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Positive Solutions for the Neumann p-Laplacian with Superdiffusive Reaction

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Abstract We consider a generalized logistic equation driven by the Neumann p-Laplacian and with a reaction that exhibits a superdiffusive kind of behavior. Using variational methods based on the critical point theory, together with truncation and comparison techniques, we show that there exists a critical value $\lambda_* > 0$ of the parameter, such that if $\lambda > \lambda_*$, the problem has at least two positive solutions, if $\lambda = \lambda_*$, the problem has at least one positive solution and it has no positive solution if $\lambda \in (0, \lambda_*)$. Finally, we show that for all $\lambda \geqslant \lambda_*$, the problem has a smallest positive solution.

Keywords p-Laplacian · Superdiffusive reaction · Local minimizers · Mountain pass theorem · Comparison principle · Bifurcation-type theorem

Mathematics Subject Classification 35J25 · 35J92

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear parametric Neumann problem:

$$\begin{cases} -\Delta_p u(z) + \beta(z)u(z)^{p-1} \;=\; \lambda g\big(z, u(z)\big) - f\big(z, u(z)\big) \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega,\ \lambda > 0,\ u > 0, \end{cases} \tag{$(P)_\lambda$}$$

with $\beta \in L^\infty(\Omega)_+$, $\beta \neq 0$. Here Δ_p denotes the p-Laplace differential operator, defined by

$$\Delta_p u \;=\; \operatorname{div}\big(\|\nabla u\|^{p-2}\nabla u\big) \quad \forall u \in W^{1,p}(\Omega),$$

with $p \in (1, +\infty)$. Also $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$. When the reaction in $(P)_\lambda$ has the particular form

$$\lambda\zeta^{q-1} - \zeta^{r-1},$$

with $q < r$, then the resulting equation is the p-logistic equation (or simply the logistic equation when $p = 2$). The logistic equation is important in mathematical biology (see Gurtin and Mac Camy [21] and Afrouzi and Brown [1]) and describes the dynamics of biological populations whose mobility is density dependent.

There are three different types of the p-logistic equation, depending on the value of the exponent q with respect to p. More precisely, we have

- the "subdiffusive" type, when $q < p < r$;
- the "equidiffusive" type, when $q = p < r$;
- the "superdiffusive" type, when $p < q < r$.

The subdiffusive and equidiffusive cases are similar, but the superdiffusive case differs essentially and it exhibits bifurcation phenomena (see Takeuchi [29,30] and Filippakis et al. [7], where the Dirichlet problem is studied).

The aim of this work, is to prove a bifurcation-type theorem for the positive solutions of $(P)_\lambda$ as the parameter $\lambda > 0$ varies in $(0, +\infty)$ and the reaction $\zeta \longmapsto \lambda g(z, \zeta) - f(z, \zeta)$ (which is more general than the standard p-logistic equation; see Afrouzi and Brown [1]), exhibits a superdiffusive kind of behavior. To the best of our konwledge, the Neumann p-logistic equation has not been studied. There is only the recent work of Marano-Papageorgiou [25], where the equidiffusive case is examined.

Our approach is variational based on the critical point theory, combined with suitable truncation and comparison techniques. In the next section, for the convenience of the reader we recall main mathematical tools which we will use in the sequel.

This work is the outgrowth of a remark made by the referee of [19]. In that paper, the authors deal with the parametric equation

$$\begin{cases} -\Delta_p u(z) \;=\; \lambda f\big(z, u(z)\big) \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

and some analogous bifurcation-type results were proved. It was pointed out by the referee that in mathematical biology, the Neumann model is a more realistic one. For some other recent results on nonlinear Neumann boundary value problems involving p-Laplacian, we refer to Gasiński and Papageorgiou [11–17].

2 Mathematical Background

Let X be a Banach space and let X^* be its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X, X^*). Let $\varphi \in C^1(X)$. We say that φ satisfies the *Palais–Smale condition*, if the following holds:

"Every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$, such that $\big\{\varphi(x_n)\big\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$\varphi'(x_n) \longrightarrow 0 \quad \text{in} \quad X^* \text{ as } \quad n \to +\infty,$$

admits a strongly convergent subsequence."

Using this compactness-type condition on φ, we can state the following theorem, known in the literature as the "mountain pass theorem".

Theorem 2.1 *If X is a Banach space, $\varphi \in C^1(X)$ satisfies the Palais–Smale condition, $x_0, x_1 \in X$, $0 < \varrho < \|x_0 - x_1\|$,*

$$\max\big\{\varphi(x_0), \varphi(x_1)\big\} < \inf\big\{\varphi(x) : \|x - x_0\| = \varrho\big\} = \eta_\varrho,$$
$$c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \varphi\big(\gamma(t)\big),$$

where

$$\Gamma = \big\{\gamma \in C\big([0,1]; X\big) : \gamma(0) = x_0,\ \gamma(1) = x_1\big\},$$

then $c \geqslant \eta_\varrho$ and c is a critical value of φ (i.e., there exists $\widehat{x} \in X$, such that $\varphi'(\widehat{x}) = 0$ and $\varphi(\widehat{x}) = c$).

In the study of problem $(P)_\lambda$, we will use the Sobolev space $W^{1,p}(\Omega)$ and the ordered Banach space $C^1(\overline{\Omega})$. The positive cone of the latter is

$$C_+ = \big\{u \in C^1(\overline{\Omega}) : u(z) \geqslant 0 \quad \text{for all} \quad z \in \overline{\Omega}\big\}.$$

This cone has a nonempty interior, given by

$$\operatorname{int} C_+ = \Big\{u \in C_+ : u(z) > 0 \quad \text{for all} \quad z \in \overline{\Omega}\Big\}.$$

The next result relates local minimizers in $W^{1,p}(\Omega)$ with local minimizers in the smaller Banach space $C^1(\overline{\Omega})$. A result of this type was first proved for the Dirichlet Laplacian by Brézis and Nirenberg [5] and was later extended to the p-Laplacian by

García Azorero et al. [8] and Guo and Zhang [20] (in the latter, for $p \geqslant 2$). Extensions to the Neumann p-Laplacian or Neumann p-Laplacian-like operators can be found in Motreanu et al. [26] and Motreanu and Papageorgiou [28].

