Bands of E -Inversive Unipotent Semigroups

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Received: 26 January 2019 / Revised: 22 August 2019 / Published online: 24 September 2019
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Abstract

We study some special types of bands of E -inversive unipotent semigroups. It has been proved that in any R -semigroup S , which is a band of E -inversive unipotent semigroups, the set of its regular elements is a retract of S . Also, some characterizations of E -inversive rectangular bands of unipotent semigroups are given. This theorem extends nearly 40-old results from the theory of epigroups. In fact, a more general result is valid in some special subclass of the class of E -inversive semigroups; this result seems to be (partially) new for all epigroups.

Keywords Band of semigroups · Unipotent semigroup · Putcha semigroup · E -inversive semigroup · Epigroup

Mathematics Subject Classification 20M10

1 Introduction and Preliminaries

The terminology and notation are that of [9] and [12].

Denote the set of idempotents of a semigroup S by E_S (and recall that S is an E -semigroup if $E_S E_S \subseteq E_S$), the set of its regular elements by $\text{Reg}(S)$ (S is called an R -semigroup if $\text{Reg}(S)$ is a semigroup) and the set of all its elements that belong to some subgroup of S by $\text{Gr}(S)$; S is an *epigroup* if every element of S has a power in $\text{Gr}(S)$ [13].

Recall also from [11] and [9] that a semigroup S is:

- (a) *unipotent* if it has exactly one idempotent;
- (b) *poor* if it is a unipotent semigroup with zero;

Communicated by Kar Ping Shum.

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- (c) *E-inversive* if for every $a \in S$ there is $x \in S$ such that $ax \in E_S$ (for example, all epigroups are *E-inversive*; in fact, the class of *E-inversive* semigroups is very large and contains *almost all* classes of semigroups which have been studied in the literature);
- (d) a *band* if $S = E_S$;
- (e) a *semilattice* if it is a commutative band;
- (f) a *rectangular band* if it is a band that meets the identity $xyz = xz$.

Let $\mathcal{C}$ be some fixed class of semigroups. We say that a congruence ρ on a semigroup S is a $\mathcal{C}$ -congruence if $S/\rho \in \mathcal{C}$ (for example, ρ is a *semilattice* congruence on S if the semigroup S/ρ is a semilattice). It is clear that the least semilattice congruence on an arbitrary semigroup always exists. Denote it by η .

Let also $\mathcal{B}$ be some variety of bands (for instance, the variety of semilattices). A semigroup S is said to be a $\mathcal{B}$ -band of $\mathcal{C}$ -semigroups if there exists a congruence ρ on S such that $S/\rho \in \mathcal{B}$ and every ρ -class of S is a $\mathcal{C}$ -semigroup (for example, S is a *band of E-inversive unipotent semigroups* if there is a band congruence ρ on S such that every ρ -class is an *E-inversive unipotent semigroup*).

In Sect. 2, we give some examples of semilattices of *E-inversive unipotent semigroups*. Further, Sect. 3 contains the main results of the paper mentioned in the abstract. We generalize there some results of [14, Sections 3.3, 3.4] and [10, Theorem 5]. It should be mentioned that the construction of the congruence ν (see below for the definition of ν) is based on a similar construction of Maity (see [10, Definition 3 and Theorem 5]; Maity “proved” that the relation ρ (from [10, Definition 3]) is a congruence on an arbitrary band of unipotent epigroups (Theorem 5 of [10]). Nevertheless, in his proof the author used the additional property that *the set of idempotents of the given semigroup forms a subsemigroup* (precisely, this happens in the first, third and fourth line on page 9 of [10]).

Before we start our study, we recall further useful facts and definitions of the algebraic theory of semigroups.

In this paper, $\leq$ is the *natural partial order* on E_S ; recall that

$$e \leq f \iff e = ef = fe$$

for all $e, f \in E_S$, and that any minimal idempotent of S with respect to $\leq$ is called *primitive*.

