Int. J. Appl. Comput. Math (2016) 2:85–96
DOI 10.1007/s40819-015-0052-8

ORIGINAL PAPER

Approximate Analytical Solution of a Coupled System of Fractional Partial Differential Equations by Bernstein Polynomials

Hasib Khan[1,2] · **Mohsen Alipour**[3] · **Hossein Jafari**[4,5] · **Rahmat Ali Khan**[1]

Published online: 22 April 2015

Abstract In this paper, we produce numerical solution for a coupled system of partial differential equations of fractional order (PDEFO) by the help of Bernstein polynomials. This method reduces the coupled system of PDEFO to a system of algebraic equations which is simple in handling and gives us good results. The accuracy of the results are examined by examples.

Keywords Coupled system of fractional order PDEs · Bernstein polynomials · Operational matrices

Mathematics Subject Classification 35R11 · 76M22

Introduction

The applications of fractional calculus can be studied in many scientific disciplines based on mathematical modeling including physics, aerodynamics, chemistry, signal and image processing, economics, Economics, and many others [13,16,18–20]. The applications of

✉ Mohsen Alipour
m.alipour2323@gmail.com; m.alipour@nit.ac.ir

1 Department of Mathematics, University of Malakand, Dir Lower, Khybar Pakhtunkhwa 18000, Pakistan

2 Shaheed Benazir Bhutto University, Sheringal, Dir Upper, Khybar Pakhtunkhwa 18000, Pakistan

3 Department of Mathematics, Faculty of Basic Science, Babol University of Technology, P.O. Box 47148-71167, Babol, Iran

4 Department of Mathematics and Computer Science, University of Mazandaran, P.O. Box 47416-95447, Babolsar, Iran

5 Department of Mathematical Sciences, University of South Africa, P.O.Box 392, UNISA 0003, South Africa

fractional calculus have caught the attention of scientists in different disciplines round the world, see for examples [1–7,9,11,12,14,15,17,21,22].

The numerical solutions of PDEFO is one of the important and hot research area which have got a good attention of the scientists round the world [7–9,11,12,14,17,21,22]. We have been influenced by the recent contributions of the scientists [1,4,6,15,21,22] and have aimed to get the numerical solution of the following PDEFO by the help of operational matrices of BPs:

$$\begin{cases} K_1 \frac{\partial^{\theta_1} U(x,t)}{\partial x^{\theta_1}} + K_2 \frac{\partial^{\theta_2} V(x,t)}{\partial t^{\theta_2}} = 0, \\ K_3 \frac{\partial^{\theta_2} V(x,t)}{\partial x^{\theta_2}} + K_4 \frac{\partial^{\theta_1} U(x,t)}{\partial t^{\theta_1}} = 0, \end{cases} \tag{1}$$

with initial conditions

$$U(0,t) = f(t), \quad V(0,t) = g(t).$$

where $x, t \in [0, 1]$, $\theta_1, \theta_2 \in (0, 1]$ and K_1, K_2, K_3, K_4, are constants.

In Eq. (2), $D^{\alpha} f(x)$ is the Caputo fractional derivative [7]:

$$D^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{1+\alpha-n}} dt, & n-1 < \alpha < n, \quad n \in \mathbb{N}, \\ \frac{d^n}{dx^n} f(x), & \alpha = n. \end{cases} \tag{2}$$

We give some properties of fractional derivative and integral from the available resources in [18–20]:

$$\text{(i)} \quad D^{\alpha} C = 0, \qquad (C \textit{ is a constant}),$$

$$\text{(ii)} \quad D^{\alpha} x^{\beta} = \begin{cases} 0, & \beta \in \mathbb{N}_0, \ \beta < \lceil \alpha \rceil \\ \frac{\Gamma(\beta+1)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, & \beta \in \mathbb{N}_0, \ \beta \geq \lceil \alpha \rceil \ or \ \beta \notin \mathbb{N}_0, \ \beta > \lfloor \alpha \rfloor \end{cases}, \tag{3}$$

$$\text{(iii)} \quad I^{\alpha} D^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!}, \qquad n-1 < \alpha \leq n, \tag{4}$$

where I^{α} is the fractional Riemann–Liouville integral:

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad \alpha > 0. \tag{5}$$

Organization of our Paper In "Bernstein Polynomials and Their Properties" section, we present BPs and approximation of functions via BPs. In "Operational Matrix for Fractional Order Derivative of BPs" section, we describe about the operational matrices for fractional order of BPs. In "Solution of the Coupled System by BPs" section, we develop a numerical scheme for the coupled system (1). In "Illustrative Examples" section, Illustrative examples are given in order to demonstrate the effectiveness and accuracy of our method.