So let $f_0 \colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function (i.e., for all $\zeta \in \mathbb{R}$, the function $z \longmapsto f_0(z,\zeta)$ is measurable and for almost all $z \in \Omega$, the function $\zeta \longmapsto f_0(z,\zeta)$ is continuous), which exhibits subcritical growth in $\zeta \in \mathbb{R}$, i.e.,

$$\left|f_0(z,\zeta)\right| \leqslant a(z) + c|\zeta|^{r-1} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R},$$

with $a \in L^\infty(\Omega)_+$, $c > 0$ and $1 < r < p^*$, where

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if} \quad p < N, \\ +\infty & \text{if} \quad p \geqslant N. \end{cases}$$

We set

$$F_0(z,\zeta)\,ds = \int_0^\zeta f_0(z,s)\,ds$$

and consider the C^1-functional $\psi_0 \colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\psi_0(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega F_0\big(z,u(z)\big)\,dz \quad \forall u \in W^{1,p}(\Omega).$$

Theorem 2.2 *If $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of ψ_0, i.e., there exists $\varrho_1 > 0$, such that*

$$\psi_0(u_0) \leqslant \psi_0(u_0 + h) \quad \forall h \in C^1(\overline{\Omega}),\ \|h\|_{C^1(\overline{\Omega})} \leqslant \varrho_1,$$

then $u_0 \in C^1(\overline{\Omega})$ and it is a local $W^{1,p}(\Omega)$-minimizer of ψ_0, i.e., there exists $\varrho_2 > 0$, such that

$$\psi_0(u_0) \leqslant \psi_0(u_0 + h) \quad \forall h \in W^{1,p}(\Omega),\ \|h\| \leqslant \varrho_2.$$

Remark 2.3 In [26,28], the result was stated in terms of $W_n^{1,p}(\Omega) = \overline{C_n^1(\overline{\Omega})}^{\|\cdot\|}$, where

$$C_n^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) : \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega\right\}.$$

Actually, there is no need for this restriction.

Let $A \colon W^{1,p}(\Omega) \longrightarrow W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\big\langle A(u), y\big\rangle = \int_\Omega \|\nabla u\|^{p-2}(\nabla u, \nabla y)_{\mathbb{R}^N}\,dz \quad \forall u, y \in W^{1,p}(\Omega). \tag{2.1}$$

The next result can be found in Aizicovici et al. [3, Proposition 2].

Proposition 2.4 *The map $A\colon W^{1,p}(\Omega) \longrightarrow W^{1,p}(\Omega)^*$ defined by (2.1) is continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$, i.e., if $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is a sequence, such that $u_n \longrightarrow u$ weakly in $W^{1,p}(\Omega)$ and*

$$\limsup_{n\to+\infty} \big\langle A(u_n),\ u_n - u\big\rangle \ \leqslant\ 0,$$

then $u_n \longrightarrow u$ in $W^{1,p}(\Omega)$.

The next simple lemma, will be useful in our estimations and can be found in Aizicovici et al. [4, Lemma 2]. Recall that by $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$, i.e.,

$$\|u\| \ =\ \big(\|u\|_p^p + \|\nabla u\|_p^p\big)^{\frac{1}{p}} \quad \forall u \in W^{1,p}(\Omega).$$

Lemma 2.5 *If $\beta \in L^\infty(\Omega)$, $\beta(z) \geqslant 0$ for almost all $z \in \Omega$ and $\beta \neq 0$, then there exists $\xi_0 > 0$, such that*

$$\|\nabla u\|_p^p + \int_\Omega \beta|u|^p\,dz \ \geqslant\ \xi_0\|u\|^p \quad \forall u \in W^{1,p}(\Omega).$$

We conclude this section by fixing some notation. By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. For every $u \in W^{1,p}(\Omega)$, we set $u^\pm = \max\{\pm u, 0\}$. We know that

$$u^\pm \in W^{1,p}(\Omega), \quad u \ =\ u^+ - u^-, \quad |u| \ =\ u^+ + u^-.$$

Finally for every measurable function $h\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$, we define

$$N_h(u)(\cdot) \ =\ h\big(\cdot, u(\cdot)\big) \quad \forall u \in W^{1,p}(\Omega)$$

(the Nemytskii map corresponding to h).

3 A Bifurcation-Type Theorem

The hypotheses on the data of problem $(P)_\lambda$ are the following:
$\underline{H_g}$ $g\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $g(z,0) = 0$ for almost all $\underline{z \in}$ Ω and

(i) we have

$$\big|g(z,\zeta)\big| \ \leqslant a(z) + c|\zeta|^{r-1} \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \in \mathbb{R},$$

with $a \in L^\infty(\Omega)_+$, $c > 0$ and $p < r < p^*$;

(ii) there exist $\vartheta > q > p$, such that

$$0 \ <\ \eta_g \ \leqslant\ \liminf_{\zeta\to+\infty} \frac{g(z,\zeta)}{\zeta^{q-1}} \ \leqslant\ \limsup_{\zeta\to+\infty} \frac{g(z,\zeta)}{\zeta^{q-1}} \ \leqslant\ \widehat{\eta}_g$$

uniformly for almost all $z \in \Omega$ and for almost all $z \in \Omega$, the function $\zeta \longmapsto \frac{g(z,\zeta)}{\zeta^{\vartheta-1}}$ is nonincreasing on $(0, +\infty)$;

(iii) we have

$$\lim_{\zeta \to 0^+} \frac{g(z,\zeta)}{\zeta^{q-1}} = 0$$

uniformly for almost all $z \in \Omega$;

(iv) there exist two functions $\sigma_0, \sigma_1 : (0, +\infty) \longrightarrow (0, +\infty)$, both upper semicontinuous, such that

$$\sigma_0(\zeta) \leqslant g(z,\zeta) \leqslant \sigma_1(\zeta) \quad \text{for almost all } z \in \Omega, \text{ all } \zeta > 0.$$

$\underline{H_f}$ $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $f(z, 0) = 0$ for almost all $z \in \Omega$ and

(i) we have

$$\left| f(z,\zeta) \right| \leqslant a(z) + c|\zeta|^{r-1} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R},$$

with $a \in L^\infty(\Omega)_+$, $c > 0$ and $p < r < p^*$;

(ii) with $\vartheta > q > p$ as in hypothesis $H_g(ii)$, we have

$$0 < \eta_f \leqslant \liminf_{\zeta \to +\infty} \frac{f(z,\zeta)}{\zeta^{\vartheta-1}} \leqslant \limsup_{\zeta \to +\infty} \frac{f(z,\zeta)}{\zeta^{\vartheta-1}} \leqslant \widehat{\eta}_f$$

uniformly for almost all $z \in \Omega$ and for almost all $z \in \Omega$, the function $\zeta \longmapsto \frac{f(z,\zeta)}{\zeta^{p-1}}$ is nondecreasing on $(0, +\infty)$;

(iii) we have

$$0 \leqslant \liminf_{\zeta \to 0^+} \frac{f(z,\zeta)}{\zeta^{q-1}} \leqslant \limsup_{\zeta \to 0^+} \frac{f(z,\zeta)}{\zeta^{q-1}} \leqslant \zeta^*$$

uniformly for almost all $z \in \Omega$;

(iv) there exists a lower semicontinuous function $\sigma_2 : (0, +\infty) \longrightarrow (0, +\infty)$, such that

$$\sigma_2(\zeta) \leqslant f(z,\zeta) \quad \text{for almost all } z \in \Omega, \text{ all } \zeta > 0.$$

$\underline{H_0}$ For every $\lambda > 0$ and $\varrho > 0$, we can find $\gamma_\varrho = \gamma_\varrho(\lambda) > 0$, such that for almost all $z \in \Omega$, the function $\zeta \longmapsto \lambda g(z,\zeta) - f(z,\zeta) + \gamma_\varrho \zeta^{\vartheta-1}$ is nondecreasing on $[0, \varrho]$ ($\vartheta > q > p$ as in the hypothesis $H_g(ii)$).

