Theoretical Error Analysis of Solution for Two-Dimensional Stochastic Volterra Integral Equations by Haar Wavelet

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Abstract

The finding an efficient way to the approximate solutions of the stochastic integral equations is an essential requirement. In this paper we discuss the convergence analysis of the two-dimensional Haar wavelet functions (2D-HWFs) method for solve 2D linear stochastic Volterra integral equation. The illustrative examples are included to demonstrate the validity and applicability of this numerical method.

Keywords Haar wavelet function · Two-dimensional integral equation · Stochastic integral equation · Volterra integral equation · Brownian motion process · Ito integral

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Introduction

In very of the engineering problems, we see the important role of 2D integral equations [1,2], where are produced from a differential equation. Also if we import the statistical noise into a general hyperbolic differential equation we can obtain 2D linear stochastic Volterra integral equation of the second kind, i.e.

$$\begin{aligned} g(x, y) = & f(x, y) + \int_0^y \int_0^x K_1(x, y, s, t) g(s, t) ds dt \\ & + \int_0^y \int_0^x K_2(x, y, s, t) g(s, t) dB(s) dB(t) \\ & (x, y) \in [0, 1] \times [0, 1], \quad s \leq x < t \leq y. \end{aligned} \quad (1)$$

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where $B(t)$ is Brownian motion process. In the last years, the numerous numerical methods have been introduced to estimate the solution of 2D ordinary integral equations [3–8] and many other papers, whereas it is worked few for 2D stochastic integral equations. Recently, some authors have proposed the methods to solve 2D stochastic integral equations [9–14]. Especially, Fallahpour et al. [9], included HWFs method to solve such equations without the error analysis. In this paper, we use HWFs method with the error and convergence analysis to derive approximate solution of (1).

This paper is organized as follows:

In next section we present HWFs. In section “Haar Wavelets Numerical Method”, the method is applied to solve Eq. (1). The error analysis of this method is discussed in section “Error Analysis”. Section “Numerical Example”, uses some numerical examples to show the convergence of the proposed method and compares it with the block-pulse functions (BPFs) method as proposed in [10,11]. Finally, conclusion is given, in section “Conclusion”.

Haar Wavelets

As we know, for HWFs we can write

$$\psi_{j,i}(z) = 2^{j/2} \psi(2^j z - i),$$

so on the interval $[0, 1)$ we have

$$h_1(z) = \begin{cases} 1, & \text{for } z \in [0, 1) \\ 0, & \text{otherwise,} \end{cases}$$

and

$$h_i(z) = \begin{cases} 1, & \text{for } z \in [\alpha, \beta) \\ -1, & \text{for } z \in [\beta, \gamma) \\ 0, & \text{otherwise,} \end{cases}$$

for $i = 2, 3, \dots$, where

$$\alpha = \frac{n}{m}, \quad \beta = \frac{(n+0.5)}{m}, \quad \gamma = \frac{(n+1)}{m},$$

so

$$m = 2^\ell, \quad \ell = 0, 1, \dots, \quad n = 0, 1, \dots, m-1.$$

The integer ℓ and n , are the wavelet level and the translation parameter, respectively. Also Haar wavelets $h_i(z)$ are pairwise orthonormal in the interval $[0, 1)$ as

$$\int_0^1 h_i(z) h_j(z) dy = \delta_{ij}, \quad (2)$$

where δ_{ij} is Kronecker delta. We display the maximum value of ℓ by J where $M = 2^J$. Therefore for any square integrable function $f(z)$ we have

$$f(z) \approx \sum_{i=1}^{2M} a_i h_i(z).$$

We show the ordinary Volterra integral of Haar wavelet as

$$p_i(z) = \int_0^z h_i(u) du,$$

where by using HWFs we get

$$p_i(z) = \begin{cases} z - \alpha, & \text{for } z \in [\alpha, \beta) \\ \gamma - z, & \text{for } z \in [\beta, \gamma) \\ 0, & \text{elsewhere.} \end{cases} \quad (3)$$

Also for the stochastic Volterra integral we can introduce

$$q_i(z) = \int_0^z h_i(u) dB(u),$$

where similarly by Haar wavelet definition we have

$$q_i(z) = \begin{cases} B(z) - B(\alpha), & \text{for } z \in [\alpha, \beta) \\ 2B(\beta) - B(\alpha) - B(z), & \text{for } z \in [\beta, \gamma) \\ 0, & \text{elsewhere.} \end{cases} \quad (4)$$