Let S be a semigroup and let $A \subseteq S$. Put

$$\sqrt{A} = \{s \in S : (\exists n \in \mathbb{Z}_+)(s^n \in A)\}.$$

Also, for each $a \in S$, let us put as usual: $J(a) = S^1 a S^1$. Recall from [12] that S is a *Putcha semigroup* if for all $a \in S$,

$$J(a) \subseteq \sqrt{J(a^2)}.$$

Recall also that a semigroup S is *archimedean* if for every ideal I of S , we get

$$S = \sqrt{I}.$$

Remark 1 Let S be an archimedean semigroup with $E_S \neq \emptyset$. Then, E_S is contained in every ideal of S . Consequently, $\text{Reg}(S)$ is a subset of a kernel of S (recall that by a *kernel* of a semigroup A we mean (if exists) the least ideal of A ; this ideal is then denoted by K_A).

The following important result is due to Putcha (see [12, Chapter 2]).

Result 1.1 *A semigroup S is a semilattice of archimedean semigroups if and only if it is a Putcha semigroup. In that case, the corresponding semilattice congruence is equal to η . Moreover,*

$$\eta = \{(a, b) \in S \times S : (\exists m, n \in \mathbb{Z}_+)(a^m \in SbS \text{ \& } b^n \in SaS)\}.$$

The following result will be used in the sequel.

Proposition 1.2 *Let a semigroup S be a band of E -inversive unipotent semigroups. Then,*

$$\text{Reg}(S) = \text{Gr}(S).$$

Moreover, S is a semilattice of rectangular bands of E -inversive unipotent semigroups.

Proof Indeed, let ρ be the corresponding band congruence on S . As any band is a semilattice of rectangular bands, S is a semilattice I of semigroups S_i ($i \in I$), where each semigroup S_i is a rectangular band of some ρ -classes of S . It is obvious that each semigroup S_i is E -inversive. Furthermore, every idempotent in any semigroup S_i is primitive (since every S_i is a rectangular band of *unipotent* semigroups), so $\text{Reg}(S_i) = \text{Gr}(S_i)$ for all $i \in I$ (see [4, 11]). On the other hand, it is clear (and well known) that for any $a \in \text{Reg}(S)$ (say: $a \in S_i$ for some $i \in I$), every inverse of a belongs to S_i . This implies (together with the above) the thesis of the proposition. $\square$

Remark 2 Remark that rectangular bands of E -inversive unipotent semigroups are characterized below in Theorem 3.6.

Recall that a semigroup is *eventually regular* if each of its elements has a regular power. Obviously, any eventually regular semigroup is E -inversive. More generally, according to Gigoń [2], any idempotent congruence class of an arbitrary eventually regular semigroup is an E -inversive semigroup. Finally, recall that each idempotent congruence class of an epigroup is an epigroup [14, Observation 2.2] and that every idempotent congruence class of a completely regular semigroup is a completely regular semigroup.

From Proposition 1.2 and from the second part of the above remark, we obtain the following corollary.

Corollary 1.3 *Every eventually regular semigroup S that is decomposable into a band of unipotent semigroups is a band of unipotent epigroups. Also, S is a semilattice of rectangular bands of unipotent epigroups.*

Every regular semigroup that is decomposable into a band of unipotent semigroups is a band of groups.

Recall that a semigroup is *simple* if it has no proper ideals (for example, a kernel of any semigroup is a simple semigroup); simple semigroups with at least one primitive idempotent are said to be *completely simple* (in fact, any idempotent of a completely simple semigroup is primitive).

It is clear that for an arbitrary ideal I of a semigroup S , the relation

$$\rho_I = (I \times I) \cup 1_S,$$

where 1_S is the identity relation on S , is a congruence on S (the so-called *Rees congruence*). We denote the (Rees) quotient semigroup induced by this congruence by S/I . In that case, we say that S is an *ideal extension* of the semigroup I by the semigroup S/I . In particular, if S/I is a poor semigroup, then S is called a *poor extension* of I . Moreover, every extension of a semigroup S by its ideal I such that I is a retract of S is called *retractive* (or a *retract extension*).

Finally, the following result already proved in [3] will be useful.