Remark 3.1 Since we are interested in positive solutions and hypotheses H_g, H_f and H_0 concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, we may (and will) assume that

$$g(z,\zeta) = f(z,\zeta) = 0 \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \leqslant 0.$$

Example 3.2 The following functions satisfy hypotheses H_g, H_f and H_0 (for the sake of simplicity we drop the z-dependence):

(a) $g(\zeta) = \begin{cases} \zeta^{s-1} & \text{if} \quad \zeta \in [0,1], \\ \zeta^{q-1} & \text{if} \quad \zeta > 1 \end{cases}$ and $f(\zeta) = \zeta^{\vartheta-1}$ for all $\zeta \geqslant 0$, with $p < q < s < \vartheta < p^*$.

(b) $g(\zeta) = \begin{cases} \zeta^{s-1} - \zeta^{\vartheta-1} & \text{if} \quad \zeta \in [0,1], \\ \zeta^{q-1} - \zeta^{p-1} & \text{if} \quad \zeta > 1 \end{cases}$ and $f(\zeta) = \zeta^{\vartheta-1} - \zeta^{s-1}$ for all $\zeta \geqslant 0$, with $p < q < s < \vartheta < p^*$.

Example (a) corresponds to the standard superdiffusive p-logistic reaction (see Afrouzi and Brown [1]).

By a positive solution of problem $(P)_\lambda$, we understand a function $u \in W^{1,p}(\Omega)$, $u \neq 0$, which is a weak solution of $(P)_\lambda$. Then $u \in L^\infty(\Omega)$ (see e.g., Gasiński and Papageorgiou [9,18] and Hu and Papageorgiou [23]). Invoking Theorem 2 of Lieberman [24], we have that $u \in C_+ \setminus \{0\}$. Let $\varrho = \|u\|_\infty$ and let $\gamma_\varrho = \gamma_\varrho(\lambda) > 0$ be as postulated by hypothesis H_0. We have

$$\begin{aligned} &-\Delta_p u(z) + \beta(z)u(z)^{p-1} + \gamma_\varrho u(z)^{\vartheta-1} \\ &\quad = \lambda g\big(z, u(z)\big) - f\big(z, u(z)\big) + \gamma_\varrho u(z)^{\vartheta-1} \geqslant 0 \quad \text{for almost all } z \in \Omega \end{aligned}$$

(see Motreanu and Papageorgiou [27]), so

$$\Delta_p u(z) \leqslant \big(\|\beta\|_\infty + \gamma_\varrho \varrho^{\vartheta-p}\big)u(z)^{p-1} \quad \text{for almost all } z \in \Omega$$

and finally

$$u \in \operatorname{int} C_+$$

(see Vázquez [31]).

So, we see that the positive solutions of problem $(P)_\lambda$, if they exist, belong in $\operatorname{int} C_+$.

Let

$$\mathcal{Y} = \big\{\lambda > 0 : \text{ problem } (P)_\lambda \text{ has a positive solution.}\big\}$$

Proposition 3.3 *If hypotheses H_g, H_f and H_0 hold, then* $\inf \mathcal{Y} > 0$.

Proof By virtue of hypothesis $H_g(ii)$, we can find $\eta_1 > 0$ and $M > 0$, such that

$$g(z,\zeta) \leqslant \eta_1 \zeta^{q-1} \quad \text{for almost all } z \in \Omega, \ \text{all } \zeta \geqslant M. \tag{3.1}$$

On the other hand, from hypothesis $H_g(iii)$, for a given $\varepsilon > 0$, we can find $\delta \in (0,1)$ small, such that

$$g(z,\zeta) \leqslant \varepsilon \zeta^{p-1} \quad \text{for almost all } z \in \Omega, \ \text{all } \zeta \in [0,\delta]. \tag{3.2}$$

The function $\zeta \longmapsto \frac{\sigma_1(\zeta)}{\zeta^{q-1}}$ is upper semicontinuous on $[\delta, M]$ and so, we can find $\widehat{\xi} \in [\delta, M]$, such that

$$\frac{\sigma_1(\zeta)}{\zeta^{q-1}} \leqslant \frac{\sigma_1(\widehat{\zeta})}{\widehat{\zeta}^{q-1}} = \eta_2(\varepsilon) \quad \forall \zeta \in [\delta, M],$$

so

$$g(z, \zeta) \leqslant \eta_2 \zeta^{q-1} \quad \text{for almost all } \zeta \in [\delta, M] \tag{3.3}$$

(see hypothesis $H_g(iv)$). From (3.1), (3.2) and (3.3), it follows that

$$g(z, \zeta) \leqslant \varepsilon \zeta^{p-1} + \widehat{\eta} \zeta^{q-1} \quad \text{for almost all } \zeta \geqslant 0, \tag{3.4}$$

with $\widehat{\eta}(\varepsilon) = \max\{\eta_1, \eta_2\} > 0$.

In a similar fashion, using hypotheses H_f (ii), (iii) and (iv), for a given $\varepsilon > 0$, we can find $\vartheta = \vartheta(\varepsilon) > 0$, such that

$$f(z, \zeta) \geqslant \vartheta \zeta^{q-1} - \varepsilon \zeta^{p-1} \quad \text{for almost all } \zeta \geqslant 0. \tag{3.5}$$

Let us fix $\varepsilon \in \left(0, \frac{\xi_0}{2}\right)$ ($\xi_0 > 0$ as in Lemma 2.5) and let $\widehat{\lambda} \leqslant \min\left\{1, \frac{\vartheta}{\widehat{\eta}}\right\}$. From (3.4) and (3.5), we have

$$\begin{aligned} \widehat{\lambda} g(z, \zeta) - f(z, \zeta) &\leqslant (\widehat{\lambda} + 1)\varepsilon \zeta^{p-1} + (\widehat{\lambda}\widehat{\eta} - \vartheta)\zeta^{q-1} \\ &\leqslant 2\varepsilon \zeta^{p-1} \quad \text{for almost all } z \in \Omega, \ \text{all } \zeta \geqslant 0. \end{aligned} \tag{3.6}$$

Suppose that for $\lambda \in (0, \widehat{\lambda})$, problem $(P)_\lambda$ has a positive solution (i.e., $\lambda \in \mathcal{Y}$). Then we can find a positive solution $u_\lambda \in \operatorname{int} C_+$ of $(P)_\lambda$. Hence

$$A(u_\lambda) + \beta u_\lambda^{p-1} = \lambda N_g(u_\lambda) - N_f(u_\lambda) \tag{3.7}$$

(see (2.1) for the definition of A). On (3.7) we act with u_λ and obtain

$$\|\nabla u_\lambda\|_p^p + \int_\Omega \beta u_\lambda^p \, dz = \int_\Omega \big(\lambda g(z, u_\lambda) - f(z, u_\lambda)\big) u_\lambda \, dz,$$

so using Lemma 2.5 and (3.6), we have

$$\xi_0 \|u_\lambda\|^p \leqslant 2\varepsilon \|u_\lambda\|^p.$$

recalling that $\varepsilon \in \left(0, \frac{\xi_0}{2}\right)$, we conclude that $u_\lambda = 0$, a contradiction. Therefore $\inf \mathcal{Y} \geqslant \widehat{\lambda} > 0$. $\square$

If $\mathcal{Y} = \emptyset$, then $\inf \mathcal{Y} = +\infty$. In the next proposition, we establish the nonemptiness of $\mathcal{Y}$.

