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ORIGINAL PAPER

Generalized Class of Estimators for Population Variance Using Auxiliary Attribute

Nitesh K. Adichwal[1] · **Prayas Sharma**[1] · **Hemant K. Verma**[1] · **Rajesh Singh**[1]

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Abstract Singh and Kumar (A family of estimators of population variance using information on auxiliary attribute. Studies in sampling techniques and time series analysis, 2011) and Singh and Malik (Appl Math Comput 235:43–49, 2014) suggested some estimators for estimating the population variance using an auxiliary attribute. This paper suggests a generalized class of estimators based on the adaption of the estimator presented by Koyuncu (Appl Math Comput 218:10900–10905, 2012) for population variance using information on an auxiliary attribute in simple random sampling. The properties of the suggested class of estimators are derived and asymptotic optimum estimator identified with its properties. The large numbers of known estimators are member of the suggested generalized class and it has been shown that proposed generalized class of estimators are more efficient than usual unbiased estimator, ratio, exponential ratio and regression estimator, estimators due to Singh and Malik (Appl Math Comput 235:43–49, 2014) and Singh and Kumar (A family of estimators of population variance using information on auxiliary attribute. Studies in sampling techniques and time series analysis, 2011) using information on auxiliary attribute. In addition, theoretical results are supported by an empirical study and findings are encouraging and support the soundness of present study.

Keywords Auxiliary information · Auxiliary attribute · Simple random sampling · Bias · Mean square error

Introduction

In the sampling literature, it is well established that efficiency of the estimator of population parameters of interest can be increased by the use of auxiliary information related auxiliary variable x, which is highly correlated with study variable y. In literature of survey sampling many authors have suggested estimators based on auxiliary information. However in many

✉ Prayas Sharma
prayassharma02@gmail.com

1 Department of Statistics, Banaras Hindu University, Varanasi, India

situations of practical importance, instead of an auxiliary variable x there exist an attribute (say, φ) which is highly correlated with study variable y. In these situations by taking the advantage of point bi-serial (see [1]) correlation between the study variable y and the auxiliary attribute φ, the efficient estimators of population parameter of interest can be constructed. Several authors including [2–10] have paid their attention towards the improved estimation using auxiliary attribute.

Let us consider a sample of size n is drawn by SRSWOR from a population of size N. Further let y_i and ϕ_i denote the observations on variable y and ϕ respectively for the ith unit (i = 1,2,3,…,N). It is assumed that attribute ϕ takes only the two values 0 and 1 according as $\phi = 1$, if ith unit of the population possesses attribute $\phi = 0$, if otherwise. The variance of the usual unbiased estimator S_y^2 is given by

$$V(S_y^2) = \frac{S_y^4}{n}(\lambda_{40} - 1) \tag{1}$$

where

$$\lambda_{rs} = \frac{\mu_{rq}}{\mu_{20}^{r/2}\mu_{02}^{q/2}}, \quad \mu_{rq} = \frac{\sum_{i=1}^{N}(y_i - \overline{Y})^r(\phi_i - P)^q}{N-1}$$

In this paper a family of estimator have been proposed for the population variance S_y^2 when the auxiliary information is available in the form of attribute. For main results we confine ourselves to sampling scheme SRSWOR ignoring the finite population correction.

Estimators in Literature

In order to have an estimate of the study variable y, assuming the knowledge of the population proportion P, [11] proposed the following estimators.

$$t_1 = s_y^2 \frac{S_\phi^2}{s_\phi^2} \tag{2}$$

$$t_2 = s_y^2 + b_\phi(S_\phi^2 - s_\phi^2) \tag{3}$$

$$t_3 = s_y^2 \exp\left[\frac{S_\phi^2 - s_\phi^2}{S_\phi^2 + s_\phi^2}\right] \tag{4}$$

The MSE expression of the estimator t_1 and variance of t_2 are given, respectively, by

$$MSE(t_1) = \frac{S_y^4[(\lambda_{40} - 1) + (\lambda_{04} - 1) - 2(\lambda_{22} - 1)]}{n} \tag{5}$$

$$V(t_2) = \frac{1}{n}\left[S_y^4(\lambda_{40} - 1) + b_\phi^2 S_\phi^4(\lambda_{04} - 1) - 2b_\phi S_y^2 S_\phi^2(\lambda_{22} - 1)\right] \tag{6}$$