Result 1.4 *Let $\varphi : A \rightarrow B$ be a mapping onto $B \subseteq A$, where the set A is equipped with some n -ary operation o_n ($n \geq 2$), such that $\varphi|_B = 1_B$ and $\ker(\varphi)$ is a congruence on A . Then, the following conditions are equivalent:*

- (a) *B is a retract of A , that is, φ is an endomorphism of A ;*
- (b) *for all $b_1, b_2, \dots, b_n \in B$:*

$$(b_1, b_2, \dots, b_n)o_n \in B.$$

2 Some Examples of Bands of Unipotent Semigroups

For definitions and elementary properties of Green's relations, the reader is referred to [9]. In this paper, for $\mathcal{H}$, we denote the equivalence $\mathcal{H}$ -class of a in S by $\mathcal{H}_a$. Recall that Green's Theorem says that either $\mathcal{H}_a \cap \mathcal{H}_a \mathcal{H}_a = \emptyset$ or $\mathcal{H}_a$ is a group. In particular, $\mathcal{H}_e$ is a group for any $e \in E_S$.

Recall from [12] that a semigroup is:

- (a) *$\mathcal{R}$ -commutative if*

$$(\forall a, b \in S)(\exists x \in S^1)(ab = bax);$$

- (b) *$\mathcal{L}$ -commutative if*

$$(\forall a, b \in S)(\exists x \in S^1)(ab = xba);$$

- (c) *$\mathcal{H}$ -commutative if*

$$(\forall a, b \in S)(\exists x \in S^1)(ab = bxa).$$

Remark 3 A semigroup is both $\mathcal{R}$ -commutative and $\mathcal{L}$ -commutative if and only if it is $\mathcal{H}$ -commutative [12, Theorem 5.1].

Also, a semigroup S is $\mathcal{H}$ -commutative if and only if $\mathcal{H}$ is a commutative congruence on S (see [12, Theorem 5.2]).

Proposition 2.1 *The set of idempotents of an arbitrary $\mathcal{H}$ -commutative semigroup S is either empty or contained in the center of S .*

Proof Indeed, let $a \in S$ and $e \in E_S$. Then, $ea = xae$ and $ae = eay$ for some $x, y \in S^1$, whence $eae = ea$ and $eae = ae$. Thus, $ea = ae$, as required. $\square$

Remark 4 Note that any quasi-commutative semigroup is $\mathcal{H}$ -commutative (see [12, Chapter 8]).

Furthermore, every $\mathcal{H}$ -commutative semigroup is a Putcha semigroup (see [12, Theorem 5.3] and Result 1.1).

Recall also from [1] that a semigroup is *idempotent-surjective* if every its idempotent congruence class contains an idempotent. Also, a semigroup S is η -*idempotent-surjective* if every η -class of S contains an idempotent.

Proposition 2.2 *Each η -class of a Putcha semigroup with central idempotents contains at most one idempotent. Thus, every η -idempotent-surjective Putcha semigroup S with central idempotents is a semilattice of unipotent epigroups. In particular, S is an epigroup.*

Proof The first statement follows from Result 1.1 and Green's theorem. Consider now an arbitrary η -class $e\eta$ of S ($e \in E_S$). We must only show that the unipotent semigroup $e\eta$ is an epigroup. We know that $e\eta$ is archimedean (Result 1.1). In particular, $\text{Reg}(e\eta) \subseteq K_{e\eta}$. Hence, $K_{e\eta}$ is a group (since $e\eta$ is unipotent). This implies that $e\eta$ is an epigroup. $\square$

Observe that a semigroup S is an epigroup if for every $a \in S$ there is a positive integer l and $e \in E_S$ such that $a^l \in \mathcal{H}_e$. In that case, denote by $\mathbf{g}(a)$ the least positive integer m for which $a^m \in \mathcal{H}_e$ and recall that $a^n \in \mathcal{H}_e$ for all natural numbers $n \geq \mathbf{g}(a)$. Finally, for any $e \in E_S$, the set

$$\mathcal{K}_e = \{a \in S : a^{\mathbf{g}(a)} \in \mathcal{H}_e\}$$

is called the *unipotency class* of S generated by e . In the light of Green's theorem, $E_S \cap \mathcal{K}_e = \{e\}$, and by above, $\{\mathcal{K}_e : e \in E_S\}$ is a partition of S . Denote the equivalence relation induced by this partition by $\mathcal{K}$.