Proposition 3.4 *If hypotheses H_g, H_f and H_0 hold, then $\mathcal{Y} \neq \emptyset$ and if $\lambda \in \mathcal{Y}$ and $\tau > \lambda$, then $\tau \in \mathcal{Y}$.*

Proof Let $\varphi_\lambda\colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the energy functional for problem $(P)_\lambda$, defined by

$$\varphi_\lambda(u) \;=\; \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{p}\int_\Omega \beta|u|^p\,dz - \lambda\int_\Omega G(z,u)\,dz + \int_\Omega F(z,u)\,dz$$

for all $u \in W^{1,p}(\Omega)$. Evidently $\varphi_\lambda \in C^1\big(W^{1,p}(\Omega)\big)$. By virtue of hypotheses $H_g(i), (ii)$, we can find $\xi_1 > 0$ and $c_1 > 0$, such that

$$G(z,\zeta) \;\leqslant\; \xi_1(\zeta^+)^q + c_1 \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \in \mathbb{R}. \tag{3.8}$$

Since $q < \vartheta$, using Young inequality with $\varepsilon > 0$, from (3.8) we see that for a given $\varepsilon > 0$, we can find $c_2 = c_2(\varepsilon) > 0$, such that

$$G(z,\zeta) \;\leqslant\; \varepsilon(\zeta^+)^\vartheta + c_2 \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \in \mathbb{R}. \tag{3.9}$$

Also, from hypotheses $H_f(i), (ii)$, we see that we can find $\xi_2 > 0$ and $c_3 > 0$, such that

$$F(z,\zeta) \;\geqslant\; \xi_2(\zeta^+)^\vartheta - c_3 \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \in \mathbb{R}. \tag{3.10}$$

Then

$$\begin{aligned}\varphi_\lambda(u) &= \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{p}\int_\Omega \beta|u|^p\,dz - \lambda\int_\Omega G(z,u)\,dz + \int_\Omega F(z,u)\,dz\\ &\geqslant \frac{\xi_0}{p}\|u\|^p + (\xi_2 - \lambda\varepsilon)\|u^+\|_\vartheta^\vartheta - c_4 \quad \forall u \in W^{1,p}(\Omega),\end{aligned} \tag{3.11}$$

for some $c_4 = c_4(\varepsilon) > 0$ (see Lemma 2.5 and (3.9), (3.10)).

We choose $\varepsilon \in \big(0, \frac{\xi_2}{\lambda}\big]$. Then, from (3.11), it follows that φ_λ is coercive. Also, it is easy to see that φ_λ is sequentially weakly lower semicontinuous. Therefore, by the Weierstrass theorem, we can find $u_\lambda \in W^{1,p}(\Omega)$, such that

$$\varphi_\lambda(u_\lambda) \;=\; \inf_{u\in W^{1,p}(\Omega)} \varphi_\lambda(u) \;=\; m_\lambda. \tag{3.12}$$

Let $\overline{u} \in \operatorname{int} C_+$. Then clearly for $\lambda > 0$ big, we have $\varphi_\lambda(\overline{u}) < 0$. Hence

$$\varphi_\lambda(u_\lambda) \;=\; m_\lambda \;<\; 0 \;=\; \varphi_\lambda(0) \quad \forall \lambda > 0,\ \text{big}$$

(see (3.12)), so

$$u_\lambda \;\neq\; 0. \tag{3.13}$$

From (3.12), we have

$$\varphi_\lambda'(u_\lambda) \;=\; 0 \quad \forall \lambda > 0,\ \text{big}$$

so

$$A(u_\lambda) + \beta|u_\lambda|^{p-2}u_\lambda \;=\; \lambda N_g(u_\lambda) - N_f(u_\lambda). \tag{3.14}$$

On (3.14) we act with $-u_\lambda^- \in W^{1,p}(\Omega)$ and we obtain

$$\|\nabla u_\lambda^-\|_p^p + \int_\Omega \beta (u_\lambda^-)^p \, dz \;=\; 0,$$

so

$$\xi_0 \|u_\lambda^-\|^p \;\leqslant\; 0$$

(see Lemma 2.5), i.e., $u_\lambda \geqslant 0$, $u_\lambda \neq 0$ (see (3.13)).

Then (3.14) becomes

$$A(u_\lambda) + \beta u_\lambda^{p-1} \;=\; \lambda N_g(u_\lambda) - N_f(u_\lambda),$$

so

$$u_\lambda \text{ solves problem } (P)_\lambda,$$

i.e., $\mathcal{Y} \neq \emptyset$.

Now suppose that $\lambda \in \mathcal{Y}$ and $\tau > \lambda$. We choose $s \in (0,1)$, such that

$$\lambda \;=\; s^{\vartheta-1}\tau \tag{3.15}$$

(recall that $\vartheta > p$ and $\lambda < \tau$). Since $\lambda \in \mathcal{Y}$, problem $(P)_\lambda$ has a solution $u_\lambda \in \operatorname{int} C_+$. We set $\underline{u} = s u_\lambda \in \operatorname{int} C_+$. Then

$$-\Delta_p \underline{u} + \beta \underline{u}^{p-1} \;=\; s^{p-1}\big(-\Delta u_\lambda + \beta u_\lambda^{p-1}\big) \;=\; s^{p-1}\big(\lambda g(z,u_\lambda) - f(z,u_\lambda)\big). \tag{3.16}$$

By virtue of hypothesis $H_g(ii)$ and since $s \in (0,1)$, we have

$$\frac{g(z,u_\lambda(z))}{u_\lambda(z)^{\vartheta-1}} \;\leqslant\; \frac{g(z,\underline{u}(z))}{\underline{u}(z)^{\vartheta-1}} \;=\; \frac{g(z,\underline{u}(z))}{s^{\vartheta-1}u_\lambda(z)^{\vartheta-1}},$$

so

$$s^{\vartheta-1} g\big(z,u_\lambda(z)\big) \;\leqslant\; g\big(z,s u_\lambda(z)\big) \;=\; g\big(z,\underline{u}(z)\big) \quad \text{for almost all } z \in \Omega. \tag{3.17}$$

Similarly, using hypothesis $H_f(ii)$, we have

$$\frac{f(z,u_\lambda(z))}{u_\lambda(z)^{p-1}} \;\geqslant\; \frac{f(z,\underline{u}(z))}{\underline{u}(z)^{p-1}} \;=\; \frac{f(z,\underline{u}(z))}{s^{p-1}u_\lambda(z)^{p-1}},$$

so

$$s^{p-1} f\big(z,u_\lambda(z)\big) \;\geqslant\; f\big(z,s u_\lambda(z)\big) \;=\; f\big(z,\underline{u}(z)\big) \quad \text{for almost all } z \in \Omega. \tag{3.18}$$