Bernstein Polynomials and Their Properties

Definition of Bernstein Polynomials

We considered the Bernstein polynomials of the mth degree on the interval on [0, 1].

We recall that [9]:

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \quad 0 \le i \le m. \tag{6}$$

The following Bernstein polynomials satisfy recursive definition:

$$B_{i,m}(x) = (1-x)B_{i,m-1}(x) + xB_{i-1,m-1}(x), \qquad i = 0, 1 \ldots, m. \tag{7}$$

By using the binomial expansion of $(1-x)^{m-i}$, Bernstein polynomials can be expressed in terms of linear combination of the basis functions

$$\begin{aligned} B_{i,m}(x) &= \binom{m}{i} x^i (1-x)^{m-i} = \binom{m}{i} x^i \left(\sum_{k=0}^{m-i} (-1)^k \binom{m-i}{k} x^k \right) \\ &= \sum_{k=0}^{m-i} (-1)^k \binom{m}{i} \binom{m-i}{k} x^{i+k}, \qquad i = 0, 1, \ldots, m. \end{aligned} \tag{8}$$

We can show the Bernstein polynomials by $B_{i,m}(x) = A_{i+1} T_m(x)$, for $i = 0, 1, \ldots, m$, where

$$A_{i+1} = \left[0, 0, \ldots, 0, (-1)^0 \binom{m}{i}, (-1)^1 \binom{m}{i} \binom{m-i}{1}, \ldots, (-1)^{m-i} \binom{m}{i} \binom{m-i}{m-i} \right],$$

and

$$T_m(x) = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^m \end{bmatrix}.$$

Now if we introduce a $(m+1) \times (m+1)$ matrix A in the form:

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{m+1} \end{bmatrix},$$

where A_i for $i = 1, 2, \ldots, m+1$ represent rows of the square matrix A. Then we have

$$\Phi(x) = AT_m(x), \tag{9}$$

where $\Phi(x) = \left[B_{0,m}(x), B_{1,m}(x), \ldots, B_{m,m}(x) \right]^T$ and matrix A is an upper triangular matrix given by:

$$A = \begin{bmatrix} (-1)^0 \binom{m}{0} & (-1)^1 \binom{m}{0} \binom{m-0}{1-0} & \ldots & (-1)^{m-0} \binom{m}{0} \binom{m-0}{m-0} \\ \ddots & \vdots & \vdots & \vdots \\ 0 & (-1)^0 \binom{m}{i} & \ldots & (-1)^{m-i} \binom{m}{i} \binom{m-i}{m-i} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & (-1)^0 \binom{m}{m} \end{bmatrix},$$

and $|A| = \Pi_{i=0}^{m} \binom{m}{i} \neq 0$, so A is an invertible matrix.

Approximation of Functions by Bernstein Polynomials

A square integrable function $F(x)$ in the interval [0, 1] can be approximated in terms of the basis of BPs $\{B_{0,m}(x), B_{1,m}(x), \ldots, B_{m,m}(x)\}$. Therefore, we have:

$$F(x) \approx \sum_{i=0}^{m} c_i B_{i,m}(x) = c^T \Phi(x), \tag{10}$$

where $c^T = [c_0, c_1, \ldots, c_m]$, $\Phi^T(x) = \left[B_{0,m}(x), B_{1,m}(x), \ldots, B_{m,m}(x)\right]$ and c^T can be obtained from (10) by the relation

$$c^T \left\langle \Phi(x), \Phi(x) \right\rangle = \left\langle F(x), \Phi(x) \right\rangle, \tag{11}$$

where

$$\left\langle F(x), \Phi(x) \right\rangle = \int_0^1 F(x)\Phi(x)^T dx = \left[\left\langle F(x), B_{0,m} \right\rangle, \left\langle F(x), B_{1,m}(x) \right\rangle, \ldots, \left\langle F(x), B_{m,m}(x) \right\rangle\right], \tag{12}$$

and $\left\langle \Phi(x), \Phi(x) \right\rangle = Q$ is known by dual matrix of $\Phi(x)$. For the dual matrix we have the following relation:

$$Q = \left\langle \Phi(x), \Phi(x) \right\rangle = \int_0^1 \Phi(x)\Phi(x)^T dx, \tag{13}$$

where Q is a symmetric $(m+1) \times (m+1)$ and is an invertible matrix whose entries are given by the following relation

$$\begin{aligned}
(Q)_{i+1,j+1} &= \int_0^1 B_{i,m}(x) B_{j,m}(x) dx \\
&= \binom{m}{i}\binom{m}{j} \int_0^1 (1-x)^{2m-(i+j)} x^{i+j} dx \\
&= \frac{\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}} \qquad i, j = 0, 1, \ldots, m.
\end{aligned}$$