Haar Wavelets Numerical Method

In this section, we solve Eq. (1) by using 2D-HWFs. For Haar wavelet approximation of a function $f(x, y)$, we use the collocation points

$$x_m = \frac{m - 0.5}{2M}, m = 1, 2, \dots, 2M, \quad (5)$$

$$y_n = \frac{n - 0.5}{2N}, n = 1, 2, \dots, 2N. \quad (6)$$

2D-HWFs System

A real-valued function $E(x, y)$ can be approximated using 2D-HWFs as

$$E(x, y) \approx \sum_{d=1}^{2M} \sum_{e=1}^{2N} b_{d,e} h_d(x) h_e(y),$$

where the unknown coefficients $b_{d,e}$'s, have calculated in [9]. Also we consider a function $E(x, y, s, t)$ where is approximated using 2D-HWFs as

$$E(x, y, s, t) \approx \sum_{d=1}^{2M} \sum_{e=1}^{2N} b_{d,e}(x, y) h_d(s) h_e(t),$$

so by substituting the collocation points

$$s_i = \frac{i - 0.5}{2M}, i = 1, 2, \dots, 2M,$$

and

$$t_j = \frac{j - 0.5}{2N}, j = 1, 2, \dots, 2N,$$

we get

$$E(x, y, s_i, t_j) \approx \sum_{d=1}^{2M} \sum_{e=1}^{2N} b_{d,e}(x, y) h_d(s_i) h_e(t_j), \quad (7)$$

where the solution of System (7) is calculated from Corollary (1) in [9].

2D-Linear Stochastic Volterra Integral Equation

We approximate the functions $K_1 g$ and $K_2 g$ in (1) by using 2D-HWFs as

$$K_1(x, y, s, t) g(s, t) \approx \sum_{i=1}^{2M} \sum_{j=1}^{2N} b_{i,j}(x, y) h_i(s) h_j(t),$$

and

$$K_2(x, y, s, t) g(s, t) \approx \sum_{i=1}^{2M} \sum_{j=1}^{2N} c_{i,j}(x, y) h_i(s) h_j(t),$$

respectively. With these approximations, Eq. (1) can be written as

$$\begin{aligned} g(x, y) = & f(x, y) + \int_0^y \int_0^x \sum_{i=1}^{2M} \sum_{j=1}^{2N} b_{i,j}(x, y) h_i(s) h_j(t) ds dt \\ & + \int_0^y \int_0^x \sum_{i=1}^{2M} \sum_{j=1}^{2N} c_{i,j}(x, y) h_i(s) h_j(t) dB(s) dB(t). \end{aligned} \quad (8)$$

By applying Eqs. (3) and (4) in (8), we have

$$g(x, y) = f(x, y) + \sum_{i=1}^{2M} \sum_{j=1}^{2N} b_{i,j}(x, y) p_i(x) p_j(y) + \sum_{i=1}^{2M} \sum_{j=1}^{2N} c_{i,j}(x, y) q_i(x) q_j(y).$$

The collocation points (5) and (6), give

$$\begin{aligned} g(x_m, y_n) = & f(x_m, y_n) + \sum_{i=1}^{2M} \sum_{j=1}^{2N} b_{i,j}(x_m, y_n) p_i(x_m) p_j(y_n) \\ & + \sum_{i=1}^{2M} \sum_{j=1}^{2N} c_{i,j}(x_m, y_n) q_i(x_m) q_j(y_n). \end{aligned}$$

Now if we replace $b_{i,j}$'s and $c_{i,j}$'s by the given expressions of Corollary 1 in [9], we get

$$\begin{aligned} g(x_m, y_n) = & f(x_m, y_n) + \frac{p_1(x_m) p_1(y_n)}{2M \times 2N} \sum_{d=1}^{2M} \sum_{e=1}^{2N} K_1(x_m, y_n, s_d, t_e) g(s_d, t_e) \\ & + \sum_{i=2}^{2M} \frac{p_i(x_m) p_1(y_n)}{\rho_1 \times 2N} \left(\sum_{d=\alpha_1}^{\beta_1} \sum_{e=1}^{2N} K_1(x_m, y_n, s_d, t_e) g(s_d, t_e) \right. \\ & \left. - \sum_{d=\beta_1+1}^{\gamma_1} \sum_{e=1}^{2N} K_1(x_m, y_n, s_d, t_e) g(s_d, t_e) \right) + \sum_{j=2}^{2N} \frac{p_1(x_m) p_j(y_n)}{2M \times \rho_2} \end{aligned}$$