Returning to (3.16) and using (3.15), (3.17) and (3.18), we have

$$\begin{aligned}
&-\Delta_p \underline{u}(z) + \beta(z)\underline{u}(z)^{p-1} \\
&= \lambda s^{p-1} g\big(z, u_\lambda(z)\big) - s^{p-1} f\big(z, u_\lambda(z)\big) \\
&\leqslant s^{\vartheta-1} \tau g\big(z, u_\lambda(z)\big) - f\big(z, \underline{u}(z)\big) \\
&\leqslant \tau g\big(z, \underline{u}(z)\big) - f\big(z, \underline{u}(z)\big) \quad \text{for almost all } z \in \Omega.
\end{aligned} \tag{3.19}$$

We consider the following truncation of the reaction in problem $(P)_\tau$:

$$h_\tau(z,\zeta) \;=\; \begin{cases} \tau g\big(z, \underline{u}(z)\big) - f\big(z, \underline{u}(z)\big) & \text{if} \quad \zeta \leqslant \underline{u}(z), \\ \tau g(z,\zeta) - f(z,\zeta) & \text{if} \quad \underline{u}(z) < \zeta. \end{cases} \tag{3.20}$$

This is a Carathéodory function. We set

$$H_\tau(z,\zeta) \;=\; \int_0^\zeta h_\tau(z,s)\,ds$$

and consider the C^1-functional $\psi_\tau\colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\psi_\tau(u) \;=\; \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{p}\int_\Omega \beta |u|^p\,dz - \int_\Omega H_\tau(z,u)\,dz \quad \forall u \in W^{1,p}(\Omega).$$

As we did for φ_λ earlier in this proof, we can check that ψ_τ is coercive and sequentially weakly lower semicontinuous. So, we can find $u_\tau \in W^{1,p}(\Omega)$, such that

$$\psi_\tau(u_\tau) \;=\; \inf_{u \in W^{1,p}(\Omega)} \psi_\tau(u),$$

so

$$\psi_\tau'(u_\tau) \;=\; 0,$$

thus

$$A(u_\tau) + \beta |u_\tau|^{p-2} u_\tau \;=\; N_{h_\tau}(u_\tau). \tag{3.21}$$

On (3.21) we act with $(\underline{u} - u_\tau)^+ \in W^{1,p}(\Omega)$ and obtain

$$\begin{aligned}
&\big\langle A(u_\tau),\ (\underline{u} - u_\tau)^+\big\rangle + \int_\Omega \beta |u_\tau|^{p-2} u_\tau (\underline{u} - u_\tau)^+\,dz \\
&= \int_\Omega h_\tau(z,u_\tau)(\underline{u} - u_\tau)^+\,dz \\
&= \int_\Omega \big(\tau g(z,\underline{u}) - f(z,\underline{u})\big)(\underline{u} - u_\tau)^+\,dz \\
&\geqslant \big\langle A(\underline{u}),\ (\underline{u} - u_\tau)^+\big\rangle + \int_\Omega \beta \underline{u}^{p-1} (\underline{u} - u_\tau)^+\,dz
\end{aligned}$$

(see (3.20) and (3.19)), so

$$\int_{\{\underline{u}>u_\tau\}} \big(\|\nabla u_\tau\|^{p-2}\nabla u_\tau - \|\nabla \underline{u}\|^{p-2}\nabla \underline{u},\ \nabla u_\tau - \nabla \underline{u}\big)_{\mathbb{R}} dz \\ + \int_{\{\underline{u}>u_\tau\}} \beta\big(|u_\tau|^{p-2}u_\tau - \underline{u}^{p-1}\big)(u_\tau - \underline{u})\, dz \leqslant 0. \tag{3.22}$$

We recall the following elementary inequalities (see e.g., Gasiński and Papageorgiou [10, Lemma 6.2.13, p. 740]). If $1 < p \leqslant 2$, then

$$(p-1)|y-v|^2(1+|y|+|v|)^{p-2} \\ \leq \big(|y|^{p-2}y - |v|^{p-2}v,\ y-v\big)_{\mathbb{R}^N} \quad \forall y, v \in \mathbb{R}^N \tag{3.23}$$

and if $2 < p$, then

$$\frac{1}{2^{p-2}}|y-v|^p \ \leq\ \big(|y|^{p-2}y - |v|^{p-2}v,\ y-v\big)_{\mathbb{R}^N} \quad \forall y, v \in \mathbb{R}^N. \tag{3.24}$$

If $1 < p \leqslant 2$, then from (3.22), (3.23) and since $u_\tau, \underline{u} \in \operatorname{int} C_+$, we have

$$\frac{p-1}{c_5} \int_{\{\underline{u}>u_\tau\}} \|\nabla u_\tau - \nabla \underline{u}\|^2\, dz \ \leqslant\ 0$$

for some $c_5 > 0$, so

$$\big|\{\underline{u} > u_\tau\}\big|_N \ =\ 0,$$

i.e., $\underline{u} \leqslant u_\tau$.

If $2 < p$, then from (3.22) and (3.24), we have

$$\frac{1}{2^{p-2}} \int_{\{\underline{u}>u_\tau\}} \|\nabla u_\tau - \nabla \underline{u}\|^p\, dz \ \leqslant\ 0,$$

so

$$\big|\{\underline{u} > u_\tau\}\big|_N \ =\ 0,$$

i.e., $\underline{u} \leqslant u_\tau$.

So, finally $\underline{u} \leqslant u_\tau$ and then (3.21) becomes

$$A(u_\tau) + \beta u_\tau^{p-1} \ =\ \tau N_g(u_\tau) - N_f(u_\tau)$$

(see (3.20)), so $u_\tau \in \operatorname{int} C_+$ is a positive solution of $(P)_\lambda$, i.e., $\tau \in \mathcal{Y}$. □

Proposition 3.5 *If hypotheses H_f, H_g and H_0 hold and $\lambda > \lambda_*$, then problem $(P)_\lambda$ has at least two positive solutions.*

Proof Let $\tau \in (\lambda_*, \lambda) \cap \mathcal{Y}$. Then, we can find $u_\tau \in \operatorname{int} C_+$, such that

$$\begin{cases} -\Delta_p u_\tau(z) + \beta(z) u_\tau(z)^{p-1} = \tau g\big(z, u_\tau(z)\big) - f\big(z, u_\tau(z)\big) & \text{in} \quad \Omega, \\ \dfrac{\partial u_\tau}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{3.25}$$

Proceeding as in the proof of Proposition 3.4, we introduce the following truncation of the reaction:

$$\widehat{h}_\lambda(z, \zeta) = \begin{cases} \lambda g\big(z, u_\lambda(z)\big) - f\big(z, u_\lambda(z)\big) & \text{if} \quad \zeta \leqslant u_\tau(z), \\ \lambda g(z, \zeta) - f(z, \zeta) & \text{if} \quad u_\tau(z) < \zeta. \end{cases} \tag{3.26}$$

This is a Carathéodory function. We set

$$\widehat{H}_\lambda(z, \zeta) = \int_0^\zeta \widehat{h}_\lambda(z, s)\, ds$$

and consider the C^1-functional $\widehat{\psi}\colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\widehat{\psi}_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{p}\int_\Omega \beta|u|^p\, dz - \int_\Omega \widehat{H}_\lambda(z, u)\, dz \quad \forall u \in W^{1,p}(\Omega).$$