ultimately by (11) and (13), we get

$$c^T = \left(\int_0^1 f(x)\phi(x)^T dx \right) Q^{-1}. \tag{14}$$

We can also approximate function $F(x,t) \in L^2([0,1] \times [0,1])$ by BPs. The approximation can be carried out in this way:

$$F(x,t) = \sum_{j=0}^{m} \sum_{i=0}^{m} c_{ij} B_{i,m}(x) B_{j,m}(t) = \Phi^T(x) C \Phi(t), \tag{15}$$

where C is an $(m+1) \times (m+1)$ matrix given by

$$C = \begin{bmatrix} c_{00} & c_{01} & \ldots & c_{0m} \\ c_{10} & c_{11} & \ldots & c_{1m} \\ \vdots & \ddots & \ddots & \vdots \\ c_{m0} & c_{m1} & \ldots & c_{mm} \end{bmatrix},$$

where C can be obtained from (15) by the relation

$$C = Q^{-1}\left(\int_0^1 \int_0^1 F(x,t)\Phi(x)\Phi^T(t)dxdt\right)Q^{-1}. \tag{16}$$

Operational Matrix for Fractional Order Derivative of BPs

In this section we get operational matrix for fractional order derivative which is followed by the definition of fractional order derivative in Caputo's sense that is for $0 \le t \le 1$

$$D^\alpha \Phi(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\mu)^{n-\alpha-1}\Phi^{(n)}(\mu)d\mu = \frac{1}{\Gamma(n-\alpha)}t^{n-\alpha-1} * \Phi^{(n)}(t), \tag{17}$$

where the operator $*$ denotes the convolution product and by (9) and (17) we can get

$$D^\alpha \Phi(t) = A\frac{1}{\Gamma(n-\alpha)}\left(t^{n-\alpha-1} * T_m^{(n)}(t)\right) = AD^\alpha T_m(t) = A\left[D^\alpha 1, D^\alpha t, \ldots, D^\alpha t^m\right]^T. \tag{18}$$

where $D^\alpha t^i$ is as defined in (3). Therefore we can write,

$$D^\alpha T_m(t) = \mathbf{H}\overline{T}, \tag{19}$$

where $\mathbf{H}$ is a square diagonal matrix of order $(m+1)\times(m+1)$ and $\overline{T}$ is a row matrix of order $(m+1)\times 1$. The entries of these matrices are given by

$$\mathbf{H} = \begin{cases} \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} & i, j = \lceil\alpha\rceil, \ldots, m \text{ and } i = j, \\ 0 & \text{otherwise,} \end{cases} \tag{20}$$

and

$$\overline{T} = \begin{cases} 0 & i = 0, \ldots, \lceil\alpha\rceil - 1, \\ t^{i-\alpha} & i = \lceil\alpha\rceil, \ldots, m. \end{cases} \tag{21}$$

Now we approximate $t^{i-\alpha}$ for $i = \lceil\alpha\rceil, \ldots, m$ with BPs by using (10) as

$$t^{i-\alpha} \approx W_i^T \Phi_m(t), \tag{22}$$

where W_i is a row matrix of order $(m+1)\times 1$ and can be obtained by the relation

$$W_i = Q^{-1}\left(\int_0^1 t^{i-\alpha}\Phi_m(t)dt\right) = Q^{-1}\overline{W_i}, \tag{23}$$

where $\overline{W_i} = \left[\overline{W}_{i,0}, \overline{W}_{i,1}, \ldots, \overline{W}_{i,m}\right]^T$ and

$$\overline{W}_{i,j} = \int_0^1 t^{i-\alpha}B_{j,m}(t)dt = \frac{m!\Gamma(j+i-\alpha+1)}{j!\Gamma(i+m-\alpha+2)}, \tag{24}$$

where $i = \lceil\alpha\rceil, \ldots, m$ and $j = 0, 1, \ldots, m$. Now we can assume a square matrix V of order $(m+1)\times(m+1)$ having zero vectors at first $\lceil\alpha\rceil$ columns and W_i in its ith columns, for $i = \lceil\alpha\rceil + 1, \ldots, m$. Consequently we have

$$D^\alpha \Phi_m(t) \approx D_\alpha \Phi_m(t), \tag{25}$$

where

$$D_\alpha \approx A\mathbf{H}V^T, \tag{26}$$

is called the operational matrix of fractional order derivative for BPs.