$$\begin{aligned}
& \times \left(\sum_{d=1}^{2M} \sum_{e=\alpha_2}^{\beta_2} K_1(x_m, y_n, s_d, t_e) g(s_d, t_e) \right. \\
& \left. - \sum_{d=1}^{2M} \sum_{e=\beta_2+1}^{\gamma_2} K_1(x_m, y_n, s_d, t_e) g(s_d, t_e) \right) \\
& + \sum_{i=2}^{2M} \sum_{j=2}^{2N} \frac{p_i(x_m) p_j(y_n)}{\rho_1 \times \rho_2} \left(\sum_{d=\alpha_1}^{\beta_1} \sum_{e=\alpha_2}^{\beta_2} K_1(x_m, y_n, s_d, t_e) g(s_d, t_e) \right. \\
& - \sum_{d=\alpha_1}^{\beta_1} \sum_{e=\beta_2+1}^{\gamma_2} K_1(x_m, y_n, s_d, t_e) g(s_d, t_e) - \sum_{d=\beta_1+1}^{\gamma_1} \sum_{e=\alpha_2}^{\beta_2} K_1(x_m, y_n, s_d, t_e) g(s_d, t_e) \\
& \left. + \sum_{d=\beta_1+1}^{\gamma_1} \sum_{e=\beta_2+1}^{\gamma_2} K_1(x_m, y_n, s_d, t_e) g(s_d, t_e) \right) + \frac{q_1(x_m) q_1(y_n)}{2M \times 2N} \\
& \times \sum_{i=1}^{2M} \sum_{j=1}^{2N} K_2(x_m, y_n, s_d, t_e) g(s_d, t_e) + \sum_{i=2}^{2M} \frac{q_i(x_m) q_1(y_n)}{\rho_1 \times 2N} \\
& \times \left(\sum_{d=\alpha_1}^{\beta_1} \sum_{e=1}^{2N} K_2(x_m, y_n, s_d, t_e) g(s_d, t_e) - \sum_{d=\beta_1+1}^{\gamma_1} \sum_{e=1}^{2N} K_2(x_m, y_n, s_d, t_e) g(s_d, t_e) \right) \\
& + \sum_{j=2}^{2N} \frac{q_1(x_m) q_j(y_n)}{2M \times \rho_2} \left(\sum_{d=1}^{2M} \sum_{e=\alpha_2}^{\beta_2} K_2(x_m, y_n, s_d, t_e) g(s_d, t_e) \right. \\
& \left. - \sum_{d=1}^{2M} \sum_{e=\beta_2+1}^{\gamma_2} K_2(x_m, y_n, s_d, t_e) g(s_d, t_e) \right) + \sum_{i=2}^{2M} \sum_{j=2}^{2N} \frac{q_i(x_m) q_j(y_n)}{\rho_1 \times \rho_2} \\
& \times \left(\sum_{d=\alpha_1}^{\beta_1} \sum_{e=\alpha_2}^{\beta_2} K_2(x_m, y_n, s_d, t_e) g(s_d, t_e) - \sum_{d=\alpha_1}^{\beta_1} \sum_{e=\beta_2+1}^{\gamma_2} K_2(x_m, y_n, s_d, t_e) \right. \\
& \times g(s_d, t_e) - \sum_{d=\beta_1+1}^{\gamma_1} \sum_{e=\alpha_2}^{\beta_2} K_2(x_m, y_n, s_d, t_e) g(s_d, t_e) \\
& \left. + \sum_{d=\beta_1+1}^{\gamma_1} \sum_{e=\beta_2+1}^{\gamma_2} K_2(x_m, y_n, s_d, t_e) g(s_d, t_e) \right), \tag{9}
\end{aligned}$$

where $\alpha_1, \beta_1, \gamma_1, \rho_1$ and $\alpha_2, \beta_2, \gamma_2, \rho_2$ are defined in [9] and Eq. (9) shows a system $2M \times 2N$.