As we did for φ_λ in the proof of Proposition 3.4, we can check that $\widehat{\psi}_\lambda$ is coercive and sequentially weakly lower semicontinuous. So, we can find $u_\lambda^0 \in W^{1,p}(\Omega)$, such that

$$\widehat{\psi}_\lambda(u_\lambda^0) = \inf_{u \in W^{1,p}(\Omega)} \widehat{\psi}_\lambda(u),$$

and

$$\widehat{\psi}_\lambda'(u_\lambda^0) = 0,$$

so

$$A(u_\lambda^0) + \beta|u_\lambda^0|^{p-2}u_\lambda^0 = N_{\widehat{h}_\lambda}(u_\lambda^0).$$

From this, as before, acting with $(u_\tau - u_\lambda^0)^+ \in W^{1,p}(\Omega)$ and using (3.25) and (3.26), we show that $u_\tau \leqslant u_\lambda^0$. Hence, we have

$$A(u_\lambda^0) + \beta(u_\lambda^0)^{p-1} = \lambda N_g(u_\lambda^0) - N_f(u_\lambda^0)$$

(see (3.26)), so $u_\lambda^0 \in \operatorname{int} C_+$ is a solution of $(P)_\lambda$ and $u_\lambda^0 \geqslant u_\tau$.

Claim 1 $u_\lambda^0 - u_\tau \in \operatorname{int} C_+$.

Let $\varrho = \|u_\lambda^0\|_\infty$. By hypothesis H_0, we can find $\gamma_\varrho = \gamma_\varrho(\lambda) > 0$, such that for all $z \in \Omega$, the function $\zeta \longmapsto \lambda g(z,\zeta) - f(z,\zeta) + \gamma_\varrho \zeta^{\vartheta-1}$ is nondecreasing on $[0,\varrho]$. For $\delta > 0$, we set

$$\overline{u}_\tau \;=\; u_\tau + \delta \;\in\; \operatorname{int} C_+.$$

Then

$$\begin{aligned}
&-\Delta_p \overline{u}_\tau + \beta \overline{u}_\tau^{p-1} + \gamma_\varrho \overline{u}_\tau^{\vartheta-1}\\
&\quad\leqslant -\Delta_p u_\tau + \beta u_\tau^{p-1} + \gamma_\varrho u_\tau^{\vartheta-1} + \xi(\delta)\\
&\quad= \tau g(z,u_\tau) - f(z,u_\tau) + \gamma_\varrho u_\tau^{\vartheta-1} + \xi(\delta)\\
&\quad= \lambda g(z,u_\tau) - f(z,u_\tau) + (\tau-\lambda) g(z,u_\tau) + \gamma_\varrho u_\tau^{\vartheta-1} + \xi(\delta)\\
&\quad\leqslant \lambda g(z,u_\tau) - f(z,u_\tau) - (\lambda-\tau)\sigma_0(u_\tau) + \gamma_\varrho u_\tau^{\vartheta-1} + \xi(\delta)
\end{aligned} \tag{3.27}$$

with $\xi(\delta) \to 0$ as $\delta \searrow 0$ (see hypothesis $H_g(iv)$ and recall that $\tau < \lambda$).

Since $u_\tau \in \operatorname{int} C_+$, the function $z \longmapsto \sigma_0\big(u_\tau(z)\big)$ is upper semicontinuous on $\overline{\Omega}$ (see hypothesis $H_g(iv)$). So, we can find $z_0 \in \overline{\Omega}$, such that

$$\sigma_0\big(u_\tau(z_0)\big) \;=\; \max_{z\in\overline{\Omega}} \sigma_0\big(u_\tau(z)\big) \;>\; 0. \tag{3.28}$$

We use (3.28) in (3.27). Since $\xi(\delta) \searrow 0$ and $\delta \searrow 0$ and $\lambda > \tau$, we infer that

$$\begin{aligned}
&-\Delta_p \overline{u}_\tau + \beta \overline{u}_\tau^{p-1} + \gamma_\varrho \overline{u}_\tau^{\vartheta-1}\\
&\quad\leqslant \lambda g(z,u_\tau) - f(z,u_\tau) + \gamma_\varrho u_\tau^{\vartheta-1}\\
&\quad\leqslant \lambda g(z,u_\lambda^0) - f(z,u_\lambda^0) + \gamma_\varrho (u_\lambda^0)^{\vartheta-1}\\
&\quad= -\Delta_p u_\lambda^0 + \beta (u_\lambda^0)^{p-1} + \gamma_\varrho (u_\lambda^0)^{\vartheta-1} \quad \text{for almost all } z \in \Omega.
\end{aligned}$$

for $\delta > 0$ small (see H_0 and recall that $u_\tau \leqslant u_\lambda^0$). Acting on this inequality with $(\overline{u}_\tau - u_\lambda^0)^+ \in W^{1,p}(\Omega)$ and using the nonlinear Green's identity (see e.g., Gasiński and Papageorgiou [9]) as above, we obtain

$$\overline{u}_\tau \;=\; u_\tau + \delta \;\leqslant\; u_\lambda^0 \quad \forall \delta > 0 \text{ small},$$

so

$$u_\lambda^0 - u_\tau \;\in\; \operatorname{int} C_+.$$

This proves Claim 1.

Let

$$[u_\tau) \;=\; \big\{ u \in W^{1,p}(\Omega) : \; u_\tau(z) \leqslant u(z) \quad \text{for almost all } z \in \Omega \big\}.$$

From (3.26), we see that

$$\widehat{\psi}_\lambda|_{[u_\tau)} = \varphi_\lambda|_{[u_\tau)} + \widehat{c}, \tag{3.29}$$

for some $\widehat{c} \in \mathbb{R}$. Then Claim 1 and (3.29) imply that u_λ^0 is a local $C^1(\overline{\Omega})$-minimizer of φ_λ. From Theorem 2.2, it follows that u_λ^0 is a local $W^{1,p}(\Omega)$-minimizer of $\varphi(\lambda)$.

By virtue of hypotheses $H_g(iii)$ and $H_f(iii)$, for a given $\varepsilon > 0$ we can find $\delta = \delta(\varepsilon) > 0$, such that

$$G(z,\zeta) \leqslant \frac{\varepsilon}{p}\zeta^p \quad \text{and} \quad F(z,\zeta) \geqslant -\frac{\varepsilon}{p}\zeta^p \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in (0,\delta]. \tag{3.30}$$

So, if $u \in C^1(\overline{\Omega})$ with $\|u\|_{C^1(\overline{\Omega})} \leqslant \delta$, then

$$\begin{aligned}
\varphi_\lambda(u) &= \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{p}\int_\Omega \beta|u|^p\,dz - \lambda\int_\Omega G(z,u)\,dz + \int_\Omega F(z,u)\,dz \\
&\geqslant \frac{\xi_0}{p}\|u\|^p - \frac{\lambda+1}{p}\varepsilon\|u^+\|^p \\
&\geqslant \frac{\xi_0 - (\lambda+1)\varepsilon}{p}\|u\|^p
\end{aligned}$$

(see Lemma 2.5) and (3.30). Choosing $\varepsilon \in \left(0, \frac{\xi_0}{\lambda+1}\right)$, we infer that

$$\varphi_\lambda(u) \geqslant 0 = \varphi_\lambda(0) \quad \forall u \in C^1(\overline{\Omega}),\ \|u\|_{C^1(\Omega)} \leqslant \delta,$$

so

$$u = 0 \quad \text{is a local } C^1(\overline{\Omega})\text{-minimizer of } \varphi_\lambda$$

and thus

$$u = 0 \quad \text{is a local } W^{1,p}(\Omega)\text{-minimizer of } \varphi_\lambda$$

(see Theorem 2.2).