On differentiating (6) with respect to b_ϕ and equating to zero we obtain

$$b_\phi = \frac{S_y^2(\lambda_{22} - 1)}{S_\phi^2(\lambda_{04} - 1)} \tag{7}$$

Substituting the optimum value of b_ϕ in (6), we get the optimum variance of estimator t_2, as

$$V(t_2)_{\min} = \frac{S_y^4}{n}\left[(\lambda_{40} - 1) - \frac{(\lambda_{22} - 1)^2}{(\lambda_{04} - 1)}\right] \tag{8}$$

The MSE expression of the estimator t_3 is given by

$$MSE(t_3) = \frac{S_y^4}{n}\left[(\lambda_{40}-1) + \frac{(\lambda_{04}-1)}{4} - (\lambda_{22}-1)\right] \tag{9}$$

Following [12,13] proposed the following variance estimator using known values of some population parameters

$$t_{KC1} = s_y^2\left(\frac{S_\phi^2 + C_p}{s_\phi^2 + C_p}\right) \tag{10}$$

$$t_{KC2} = s_y^2\left(\frac{S_\phi^2 + \beta_{2\phi}}{s_\phi^2 + \beta_{2\phi}}\right) \tag{11}$$

$$t_{KC3} = s_y^2\left(\frac{S_\phi^2\beta_{2\phi} + C_p}{s_\phi^2\beta_{2\phi} + C_p}\right) \tag{12}$$

$$t_{KC4} = s_y^2\left(\frac{S_\phi^2 C_p + \beta_{2\phi}}{s_\phi^2 C_p + \beta_{2\phi}}\right) \tag{13}$$

where s_y^2 and s_ϕ^2 are unbiased estimator of population variances S_y^2 and S_ϕ^2, respectively.

To obtain the bias and MSE, we write-
$s_y^2 = S_y^2\,(1+e_0)$ and $s_\phi^2 = S_\phi^2\,(1+e_1)$.
Such that $E\,(e_0) = E\,(e_1) = 0$
and $E\left(e_0^2\right) = \frac{(\lambda_{40}-1)}{n}$, $E\left(e_1^2\right) = \frac{(\lambda_{04}-1)}{n}$, $E\,(e_0e_1) = \frac{(\lambda_{22}-1)}{n}$
and $\kappa_{pb} = \rho_{pb}\frac{C_y}{C_p}$.

The MSE expression of t_{KC_i} (i = 1,2,3,4) to the first order of approximation are respectively given by

$$MSE(t_{KC_I}) = \frac{S_y^4}{n}\left[(\lambda_{40}-1) + w_i^2(\lambda_{04}-1) - 2w_i(\lambda_{22}-1)\right],\ (i = 1, 2, 3, 4) \tag{14}$$

where

$$w_1 = \frac{S_\phi^2}{S_\phi^2 + C_p},\quad w_2 = \frac{S_\phi^2}{S_\phi^2 + \beta_{2\phi}},\quad w_3 = \frac{S_\phi^2\beta_{2\phi}}{S_\phi^2\beta_{2\phi} + C_p},\quad w_4 = \frac{S_\phi^2 C_p}{S_\phi^2 C_p + \beta_{2\phi}}$$

Following [5], Singh and Malik proposed the following variance estimator.

$$t_S = \frac{s_y^2 + b_\phi(S_\phi^2 - s_\phi^2)}{(n_1 s_\phi^2 + n_2)}(n_1 S_\phi^2 + n_2) \tag{15}$$

where $n_1,\ n_2$ are either real numbers or the functions of the known parameters of attribute such as $C_p,\ \rho_{pb},\ \beta_{2\phi}$ and κ_{pb}.

The MSE expression of t_s to the first order of approximation are respectively given by

$$\begin{aligned} MSE(t_s) = \frac{1}{n}\Big[&S_y^4(\lambda_{40}-1) + (\lambda_{04}-1)\left\{b_\phi^2 S_\phi^4 + A_1^2 S_y^4 + 2A_1 b_\phi S_y^2 S_\phi^2\right\} \\ &-2S_y^2(\lambda_{22}-1)\left\{b_\phi S_\phi^2 + A_1 S_y^2\right\}\Big] \end{aligned} \tag{16}$$

where $A_1 = \frac{n_1 S_\phi^2}{n_1 S_\phi^2 + n_2}$.