Error Analysis

In this section, we investigate the convergence and error analysis of HWFs method for solution (1). Here norm 2 is defined as

$$\|f\|_2 = \left[\underbrace{\int_a^b \int_a^b \dots \int_a^b}_{m \text{ times}} |f(x_1, x_2, \dots, x_m)|^2 dx_1 dx_2 \dots dx_m \right]^{1/2}.$$

To obtaining the typical convergence rate of HWFs method, we have:

Theorem 1 [15] For the differentiable function $p : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$, with

$$\|p'\| \leq M,$$

where D and M are a convex open set and a real number, respectively, we have

$$|p(b) - p(a)| \leq M|b - a|$$

for all $a, b \in D$.

Theorem 2 [16] Suppose that $f(s, t) \in L^2([0, 1) \times [0, 1))$ with bounded partial derivatives,

$$\left| \frac{\partial^2 f}{\partial s \partial t} \right| \leq M, \text{ and}$$

$$e_m(s, t) = f(s, t) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} f_{i,j} h_i(s) h_j(t).$$

Then for $(s, t) \in D^2$, we have

$$\|e_m\| \leq \frac{M}{3m^2}.$$

Theorem 3 Consider $K(x, y, s, t) \in L^2([0, 1) \times [0, 1) \times [0, 1) \times [0, 1))$ with bounded partial derivatives,

$$\left| \frac{\partial^4 f}{\partial x \partial y \partial s \partial t} \right| \leq U,$$

and

$$\hat{K}_m(x, y, s, t) = \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \sum_{c=0}^{m-1} \sum_{d=0}^{m-1} K_{abcd} h_a(x) h_b(y) h_c(s) h_d(t),$$

is 4D-HWFs expansion of $K(x, y, s, t)$ where

$$e_{a,b,c,d}(x, y, s, t) = K(x, y, s, t) - \hat{K}_m(x, y, s, t),$$

then

$$\|e_{a,b,c,d}\| \leq \frac{U}{9m^4},$$

for every $(x, y, s, t) \in D^4$.

Proof We can write

$$\begin{aligned} \|e_{a,b,c,d}\|^2 &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \\ &\quad \left(K(x, y, s, t) - \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} \sum_{c=0}^{m-1} \sum_{d=0}^{m-1} K_{abcd} h_a(x) h_b(y) h_c(s) h_d(t) \right)^2 dt ds dy dx, \\ &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \left(\sum_{a=m}^{\infty} \sum_{b=m}^{\infty} \sum_{c=m}^{\infty} \sum_{d=m}^{\infty} K_{abcd} h_a(x) h_b(y) h_c(s) h_d(t) \right)^2 dt ds dy dx, \end{aligned}$$

$$= \sum_{a=m}^{\infty} \sum_{b=m}^{\infty} \sum_{c=m}^{\infty} \sum_{d=m}^{\infty} \int_0^1 \int_0^1 \int_0^1 \int_0^1 K_{abcd}^2 h_a^2(x) h_b^2(y) h_c^2(s) h_d^2(t) dt ds dy dx, \quad (10)$$

where $a = 2^{j_1} + k$, $b = 2^{j_2} + k$, $c = 2^{j_3} + k$, $d = 2^{j_4} + k$, $m = 2^J$, $J > 0$ and

$$K_{abcd} = \int_0^1 \int_0^1 \int_0^1 \int_0^1 h_a(x) h_b(y) h_c(s) h_d(t) K(x, y, s, t) dt ds dy dx.$$

Based on Haar wavelet definition, mean value theorem, Theorems 1 and 2 there are

$$\eta_{j_4}, \alpha, \alpha', \quad \eta_{j_3}, \beta, \beta', \quad \eta_{j_2}, \gamma, \gamma', \quad \eta_{j_1}, \theta, \theta',$$

where

$$\begin{aligned} \alpha, \alpha' &\in \left[k2^{-j_4}, \left(k + \frac{1}{2}\right) 2^{-j_4} \right], \quad \beta, \beta' \in \left[k2^{-j_3}, \left(k + \frac{1}{2}\right) 2^{-j_3} \right], \\ \gamma, \gamma' &\in \left[k2^{-j_2}, \left(k + \frac{1}{2}\right) 2^{-j_2} \right], \quad \theta, \theta' \in \left[k2^{-j_1}, \left(k + \frac{1}{2}\right) 2^{-j_1} \right], \end{aligned}$$

