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The Bound of the Hankel Determinant of the Third Kind for Starlike Functions

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Abstract
In the present paper, the estimate of the third Hankel determinant

$$H_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

for the class of starlike functions, i.e., for the class of analytic functions f standardly normalized such that $\mathrm{Re}(zf'(z)/f(z)) > 0,\ z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, is improved.

Keywords Univalent functions · Starlike functions · Carathéodory functions · Hankel determinant · Fourth coefficient

Mathematics Subject Classification Primary 30C45

1 Introduction

Let $\mathcal{H}$ be a class of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}$ be its subclass normalized by $f(0) := 0$, $f'(0) := 1$, i.e., of the form

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$$f(z)=\sum_{n=1}^{\infty}a_nz^n,\quad a_1:=1,\ z\in\mathbb{D}. \tag{1.1}$$

Let $\mathcal{S}^*$ denote the class of starlike functions, namely, the subclass of $\mathcal{A}$ consisting of functions f such that

$$\operatorname{Re}\frac{zf'(z)}{f(z)}>0,\quad z\in\mathbb{D}. \tag{1.2}$$

Given $q,n\in\mathbb{N}$, the Hankel determinants $H_{q,n}(f)$ of Taylor's coefficients of functions $f\in\mathcal{A}$ of the form (1.1) are defined as

$$H_{q,n}(f):=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1}\\ a_{n+1} & a_{n+2} & \cdots & a_{n+q}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

Particularly, the third Hankel determinant $H_{3,1}(f)$ is given by

$$H_{3,1}(f):=\begin{vmatrix} a_1 & a_2 & a_3\\ a_2 & a_3 & a_4\\ a_3 & a_4 & a_5\end{vmatrix}=a_3(a_2a_4-a_3^2)-a_4(a_4-a_2a_3)+a_5(a_3-a_2^2). \tag{1.3}$$

To find the growth of the Hankel determinant $H_{q,n}(f)$ dependent on q and n for the whole class $\mathcal{S}\subset\mathcal{A}$ of univalent functions as well as for its subclasses is an interesting problem to study. For the class $\mathcal{S}$ some important result was shown by Pommerenke [13]. For fixed q and n the growth problem can be reduced to an estimate of the Hankel determinant for the selected subclasses of $\mathcal{A}$. Recently many authors examined the Hankel determinant $H_{2,2}(f)=a_2a_4-a_3^2$ of order 2 (see, e.g., [3,4,6,8,12]). Note also that $H_{2,1}(f)=a_3-a_2^2$. Thus the Hankel determinant $H_{2,1}(f)$ reduces to the well-known coefficient functional which for $\mathcal{S}$ was estimated in 1916 by Bieberbach (see, e.g., [5, Vol. I, p. 35]).

The problem to find the upper bound of the Hankel determinant $H_{3,1}(f)$ of order 3 is more sophisticated if we expect to get sharp result. From (1.3) by using the triangle inequality we get at once the following inequality

$$|H_{3,1}(f)|\le|a_3||H_{2,2}(f)|+|a_4||a_4-a_2a_3|+|a_5||H_{2,1}(f)|. \tag{1.4}$$

This simple observation allowed to estimate of $|H_{3,1}(f)|$ for compact subclasses $\mathcal{F}$ of $\mathcal{A}$ by various authors (see, e.g., [2,15–18]). However, these results are far from sharpness. If case when a given subclass $\mathcal{F}$ of $\mathcal{A}$ has a representation with using the Carathéodory class $\mathcal{P}$, i.e., the class of functions $p\in\mathcal{H}$ of the form

$$p(z)=1+\sum_{n=1}^{\infty}c_nz^n,\quad z\in\mathbb{D}, \tag{1.5}$$

having a positive real part in $\mathbb{D}$, the coefficients of functions in $\mathcal{F}$ have a suitable representation expressed by the coefficients of functions in $\mathcal{P}$. Therefore to get the upper bound of each term in (1.4) cited authors based their computing on the well-known formulas on coefficient c_2 (e.g., [14, p. 166]) and on the formula c_3 due to Libera and Zlotkiewicz [9].

In order to improve the bound of $|H_{3,1}(f)|$ we have to use directly formula (1.3), where we need to apply a formula for c_4, similar to the formulas (2.1) and (2.2). In a recent paper [7] the authors found such a formula for c_4. According to the authors' knowledge, formulas for the coefficients c_n for $n \geq 5$ analogous to the formulas (2.1) and (2.2) are not known.