The minimum MSE of t_s is observed at $n_1 = \rho_{pb}$ and $n_2 = \beta_{2\phi}$.

Following [5], Singh and Malik proposed another improve ratio type estimator t_{rs} for the population variance as

$$t_{rs} = s_y^2 \frac{(\eta S_\phi^2 - v)}{\left[\alpha(\eta s_\phi^2 - v) + (1-\alpha)(\eta S_\phi^2 - v)\right]} \tag{17}$$

where η, v are either real numbers or the functions of the known parameters of attributes such as C_p, $\beta_{2\phi}$, ρ_{pb} and κ_{pb}.

Up to the first order approximation, the minimum MSE of t_{rs} is given by,

$$\mathrm{MSE}_{min}(t_{rs}) = \frac{S_y^4}{n}\left\{(\lambda_{40}-1) + A_2^2\alpha_0^2(\lambda_{04}-1) - 2A_2\alpha_0(\lambda_{22}-1)\right\} \tag{18}$$

where $\alpha_0 = \frac{(\lambda_{22}-1)}{A_2(\lambda_{04}-1)}$ and $A_2 = \frac{\eta S_\phi^2}{(\eta S_\phi^2 - v)}$.

Another improved class of estimator suggested by [13] is as follows

$$t_n = s_y^2\left[m_1 + m_2(S_\phi^2 - s_\phi^2)\right]\exp\left(\gamma\frac{\left[\delta S_\phi^2 + \mu\right] - \left[\delta s_\phi^2 + \mu\right]}{\left[\delta S_\phi^2 + \mu\right] + \left[\delta s_\phi^2 + \mu\right]}\right) \tag{19}$$

where γ and μ are either real numbers or function of known parameters of the auxiliary attribute ϕ such as C_p, $\beta_{2\phi}$, ρ_{pb} and κ_{pb}. The scalar γ takes value -1 and +1 for ratio and product type estimators, respectively.

The min MSE of estimator t_n up to the first order of approximation is given by,

$$\mathrm{MSE}(t_n) = S_y^4\left[1 + m_1^2R_1 + m_2^2R_2 + 2m_1m_2R_3 - 2m_1R_4 - 2m_2R_5\right] \tag{20}$$

where,

$$R_1 = 1 + \frac{1}{n}\left[(\lambda_{40}-1) + \gamma^2\theta^2(\lambda_{04}-1) + 2\gamma\left(1+\frac{\gamma}{2}\right)\theta^2(\lambda_{04}-1) - 4\gamma\theta(\lambda_{22}-1)\right]$$
$$R_2 = \frac{1}{n}S_\phi^4(\lambda_{40}-1)$$
$$R_3 = \frac{1}{n}S_\phi^4\left[2(\lambda_{22}-1) + 2\gamma\theta(\lambda_{04}-1)\right]$$
$$R_4 = 1 + \frac{1}{n}\left[\gamma\left(1+\frac{\gamma}{2}\right)\theta^2(\lambda_{04}-1) - \gamma\theta(\lambda_{22}-1)\right]$$
$$R_5 = \frac{1}{n}S_\phi^2\left[\gamma\theta(\lambda_{04}-1) - (\lambda_{22}-1)\right]$$
$$\theta = \frac{\delta S_\phi^2}{2(\delta S_\phi^2 + \mu)}$$
$$m_1 = \frac{(R_2R_4 - R_3R_5)}{(R_1R_2 - R_3^2)}$$
$$m_2 = \frac{(R_1R_5 - R_3R_4)}{(R_1R_2 - R_3^2)}$$