Finally, consider an arbitrary epigroup S with central idempotents. (It is well known that any epigroup is idempotent-surjective.) It is clear that in such a case, S meets condition (5a) of [13, Theorem 3]; so S is a Putcha semigroup (by [13, Theorem 3.16], see condition (6b)). In a view of the above proposition, S is the semilattice S/η of the unipotent epigroups. Finally, it is clear that in such a case,

$$\eta = \mathcal{K}.$$

We have just proved the following known result (see [13, Corollary 3.37]).

Corollary 2.3 Every epigroup S with central idempotents is the semilattice E_S of the unipotent epigroups $\mathcal{K}_e$ ($e \in E_S$).

From the above two results, we obtain the following corollary.

Corollary 2.4 The following conditions concerning a semigroup S with central idempotents are equivalent:

- (a) S is an epigroup;
- (b) S is an η -idempotent-surjective Putcha semigroup.

Corollary 2.5 Let S be a Putcha semigroup with central idempotents (we assume that $E_S \neq \emptyset$). Then,

$$\bigcup_{e \in E_S} e\eta$$

is a semilattice of unipotent epigroups.

The following result is known (see [4] or [11]).

Result 2.6 Let S_e be an E -inversive unipotent semigroup with $E_{S_e} = \{e\}$. Then, the set G_e of all regular elements of S_e is a group kernel of S_e .

Moreover, the map $\varphi_e : S_e \rightarrow G_e$ defined for all a in S_e , as follows:

$$a\varphi_e = ea,$$

is an endomorphism of S_e leaving the elements of G_e fixed, i.e., φ_e is a retraction of S_e onto G_e .

Finally, put

$$\ker(\varphi_e) := \sigma_e.$$

Then, σ_e is the least group congruence on S_e and

$$S_e/\sigma_e \cong G_e.$$

Proof We give a simple proof for the sake of completeness. Clearly, any ideal of S_e contains an idempotent. Hence, S_e has a kernel which contains e , say G_e . As S_e is unipotent, G_e is a unipotent simple semigroup, that is, it is a group. Obviously, $G_e \subseteq \text{Reg}(S_e)$. Conversely, take an arbitrary regular element a of S_e . Then, $a = axa$ for some $x \in S_e$. Thus,

$$a \in J(a) = J(axa) = J(ea) \subseteq J(e) = G_e,$$

Consequently, $G_e = \text{Reg}(S_e)$.

Further, note that $ea = eae = ae$ for every $a \in S_e$. This implies, together with the first part of the result, its second part.

Finally, we must only show that σ_e is the *least* group congruence on S_e . For this, let ρ be any group congruence on S_e and let $a \sigma_e b$. Then, $ea = eb$. Hence, we have $(e\rho)(a\rho) = (e\rho)(b\rho)$. Thus, $a \rho b$ by cancellation, so $\sigma_e \subseteq \rho$. $\square$

Let ρ be a such band congruence on a semigroup S that every ρ -class of S is an E -inversive unipotent semigroup (that is, S is a band of E -inversive unipotent semigroups). For every $e \in E_S$, put:

$$S_e := e\rho, \quad G_e := \text{Reg}(e\rho).$$

Let also $\varphi_e : S_e \rightarrow G_e$ be defined as in Result 2.6 and put:

$$\sigma_e := \ker(\varphi_e).$$

Then (we treat all functions φ_e as relations)

$$\varphi = \bigcup_{e \in E_S} \varphi_e$$

is a mapping from S onto

$$\bigcup_{e \in E_S} G_e = \text{Gr}(S) = \text{Reg}(S).$$

Finally, put:

$$\nu := \ker(\varphi).$$

Remark 5 The above notation will be used below. If, in addition, S is an E -semigroup, then we will say that S is the band E_S of the E -inversive unipotent semigroups S_e ($e \in E_S$). A similar notation and nomenclature will be used for semilattices or rectangular bands of E -inversive unipotent semigroups.