Solution of the Coupled System by BPs

In this section we develop a numerical scheme for the Coupled System of PDEFO (1) based on BPs. By the use of (15), we have

$$U(x,t) = \Phi^T(x)C_1\Phi(t), \tag{27}$$
$$V(x,t) = \Phi^T(x)C_2\Phi(t), \tag{28}$$

by the use of (25) and (27) ,(28) we have

$$\frac{\partial^{\theta_1}}{\partial x^{\theta_1}}U(x,t) \approx \left(\frac{\partial^{\theta_1}}{\partial x^{\theta_1}}\Phi(x)\right)^T C_1\Phi(t) \approx (D_{\theta_1}\Phi(x))^T C_1\Phi(t) = \Phi(x)^T D_{\theta_1}^T C_1\Phi(t), \tag{29}$$

$$\frac{\partial^{\theta_2}}{\partial x^{\theta_2}}V(x,t) \approx \left(\frac{\partial^{\theta_2}}{\partial x^{\theta_2}}\Phi(x)\right)^T C_2\Phi(t) \approx (D_{\theta_2}\Phi(x))^T C_2\Phi(t) = \Phi(x)^T D_{\theta_2}^T C_2\Phi(t), \tag{30}$$

$$\frac{\partial^{\theta_1}}{\partial t^{\theta_1}}U(x,t) \approx \Phi(x)^T C_1 D_{\theta_1}\Phi(t), \tag{31}$$

$$\frac{\partial^{\theta_2}}{\partial t^{\theta_2}}V(x,t) \approx \Phi(x)^T C_2 D_{\theta_2}\Phi(t), \tag{32}$$

and

$$U(0,t) = f(t) \approx F^T\Phi(t), \tag{33}$$
$$V(0,t) = g(t) \approx G^T\Phi(t), \tag{34}$$

by the use of (29)–(34), in (1) we have

$$\begin{aligned} K_1\Phi(x)^T D_{\theta_1}^T C_1\Phi(t) + K_2\Phi(x)^T C_2 D_{\theta_2}\Phi(t) = 0,\\ K_3\Phi(x)^T D_{\theta_2}^T C_2\Phi(t) + K_4\Phi(x)^T C_1 D_{\theta_1}\Phi(t) = 0, \end{aligned} \tag{35}$$

with initial conditions

$$\Phi(0)^T C_1\Phi(t) = F^T\Phi(t), \tag{36}$$
$$\Phi(0)^T C_2\Phi(t) = G^T\Phi(t). \tag{37}$$

Also, by using Tau method [10] we can generate algebraic equations from (35) to (37) as follows

$$\begin{aligned} \int_0^1\int_0^1 B_i(x)\Phi(x)^T\left(K_1D_{\theta_1}^T C_1 + K_2C_2D_{\theta_2}\right)\Phi(t)B_j(t)dxdt = 0,\\ \int_0^1\int_0^1 B_i(x)\Phi(x)^T\left(K_3D_{\theta_2}^T C_2 + K_4C_1D_{\theta_1}\right)\Phi(t)B_j(t)dxdt = 0,\\ \left(\Phi(0)^T C_1 - F^T\right)Q = 0,\\ \left(\Phi(0)^T C_2 - G^T\right)Q = 0, \end{aligned} \tag{38}$$

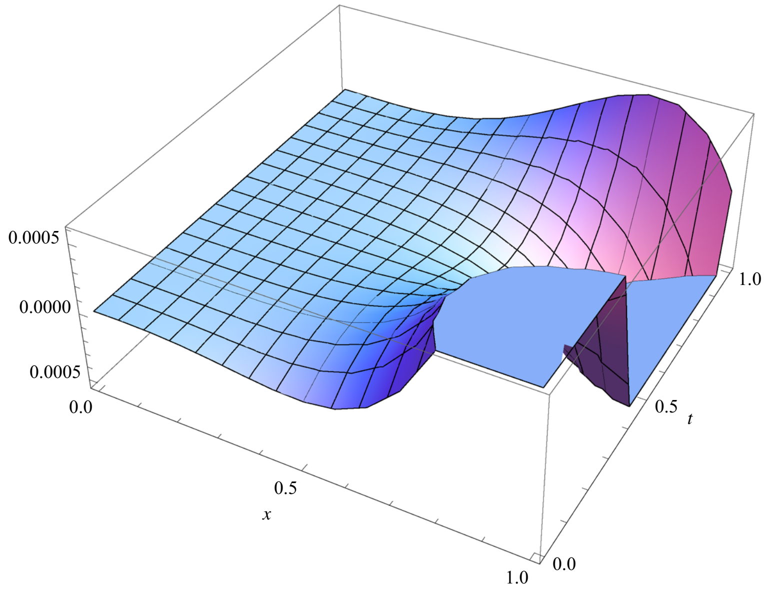

Fig. 1 Error in our approximation for $U(x, t)$, with $m = 5$ and $\theta_1, \theta_2 = 1$