Basing on the formulas for c_2, c_3 and c_4, we improve the known estimate of the Hankel determinant $H_{3,1}(f)$ in the class $\mathcal{S}^*$ of starlike functions. We show that $|H_{3,1}(f)| \leq 8/9$. Estimating each term of the right hand of (1.4) Babalola [1] showed that $|H_{3,1}(f)| \leq 16$. In [19] Zaprawa by a suitable grouping and using Lemma 1 due to Livingston [11] proved that $|H_{3,1}(f)| \leq 1$.

2 Main Result

The basis for proof of the main result is the following lemma. It contains the well-known formula for c_2 (e.g., [14, p. 166]), the formula for c_3 due to Libera and Zlotkiewicz [9,10] and the formula for c_4 found by the authors [7].

Lemma 2.1 *If $p \in \mathcal{P}$ is of the form* (1.5) *with $c_1 \geq 0$, then*

$$2c_2 = c_1^2 + (4 - c_1^2)\zeta, \tag{2.1}$$
$$4c_3 = c_1^3 + (4 - c_1^2)c_1\zeta(2 - \zeta) + 2(4 - c_1^2)(1 - |\zeta|^2)\eta \tag{2.2}$$

and

$$\begin{aligned} 8c_4 = c_1^4 &+ (4 - c_1^2)\zeta\left[c_1^2(\zeta^2 - 3\zeta + 3) + 4\zeta\right] \\ &- 4(4 - c_1^2)(1 - |\zeta|^2)\left[c_1(\zeta - 1)\eta + \overline{\zeta}\eta^2 - \left(1 - |\eta|^2\right)\xi\right] \end{aligned} \tag{2.3}$$

for some $\zeta, \eta, \xi \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$.

Now, we will estimate the third-order Hankel determinant $H_{3,1}(f)$ for $f \in \mathcal{S}^*$. To this end, the following propositions are required.

Proposition 2.2 *Let $\Theta : [0, 3] \times [0, 1] \to \mathbb{R}$ be a function defined by*

$$\Theta(t, x) := 96\theta_1(x) - 8\theta_2(x)t + 3\theta_3(x)t^2, \tag{2.4}$$

where for $x \in [0, 1]$,

$$\theta_1(x) := 2 + 8x - x^2 - 6x^3,$$
$$\theta_2(x) := 16 + 67x - 34x^2 - 53x^3 + 2x^4$$

and

$$\theta_3(x) := 12 + 19x - 21x^2 - 17x^3 + 3x^4.$$

Then $\Theta(t, x) > 0$ *for* $0 \le t \le 3$ *and* $0 \le x \le 1$.

Proof At first, note that the polynomial θ_3 has a unique zero $x =: x_1 \approx 0.9314$ in $(0, 1)$. Since $x_1 \in (0.92, 0.95)$ and for $x \in (0.92, 0.95)$,

$$\begin{aligned}\theta_2(x) &> 16 + 67 \cdot (0.92) - 34 \cdot (0.95)^2 - 53 \cdot (0.95)^3 + 2 \cdot (0.92)^4 \\ &= 2.94691092 > 0,\end{aligned}$$

it follows that

$$\frac{\partial}{\partial t}\Theta(t, x_1) = -8\theta_2(x_1) \neq 0.$$

For $x \neq x_1$, $(\partial/\partial t)\Theta(t, x) = 0$ occurs at

$$t = \frac{4\theta_2(x)}{3\theta_3(x)} =: t_0(x).$$

We have

$$\frac{\partial}{\partial x}\Theta(t, x)\bigg|_{t=t_0(x)} = \frac{16\theta_4(x)}{9\theta_3^2(x)},$$

where

$$\begin{aligned}\theta_4(x) &:= 54\theta_1'(x)\theta_3^2(x) - 6\theta_2'(x)\theta_2(x)\theta_3(x) + 3\theta_3'(x)\theta_2^2(x) \\ &= -3\Big(128 + 31896x - 18709x^2 - 133828x^3 - 3737x^4 + 198602x^5 \\ &\quad +74185x^6 - 91136x^7 - 54071x^8 - 2774x^9 + 668x^{10} \\ &\quad +48x^{11}\Big), \quad x \in (0, 1).\end{aligned}$$