Proposition 2.7 *Let S be an η -idempotent-surjective Putcha semigroup with central idempotents, i.e., S is the semilattice E_S of the unipotent epigroups $S_e = \mathcal{K}_e$ ($e \in E_S$). Then, the above map φ is a retraction of S onto $\text{Reg}(S)$. In particular,*

$$S/\nu \cong \text{Reg}(S).$$

Finally, ν is the least regular congruence on S .

Proof The first two statements are clear.

Finally, as in the proof of [5, Theorem 2.7], one can prove that ν is the least semilattice of groups congruence on S . Consequently, ν is the least regular congruence on S by [3, Lemma 1.7]. $\square$

Remark 6 In any $\mathcal{H}$ -commutative epigroup S , $\mathcal{H}$ is a (commutative) congruence. Thus, $\mathcal{H} \subseteq \mathcal{K}$. Moreover, put $T := S/\mathcal{H}$. Then, T is the (commutative) semilattice E_T of the (commutative) nil semigroups $\mathcal{K}_e$ ($e \in E_T = \text{Reg}(T)$).

3 The Main Results

Recall from [11] that a semigroup S is E -inverse if and only if the set

$$W_S(a) = \{x \in S : x = xax\}$$

is non-empty for every $a \in S$. In that case, $ax, xa \in E_S$.

The following lemma will be useful (see Result 2.6 for the notation).

Lemma 3.1 *Let S_e be an E -inverse unipotent semigroup with $E_{S_e} = \{e\}$. Then, the following conditions are equivalent ($a, b \in S_e$):*

- (a) $a \sigma_e b$;
- (b) $ab^* = e$ for every $b^* \in W_{S_e}(b)$;
- (c) $ab^* = e$ for some $b^* \in W_{S_e}(b)$.

Furthermore, for all $x, y \in S_e$ and any $y^* \in W_{S_e}(y)$,

$$xy^* = e \iff y^*x = e.$$

Proof (a) $\implies$ (b). Let $a \sigma_e b$ and let $b^* \in W_{S_e}(b)$. Then, $ea = eb$. Hence,

$$eab^* = ebb^* = ee = e.$$

Thus, $ab^* = e$ (since ab^* belongs to the group G_e with identity e).

(c) $\implies$ (a). Let $ab^* = e$ for some $b^* \in W_{S_e}(b)$. Then,

$$eb = ab^*b = ae = ea.$$

Finally, let $x, y \in S_e$ and let $y^* \in W_{S_e}(y)$. Suppose first that $xy^* = e$. Then,

$$(y^*x)^2 = y^*(xy^*)x = (y^*e)x = y^*x = e.$$

The converse implication follows in a similar way. □

Further, recall from [6, Corollary 4.2] that in an arbitrary E -semigroup S , for every $e \in E_S$, $a \in S$ and any $a^* \in W_S(a)$, we have:

$$aea^*, a^*ea \in E_S.$$

The following proposition is less general than the theorem that occurs after it. We quote it here because of the other methods of its proof.

Proposition 3.2 *Let an E -semigroup S be the band E_S of the E -inverse unipotent semigroups S_e ($e \in E_S$). Then, ν is a congruence on S . Also, φ is a retraction of S onto $\text{Reg}(S)$. In particular,*

$$S/\nu \cong \text{Reg}(S)$$

is an orthodox band of groups.

Proof It is clear that ν is an equivalence relation on S . Let $e, f \in E_S$ and let $a, b \in S_e$ be such that $a \nu b$. Finally, take any $c \in S_f$ and let $(bc)^* \in W_{S_{ef}}(bc)$, $(cb)^* \in W_{S_{fe}}(cb)$. Then, $ba^* = e = a^*b$ for some $a^* \in W_{S_e}(a)$ (Lemma 3.1). Observe also that $c(bc)^*b \in E_S \cap S_{fe}$ and $b(cb)^*c \in E_S \cap S_{ef}$. Hence,

$$c(bc)^*b = fe, \quad b(cb)^*c = ef.$$

We show first that ν is a right congruence on S . We have:

$$(ac)(bc)^*e = a(c(bc)^*b)a^* = afea^* = afa^* = efe,$$

since $afa^* \in E_S \cap S_{efe}$. Hence,

$$(ac)(bc)^*ef = efef = ef.$$

Thus, $(ac)(bc)^* = ef$, that is, $ac \nu bc$.