such that

$$\begin{aligned} K_{abcd} &= \int_0^1 \int_0^1 \int_0^1 h_a(x) h_b(y) h_c(s) \left(\int_0^1 h_d(t) K(x, y, s, t) dt \right) ds dy dx \\ &= \int_0^1 \int_0^1 \int_0^1 h_a(x) h_b(y) h_c(s) \left(2^{\frac{-j_4}{2}-1} (\alpha - \alpha') \frac{\partial K(x, y, s, \eta_{j_4})}{\partial t} \right) ds dy dx \\ &= \int_0^1 \int_0^1 h_a(x) h_b(y) \times 2^{\frac{-j_4}{2}-1} (\alpha - \alpha') \left(\int_0^1 h_c(s) \frac{\partial K(x, y, s, \eta_{j_4})}{\partial t} ds \right) dy dx \\ &= \int_0^1 \int_0^1 2^{\frac{-j_4}{2}-\frac{-j_3}{2}-2} \times (\alpha - \alpha') (\beta - \beta') (\gamma - \gamma') \frac{\partial^2 K(x, y, \eta_{j_3}, \eta_{j_4})}{\partial s \partial t} h_a(x) h_b(y) dy dx. \end{aligned}$$

Finally we conclude

$$\begin{aligned} K_{abcd} &= 2^{\frac{-j_4}{2}-\frac{-j_3}{2}-\frac{-j_2}{2}-\frac{-j_1}{2}-4} \\ &\quad \times (\alpha - \alpha') (\beta - \beta') (\gamma - \gamma') (\theta - \theta') \frac{\partial^4 K(\eta_{j_1}, \eta_{j_2}, \eta_{j_3}, \eta_{j_4})}{\partial x \partial y \partial s \partial t}. \end{aligned} \quad (11)$$

By using Eq. (2) and substituting Eq. (11) in (10) we get

$$\begin{aligned} \|e_{a,b,c,d}\|^2 &= \sum_{a=m}^{\infty} \sum_{b=m}^{\infty} \sum_{c=m}^{\infty} \sum_{d=m}^{\infty} 2^{-j_4-j_3-j_2-j_1-8} \\ &\quad \times (\alpha - \alpha')^2 (\beta - \beta')^2 (\gamma - \gamma')^2 (\theta - \theta')^2 \left| \frac{\partial^4 K(\eta_{j_1}, \eta_{j_2}, \eta_{j_3}, \eta_{j_4})}{\partial x \partial y \partial s \partial t} \right|^2 \\ &\leq \sum_{a=m}^{\infty} \sum_{b=m}^{\infty} \sum_{c=m}^{\infty} \sum_{d=m}^{\infty} 2^{-j_4-j_3-j_2-j_1-8} \times 2^{-2j_1} \times 2^{-2j_2} \times 2^{-2j_3} \times 2^{-2j_4} \times U^2 \end{aligned}$$

$$\begin{aligned}
&= \sum_{a=m}^{\infty} \sum_{b=m}^{\infty} \sum_{c=m}^{\infty} \sum_{d=m}^{\infty} 2^{-3j_4-3j_3-3j_2-3j_1-8} \times U^2 \\
&= U^2 \sum_{a=m}^{\infty} \sum_{b=m}^{\infty} 2^{-3j_2-3j_1-4} \sum_{c=m}^{\infty} \sum_{d=m}^{\infty} 2^{-3j_4-3j_3-4},
\end{aligned}$$

therefore we can derive

$$\|e_{a,b,c,d}\|^2 \leq U^2 \times \frac{1}{9m^4} \times \frac{1}{9m^4} = \frac{U^2}{81m^8}.$$

In other words

$$\|e_{a,b,c,d}\| \leq \frac{U}{9m^4}.$$

□

Theorem 4 If $g(s, t)$ be the exact solution of Eq. (1) and $\hat{g}_m(s, t)$ be HWFs approximate solution of (1) that is obtained by (9) and U_i 's be defined in Theorem 3, then by using hypotheses