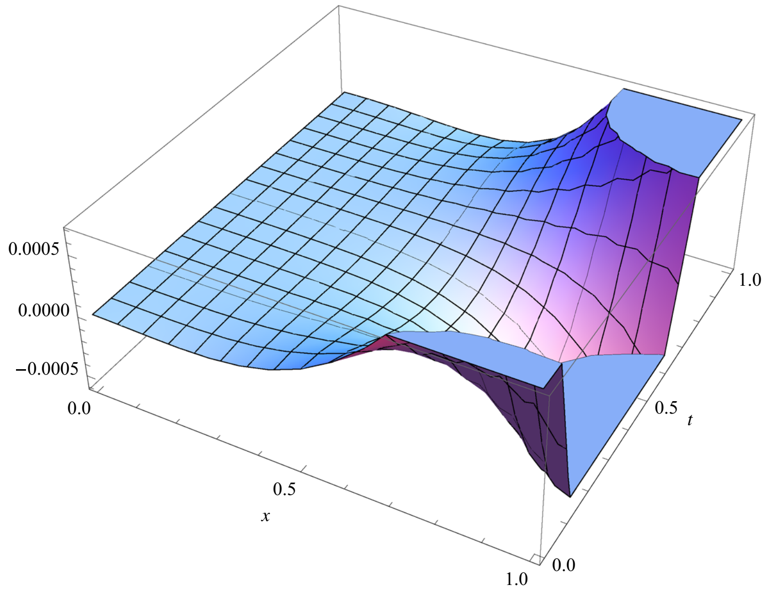

Fig. 2 Error in our approximation for $V(x, t)$, with $m = 5$ and $\theta_1, \theta_2 = 1$

where $i = 0, 1, \ldots, m-1$ and $j = 0, 1, 2, \ldots, m$. Solving this system of algebraic equations for the unknown C_1, C_2 and putting in (27), (28) giving our required approximations.

Illustrative Examples

In this section, the applications of the prescribed technique are presented.

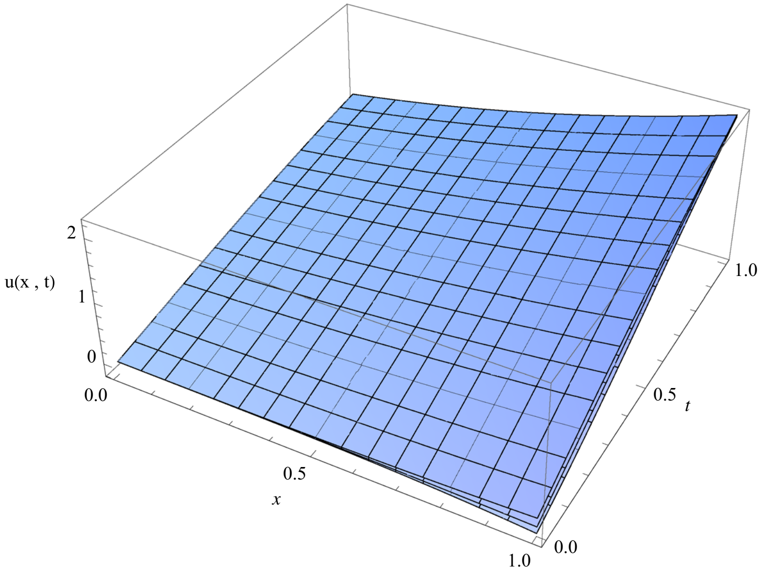

Fig. 3 Plot of the approximation solutions $u(x, t)$, with $m = 3$, $\theta_1 = 1$ and $\theta_2 = 0.8, 0.9, 1$