Also, we prove that ν is a left congruence on S . We have:

$$e(cb)^*(ca) = a^*(b(cb)^*c)a = a^*efa = a^*fa = efe,$$

as $a^*fa \in E_S \cap S_{efe}$. Hence, $(cb)^*(ca) = fe$. Thus, $ca \nu cb$ by Lemma 3.1.

Further, φ is a retraction of S onto $\text{Reg}(S)$ by the above and Result 1.4.

Finally, the last statement of the proposition is clear. $\square$

Recall the known fact due to Hall (see [8, Result 7]) that a semigroup S with $E_S \neq \emptyset$ is an R -semigroup if and only if $E_S E_S \subseteq \text{Reg}(S)$. Therefore, the following theorem is more general than the above proposition.

Theorem 3.3 *Let an R -semigroup S be the band S/ρ of the E -inversive unipotent semigroups $S_e = e\rho$ ($e \in E_S$). Then, the map φ is a retraction of S onto $\text{Reg}(S)$. In particular,*

$$S/\nu \cong \text{Reg}(S)$$

is a band of groups.

Proof Let $a, b \in S$, $e, f \in E_S$ and let $ef \in S_g$, where $g \in E_S$. Then,

$$eg = (eg)g \in S_g G_g = G_g.$$

Hence,

$$(eg)^2 = e(g \cdot eg) = eeg = eg.$$

Thus, $eg = g$. We may equally well show that $gf = g$. Assume furthermore that $a \in S_e, b \in S_f$. Then,

$$(ea)(fb) = (ea)(bf) \in G_e G_f \subseteq G_g,$$

since S is an R -semigroup. Finally, observe that

$$g \cdot eab \in G_g S_g = G_g, \quad ab \cdot g \in S_g G_g = G_g.$$

Consequently, we have:

$$(ea)(fb) = geabf = (geab \cdot g)f = geab \cdot gf = ge(g \cdot abg) = (g \cdot eg)abg = g(ab).$$

Hence, φ is a retraction of S onto $\text{Reg}(S)$. $\square$

Recall from [1] that a congruence on a semigroup is *idempotent-separating* if every its equivalence class contains at most one idempotent. It has been shown in [1] that every idempotent-surjective semigroup has the maximum idempotent-separating congruence.

Remark 7 Note that if the semigroup S from Theorem 3.3 is idempotent-surjective, then the band congruence from the above theorem is the *least* band and the *maximum* idempotent-separating congruence on S .

The following result will be useful.

Lemma 3.4 *Let θ be a regular congruence on an idempotent-surjective semigroup S which is the band S/ρ of the unipotent E -inversive semigroups $e\rho$ ($e \in E_S$). Then S/θ is the band S/ρ of the groups $G_e = \{a\theta_e : a \in e\rho\}$ ($e \in E_S$), where θ_e is the restriction of θ to $e\rho$.*

Proof Indeed, each G_e is a unipotent (E -inversive) semigroup, so S/θ is the band S/ρ of the unipotent (E -inversive) semigroups G_e ($e \in E_S$). The thesis of the lemma follows now from the second part of Corollary 1.3. $\square$

Corollary 3.5 *Let an idempotent-surjective R -semigroup S be the band S/ρ of the unipotent E -inversive semigroups $e\rho$ ($e \in E_S$). Then the relation*

$$\nu = \{(a, b) \in S \times S : (\exists e \in E_S)(a, b \in e\rho \text{ \& } ea = eb)\}$$

is the least regular congruence on S .

