)_\omega))$ is a solution of the inclusion (1).

The proof is completed. $\square$

Remark 5 From the above theorem it follows that if $u_* \in D(w((-\Delta)_\omega))$ is a solution of the problem (1) then it is a minimizer of J . Consequently, only minimizers of J can be solutions of (1) (they are in effect solutions of (1) if additionally $J(u_*) = 0$, where u_* denotes a minimizer of J).

Now, we formulate and prove the main result of this paper, namely a theorem on the existence of a minimizer of the functional J on $D(w((-\Delta)_\omega))$. We have

Theorem 4 Assume that conditions (F1)–(F3) are satisfied and

$$c < \frac{\sqrt{M_{\beta_k}} \alpha_k}{4}, \quad (20)$$

where λ_1 is the first eigenvalue of $(-\Delta)_\omega$ and c is a constant from the assumption (F3). Then there exists a minimizer $u_* \in D(w((-\Delta)_\omega))$ of the functional J . Moreover, if $F(x, \cdot)$ is strictly convex on $\mathbb{R}$ for a.e. $x \in \Omega$, then the minimizer is unique.

Proof Since the space $D(w((-\Delta)_\omega))$ is reflexive (as the Hilbert space), therefore it is sufficient to show that the functional J is coercive and sequentially weakly lower semicontinuous on $D(w((-\Delta)_\omega))$.

Coercivity: Let $(u_l)_{l \in \mathbb{N}} \subset D(w((-\Delta)_\omega))$ be any sequence such that

$$\|u_l\|_{w((-\Delta)_\omega)} \xrightarrow{l \rightarrow \infty} \infty.$$

From the proof of Theorem 3 it follows that $J(u_l) \geq 0$. Moreover, using nonnegativity of F , conditions (16), (20), the Hölder inequality and the Poincaré inequality (10) we obtain

$$\begin{aligned}
 J(u_l) &= \int_{\Omega} (F(x, u_l(x)) + F^*(x, w((-\Delta)_{\omega})u_l(x)) - \langle u_l(x), w((-\Delta)_{\omega})u_l(x) \rangle_{\mathbb{R}}) dx \\
 &\geq \int_{\Omega} F^*(x, w((-\Delta)_{\omega})u_l(x)) dx - \int_{\Omega} \langle u_l(x), w((-\Delta)_{\omega})u_l(x) \rangle_{\mathbb{R}} dx \\
 &\geq - \int_{\Omega} v(x) dx + \frac{1}{4c} \|w((-\Delta)_{\omega})u_l\|_{L^2}^2 - \|u_l\|_{L^2} \|w((-\Delta)_{\omega})u_l\|_{L^2} \\
 &\geq C + \frac{1}{4c} \|u_l\|_{w((-\Delta)_{\omega})}^2 - \frac{1}{\sqrt{M_{\beta_k} \alpha_k}} \|u_l\|_{w((-\Delta)_{\omega})}^2 \\
 &= C + \left(\frac{1}{4c} - \frac{1}{\sqrt{M_{\beta_k} \alpha_k}} \right) \|u_l\|_{w((-\Delta)_{\omega})}^2 \xrightarrow{l \rightarrow \infty} \infty,
 \end{aligned}$$

where $C = - \int_{\Omega} v(x) dx$. This means that J is coercive.

Sequential weak lower semicontinuity of J : Assume that $(u_l)_{l \in \mathbb{N}} \subset D(w((-\Delta)_{\omega}))$ is a sequence such that

$$u_l \xrightarrow{l \rightarrow \infty} u_* \quad \text{weakly in } D(w((-\Delta)_{\omega}))$$

and let

$$J(u) = J_1(u) + J_2(u) - J_3(u),$$

where

$$\begin{aligned}
 J_1(u) &= \int_{\Omega} F(x, u(x)) dx, \quad J_2(u) = \int_{\Omega} F^*(x, w((-\Delta)_{\omega})u(x)) dx, \\
 J_3(u) &= \int_{\Omega} \langle u(x), w((-\Delta)_{\omega})u(x) \rangle_{\mathbb{R}} dx.
 \end{aligned}$$