Table 1 Set of estimators generated from the class of estimators t_M

Subset of proposed estimator	γ	η	λ
$t_{M1}=\left(\omega_1 s_y^2+\omega_2\right)\exp\left(\frac{S_\phi^2-s_\phi^2}{S_\phi^2+s_\phi^2}\right)$	0	1	0
$t_{M2}=\left\{\omega_1 s_y^2+\omega_2\left(\frac{s_\phi^2}{S_\phi^2}\right)\right\}\exp\left(\frac{S_\phi^2-s_\phi^2}{S_\phi^2+s_\phi^2}\right)$	-1	1	0
$t_{M3}=\left\{\omega_1 s_y^2+\omega_2\left(\frac{s_\phi^2}{S_\phi^2}\right)\right\}\exp\left\{\frac{S_\phi^2-s_\phi^2}{\left(S_\phi^2+s_\phi^2\right)+2}\right\}$	-1	1	1
$t_{M4}=\left\{\omega_1 s_y^2+\omega_2\left(\frac{s_\phi^2}{S_\phi^2}\right)\right\}\exp\left\{\frac{S_\phi^2-s_\phi^2}{\left(S_\phi^2+s_\phi^2\right)+2S_\phi}\right\}$	-1	1	S_ϕ
$t_{M5}=\left\{\omega_1 s_y^2+\omega_2\left(\frac{s_\phi^2}{S_\phi^2}\right)\right\}\exp\left\{\frac{S_\phi^2-s_\phi^2}{\left(S_\phi^2+s_\phi^2\right)+2\rho_{pb}}\right\}$	-1	1	ρ_{pb}
$t_{M6}=\left\{\omega_1 s_y^2+\omega_2\left(\frac{s_\phi^2}{S_\phi^2}\right)\right\}\exp\left\{\frac{S_\phi^2-s_\phi^2}{\left(S_\phi^2+s_\phi^2\right)+2C_p}\right\}$	-1	1	C_p
$t_{M7}=\left(\omega_1 s_y^2+\omega_2\right)\exp\left\{\frac{S_\phi^2-s_\phi^2}{\left(S_\phi^2+s_\phi^2\right)+2C_p}\right\}$	0	1	C_p
$t_{M8}=\left(\omega_1 s_y^2+\omega_2\right)\exp\left\{\frac{S_\phi^2-s_\phi^2}{\left(S_\phi^2+s_\phi^2\right)+2S_\phi}\right\}$	0	1	S_ϕ
$t_{M9}=\left\{\omega_1 s_y^2+\omega_2\left(\frac{s_\phi^2}{S_\phi^2}\right)\right\}\exp\left\{\frac{\rho_{pb}\left(S_\phi^2-s_\phi^2\right)}{\rho_{pb}\left(S_\phi^2+s_\phi^2\right)+2S_\phi}\right\}$	-1	ρ_{pb}	S_ϕ
$t_{M10}=\left\{\omega_1 s_y^2+\omega_2\left(\frac{S_\phi^2}{s_\phi^2}\right)\right\}\exp\left(\frac{S_\phi^2-s_\phi^2}{S_\phi^2+s_\phi^2}\right)$	1	1	0

The Suggested Class of Estimators

Motivated by [14] we propose generalized class of estimators t_M for estimating the population variance S_y^2, as

$$t_M=\left[\omega_1 s_y^2+\omega_2\left(\frac{s_\phi^2}{S_\phi^2}\right)^{\gamma}\right]\exp\left[\frac{\eta\left(S_\phi^2-s_\phi^2\right)}{\eta\left(S_\phi^2+s_\phi^2\right)+2\lambda}\right] \tag{21}$$

where ω_1 and ω_2 $(\omega_1+\omega_2\neq 1)$ are suitable constants to be determined such that MSE of t_M is minimum, η, γ and λ are either real numbers or the functions of the known parameter associated with auxiliary attribute (See [15]).

γ and η are chosen in such a way that they generate ratio type and product type estimators for variance estimators for particular values as +1 and -1.

A set of new estimators generated from (21) using suitable values of ω_1, ω_2, γ, η and λ are listed in Table 1.