Proof As in the proof of [5, Theorem 2.7], one can show that ν is the least band of groups congruence on S . Consequently, ν is the least regular congruence on S by the above lemma. $\square$

Recall that every equivalence class of a rectangular band congruence on an arbitrary E -inversive semigroup is an E -inversive semigroup [7, Corollary 2.7]. This implies that if S is an E -inversive semigroup which is the rectangular band S/ρ of the unipotent semigroups $e\rho$ ($e \in E_S$), then ρ is the *least* rectangular band congruence on S .

It is very well known that a semigroup is completely simple if and only if Green's relation $\mathcal{H}$ is a (least) rectangular band congruence on it. Observe also that any idempotent of a rectangular band of unipotent semigroups is primitive (such semigroups are said to be *primitive*, see [11]).

Further, we have the following important characterization.

Theorem 3.6 *The following conditions concerning an E -inversive semigroup S are equivalent:*

- (a) S is a rectangular band of unipotent semigroups;
- (b) S is primitive and $\text{Reg}(S)$ is a retract of S ;
- (c) S is a retractive poor extension of a completely simple semigroup;
- (d) S is a subdirect product of a poor semigroup and a completely simple semigroup.

Proof (a) $\implies$ (b). We know that any idempotent of S is primitive. This implies that $\text{Reg}(S)$ is a completely simple kernel of S (see [4] or [11]). Hence, $\text{Reg}(S)$ is also a retract of S by Theorem 3.3.

(b) $\implies$ (c). This is clear by the above proof.

(c) $\implies$ (d). This is also clear.

(d) $\implies$ (a). Suppose that S is a subdirect product of a poor semigroup A and a completely simple semigroup B (we may assume that $S \subseteq A \times B$). Define the relation ρ on S , as follows $((a, b), (c, d) \in S)$:

$$(a, b) \rho (c, d) \iff b \mathcal{H} d.$$

Since $\mathcal{H}$ is a rectangular band congruence on B , ρ is a rectangular band congruence on S . Hence, every ρ -class of S is an E -inversive semigroup. Finally, each ρ -class of S is unipotent (as A is poor and $\mathcal{H}$ is an idempotent-separating congruence on B by Green's theorem). Note that

$$\text{Reg}(S) = \{(0, b) : b \in B\},$$

where 0 is a zero of A (we have $E_A = \{0\}$).

□

Remark 8 One can easily modify the above theorem for *left zero bands* of unipotent semigroups. Note that in any E -inversive left zero band S/ρ of the unipotent semigroups $e\rho$ ($e \in E_S$) the set E_S forms a left zero band. Consequently, $\text{Reg}(S)$ is a left group. Indeed, $e \rho ef$ for all $e, f \in E_S$ and $ef \in \text{Reg}(S)$. Hence,

$$(ef)^2 = (ef \cdot e)f = eff = ef = e.$$

Remark 9 Let us note that the condition (d) from the above theorem is a source of examples of semigroups from the condition (a). For instance, we may consider the semigroup $A = \{0, 2, 3, 4, \dots\}$ with multiplication and an arbitrary completely simple semigroup B . Then, the direct product $T = A \times B$ is a rectangular band of unipotent E -inversive semigroups. Note that T is *not* an epigroup.

We say that a semigroup A is β -fruitful if every β -class of the least band congruence β on A is an E -inversive semigroup. Observe that in any β -fruitful semigroup S which is the band S/ρ of unipotent semigroups, we get $\rho = \beta$.

Moreover, the following “more general” result is also valid.

Theorem 3.7 *The following conditions concerning a β -fruitful semigroup S are equivalent:*

- (a) S is a band of unipotent semigroups and $\text{Reg}(S)$ is an ideal of S ;
- (b) $\text{Reg}(S)$ is a band of groups and a retract ideal of S ;
- (c) S is a retractive poor extension of a band of groups;
- (d) S is a subdirect product of a poor semigroup and a band of groups.

Proof (a) $\implies$ (b). This follows from Theorem 3.3.

(b) $\implies$ (c) $\implies$ (d). This is clear.