1. $\|g\| \leq A, \quad (s, t) \in [0, 1) \times [0, 1),$
2. $\|K_i\| \leq Q_i, \quad i = 1, 2, \quad (x, y, s, t) \in [0, 1) \times [0, 1) \times [0, 1) \times [0, 1),$
3. $O(x) = \sup_{x \in [0, 1)} |B(x)|,$
4. $\left(Q_1 + \frac{U_1}{9m^4} + \left(Q_2 + \frac{U_2}{9m^4} \right) \times O(x) \times O(y) \right) < 1,$

we have

$$\|g - \hat{g}_m\| \leq \frac{\frac{M}{3m^2} + \frac{U_1 A}{9m^4} + \frac{U_2 A}{9m^4} \times O(x) \times O(y)}{1 - \left[Q_1 + \frac{U_1}{9m^4} + \left(Q_2 + \frac{U_2}{9m^4} \right) \times O(x) \times O(y) \right]}.$$

Proof We can obtain

$$\begin{aligned}
&g(x, y) - \hat{g}_m(x, y) \\
&= f(x, y) - \hat{f}_m(x, y) \\
&\quad + \int_0^y \int_0^x \left(K_1(x, y, s, t)g(s, t) - \hat{K}_{1,m}(x, y, s, t)\hat{g}_m(s, t) \right) ds dt \\
&\quad + \int_0^y \int_0^x \left(K_2(x, y, s, t)g(s, t) - \hat{K}_{2,m}(x, y, s, t)\hat{g}_m(s, t) \right) dB(s)dB(t).
\end{aligned}$$

The mean value theorem conclude

$$\begin{aligned}
\|g - \hat{g}_m\| &\leq \|f - \hat{f}_m\| + xy \|K_1 g - \hat{K}_{1,m} \hat{g}_m\| \\
&\quad + B(x)B(y) \|K_2 g - \hat{K}_{2,m} \hat{g}_m\|.
\end{aligned} \tag{12}$$

By using two first hypothesis and Theorem 3 we obtain

$$\begin{aligned}
\|K_1 g - \hat{K}_{1,m} \hat{g}_m\| &\leq \|K_1\| \|g - \hat{g}_m\| + \|K_1 - \hat{K}_{1,m}\| (\|g - \hat{g}_m\| + \|g\|) \\
&\leq Q_1 \|g - \hat{g}_m\| + \frac{U_1}{9m^4} (\|g - \hat{g}_m\| + A)
\end{aligned}$$

$$= \left(Q_1 + \frac{U_1}{9m^4} \right) \|g - \hat{g}_m\| + \frac{U_1}{9m^4} A. \quad (13)$$

Similarly for the stochastic case we get

$$\begin{aligned} \|K_2 g - \hat{K}_{2,m} \hat{g}_m\| &\leq \|K_2\| \|g - \hat{g}_m\| + \|K_2 - \hat{K}_{2,m}\| (\|g - \hat{g}_m\| + \|g\|) \\ &= \left(Q_2 + \frac{U_2}{9m^4} \right) \|g - \hat{g}_m\| + \frac{U_2}{9m^4} A. \end{aligned} \quad (14)$$

By Theorem 2 and substituting Eqs. (12) and (13) in (14), we have

$$\begin{aligned} \|g - \hat{g}_m\| &\leq \frac{M}{3m^2} + xy \left[\left(Q_1 + \frac{U_1}{9m^4} \right) \|g - \hat{g}_m\| + \frac{U_1}{9m^4} A \right] \\ &\quad + B(x)B(y) \left[\left(Q_2 + \frac{U_2}{9m^4} \right) \|g - \hat{g}_m\| + \frac{U_2}{9m^4} A \right]. \end{aligned} \quad (15)$$

By taking sup we have

$$\begin{aligned} \|g - \hat{g}_m\| &\leq \frac{M}{3m^2} + \sup_{x \in [0,1]} x \times \sup_{y \in [0,1]} y \left[\left(Q_1 + \frac{U_1}{9m^4} \right) \sup_{s \leq x, t \leq y} \|g - \hat{g}_m\| + \frac{U_1}{9m^4} A \right] \\ &\quad + \sup_{x \in [0,1]} |B(x)| \times \sup_{y \in [0,1]} |B(y)| \left[\left(Q_2 + \frac{U_2}{9m^4} \right) \sup_{s \leq x, t \leq y} \|g - \hat{g}_m\| + \frac{U_2}{9m^4} A \right], \end{aligned}$$