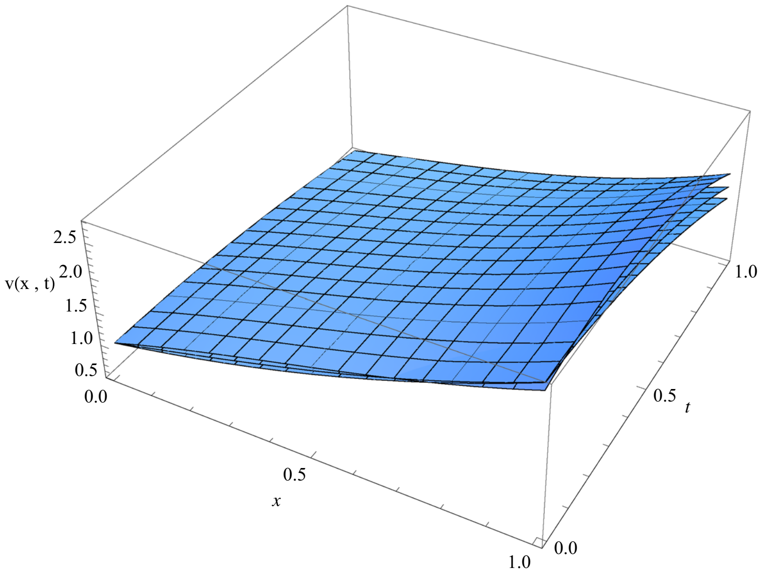

Fig. 4 Plot of the approximation solutions $v(x, t)$, with $m = 3$, $\theta_1 = 1$ and $\theta_2 = 0.8, 0.9, 1$

Example 1 Here we consider our coupled system with initial conditions $f(t) = \sin(t)$, $g(t) = \cos(t)$ and the constants $K_1 = 1$, $K_2 = 1$, $K_3 = 1$, $K_4 = -1$. Since with $\theta_1 = \theta_2 = 1$ our system has the exact solutions for $U(x, t) = e^x \sin(t)$ and $V(x, t) = e^x \cos(t)$. In Figs. 1 and 2, we show the errors of obtained results for $m = 5$ and $\theta_1 = \theta_2 = 1$. So, we can see the approximate solutions are good agreement with analytical solutions. In [15], this example has been solved and errors in their approximate solutions for U and V have been recorded

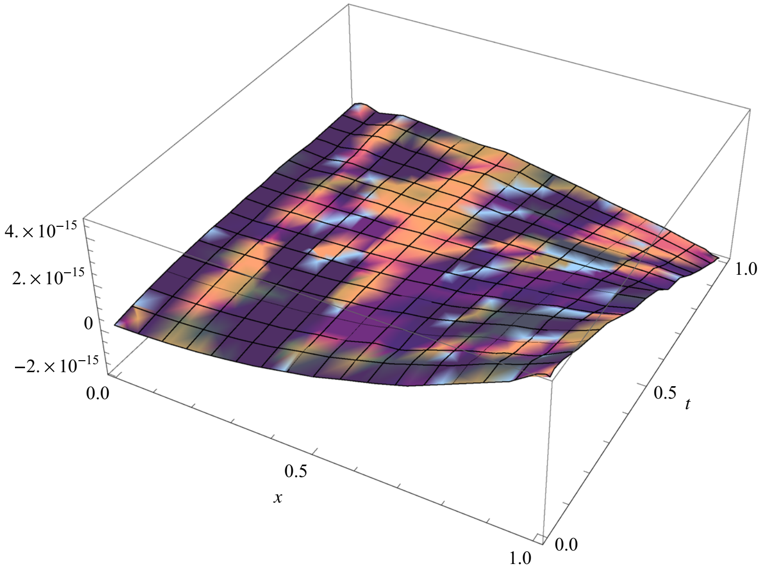

Fig. 5 Error in our approximation for $U(x,t)$, with $m = 3$ and $\theta_1, \theta_2 = 1$

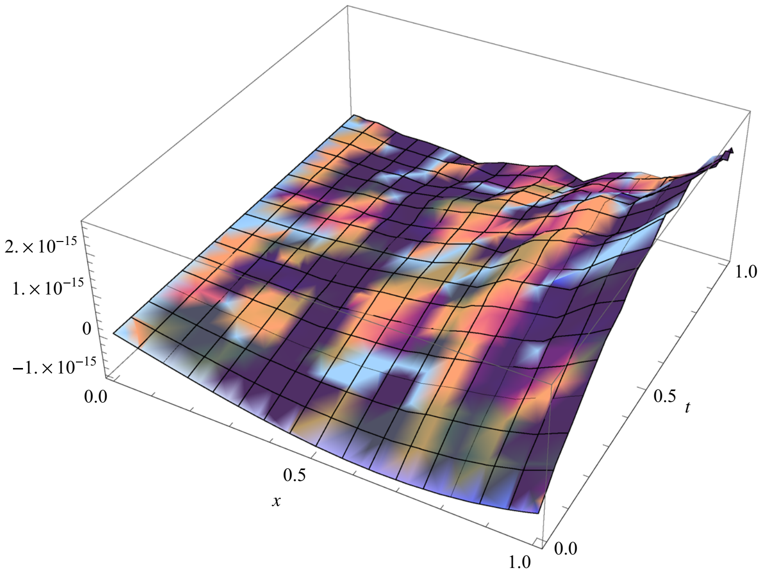

Fig. 6 Error in our approximation for $V(x,t)$, with $m = 3$ and $\theta_1, \theta_2 = 1$

by 10^{-3} with $m = 5$. While in Our results which are based on Bernstein Polynomial, are better as compared to [15]. In present work, the errors in the approximate solutions for U, V are rounded by 10^{-4} for the same m. Also, Figs. 3 and 4 show the approximate solutions for $\theta_1 = 1, \theta_2 = 0.8, 0.9, 1$ and $m = 3$. These figures show that for fixed $\theta_1 = 1$, as θ_2 approaches close to 1, the approximate solutions approach to the solutions for $\theta_2 = 1$ as expected.