Expanding Eq. (21) in terms of e's up to the first order of approximation, we have,

$$\begin{aligned} t_M - S_y^2 &= S_y^2(\omega_1 - 1) + \omega_1 S_y^2 e_0 + \left(\omega_2 + \omega_2 \gamma e_1 + \omega_2 \frac{\gamma(\gamma-1)}{2} e_1^2\right) - \frac{1}{2}\omega_1 S_y^2 v e_1 \\ &\quad -\frac{1}{2}\omega_1 S_y^2 v e_0 e_1 - \frac{1}{2}\omega_2 v e_1 - \frac{1}{2}\omega_2 v\gamma e_1^2 + \frac{3}{8}\omega_1 S_y^2 v^2 e_1^2 + \frac{3}{8}\omega_2 v^2 e_1^2 \end{aligned} \tag{22}$$

where, $e_0 = \frac{s_y^2 - S_y^2}{S_y^2}$, $e_1 = \frac{s_\phi^2 - S_\phi^2}{S_\phi^2}$ and $v = \frac{\eta S_\phi^2}{\eta S_\phi^2 + \lambda}$.

To obtain the bias and MSE of the estimator t_M to the first degree of approximation, we write

Such that $E(e_0) = E(e_1) = 0$

Also $E(e_0) = \frac{\lambda_{04}-1}{n}$, $E(e_1) = \frac{\lambda_{40}-1}{n}$ and $E(e_0 e_1) = \frac{\lambda_{22}-1}{n}$

Taking expectation both sides of Eq. (22), we get the bias expression of estimator t_M as

$$\begin{aligned} \mathrm{Bias}(t_M) &= -S_y^2 + \omega_1 S_y^2 \left[1 - \frac{1}{2}v\frac{(\lambda_{22}-1)}{n} + \frac{3}{8}v^2\frac{(\lambda_{22}-1)}{n}\right] \\ &\quad + \omega_2\left[1 + \left\{\frac{1}{2}\gamma(\gamma-1) - \frac{1}{2}v\gamma + \frac{3}{8}v^2\right\}\frac{(\lambda_{40}-1)}{n}\right] \end{aligned} \tag{23}$$

Squaring both sides of Eq. (23) and taking expectation we get the MSE expression of estimator t_M as

$$\mathrm{MSE}\,(t_M) = \left[S_y^4 + \omega_1^2 S_y^4 A + \omega_2^2 B + \omega_1 S_y^4 D + \omega_2 S_y^2 G + \omega_1\omega_2 S_y^2 F\right] \tag{24}$$

where

$$\begin{aligned} A &= \left[1 + \frac{(\lambda_{40}-1)}{n} + v^2\frac{(\lambda_{04}-1)}{n} - 2v\frac{(\lambda_{22}-1)}{n}\right] \\ B &= \left[1 + \left\{v^2 + \gamma^2 - 2v\gamma + \gamma(\gamma-1)\right\}\frac{(\lambda_{04}-1)}{n}\right] \\ D &= \left[-2 - \frac{3}{4}v^2\left\{\frac{(\lambda_{04}-1)}{n}\right\} + v\left\{\frac{(\lambda_{22}-1)}{n}\right\}\right] \\ G &= \left[-2 + \left\{v\gamma - \frac{3}{4}v^2 - \gamma(\gamma-1)\right\}\frac{(\lambda_{04}-1)}{n}\right] \\ F &= \left[2 + \left\{2v^2 - 2v\gamma + \gamma(\gamma-1)\right\}\frac{(\lambda_{04}-1)}{n} + 2(\gamma - v)\frac{(\lambda_{22}-1)}{n}\right] \end{aligned}$$

Partially differentiating Eq. (24) with respect to ω_1 and ω_2 and equating to zero, we get the optimum value of ω_1 and ω_1 as

$$\begin{aligned} \omega_1(\mathrm{opt}) &= \left\{\frac{GF - 2BD}{4BA - F^2}\right\} \\ \omega_2(\mathrm{opt}) &= \left\{\frac{DF - 2GA}{4BA - F^2}\right\} \end{aligned}$$

Substituting the optimal values of ω_i (i = 1,2) we obtain the minimum MSE associated with t_M,

$$\mathrm{MSE}_{\min}(t_M) = S_y^4\left[1 - \frac{BD^2 - DFG + AG^2}{(4AB - F^2)}\right] \tag{25}$$