so

$$\|g - \hat{g}_m\| \leq \frac{\frac{M}{3m^2} + \frac{U_1 A}{9m^4} + \frac{U_2 A}{9m^4} \times O(x) \times O(y)}{1 - \left[Q_1 + \frac{U_1}{9m^4} + \left(Q_2 + \frac{U_2}{9m^4} \right) \times O(x) \times O(y) \right]},$$

finally Theorem 3 of [11] complete the proof. $\square$

Numerical Example

To illustrate the effectiveness of the proposed method, are carried out three numerical examples in this section as the solution mean, the error mean, %95 confidence interval and the length of the confidence interval are shown by $\bar{g}(x, y)$, $\bar{e}(x, y)$, CI and LCI , respectively in Tables 1, 2 and 3. According to these tables by applying HWFs method, as J increases, $\bar{e}(x, y)$ and LCI decrease, nearly. Also the results of this method are compared with the BPFs method [10,11] and are displayed in Table 4. According to Table 4 we can say that HWFs method is better than the BPFs method at some points and also there are some other points that the BPFs method is an able method in those. In addition, 3D graphs of the these examples are shown in Figs. 1, 2 and 3. Consider the following 2D-linear stochastic Volterra integral equations of the second kind:

Example 1

$$g(x, y) = f(x, y) + \int_0^y \int_0^x xystg(s, t)dsdt + \int_0^y \int_0^x (x + y + s + t)g(s, t)dB(s)dB(t),$$

Table 1 The numerical results to Example 1

J	$2M$	(x, y)	$\bar{g}(x, y)$	$\bar{e}(x, y)$	(L, U)	LCI
1	2	(0.12, 0.37)	0.9136	0.52231	(0.82015, 2.12111)	1.30096
		(0.62, 0.87)	1.0256	0.61874	(0.90361, 2.52031)	1.61670
2	8	(0.19, 0.31)	0.8564	0.58270	(0.36914, 1.50694)	1.13780
		(0.69, 0.94)	0.6846	0.49231	(0.53103, 1.14062)	0.60959
3	16	(0.16, 0.34)	1.6913	0.21044	(1.54109, 1.90764)	0.36655
		(0.53, 0.91)	0.7318	0.17455	(0.52810, 0.85127)	0.32317
4	32	(0.30, 0.76)	0.8456	0.22661	(0.81154, 1.20124)	0.38970
		(0.58, 0.92)	0.5962	0.09197	(0.54120, 0.71061)	0.16941

Table 2 The numerical results to Example 2

J	$2M$	(x, y)	$\bar{g}(x, y)$	$\bar{e}(x, y)$	(L, U)	LCI
1	2	(0.12, 0.37)	0.5201	0.62163	(0.48156, 1.85401)	1.37245
		(0.62, 0.87)	0.4545	0.34438	(0.41086, 1.71244)	1.30158
2	8	(0.19, 0.31)	0.4891	0.44962	(0.25116, 1.34201)	1.09085
		(0.69, 0.94)	0.5709	0.45031	(0.22164, 1.31132)	1.08963
3	16	(0.16, 0.34)	0.5228	0.16420	(0.48511, 0.92120)	0.43609
		(0.53, 0.91)	0.6814	0.18010	(0.43160, 0.87492)	0.44332
4	32	(0.30, 0.76)	0.7001	0.08460	(0.65320, 0.89418)	0.24098
		(0.58, 0.92)	0.4667	0.11020	(0.42293, 0.69409)	0.27116

Table 3 The numerical results to Example 3

J	$2M$	(x, y)	$\bar{g}(x, y)$	$\bar{e}(x, y)$	(L, U)	LCI
1	2	(0.12, 0.37)	0.6143	0.21438	(0.51617, 0.91564)	0.39947
		(0.62, 0.87)	0.2182	0.25460	(0.19851, 0.75694)	0.55843
2	8	(0.19, 0.31)	0.5149	0.19460	(0.36180, 0.50934)	0.14754
		(0.69, 0.94)	0.3939	0.18281	(0.34108, 0.58447)	0.24339
3	16	(0.16, 0.34)	0.5196	0.19400	(0.48156, 0.71601)	0.23445
		(0.53, 0.91)	0.5994	0.13327	(0.48485, 0.75462)	0.26977
4	32	(0.30, 0.76)	0.6996	0.14025	(0.66451, 0.73124)	0.06673
		(0.58, 0.92)	0.4984	0.05612	(0.44640, 0.52681)	0.08041

where

$$\begin{aligned}
 f(x, y) = & (x + y) \left[1 - \frac{1}{6}x^3y^3 - \left(xB(x) - \int_0^x B(s)ds \right) B(y) \right. \\
 & \left. - \left(yB(y) - \int_0^y B(t)dt \right) B(x) \right] - \left(x^2B(x) - 2 \int_0^x sB(s)ds \right) B(y) \\
 & - 2 \left(xB(x) - \int_0^x B(s)ds \right) \left(yB(y) - \int_0^y B(t)dt \right)
 \end{aligned}$$