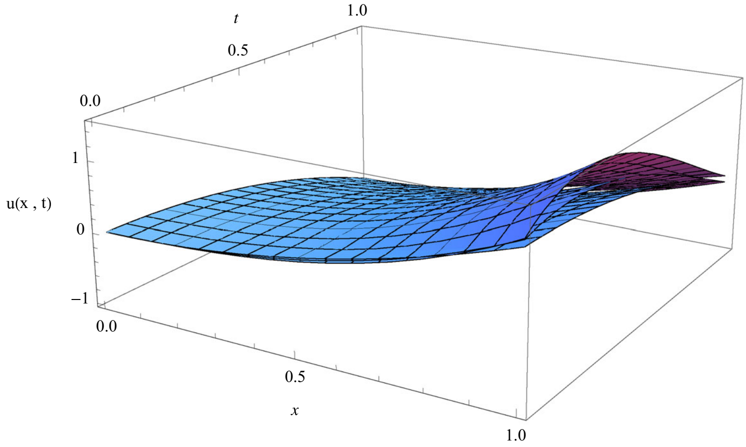

Fig. 7 Plot of the approximation solutions $u(x, t)$, with $m = 3$, $\theta_1 = 1$ and $\theta_2 = 0.8, 0.9, 1$

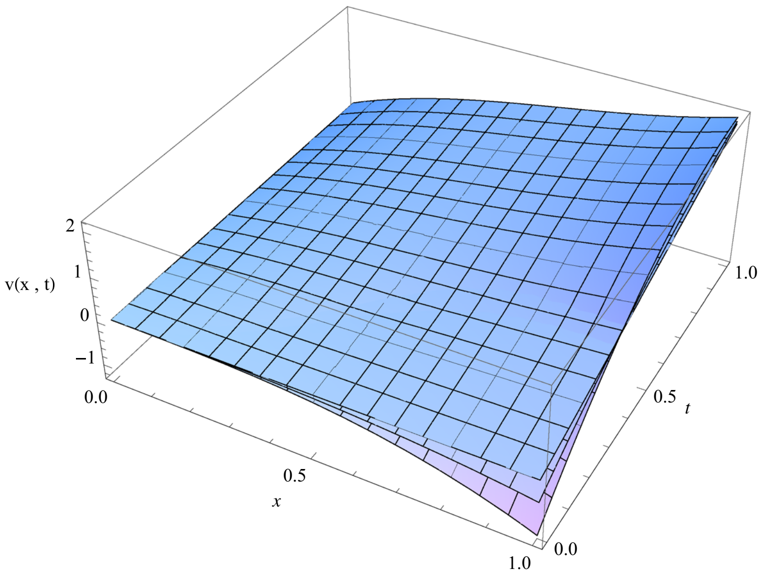

Fig. 8 Plot of the approximation solutions $v(x, t)$, with $m = 3$, $\theta_1 = 1$ and $\theta_2 = 0.8, 0.9, 1$

Example 2 Here we consider our coupled system with initial conditions $f(t) = -t^2$, $g(t) = 0$ and the constants $K_1 = 1$, $K_2 = -1$, $K_3 = 1$, $K_4 = 1$. Since with $\theta_1 = \theta_2 = 1$ our system has the exact solutions for $U(x, t) = x^2 - t^2$ and $V(x, t) = 2xt$. The errors of approximate solutions with $m = 3$ and $\theta_1 = \theta_2 = 1$ are shown in Figs. 5 and 6. The results show the approximate solutions have high accuracy. In [15], this example has been solved and errors in their approximate solutions for U and V have been approximated by 10^{-3}. While in Our results which are based on Bernstein Polynomial, are too better as compared to [15]. In present work, the errors in the approximate solutions for U, V are rounded by 10^{-15}. Also, Figs. 7 and 8 show the approximate solutions for $\theta_1 = 1$, $\theta_2 = 0.8, 0.9, 1$ and $m = 3$.