(d) $\implies$ (a). Let S be a subdirect product of a poor semigroup A and a band of groups B (we may assume that $S \subseteq A \times B$). Define the relation ρ on S as in the proof of Theorem 3.6. As $\mathcal{H}$ is a band congruence on B , ρ is a band congruence on S . Hence, every ρ -class of S is an E -inverse semigroup (as S is β -fruitful). Further, each ρ -class of S is unipotent. Finally, the set

$$\text{Reg}(S) = \{(0, b) : b \in B\}$$

is obviously an ideal of S .

□

4 Application to Epigroup Theory

Recall that a poor epigroup is called a *nil semigroup*. Let I be an arbitrary ideal of a semigroup S . We say that S is a *nil-extension* of I if S/I forms a nil semigroup [13].

As every idempotent congruence class of an epigroup is an epigroup [14, Observation 2.2], the following corollary which seems to be (partially) new is valid (in fact, the implication “(d) $\implies$ (a)” in this corollary is new; see also [14, Proposition 3.5] and [14, Theorem 3.48]).

Corollary 4.1 *The following conditions on an epigroup S are equivalent:*

- (a) S is a band of unipotent semigroups and $\text{Reg}(S)$ is an ideal of S ;
- (b) $\text{Reg}(S)$ is a band of groups and a retract ideal of S ;
- (c) S is a retractive nil-extension of a band of groups;
- (d) S is a subdirect product of a nil semigroup and a band of groups.

Recall that a *completely regular* semigroup S is a semigroup in which every element is contained in some subgroup of S .

Remark 10 Let a semigroup S be a subdirect product of a nil semigroup A and a completely regular semigroup B (we assume *below* that $S \subseteq A \times B$) and let b be an arbitrary element of B (say: $b \in \mathcal{H}_e$, where $e \in E_B$). Then, $(a, b) \in S$ for some $a \in A$. As A is a nil semigroup, there is a positive integer n such that $a^n = 0$. Hence, $(a, b)^n = (0, b^n) \in S$. Thus, $(0, e) \in S$. Therefore, $(0, b) \in S$. Consequently,

$$\text{Reg}(S) = \text{Gr}(S) = \{(0, b) : b \in B\} \cong B$$

is a retract ideal of S (by the map $\varphi : (a, b) \rightarrow (0, b)$; in fact, if $b \in \mathcal{H}_e$, where $e \in E_B$, then $(0, b) = (0, e)(a, b)$). Finally, observe that S is an epigroup.

It is not possible to prove in a similar way an analogous theorem for a semigroup that is a subdirect product of a poor semigroup and a completely regular semigroup.

On the other hand, in a semigroup S in which the set $\text{Gr}(S)$ is its ideal,

$$\text{Gr}(S) = \text{Reg}(S)$$

(in such a case, S is E -inversive).

From the above considerations, we get the following “stronger” version of [14, Proposition 3.5].

Proposition 4.2 *In any semigroup S which is a subdirect product of a nil semigroup A and a completely regular semigroup B , the set $\text{Gr}(S)$ is a retract ideal of S . In that case, S is an epigroup and*

$$\text{Reg}(S) = \text{Gr}(S) = \{(0, b) : b \in B\} \cong B,$$

where 0 is a zero of the nil semigroup A .

Conversely, each eventually regular semigroup S , in which the set $\text{Gr}(S)$ is its retract ideal, is a subdirect product of the nil semigroup $S/\text{Gr}(S)$ and the completely regular semigroup $\text{Gr}(S)$.

Observe that if an epigroup is a band of unipotent semigroups, then the corresponding band congruence is equal to $\mathcal{K}$ (as clearly $\mathcal{K} \subseteq \beta$ in any epigroup, whence in parenthesis, $\beta = \mathcal{K}^\#$). For brevity, we shall say that an epigroup is an R -epigroup if it is an R -semigroup.

Finally, from Corollary 3.5 we obtain the following corollary.

Corollary 4.3 *The relation ν is the least regular congruence on any R -epigroup S which is a band of unipotent semigroups. Moreover,*

$$\nu = \{(a, b) \in S \times S : (\exists e \in E_S)(a, b \in \mathcal{K}_e \text{ \& } ea = eb)\}.$$

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