First, let us note that since F is a Carathéodory function, therefore $(F(\cdot, u_l(\cdot)))_{l \in \mathbb{N}}$ is the sequence of measurable functions. Moreover all terms of this sequence are nonnegative. Consequently, using Fatou's Lemma, we conclude

$$\int_{\Omega} \liminf_{l \rightarrow \infty} F(x, u_l(x)) dx \leq \liminf_{l \rightarrow \infty} \int_{\Omega} F(x, u_l(x)) dx. \quad (21)$$

[21, Proposition 3.10] implies

$$u_l \xrightarrow{l \rightarrow \infty} u_* \text{ strongly in } L^2(\Omega, \mathbb{R}). \quad (22)$$

So, there exists a subsequence $(u_{l_j})_{j \in \mathbb{N}}$ such that

$$u_{l_j}(x) \xrightarrow{j \rightarrow \infty} u_*(x), \quad x \in \Omega \text{ a.e.}$$

Hence and from the fact that F is continuous with respect to the second variable we get

$$\lim_{j \rightarrow \infty} F(x, u_{l_j}(x)) = F(x, u_*(x)), \quad x \in \Omega \text{ a.e.}$$

Supposing contrary and repeating the above argumentation we assert that

$$\lim_{l \rightarrow \infty} F(x, u_l(x)) = F(x, u_*(x)), \quad x \in \Omega \text{ a.e.}$$

Consequently, the inequality (21) can be written as

$$\int_{\Omega} F(x, u_*(x)) dx \leq \liminf_{l \rightarrow \infty} \int_{\Omega} F(x, u_l(x)) dx.$$

This means that J_1 is sequentially weakly lower semicontinuous on $D(w((-\Delta)_{\omega}))$.

The functional J_2 has also such a property due to Lemma 3.

Now, we show that J_3 is sequentially weakly continuous on $D(w((-\Delta)_{\omega}))$. Indeed, using once again [21, Proposition 3.10] we assert that

$$w((-\Delta)_{\omega})u_l \xrightarrow{l \rightarrow \infty} w((-\Delta)_{\omega})u_* \text{ weakly in } L^2(\Omega, \mathbb{R}).$$

This fact and convergence (22) imply

$$\lim_{l \rightarrow \infty} \int_{\Omega} \langle u_l(x), w((-\Delta)_{\omega})u_l(x) \rangle_{\mathbb{R}} dx = \int_{\Omega} \langle u_*(x), w((-\Delta)_{\omega})u_*(x) \rangle_{\mathbb{R}} dx.$$

Finally, we conclude that J is sequentially weakly lower semicontinuous on $D(w((-\Delta)_{\omega}))$.

Uniqueness of a solution: It is clear that since F and F^* are convex with respect to the second variable, therefore J is convex on $D(w((-\Delta)_{\omega}))$. Let us suppose that u_* and v_* are two different minimizers of J on $D(w((-\Delta)_{\omega}))$ and assume that $F(x, \cdot)$ is strictly convex on $\mathbb{R}$ for a.e. $x \in \Omega$. This implies that J_1 , so also J , are strictly convex on $D(w((-\Delta)_{\omega}))$. This means in particular that

$$J\left(\frac{u_* + v_*}{2}\right) < J(u_*) = J(v_*).$$

This contradicts the assumption that $u_*(v_*)$ is a minimizer of J on $D(w((-\Delta)_\omega))$. The proof is completed. $\square$

Remark 6 Let us note that only the order β_k and the value α_k impact the existence of a minimizer of J . This is a consequence of the fact that domains $D(((\Delta)_\omega)^{\beta_k})$ and $D(w((-\Delta)_\omega))$ coincide.

4 Illustrative Examples

In this section we present two simple theoretical examples which illustrate results obtained in Sect. 3.