Similar to previous example, These figures show that for $\theta_1 = 1$, as θ_2 approaches close to 1, the approximate solutions approach to the solutions for $\theta_2 = 1$ as expected.

Conclusion

In this research work, we have produced a more efficient scheme for the approximate solution of a coupled system of PDEFO via operational matrices for fractional order based on BPs. In this method, we get the problem to a system of algebraic equations that can be solved easily. Numerical examples are simulated to demonstrate the high performance of the proposed method. We observed that the results are in good agreement with the exact solution with a low number of approximating term. Also, we saw that the solutions approach to the solutions for problems as the order of the fractional derivative approaches to 1, for fixed m. This procedure can be applied to many other linear and nonlinear problems in PDEFO. For the simulations we used the software Mathematica.

References

1. Alipour, M., Beghin, L., Rostamy, D.: Generalized fractional nonlinear birth processes. Methodol. Comput. Appl. Probab. (2013). doi:10.1007/s11009-013-9369-0
2. Alipour, M., Rostamy, D.: Bernstein polynomials for solving Abel's integral equation. J. Math. Comput. Sci. **3**(4), 403–412 (2011)
3. Alipour, M., Rostamy, D.: Solving nonlinear fractional differential equations by Bernstein polynomials operational matrices. J. Math. Comput. Sci. **5**(3), 185–196 (2012)
4. Alipour, M., Rostamy, D., Baleanu, D.: Solving multidimensional FOCPs with inequality constraint by BPs operational matrices. J. Vib. Control (2012). doi:10.1177/1077546312458308
5. Bagley, R.L., Torvik, P.J.: A theoretical basis for the application of fractional calculus to viscoelasticity. J. Rheol. **27**, 201–210 (1983)
6. Baleanu, D., Alipour, M., Jafari, H.: The Bernstein operational matrices for solving the fractional quadratic Riccati differential equations with the Riemann–Liouville derivative. Abstract and Applied Analysis, vol. 2013. Hindawi Publishing Corporation, Cairo (2013). doi:10.1155/2013/461970. Article ID 461970
7. Baleanu, D., Diethelm, K., Scalas, E., Trujillo, J.J.: Fractional Calculus Models and Numerical Methods. Series on Complexity Nonlinearity and Chaos. World Scientific, Singapore (2012)
8. Baleanu, D., Srivastava, H.M., Yang, X.J.: Local fractional variational iteration algorithms for the parabolic Fokker–Planck equation defined on cantor sets. Prog. Fract. Differ. Appl. **1**(1), 1–10 (2015)
9. Bhatti, M., Bracken, P.: Solutions of differential equations in a Bernstein polynomial basis. J. Comput. Appl. Math. **205**, 272–280 (2007)
10. Canuto, C., Hussaini, M.Y., Quarteroni, A., Zang, T.A.: Spectral Methods in Fluid Dynamic. Prentice-Hall, Englewood Cliffs (1988)
11. Chow, T.S.: Fractional dynamics of interfaces between soft-nanoparticles and rough substrates. Phys. Lett. A **342**, 148–155 (2005)
12. Doha, E.H., Bhrawy, A.H., Saker, M.A.: Integrals of Bernstein polynomials: an application for the solution of high even-order differential equations. Appl. Math. Lett. **24**, 559–565 (2011)
13. Hilfer, R. (ed.): Application of Fractional Calculus in Physics. World scientific publishing Co., Singapore (2000)
14. Jafari, H., Sayevand, K., Tajadodi, H., Baleanu, D.: Homotopy analysis method for solving Abel differential equation of fractional order. Cent. Eur. J. Phys. **11**(10), 1523–1527 (2013)
15. Khan, R.A., Khalil, H.: A new method based on legendre polynomials for solution of system of fractional order differential equations. Int. J. Comput. Math. (2014). doi:10.1080/00207160.2014.880781
16. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations, vol. 24. North-Holland Mathematics Studies, Amsterdam (2006)
17. Luchko, A., Groneflo, R.: The initial value problem for some fractional differential equations with the Caputo derivative, Preprint series A08–98. Fachbreich Mathematik und Informatik, Freic Universitat Berlin (1998)

18. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
19. Oldhalm, K.B., Spainer, J.: The Fractional Calculus. Academic Press, New York (1974)
20. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1999)
21. Rostamy, D., Alipour, M., Jafari, H., Baleanu, D.: Solving multi-term orders fractional differential equations by operational matrices of BPs with convergence analysis. Rom. Rep. Phys. **65**(2), 334–349 (2013)
22. Yousefi, S.A., Behroozifar, M.: Operational matrices of Bernstein polynomials and their applications. Int. J. Syst. Sci. **41**(6), 709–716 (2010)