The polynomial θ_4 has exactly two zeros in $(0, 1)$, namely, $x =: x_2 \approx 0.533701$ and $x =: x_3 \approx 0.811327$. We will now show that

$$t_0(x) > 3, \quad x \in [0.5, 0.9]. \tag{2.5}$$

Since $x_1 > 0.9$, so $\theta_3(x) > 0$, for $x \in [0.5, 0.9]$ and the inequality (2.5) is equivalent to

$$4\theta_2(x) - 9\theta_3(x) > 0, \quad x \in [0.5, 0.9].$$

The above one can be equivalently written as

$$19x^4 + 59x^3 - 53x^2 - 97x + 44 < 0, \quad x \in [0.5, 0.9].$$

As the polynomial on the left hand of the above inequality has a unique zero $x \approx 0.40928$ in $[0, 1]$, the above inequality is true, so is the inequality (2.5). Thus the function Θ has no critical point in $(0, 3) \times (0, 1)$. Hence it is sufficient to show that $\Theta > 0$ on the boundary of $[0, 3] \times [0, 1]$. We can easily check that the following inequalities hold:

$$\begin{aligned}\Theta(t, 0) &= 4(48 - 32t + 9t^2) \geq \frac{704}{9}, \quad t \in [0, 3],\\ \Theta(t, 1) &= 4(72 + 4t - 3t^2) \geq 228, \quad t \in [0, 3],\\ \Theta(0, x) &= 96(2 + 8x - x^2 - 6x^3) \geq 192, \quad x \in [0, 1],\end{aligned}$$

and

$$\begin{aligned}\Theta(3, x) &= 3\left(44 - 109x + 51x^2 + 79x^3 + 11x^4\right)\\ &= 3\left(44(1 - x)^3 + x(23 - 81x + 123x^2) + 11x^4\right)\\ &\geq 3\left(44(1 - x)^3 + \frac{1585}{164}x + 11x^4\right) > 0, \quad x \in [0, 1].\end{aligned}$$

Thus the proof of the proposition is completed. □

Proposition 2.3 *Let* $\Psi : [1, 4] \times [0, 1] \to \mathbb{R}$ *be a function defined by*

$$\Psi(t, x) := 16\psi_1(x) + 8\psi_2(x)t + 3\psi_3(x)t^2, \tag{2.6}$$

where for $x \in [0, 1]$,

$$\begin{aligned}\psi_1(x) &:= -2 + 27x + 21x^2 - 37x^3 + x^4,\\ \psi_2(x) &:= 10 - 12x - 9x^2 + 20x^3 + x^4\end{aligned}$$

and

$$\psi_3(x) := x(3 - 5x - x^2 - x^3).$$

Then $\Psi(t, x) > 0$ *for* $1 \leq t \leq 4$ *and* $0 \leq x \leq 1$.

Proof At first, note that the function ψ_3 has a unique zero $x =: x_1 \approx 0.51839$ in $(0, 1)$. Since $x_1 \in (0.5, 0.6)$ and for $x \in (0.5, 0.6)$,

$$\psi_2(x) > 10 - 12 \cdot (0.6) - 9 \cdot (0.6)^2 + 20 \cdot (0.5)^3 + (0.5)^4 = 2.1225 > 0, \tag{2.7}$$

it follows that

$$\frac{\partial}{\partial t}\Psi(t, x_1) = 8\psi_2(x_1) \neq 0.$$

For $x \neq x_1$, $(\partial/\partial t)\Psi(t,x) = 0$ occurs at

$$t = \frac{-4\psi_2(x)}{3\psi_3(x)} =: t_0(x).$$

We have

$$\frac{\partial}{\partial x}\Psi(t,x)\Big|_{t=t_0(x)} = \frac{16\psi_4(x)}{3\psi_3^2(x)},$$

where

$$\begin{aligned}\psi_4(x) &:= 3\psi_1'(x)\psi_3^2(x) - 2\psi_2'(x)\psi_2(x)\psi_3(x) + \psi_3'(x)\psi_2^2(x)\\ &= 300 - 1000x + 1737x^2 - 4912x^3 + 2009x^4 + 13706x^5 - 17777x^6\\ &\quad + 6596x^7 - 1541x^8 + 546x^9 - 184x^{10} + 16x^{11}, \quad x \in (0,1).\end{aligned}$$