Example 1 Let us consider the following problem:

$$((-\Delta)_\omega)^{\frac{1}{3}}u(x) + 5((-\Delta)_\omega)^{\frac{1}{2}}u(x) \in \partial_u F(x, u(x)), \quad x = (x_1, x_2) \in \Omega \text{ a.e.,} \quad (23)$$

where $\Omega = (0, \pi) \times (0, \pi) \subset \mathbb{R}^2$ and $F : \Omega \times \mathbb{R} \rightarrow [0, \infty)$ is given by

$$F(x, u) := \begin{cases} |u| & \text{if } |u| \leq 1 \\ u^2 & \text{if } |u| > 1. \end{cases}$$

It is easy to check that

$$\partial_u F(x, u) = \begin{cases} [-2, -1] & \text{if } u = -1 \\ \{-1\} & \text{if } u \in (-1, 0) \\ [-1, 1] & \text{if } u = 0 \\ \{1\} & \text{if } u \in (0, 1) \\ [1, 2] & \text{if } u = 1 \\ \{2u\} & \text{if } |u| > 1. \end{cases}$$

Moreover,

$$\begin{aligned} F^*(x, z) &= \sup_{u \in \mathbb{R}} [\langle u, z \rangle_{\mathbb{R}} - F(x, u)] \\ &= \max \left\{ \sup_{|u| \leq 1} [\langle u, z \rangle_{\mathbb{R}} - |u|], \sup_{|u| > 1} [\langle u, z \rangle_{\mathbb{R}} - u^2] \right\}. \end{aligned}$$

We check that

$$\sup_{|u| \leq 1} [\langle u, z \rangle_{\mathbb{R}} - |u|] = \begin{cases} -z - 1 & \text{if } z < -1 \\ z - 1 & \text{if } z > 1 \\ 0 & \text{if } |z| \leq 1 \end{cases}$$

and

$$\sup_{|u| > 1} [\langle u, z \rangle_{\mathbb{R}} - u^2] = \begin{cases} -z - 1 & \text{if } z \in [-2, 0] \\ z - 1 & \text{if } z \in [0, 2] \\ \frac{1}{4}z^4 & \text{if } |z| > 2. \end{cases}$$

Consequently,

$$F^*(x, z) = \begin{cases} 0 & \text{if } |z| < 1 \\ -z - 1 & \text{if } z \in [-2, -1] \\ z - 1 & \text{if } z \in [1, 2] \\ \frac{1}{4}z^4 & \text{if } |z| > 2. \end{cases}$$

Of course assumptions (F1)–(F3) are satisfied. In particular,

$$F(x, u) \leq 1 + |u|^2,$$

so the condition (13) holds with $c = 1$. Moreover, it is well known (cf. [2, Proposition 8.5.3]) that the first eigenvalue of $(-\Delta)_{\omega}$ equals to 2, so the assumption (20) is also satisfied. Consequently, using Theorem 4, we assert that there exists a minimizer (not necessarily unique) of the functional J given by (17) which can be the only possible solution of the problem (23) (cf. Remark 5).

Example 2 Let us consider the following boundary value problem:

$$\sum_{i=0}^k \alpha_i ((-\Delta)_{\omega})^{\beta_i} u(x) = D_u F(x, u(x)), \quad x \in \Omega \text{ a.e.}, \quad (24)$$

where $\alpha_i > 0$, $i = 0, \dots, k$ ($k \in \mathbb{N} \cup \{0\}$), $0 \leq \beta_0 < \beta_1 < \dots < \beta_k$, $\Omega \subset \mathbb{R}^N$, is any open and bounded set, $F : \Omega \times \mathbb{R} \rightarrow [0, \infty)$ is given by

$$F(x, u) := \frac{a(x)}{p} |u|^p,$$

where $1 < p < 2$ and $a \in L^{\infty}(\Omega, \mathbb{R}_+)$.

We shall show that the above problem has the only trivial solution. Indeed, first let us note that

$$F^*(x, z) = \frac{1}{qa(x)}|z|^q, \quad \frac{1}{p} + \frac{1}{q} = 1$$

and the partial derivative of F with respect to u is given by

$$D_u F(x, u) = a(x)|u|^{p-2}u.$$

Moreover, assumptions (F1)–(F3) hold. In particular, for any $c > 0$ there exists a sufficiently large constant $R > 0$ (dependent on the constants c , p and the function a) such that

$$\frac{a(x)}{p}|u|^p \leq \frac{\|a\|_{L^\infty}}{p}|u|^p \leq \begin{cases} c|u|^2, & |u| > R \\ \frac{\|a\|_{L^\infty}}{p}R^p + c|u|^2, & |u| \leq R, \end{cases}$$

so, the condition (13) is satisfied for any $c > 0$.