Table 4 Comparison of the numerical results for HWFs method with BPFs method for Examples 1, 2 and 3

Example	J	(x, y)	Method	$\bar{e}(x, y)$	LCI
1	1	(0.37, 0.87)	BPFs	0.12101	1.23060
			HWFs	0.11569	0.85423
	3	(0.16, 0.91)	BBFs	0.01892	0.85446
			HWFs	0.21520	0.23148
	4	(0.64, 0.89)	BBFs	0.20236	0.75129
			HWFs	0.13740	0.81453
2	1	(0.37, 0.87)	BPFs	0.23001	0.46501
			HWFs	0.01475	0.15047
	3	(0.16, 0.91)	BBFs	0.11254	0.42718
			HWFs	0.23160	0.08297
	4	(0.64, 0.89)	BBFs	0.04156	0.13451
			HWFs	0.01266	0.02828
3	1	(0.37, 0.87)	BPFs	0.19007	0.25425
			HWFs	0.10105	0.36061
	3	(0.16, 0.91)	BBFs	0.00892	0.56490
			HWFs	0.05141	0.54001
	4	(0.64, 0.89)	BBFs	0.15230	0.31640
			HWFs	0.05857	0.03114

Fig. 1 Solutions of Example 1

Fig. 2 Solutions of Example 2

Fig. 3 Solutions of Example 3

$$-\left(y^2 B(y) - 2 \int_0^y t B(t) dt\right) B(x),$$

with the exact solution $g(x, y) = x + y$.

Example 2

$$g(x, y) = f(x, y) + \int_0^y \int_0^x (x + y + s + t) g(s, t) ds dt + \int_0^y \int_0^x x y s t g(s, t) dB(s) dB(t),$$

where

$$\begin{aligned} f(x, y) = & B(x)B(y) - (x + y) \int_0^y B(s) ds \int_0^x B(s) ds - y \int_0^x s B(s) ds \\ & - x \int_0^y s B(s) ds - xy \left(\frac{y}{2} B^2(y) - \frac{y^2}{4} - \frac{1}{2} \int_0^y B^2(s) ds \right) \\ & \times \left(\frac{x}{2} B^2(x) - \frac{x^2}{4} - \frac{1}{2} \int_0^x B^2(s) ds \right), \end{aligned}$$

by the exact solution $g(x, y) = B(x)B(y)$.

Example 3

$$\begin{aligned} g(x, y) = & f(x, y) + \int_0^y \int_0^x t \sqrt{x + y} \sin(s) g(s, t) ds dt \\ & + \int_0^y \int_0^x s t \cos(xy) g(s, t) dB(s) dB(t), \end{aligned}$$

where

$$\begin{aligned} f(x, y) = & (xy) e^{B(x)+B(y)} - \int_0^y \int_0^x t^2 s \sqrt{x + y} \sin(s) e^{B(s)+B(t)} ds dt \\ & - \cos(xy) \left(x e^{B(x)} - \int_0^x e^{B(s)} \left(1 + \frac{s}{2} \right) ds \right) \\ & \left(y e^{B(y)} - \int_0^y e^{B(s)} \left(1 + \frac{s}{2} \right) ds \right), \end{aligned}$$

by the exact solution $g(x, y) = (xy) e^{B(x)+B(y)}$.

Conclusion

In this paper, we developed HWFs numerical method for approximate a solution of Eq. (1). The error analysis and the numerical examples show accuracy of this method. Furthermore, the current method can be run with increasing J until the results settle down to an appropriate accuracy that it leads to solving 2^{2J+2} linear systems, that have its difficulties. Finally, this method can be extended and applied to 2D linear or non-linear multi-noise stochastic Volterra–Fredholm integral equations of the first or second kind.

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