The polynomial ψ_4 has a unique zero $x =: x_2 \approx 0.388025$ in $(0,1)$. Since $x_1 > 0.5$, so $\psi_3(x) > 0$, for $x \in (0, 0.5)$. Additionally, since ψ_2 has no zero in $(0,1)$, the inequality (2.7) is true on $[0,1]$. Thus $t_0(x) < 0$ for $x \in (0, 0.5)$ and in consequence, the function Ψ has no critical point in $(1,4) \times (0,1)$. Hence it is sufficient to show that $\Psi > 0$ on the boundary of $[1,4] \times [0,1]$. We can easily check that the following inequalities hold:

$$\begin{aligned}\Psi(t,0) &= -32 + 80t \geq 48, \quad t \in [1,4],\\ \Psi(t,1) &= 160 + 80t - 12t^2 \geq 228, \quad t \in [1,4],\\ \Psi(4,x) &= 96(3 + 2x - 2x^2) \geq 288, \quad x \in [0,1],\end{aligned}$$

and
$$\begin{aligned}\Psi(1,x) &= 3\left(16 + 115x + 83x^2 - 145x^3 + 7x^4\right)\\ &= 3\left(16 + 53x^2 + 7x^4 + 115x(1-x^2) + 30x^2(1-x)\right)\\ &\geq 48, \quad x \in [0,1].\end{aligned}$$

Thus the proof of the proposition is completed. □

Proposition 2.4 *Let* $\Phi : [3,4] \times [0,1] \to \mathbb{R}$ *be a function defined by*

$$\Phi(t,x) := 48\phi_1(x) + 8\phi_2(x)t - 3\phi_3(x)t^2, \tag{2.8}$$

where for $x \in [0,1]$,

$$\begin{aligned}\phi_1(x) &:= 1 + 7x + x^2 - 3x^3,\\ \phi_2(x) &:= 5 - 19x + 10x^2 + 5x^3 + x^4\end{aligned}$$

and

$$\phi_3(x) := x(-3 + 5x + x^2 + x^3).$$

Then $\Phi(t,x) > 0$ *for* $3 \leq t \leq 4$ *and* $0 \leq x \leq 1$.

Proof Since $\phi_3 = -\psi_3$, by the part of proof of Proposition 2.3, we at once have

$$\frac{\partial}{\partial t}\Phi(t, x_1) = 8\phi_2(x_1) \neq 0.$$

For $x \neq x_1$, $(\partial/\partial t)\Phi(t, x) = 0$ occurs at

$$t = \frac{4\phi_2(x)}{3\phi_3(x)} =: t_0(x).$$

We have

$$\frac{\partial}{\partial x}\Phi(t, x)\Big|_{t=t_0(x)} = \frac{16\phi_4(x)}{3\phi_3^2(x)},$$

where

$$\begin{aligned}\phi_4(x) &:= 9\phi_1'(x)\phi_3^2(x) + 2\phi_2'(x)\phi_2(x)\phi_3(x) - \phi_3'(x)\phi_2^2(x)\\ &= 75 - 250x + 59x^2 + 532x^3 - 893x^4 + 558x^5 - 269x^6 + 844x^7\\ &\quad - 366x^8 + 16x^9 - 46x^{10} + 4x^{11}, \quad x \in (0, 1).\end{aligned}$$

The polynomial ϕ_4 has exactly two zeros in $(0, 1)$, namely $x =: x_2 \approx 0.414034$ and $x =: x_3 \approx 0.663886$. We have $t_0(x_2) \approx 3.59845$ and $t_0(x_3) = -2.95522$. Therefore the function Φ has a unique critical point $(t_0(x_2), x_2)$ in $(3, 4) \times (0, 1)$. For $(t, x) \in [3.58, 3.61] \times [0.39, 0.43]$ by simple computing, we show that $\Phi(t, x) > 0$. Thus, particularly $\Phi(t_0(x_2), x_2) > 0$. Therefore it is sufficient to show that $\Phi > 0$ on the boundary of $[3, 4] \times [0, 1]$. We can easily check that the following inequalities hold:

$$\begin{aligned}\Phi(t, 0) &= 8(6 + 5t) \geq 168, \quad t \in [3, 4],\\ \Phi(t, 1) &= 4\left(72 + 4t - 3t^2\right) \geq 160, \quad t \in [3, 4],\\ \Phi(4, x) &= 16\left(13 - 8x + 8x^2 - 2x^3 - x^4\right) \geq 16(2 + 8x^2) \geq 32, \quad x \in [0, 1],\end{aligned}$$

and

$$\Phi(3, x) = 3\left(56 - 13x + 51x^2 - 17x^3 - x^4\right) \geq 3(25 + 51x^2) \geq 75, \quad x \in [0, 1].$$

Thus the proof of the proposition is completed. □

Finally, we estimate now the third-order Hankel determinant $H_{3,1}(f)$ for $f \in \mathcal{S}^*$.