Efficiency Comparisons

We compare the efficiency of the proposed estimator t_M under optimum condition with the usual unbiased estimator, ratio estimator, exponential ratio estimator, regression estimator for variance estimation using an auxiliary attribute:

$$\begin{aligned} V(S_y^2) - \mathrm{MSE}(t_M) &= \frac{S_y^4}{n}\left[(\lambda_{40} - 1)\right] \\ &\quad - \left[S_y^4\left(1 + \omega_1^2 A + \omega_1 D\right) + S_y^2\left(\omega_2 G + \omega_1\omega_2 F\right) + \omega_2^2 B\right] \geq 0 \end{aligned} \tag{26}$$

$$\begin{aligned} \mathrm{MSE}(t_1) - \mathrm{MSE}(t_M) &= \frac{S_y^4\left[(\lambda_{40} - \lambda_{04} - 2\lambda_{22})\right]}{n} \\ &\quad - \left[S_y^4\left(1 + \omega_1^2 A + \omega_1 D\right) + S_y^2\left(\omega_2 G + \omega_1\omega_2 F\right) + \omega_2^2 B\right] \geq 0 \end{aligned} \tag{27}$$

$$\begin{aligned} \mathrm{MSE}(t_2)_{\min} - \mathrm{MSE}(t_M) &= \frac{S_y^4}{n}\left[(\lambda_{40} - 1) - \frac{(\lambda_{22} - 1)^2}{(\lambda_{04} - 1)}\right] \\ &\quad - \left[S_y^4\left(1 + \omega_1^2 A + \omega_1 D\right) + S_y^2\left(\omega_2 G + \omega_1\omega_2 F\right) + \omega_2^2 B\right] \geq 0 \end{aligned} \tag{28}$$

$$\begin{aligned} \mathrm{MSE}(t_3) - \mathrm{MSE}(t_M) &= \frac{S_y^4}{n}\left[\lambda_{40} - \lambda_{22} + \frac{(\lambda_{04} - 1)}{4}\right] \\ &\quad - \left[S_y^4\left(1 + \omega_1^2 A + \omega_1 D\right) + S_y^2\left(\omega_2 G + \omega_1\omega_2 F\right) + \omega_2^2 B\right] \geq 0 \end{aligned} \tag{29}$$

$$\begin{aligned} \mathrm{MSE}(t_{KC_I}) - \mathrm{MSE}(t_M) &= \frac{S_y^4}{n}\left[(\lambda_{40} - 1) + w_i^2(\lambda_{04} - 1) - 2w_i(\lambda_{22} - 1)\right] \\ &\quad - \left[S_y^4\left(1 + \omega_1^2 A + \omega_1 D\right) + S_y^2\left(\omega_2 G + \omega_1\omega_2 F\right) + \omega_2^2 B\right] \geq 0 \end{aligned} \tag{30}$$

$$\begin{aligned} \mathrm{MSE}(t_S) - \mathrm{MSE}(t_M) &= \frac{1}{n}\left[S_y^4(\lambda_{40} - 1) + (\lambda_{04} - 1)\left\{b_\phi^2 S_\phi^4 + A_1^2 S_y^4 + 2A_1 b_\phi S_y^2 S_\phi^2\right\}\right. \\ &\quad \left. - 2S_y^2\left\{b_\phi S_\phi^2 + A_1 S_y^2\right\}\right] \\ &\quad - \left[S_y^4\left(1 + \omega_1^2 A + \omega_1 D\right) + S_y^2\left(\omega_2 G + \omega_1\omega_2 F\right) + \omega_2^2 B\right] \geq 0 \end{aligned} \tag{31}$$

$$\begin{aligned} \mathrm{MSE}(t_{rs}) - \mathrm{MSE}(t_M) &= \frac{S_y^4}{n}\left\{(\lambda_{40} - 1) + A_2^2\alpha^2(\lambda_{04} - 1) - 2A_2\alpha(\lambda_{22} - 1)\right\} \\ &\quad - \left[S_y^4\left(1 + \omega_1^2 A + \omega_1 D\right) + S_y^2\left(\omega_2 G + \omega_1\omega_2 F\right) + \omega_2^2 B\right] \geq 0 \end{aligned} \tag{32}$$