Consequently, since all assumptions of Theorem 4 are satisfied and F is strictly convex with respect to u for a.e. $x \in \Omega$, therefore the functional J given by

$$J(u) = \int_{\Omega} \left(\frac{a(x)}{p}|u(x)|^p + \frac{1}{qa(x)} \left| [w((-\Delta)_{\omega})u(x)] \right|^q - u(x)w((-\Delta)_{\omega})u(x) \right) dx \quad (25)$$

has a unique minimizer $u_* \in D(w((-\Delta)_{\omega}))$.

On the other hand it is clear that $u_* = 0$ is a solution of the problem (24), so it must minimize J on $D(w((-\Delta)_{\omega}))$. This means that $u_* = 0$ is the only solution of (24).

Remark 7 If $p = 2$ in the above example then the condition (13) holds for $c = \frac{\|a\|_{L^\infty}}{2}$. Consequently, the linear problem of the form

$$\sum_{i=0}^k \alpha_i ((-\Delta)_{\omega})^{\beta_i} u(x) = a(x)u(x), \quad x \in \Omega \text{ a.e.}$$

has the only trivial solution provided that

$$\|a\|_{L^\infty} < \frac{\sqrt{M_{\beta_k}} \alpha_k}{2}. \quad (26)$$

Here it is worth to note that replacing the functional J given by (25) (for $p = q = 2$) with the following one

$$J_1(u) = \int_{\Omega} \left(\frac{a(x)}{2} |u(x)|^2 + \frac{1}{2a(x)} \left| w((-\Delta)_{\omega})u(x) - f(x) \right|^2 - u(x) \left(w((-\Delta)_{\omega})u(x) - f(x) \right) \right) dx$$

we can prove (similarly as in Sect. 3) counterparts of Theorems 3 and 4 for the functional J_1 (in particular, under assumption (26) one can obtain the existence of a unique minimizer of J_1 on $D(w((-\Delta)_{\omega}))$) and the following, more general linear problem

$$\sum_{i=0}^k \alpha_i ((-\Delta)_{\omega})^{\beta_i} u(x) = a(x)u(x) + f(x), \quad x \in \Omega \text{ a.e.}, \quad (27)$$

where $f \in L^2(\Omega, \mathbb{R})$. Additionally, if a is a constant function, then the problem (27) has a unique solution. Indeed, since J_1 has a unique minimizer u_* on $D(w((-\Delta)_{\omega}))$, therefore

$$\delta J_1(u_*, h) = 0, \quad h \in D(w((-\Delta)_{\omega})), \quad (28)$$

where $\delta J_1(u_*, h)$ denotes the first variation of J_1 at u_* in the direction h . So,

$$\begin{aligned} \delta J_1(u_*, h) &= \int_{\Omega} \frac{1}{a} \left(w((-\Delta)_{\omega})u_*(x) - au_*(x) - f(x) \right) \\ &\quad \left(w((-\Delta)_{\omega})h(x) - ah(x) \right) dx = 0 \end{aligned} \quad (29)$$

for any $h \in D(w((-\Delta)_{\omega}))$.

Since $a < \frac{\sqrt{M_{\beta_k} \alpha_k}}{2}$, therefore the constant a is not the eigenvalue of the operator $w((-\Delta)_{\omega})$. Consequently, the kernel of the self-adjoint operator $L: D(w((-\Delta)_{\omega})) \rightarrow L^2(\Omega, \mathbb{R})$, defined as

$$Lh := w((-\Delta)_{\omega})h - ah, \quad a > 0,$$

is trivial, so the image of L satisfies the equality $R(L) = L^2(\Omega, \mathbb{R})$. Hence and from (29) we conclude

$$w((-\Delta)_{\omega})u_*(x) - au_*(x) - f(x) = 0, \quad x \in \Omega \text{ a.e.}$$

This means that u_* is a unique solution of (27).

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