Theorem 2.5 *If $f \in \mathcal{S}^*$ is the form (1.1), then*

$$|H_{3,1}(f)| \leq \frac{8}{9}. \tag{2.9}$$

Proof Let $f \in \mathcal{S}^*$ be of the form (1.1). Then by (1.2) we have

$$zf'(z) = f(z)p(z), \quad z \in \mathbb{D}, \tag{2.10}$$

for some function $p \in \mathcal{P}$ of the form (1.5). Since the class $\mathcal{P}$ is invariant under the rotations, we may assume that $c := c_1 \in [0, 2]$ (e.g., [5, Vol. I, p. 80, Theorem 3]). Putting the series (1.1) and (1.5) into (2.10) and by equating the coefficients we get

$$a_2 = c, \quad a_3 = \frac{1}{2}\left(c^2 + c_2\right), \quad a_4 = \frac{1}{6}\left(c^3 + 3cc_2 + 2c_3\right)$$

and

$$a_5 = \frac{1}{24}\left(c^4 + 6c^2c_2 + 8cc_3 + 3c_2^2 + 6c_4\right).$$

Hence

$$\begin{aligned} H_{3,1}(f) &= -a_3^3 + 2a_2a_3a_4 - a_4^2 - a_2^2a_5 + a_3a_5 \\ &= \frac{1}{144}\Big(-c^6 + 3c^4c_2 - 9c_2^3 + 8c^3c_3 + 24cc_2c_3 - 16c_3^2 \\ &\quad +18c_2c_4 - 9c^2c_2^2 - 18c^2c_4\Big). \end{aligned}$$

Now using the equalities (2.1)–(2.3), by straightforward algebraic computation we have

$$H_{3,1}(f) = \frac{1}{1152}(c^2 - 4)\left[\gamma_1(c, \zeta) + \gamma_2(c, \zeta)\eta + \gamma_3(c, \zeta)\eta^2 + \Gamma(c, \zeta, \eta)\xi\right], \tag{2.11}$$

where for $\zeta, \eta, \xi \in \overline{\mathbb{D}}$,

$$\begin{aligned} \gamma_1(c, \zeta) &:= c^2\zeta\left[-3c^2 + \left(44 - 5c^2\right)\zeta + \left(40 - c^2\right)\zeta^2\right] - c^2\left(4 - c^2\right)\zeta^4, \\ \gamma_2(c, \zeta) &:= -4c\left(1 - |\zeta|^2\right)\left[3c^2 + 4\left(5 + c^2\right)\zeta - \left(4 - c^2\right)\zeta^2\right], \\ \gamma_3(c, \zeta) &:= 32\left(4 - c^2\right) - 28|\zeta|^2\left(4 - c^2\right) - 36c^2\overline{\zeta}\left(1 - |\zeta|^2\right) - 4\left(4 - c^2\right)|\zeta|^4, \end{aligned}$$

and

$$\Gamma(c, \zeta, \eta) := 36\left[c^2 + \left(c^2 - 4\right)\zeta\right]\left(1 - |\zeta|^2\right)\left(1 - |\eta|^2\right).$$

Setting $x := |\zeta| \in [0, 1]$, $y := |\eta| \in [0, 1]$ and taking into account that $|\xi| \leq 1$, from (2.11) we get

$$\begin{aligned} &|H_{3,1}(f)| \\ &\leq \frac{1}{1152}\left(4-c^2\right)\left[|\gamma_1(c,\zeta)| + |\gamma_2(c,\zeta)||\eta| + |\gamma_3(c,\zeta)||\eta|^2 + |\Gamma(c,\zeta,\eta)|\right] \\ &\leq \frac{1}{1152}(4-c^2)F(c,x,y), \end{aligned} \tag{2.12}$$

where

$$F(c,x,y) := f_1(c,x) + f_2(c,x)y + f_3(c,x)y^2 + f_4(c,x)\left(1-y^2\right), \tag{2.13}$$