Table 2 PRE's of various estimators w.r.t. S_y^2

Estimators	PRE's	
	Population I	Population II
S_y^2	100.00	100.00
t_1	141.89	96.41
$t_{2(min)}$	262.19	101.13
t_3	254.27	98.49
t_{KC_1}	108.84	99.40
t_{KC_2}	103.82	99.49
t_{KC_3}	155.19	99.37
t_{KC_4}	110.31	99.54
$t_{S(min)}$	118.23	100.44
$t_{rs(min)}$	262.19	101.13
$t_{n(min)}$	284.63	324.94
t_{M1}	728.53	31103.76
t_{M2}	468.79	3488.95
t_{M3}	1427.47	6411.99
t_{M4}	1032.11	5709.92
t_{M5}	1335.77	6162.57
t_{M6}	1692.00	6289.62
t_{M7}	367005.50	623329.90
t_{M8}	9342.04	277128.00
t_{M9}	14050.20	5983.66
t_{M10}	413.18	32095.56

$$
\begin{aligned}
MSE(t_n) - MSE(t_M) = {} & S_y^4\left[1 + m_1^2R_1 + m_2^2R_2 + 2m_1m_2R_3 - 2m_1R_4 - 2m_2R_5\right] \\
& - \left[S_y^4\left(1 + \omega_1^2A + \omega_1D\right) + S_y^2\left(\omega_2G + \omega_1\omega_2F\right) + \omega_2^2B\right] \geq 0
\end{aligned} \tag{33}
$$

From Eqs. (26) to (33), we conclude that the proposed estimator t_M under aforesaid conditions performs better than the other existing estimator for the same scenario discussed in this paper.

Empirical Study

In this section we compare the performance of different estimators considered in this paper using two population data sets. The description of population data sets are as follows.

Population I [Source: [16], p. 256].

y = Number of villages in the circle.
ϕ = A circle consisting more than five villages.
$N=89$, $n=23$, $S_y^2 = 4.074$, $S_\phi^2 = 0.11$, $C_y = 0.601$, $C_p = 2.678$, $\rho_{pb} = 0.766$, $\beta_{2\phi} = 6.162$, $\lambda_{22} = 3.996$, $\lambda_{40} = 3.811$, $\lambda_{04} = 6.162$.

Population II [Source: [17], p. 203].

y = Household size in each household of village.
ϕ = Household consisting size more than five.
N=35, n=15, $S_y^2 = 4.232$, $S_\phi^2 = 0.252$, $C_y = 0.346$, $C_p = 0.879$, $\rho_{pb} = 0.773$, $\beta_{2\phi} = 1.052$, $\lambda_{22} = 0.952$, $\lambda_{40} = 4.977$, $\lambda_{04} = 1.052$.

Table 2 exhibits that the PRE's of the proposed estimators including the different members of the proposed class along with the PRE's of the existing estimators with respect to S_y^2 for two real population data sets. Estimators t_{M_i} (i=1,2, …, 10) are obtained from class of estimators t_M by taking different values of η and γ and percent relative efficiency shown in the table. The highest PRE is obtained for $\gamma = 0$, $\eta = 1$ and $\lambda = C_p$ of estimator t_M. It has also been observed that the suggested class of estimators t_M under optimum condition is more efficient than the usual unbiased estimator, ratio estimator, regression estimator, [11,13] estimator and other estimators discussed in this paper. Hence for observed choice of parameters the proposed estimator t_M is best among the entire estimators considered in this paper.

Conclusion

In this article we have suggested a generalized class of estimators for the population variance of study variable y when information is available on an auxiliary attribute in simple random sampling without replacement (SRSWOR). In addition, some known estimators of population variance such as usual unbiased estimator, ratio, and exponential ratio type estimators are found to be members of the proposed generalized class of estimators. Some new members are also generated from the proposed generalized class of estimators. We have determined the biases and mean square errors of the proposed class of estimators up to the first order of approximation. The proposed generalized class of estimators is advantageous in the sense that the properties of the estimators, which are members of the proposed class of estimators, can be easily obtained from the properties of the proposed generalized class. Thus the study unifies properties of several estimators for population variance using information on an auxiliary attribute. In theoretical and empirical efficiency comparisons, it has been shown that almost all the members of the proposed generalized class of estimators are more efficient than the usual unbiased estimator, ratio, exponential ratio, regression estimator, estimators due to [11,13] and all other estimators considered here using information on an auxiliary attribute, scrupulously, estimator t_{M7} is best among all the members of generalized class in the sense of having least mean square error.

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