Without any loss of generality, we may assume that

$$\varphi_\lambda(0) = 0 \leqslant \varphi_\lambda(u_\lambda^0)$$

(the analysis is similar if the opposite inequality is true). Moreover, we may assume that both local minimizers $u = 0$ and $u = u_\lambda^0$ are isolated (otherwise it is clear that we have a whole sequence of positive solutions of $(P)_\lambda$ and so we are done). Reasoning as in Aizicovici et al. [2, Proposition 29], we can find $\varrho \in \left(0, \|u_\lambda^0\|\right)$ small, such that

$$\varphi_\lambda(0) = 0 \leqslant \varphi_\lambda(u_\lambda^0) < \inf\left\{\varphi_\lambda(u) : \|u - u_\lambda^0\| = \varrho\right\} = \eta_0^\lambda. \tag{3.31}$$

Recall that φ_λ is coercive (see the proof of Proposition 3.4). Hence it satisfies the Palais–Smale condition. This fact and (3.1) permit the use of the mountain pass theorem (see Theorem 2.1) and so, we obtain $\widehat{u}_\lambda \in W^{1,p}(\Omega)$, such that

$$\eta_\varrho \leqslant \varphi_\lambda(\widehat{u}_\lambda) \tag{3.32}$$

and

$$\varphi'_\lambda(\widehat{u}_\lambda) = 0. \tag{3.33}$$

From (3.31) and (3.32), it follows that $\widehat{u}_\lambda \notin \{0, u^0_\lambda\}$. From (3.33), we have

$$A(\widehat{u}_\lambda) + \beta \widehat{u}_\lambda^{p-1} = \lambda N_g(\widehat{u}_\lambda) - N_f(\widehat{u}_\lambda),$$

so $\widehat{u}_\lambda \in \operatorname{int} C_+$ is a solution of $(P)_\lambda$.

So, we conclude that $(P)_\lambda$ $(\lambda > \lambda_*)$ has at least two positive solutions $u^0_\lambda, \widehat{u}_\lambda \in \operatorname{int} C_+$. □

Next we examine what happens in the critical case $\lambda = \lambda_*$.

Proposition 3.6 *If hypotheses H_f, H_g and H_0 hold, then $\lambda_* \in \mathcal{Y}$.*

Proof Let $\lambda_n > \lambda_*$ for $n \geqslant 1$ be such that $\lambda_n \searrow \lambda_*$ and let $u_n = u_{\lambda_n} \in \operatorname{int} C_+$ be positive solutions for problem $(P)_\lambda$ for $n \geqslant 1$ (see Proposition 3.4). We have

$$A(u_n) + \beta u_n^{p-1} = \lambda_n N_g(u_n) - N_f(u_n) \quad \forall n \geqslant 1. \tag{3.34}$$

By virtue of hypothesis $H_g(ii)$ and since $\vartheta > q$, we have

$$\lim_{\zeta \to +\infty} \frac{g(z, \zeta)}{\zeta^{\vartheta - 1}} = 0 \quad \text{uniformly for almost all } z \in \Omega.$$

This fact combined with hypothesis $H_g(i)$, implies that for a given $\varepsilon > 0$, we can find $c_6 = c_6(\varepsilon) > 0$, such that

$$g(z, \zeta)\zeta \leqslant \frac{\varepsilon}{\vartheta}(\zeta^+)^\vartheta + c_6 \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R}. \tag{3.35}$$

In a similar fashion, using hypotheses $H_f(i)$ and (ii), we see that we can find $\eta > 0$ and $c_7 > 0$, such that

$$f(z, \zeta)\zeta \geqslant \frac{\eta}{\vartheta}(\zeta^+)^\vartheta - c_7 \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R}. \tag{3.36}$$

On (3.34) we act with $u_n \in W^{1,p}(\Omega)$ and obtain

$$\begin{aligned} \|\nabla u_n\|_p^p + \int_\Omega \beta u_n^p \, dz &= \lambda_n \int_\Omega g(z, u_n) u_n \, dz - \int_\Omega f(z, u_n) u_n \, dz \\ &\leqslant \frac{\lambda_n \varepsilon - \eta}{\vartheta} \|u_n\|_\vartheta^\vartheta + c_8 \quad \forall n \geqslant 1, \end{aligned} \tag{3.37}$$

for some $c_8 > 0$ (see (3.35) and (3.36)).

We choose $\varepsilon \in \left(0, \frac{\eta}{\lambda_1}\right)$ (recall that $\lambda_n \leqslant \lambda_1$ for all $n \geqslant 1$). Then from (3.37) and Lemma 2.5, it follows that

$$\xi_0 \|u_n\|^p \leqslant c_8 \quad \forall n \geqslant 1$$

and so the sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded.

So, passing to a subsequence if necessary, we may assume that

$$u_n \longrightarrow u_* \quad \text{weakly in } W^{1,p}(\Omega), \tag{3.38}$$

$$u_n \longrightarrow u_* \quad \text{in } L^\theta(\Omega), \tag{3.39}$$

with $\theta < p^*$. On (3.34) we act with $u_n - u_*$, pass to the limit as $n \to +\infty$ and use (3.38). We obtain

$$\lim_{n\to+\infty} \langle A(u_n),\ u_n - u_* \rangle = 0,$$

so

$$u_n \longrightarrow u_* \quad \text{in } W^{1,p}(\Omega) \tag{3.40}$$

(see Proposition 2.4).

So, if in (3.34) we pass to the limit as $n \to +\infty$ and use (3.40), we obtain

$$A(u_*) + \beta u_*^{p-1} = \lambda_* N_g(u_*) - N_f(u_*),$$

so $u_* \in C_+$ and it solves problem $(P)_{\lambda_*}$.

It remains to show that $u_* \neq 0$. Arguing by contradiction, suppose that $u_* = 0$. From (3.34), we have

$$\begin{cases} -\Delta_p u_n(z) + \beta(z) u_n(z)^{p-1} = \lambda g\big(z, u_n(z)\big) - f\big(z, u_n(z)\big) & \text{in} \quad \Omega, \\ \dfrac{\partial u_n}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{3.41}$$

From (3.41) and Theorem 2 of Lieberman [24], we know that we can find $\alpha \in (0, 1)$ and $M > 0$, such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leqslant M \quad \forall n \geqslant 1.$$

From the compactness of the embedding $C^{1,\alpha}(\overline{\Omega}) \subseteq C^1(\overline{\Omega})$, we have

$$u_n \longrightarrow u_* \quad \text{in } C^1(\overline{\Omega}).$$

Let

$$y_n = \frac{u_n}{\|u_n\|} \quad \forall n \geqslant 1.$$

Then

$$y_n \geqslant 0, \quad \|y_n\| = 1 \quad \forall n \geqslant 1.$$

So, passing to a subsequence if necessary, we may assume that

$$y_n \longrightarrow y_* \quad \text{weakly in } W^{1,p}(\Omega), \tag{3.42}$$

$$y_n \longrightarrow y_* \quad \text{in } L^{\vartheta}(\Omega). \tag{3.43}$$

From (3.34), we have

$$A(y_n) + \beta y_n^{p-1} = \lambda_n \frac{N_g(u_n)}{\|u_n\|^{p-1}} - \frac{N_f(u_n)}{\|u_n\|^{p-1}} \quad \forall n \geqslant 1. \tag{3.44}$$

From hypotheses $H_g(i)$, (iii) and $H_f(i)$, (iii), it follows that

$$\text{the sequences } \left\{\frac{N_g(u_n)}{\|u_n\|^{p-1}}\right\}_{n\geqslant 1}, \left\{\frac{N_f(u_n)}{\|u_n\|^{p-1}}\right\}_{n\geqslant 1} \subseteq L^{p'}(\Omega) \text{ are bounded}$$

(where $\frac{1}{p} + \frac{1}{p'} = 1$).