with

$$\begin{aligned} f_1(c,x) &:= c^2x\left[3c^2 + (44-5c^2)x + (40-c^2)x^2\right] + c^2(4-c^2)x^4, \\ f_2(c,x) &:= 4c(1-x^2)\left[3c^2 + 4(5+c^2)x + (4-c^2)x^2\right], \\ f_3(c,x) &:= 32(4-c^2) + 28x^2(4-c^2) + 36c^2x(1-x^2) + 4(4-c^2)x^4 \end{aligned}$$

and

$$f_4(c,x) := 36\left[c^2 + (4-c^2)x\right](1-x^2).$$

Now, we will show that

$$(4-c^2)F(c,x,y) \leq 1024 \tag{2.14}$$

for $c \in [0, 2]$, $x \in [0, 1]$ and $y \in [0, 1]$.

I. Assume first that $c \in [1, 2]$. Then by (2.13) we have

$$\begin{aligned} &F(c,x,y) \\ &\quad\leq f_1(c,x) + cf_2(c,x)y + f_3(c,x)y^2 + f_4(c,x)\left(1-y^2\right) \\ &\quad= f_1(c,x) + f_4(c,x) + cf_2(c,x)y + (f_3(c,x) - f_4(c,x))y^2 \\ &\quad=: F_1(c,x,y). \end{aligned} \tag{2.15}$$

(a) Consider the case $f_3(c,x) \geq f_4(c,x)$ in $[1, 2] \times [0, 1]$. Let

$$\Omega_1 := \{(c,x) \in [1,2] \times [0,1] : f_3(c,x) \geq f_4(c,x)\}.$$

By (2.15) we get

$$\begin{aligned} F_1(c,x,y) &\leq F_1(c,x,1) \\ &= f_1(c,x) + cf_2(c,x) + f_3(c,x), \quad (c,x) \in \Omega_1,\ y \in [0,1]. \end{aligned}$$

Set $t := 4 - c^2$. Clearly, $t \in [0, 3]$. Define

$$\tilde{F}_1(t, x) := t F_1(\sqrt{4-t}, x, 1), \quad (\sqrt{4-t}, x) \in \Omega_1.$$

A simple computing yields

$$\begin{aligned}\tilde{F}_1(t, x) = t &\Big\{(4-t)x\left[12 - 3t + (24+5t)x + (36+t)x^2\right]\\ &+ t(4-t)x^4 + 32t + 28tx^2 + 36(4-t)x(1-x^2) + 4tx^4\\ &+4(4-t)(1-x^2)\left[12 - 3t + 4(9-t)x + tx^2\right]\Big\}\\ =\ & 96(2 + 8x - x^2 - 6x^3)t - 4(16 + 67x - 34x^2 - 53x^3 + 2x^4)t^2\\ &+ (12 + 19x - 21x^2 - 17x^3 + 3x^4)t^3, \quad (\sqrt{4-t}, x) \in \Omega_1.\end{aligned}$$

Hence and by Proposition 2.2 we have

$$\frac{\partial}{\partial t}\tilde{F}_1(t, x) = \Theta(t, x) > 0, \quad (\sqrt{4-t}, x) \in \Omega_1,$$

where the function Θ is defined by (2.4). Thus the function $[0, 3] \ni t \mapsto \tilde{F}_1(t, \cdot)$ is increasing, and therefore we have

$$\tilde{F}_1(t, x) \le \tilde{F}_1(3, x) = 9\left(36 + 45x + 41x^2 - 31x^3 + x^4\right) < 1024, \quad x \in [0, 1]. \tag{2.16}$$

Indeed, the last inequality is true since, as easy to verify the inequality

$$-700 + 405x + 369x^2 - 279x^3 + 9x^4 < 0, \quad x \in [0, 1],$$

holds. Thus the inequality (2.16) confirms the inequality (2.14).

(b) Consider the case $f_3(c, x) < f_4(c, x)$ in $[1, 2] \times [0, 1]$. Let

$$\Omega_2 := \{(c, x) \in [1, 2] \times [0, 1] : f_3(c, x) < f_4(c, x)\}.$$

Since $f_2(c, x) \ge 0$ in $[1, 2] \times [0, 1]$, so

$$\sigma := \frac{-cf_2(c, x)}{2(f_3(c, x) - f_4(c, x))} \ge 0, \quad (c, x) \in \Omega_2.$$

If $\sigma \ge 1$, i.e., if $cf_2(c, x) + 2(f_3(c, x) - f_4(c, x)) \ge 0$, then

$$\begin{aligned}F_1(c, x, y) &\le F_1(c, x, 1)\\ &= f_1(c, x) + cf_2(c, x) + f_3(c, x), \quad (c, x) \in \Omega_2,\ y \in [0, 1].\end{aligned}$$

and repeating the argumentation of Part (a) we get the inequality (2.14).

If $\sigma < 1$, i.e., if $cf_2(c,x) + 2(f_3(c,x) - f_4(c,x)) < 0$, then

$$\begin{aligned}F_1(c,x,y) \le F_1(c,x,\sigma) &= \frac{-c^2 f_2^2(c,x)}{4(f_3(c,x)-f_4(c,x))} + f_1(c,x) + f_4(c,x) \\ &\le \frac{[-2(f_3(c,x)-f_4(c,x))]^2}{4(f_3(c,x)-f_4(c,x))} + f_1(c,x) + f_4(c,x) \\ &\le f_1(c,x) + f_3(c,x) + 2f_4(c,x) =: F_2(c,x), \quad (c,x) \in \Omega_2.\end{aligned}$$

Set $t := c^2$. Clearly, $t \in [1,4]$. Define

$$\tilde{F}_2(t,x) := (4-t)F_2(\sqrt{t},x), \quad (\sqrt{t},x) \in \Omega_2.$$

A simple computing yields

$$\begin{aligned}\tilde{F}_2(t,x) = (4-t)\Big\{&tx\left[3t + (44-5t)x + (40-t)x^2\right] + t(4-t)x^4 \\ &+ 32(4-t) + 28x^2(4-t) + 36tx(1-x^2) + 4(4-t)x^4 \\ &+72\left[t + (4-t)x\right](1-x^2)\Big\} \\ = -\Big\{&-64\left(8 + 18x + 7x^2 - 18x^3 + x^4\right) \\ &+ 16\left(-2 + 27x + 21x^2 - 37x^3 + x^4\right)t \\ &+ 4\left(10 - 12x - 9x^2 + 20x^3 + x^4\right)t^2 \\ &+x\left(3 - 5x - x^2 - x^3\right)t^3\Big\}, \quad (\sqrt{t},x) \in \Omega_2.\end{aligned}$$

Hence and by Proposition 2.3 we have

$$\frac{\partial}{\partial t}\tilde{F}_2(t,x) = -\Psi(t,x) < 0, \quad (\sqrt{t},x) \in \Omega_2,$$

where the function Ψ is defined by (2.6). Thus the function $[1,4] \ni t \mapsto \tilde{F}_2(t,\cdot)$ is decreasing, and therefore we have

$$\tilde{F}_2(t,x) \le \tilde{F}_2(1,x) = 9\left(56 + 85x + 17x^2 - 71x^3 + 5x^4\right) < 1024, \quad x \in [0,1]. \tag{2.17}$$

Indeed, the last inequality is true since, as easy to verify the inequality

$$-520 + 765x + 153x^2 - 639x^3 + 45x^4 < 0, \quad x \in [0,1],$$

holds. Thus the inequality (2.17) confirms the inequality (2.14).

II. Assume that $c \in (0, 1)$. Then by (2.13) we have

$$\begin{aligned} & F(c, x, y) \\ & \le f_1(c, x) + \frac{1}{c} f_2(c, x) y + f_3(c, x) y^2 + f_4(c, x) \left(1 - y^2\right) \\ & = f_1(c, x) + f_4(c, x) + \frac{1}{c} f_2(c, x) y + (f_3(c, x) - f_4(c, x)) y^2 \\ & =: F_3(c, x, y). \end{aligned} \tag{2.18}$$

(a) Consider the case $f_3(c, x) \ge f_4(c, x)$ in $(0, 1) \times [0, 1]$. Let

$$\Omega_3 := \{(c, x) \in (0, 1) \times [0, 1] : f_3(c, x) \ge f_4(c, x)\}.$$

By (2.18) we get

$$\begin{aligned} F_3(c, x, y) & \le F_3(c, x, 1) \\ & = f_1(c, x) + \frac{1}{c} f_2(c, x) + f_3(c, x), \quad (c, x) \in \Omega_3, \ y \in [0, 1]. \end{aligned}$$