Acting on (3.44) with $y_n - y_*$, passing to the limit as $n \to +\infty$ and using (3.42), we obtain

$$\lim_{n\to+\infty} \langle A(y_n),\ y_n - y_*\rangle = 0,$$

so

$$y_n \longrightarrow y_* \quad \text{in } W^{1,p}(\Omega) \tag{3.45}$$

(see Proposition 2.4) and so $\|y_*\| = 1$.

Note that by virtue of hypotheses $H_g(iii)$ and $H_f(iii)$, we have

$$\frac{N_f(u_n)}{\|u_n\|^{p-1}} \longrightarrow 0 \quad \text{and} \quad \frac{N_g(u_n)}{\|u_n\|^{p-1}} \longrightarrow \widehat{\zeta} y_*^{p-1} \quad \text{weakly in } L^p(\Omega), \tag{3.46}$$

with $0 \leqslant \widehat{\zeta}(z) \leqslant \zeta^*$ for almost all $z \in \Omega$. So, if in (3.44) we pass to the limit as $n \to +\infty$ and we use (3.45) and (3.46), we obtain

$$A(y_*) + \beta y_*^{p-1} = -\widehat{\zeta} y_*^{p-1},$$

so

$$\|\nabla y_*\|_p^p + \int_\Omega \beta y_*^p \, dz \leqslant -\int_\Omega \widehat{\zeta} y_*^p \, dz \leqslant 0,$$

thus

$$\xi_0 \|y_*\|^p \leqslant 0$$

(see Lemma 2.5) and finally, we have that $y_* = 0$, which contradicts to (3.45).

This proves that $u_* \neq 0$. Hence $u_* \in \operatorname{int} C_+$ is a solution of problem $(P)_{\lambda_*}$. Therefore $\lambda_* \in \mathcal{Y}$. $\square$

We show that for every $\lambda \geqslant \lambda_*$, problem $(P)_\lambda$ has an extremal (smallest) positive solution.

Proposition 3.7 *If hypotheses H_f, H_g, and H_0 hold and $\lambda \geqslant \lambda_*$, then problem $(P)_\lambda$ has a smallest positive solution $u_\lambda^* \in \operatorname{int} C_+$.*

Proof Let $S(\lambda)$ be the set of positive solutions for problem $(P)_\lambda$. Since $\lambda \geqslant \lambda_*$, $S(\lambda) \neq 0$ and $S(\lambda) \subseteq \operatorname{int} C_+$. Let $C \subseteq S(\lambda)$ be a chain (i.e., a nonempty linearly ordered subset of $S(\lambda)$). From Dunford and Schwartz [6, p.336], we know that we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq C$, such that

$$\inf_{n\geqslant 1} u_n = \inf C.$$

Moreover, from Lemma 11.5(a) of Heikkilä and Lakshmikantham [22, p. 15], we know that we may assume that the sequence $\{u_n\}_{n\geqslant 1}$ is decreasing. We have

$$A(u_n) + \beta u_n^{p-1} = \lambda N_g(u_n) - N_f(u_n) \quad \forall n \geqslant 1, \tag{3.47}$$

so

$$\|\nabla u_n\|_p^p + \int_\Omega \beta u_n^p \, dz = \int_\Omega \big(\lambda g(z, u_n) - f(z, u_n)\big) u_n \, dz \leqslant M_1 \quad \forall n \geqslant 1,$$

for some $M_1 > 0$ (see hypotheses $H_g(i)$, $H_f(i)$ and recall that $u_n \leqslant u_1$ for all $n \geqslant 1$). So,

$$\xi_0 \|u_n\|^p \leqslant M_1 \quad \forall n \geqslant 1$$

(see Lemma 2.5) and thus the sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded.

So, passing to a subsequence if necessary, we may assume that

$$u_n \longrightarrow u_* \quad \text{weakly in } W^{1,p}(\Omega), \tag{3.48}$$

$$u_n \longrightarrow u_* \quad \text{in } L^\theta(\Omega), \tag{3.49}$$

with $\theta < p^*$. On (3.47) we act with $u_n - u_*$, pass to the limit as $n \to +\infty$ and use (3.48). Then

$$\lim_{n\to+\infty} \langle A(u_n),\ u_n - u_* \rangle = 0,$$

so

$$u_n \longrightarrow u_* \quad \text{in } W^{1,p}(\Omega)$$

(see Proposition 2.4).

Reasoning as in the proof of Proposition 3.6, we show that $u_* \neq 0$ and so $u_* \in \operatorname{int} C_+$ is a positive solution of $(P)_\lambda$. Hence $u_* = \inf C \in S(\lambda)$ and since C was an arbitrary chain, from the Kuratowski-Zorn lemma, we infer that $S(\lambda)$ has a minimum element $u_\lambda^* \in \operatorname{int} C_+$. But $S(\lambda)$ is downward directed (i.e., if $u, v \in S(\lambda)$, then there exists $y \in S(\lambda)$, such that $y \leqslant \min\{u, v\}$; see Aizicovici et al. [3]). So, it follows that $u_\lambda^* \leqslant u$ for all $u \in S(\lambda)$, i.e., $u_\lambda^* \in \operatorname{int} C_+$ is the smallest positive solution of problem $(P)_\lambda$. $\square$

Summarizing the situation, we have the following bifurcation-type theorem describing the dependence of positive solutions of $(P)_\lambda$ on the parameter $\lambda > 0$.

Theorem 3.8 *If hypotheses H_f, H_g and H_0 hold, then there exists $\lambda_* > 0$, such that:*

(a) *for all $\lambda > \lambda_*$, problem $(P)_\lambda$ has at least two positive solutions*

$$u_0,\ \widehat{u} \in \operatorname{int} C_+;$$

(b) *for $\lambda = \lambda_*$, problem $(P)_\lambda$ has at least one positive solution $u_* \in \operatorname{int} C_+$;*
(c) *for all $\lambda \in (0, \lambda_*)$, problem $(P)_\lambda$ has no positive solution.*

Moreover, if $\lambda \geqslant \lambda_*$, then problem $(P)_\lambda$ has a smallest positive solution $u_\lambda^* \in \operatorname{int} C_+$.

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