Set $t := 4 - c^2$. Clearly, $t \in (3, 4)$. Define

$$\tilde{F}_3(t, x) := t F_3(\sqrt{4 - t}, x, 1), \quad (\sqrt{4 - t}, x) \in \Omega_3.$$

A simple computing yields

$$\begin{aligned} \tilde{F}_3(t, x) = t & \Big\{ 3(4 - t)^2 x + (4 - t)(24 + 5t) x^2 + (4 - t)(36 + t) x^3 + (4 - t) t x^4 \\ & + 4(1 - x^2) \left[12 - 3t + 4(9 - t) x + t x^2 \right] \\ & + 32t + 28 t x^2 + 36(4 - t) x (1 - x^2) + 4 t x^4 \Big\} \\ = {} & 48 \left(1 + 7x + x^2 - 3x^3\right) t + 4 \left(5 - 19x + 10x^2 + 5x^3 + x^4\right) t^2 \\ & - x \left(-3 + 5x + x^2 + x^3\right) t^3, \quad (\sqrt{4 - t}, x) \in \Omega_3. \end{aligned}$$

Hence and by Proposition 2.4 we have

$$\frac{\partial}{\partial t} \tilde{F}_3(t, x) = \Phi(t, x) > 0, \quad (\sqrt{4 - t}, x) \in \Omega_3,$$

where the function Φ is defined by (2.8). Thus the function $(3, 4) \ni t \mapsto \tilde{F}_1(t, \cdot)$ is increasing, and therefore we have

$$\tilde{F}_3(t, x) \le 512 + 320x + 512x^2 - 320x^3 \le 1024. \tag{2.19}$$

Indeed, the last inequality is true since so is the following one

$$-512 + 320x + 512x^2 - 320x^3 = (1 - x^2)(320x - 512) \leq 0, \quad x \in [0, 1].$$

Thus the inequality (2.19) confirms the inequality (2.14).

(b) Consider the case $f_3(c, x) < f_4(c, x)$ in $(0, 1) \times [0, 1]$ which is equivalent to

$$\begin{aligned} &32 - 8c^2 + 28x^2 - 7c^2x^2 + 9c^2x - 9c^2x^3 + 4x^4 - c^2x^4 \\ &\quad < 9c^2 - 9c^2x^2 + 36x - 9c^2x - 36x^3 + 9c^2x^3 \end{aligned} \tag{2.20}$$

for $c \in (0, 1)$ and $x \in [0, 1]$. Note that

$$17 - 18x - 2x^2 + 18x^3 + x^4 > 0, \quad x \in [0, 1].$$

Thus the inequality (2.20) can be written as

$$c^2 > \frac{32 - 36x + 28x^2 + 36x^3 + 4x^4}{17 - 18x - 2x^2 + 18x^3 + x^4}, \quad c \in (0, 1), \; x \in [0, 1]. \tag{2.21}$$

However,

$$\frac{32 - 36x + 28x^2 + 36x^3 + 4x^4}{17 - 18x - 2x^2 + 18x^3 + x^4} \geq 1, \quad x \in [0, 1]. \tag{2.22}$$

Indeed, the above inequality is equivalent to

$$32 - 36x + 28x^2 + 36x^3 + 4x^4 \geq 17 - 18x - 2x^2 + 18x^3 + x^4, \quad x \in [0, 1],$$

which by simplifying is equivalent to the true inequality

$$(x - 1)^4 + 10x^3 + 4(x - 1)^2 + 6x \geq 0, \quad x \in [0, 1].$$

Thus by (2.21) and (2.22) it follows that $c \geq 1$ which contradicts the assumption.

III. At the end assume that $c = 0$. Then by (2.13) we have

$$\begin{aligned} F(0, x, y) &= 16\left((8 - 9x + 7x^2 + 9x^3 + x^4)y^2 + 9x(1 - x^2)\right) \\ &\leq 16(8 + 7x^2 + x^4) \leq 256, \quad x \in [0, 1], \; y \in [0, 1]. \end{aligned}$$

Summarizing, from all considering cases it follows that the inequality (2.14) holds which together with (2.12) shows (2.9). □

Remark 2.6 Although the constant $8/9$ improves essentially the estimates found in [1] and [19], it is not the best possible. To find the sharp estimate of the Hankel determinant $H_{3,1}(f)$ for starlike functions is still an open problem.

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