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ORIGINAL PAPER

Analytical Treatment for Solving a Class of Lane–Emden Equations

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Abstract Analytical and numerical results are reported on the approximate solution of the Lane–Emden equation by Adomian decomposition method. Considering only a few terms of the series solution, the result has been compared with the exact solution for a particular type of nonlinearity. The present method performs extremely well in terms of accuracy, efficiency and simplicity.

Keywords Lane–Emden equation · ADM · Nonlinearity

1 Introduction

Recently, a great deal of interest has been focused on Adomian's decomposition method (ADM) and its applications to a wide class of physical problems [1–5]. The decomposition method employed here is adequately discussed in the published literature [6,7], but it still deserves emphasis to point out the very significant advantages over other methods. The said method can also be an effective procedure for the analytical solution of Lane–Emden equation.

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The solution obtained by this method is compared with that of exact solution derived from a particular type of nonlinearity.

First of all, Adomian decomposition method (ADM) will be discussed as given in [3,6,7] and for the sake of convergence, the analysis given in [9,10] will be followed.

1.1 The Decomposition Method

Let us discuss a brief outline of the Adomian decomposition method, in general. For this, let us consider an equation in the form

$$Lu + Ru + Nu = g, \tag{1.1}$$

where L is an easily or trivially invertible linear operator, R is the remaining linear part and N represents a non-linear operator.

The general solution of the given equation is decomposed into sum

$$u = \sum_{n=0}^{\infty} u_n, \tag{1.2}$$

Where u_0 is the complete solution of $Lu = g$. From Eq. (1.1), one can write

$$Lu = g - Ru - Nu.$$

Because L is invertible, an equivalent expression is

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu.$$

For initial value problems, we conveniently define L^{-1} for $L = \frac{d^n}{dt^n}$ as the n-fold definite integration operator from 0 to t.

For the operator $L = \frac{d^2}{dt^2}$, for example, we have, $L^{-1}Lu = u - u(0) - tu'(0)$ and therefore,

$$u = u(0) + tu'(0) + L^{-1}g - L^{-1}Ru - L^{-1}Nu \tag{1.3}$$

For boundary value problems (and, if described, for initial value problems as well), if indefinite integrations are used, considered constants are evaluated from the given conditions. Solving for u yields

$$u = A + Bt + L^{-1}g - L^{-1}Ru - L^{-1}Nu \tag{1.4}$$

The first three terms in Eq. (1.3) or (1.4) are identified as u_0 in the assumed decomposition method $u = \sum_{n=0}^{\infty} u_n$.

Finally, assuming Nu is analytic, we write $Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, ..., u_n)$, where A_n are special set of polynomials obtained for the particular non-linearity $Nu = f(u)$ and were generated by Adomian [3,6,7]. These A_n polynomials depend, of course, on the particular non-linearity. The A_n's are given as below:

$$
\begin{aligned}
A_0 &= f(u_0),\\
A_1 &= u_1 \frac{d}{du_0} f(u_0),\\
A_2 &= u_2 \frac{d}{du_0} f(u_0) + \frac{u_1^2}{2!}\frac{d^2}{du_0^2} f(u_0),\\
A_3 &= u_3 \frac{d}{du_0} f(u_0) + u_1 u_2 \frac{d^2}{du_0^2} f(u_0) + \frac{u_3^3}{3!}\frac{d^3}{du_0^3} f(u_0), \ldots
\end{aligned}
$$

and can be found from the formula (for $n \geq 1$).

$$A_n = \sum_{i=1}^{\infty} C(i, n) f^i(u_0), \tag{1.5}$$

where the $C(i, n)$ are products (or sums or products) of i components of u, where subscripts sum to n, divided by the factorial of the number of repeated subscripts [3,6,7].

Recently, the Adomian decomposition method (ADM) is reviewed and a mathematical model of Adomian polynomials is introduced [6,7].

Therefore, the general solution becomes

$$u = u_0 - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}Nu, \tag{1.6}$$

$$= u_0 - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n \tag{1.7}$$

where

$$u_0 = \Phi + L^{-1}g \quad \text{and} \quad L\Phi = 0, \tag{1.8}$$

so that

$$u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n, \quad n \geq 0 \tag{1.9}$$

Using the known u_0, all components $u_1, u_2, \ldots, u_n, \ldots$ are determinable by using Eq. (1.9). Substituting these $u_0, u_1, u_2, \ldots, u_n, \ldots$ etc. in Eq. (1.2), u is obtained completely.

Convergence of this method has been rigorously established by Cherruault [8], Abbaoui and Cherruault [9] and Himoun et al. [10].

2 Analysis

The Lane–Emden equation is one of the basic equations in the theory of stellar structure and has been the focus of many studies. This non-linear differential equation describes the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of classical thermodynamics [11–13]. It also describes the variation of density as a function of the radial distance for a polytrope [14]. The Lane–Emden equation has the general form

$$y'' + \left(\frac{2}{x}\right) y' + f(y) = 0, \quad 0 < x \leq 1$$

subject to the initial conditions $y(0) = 1$, $y'(0) = 0$,
where the prime ($'$) denotes differentiation with respect to the independent variable x.

2.1 Case-1 ($f(y) = y^\alpha$)

The Lane–Emden equation has the form [15]

$$y'' + \left(\frac{2}{x}\right) y' + y^\alpha = 0, \quad 0 < x \leq 1$$
$$y(0) = 1, \quad y'(0) = 0 \tag{2.1}$$

In Eq. (2.1), y and x are dimensionless variables and α is an index related to the ratio of specific heats of the gases comprising the star. Hence the parameter α corresponds to the particular choice of the equation of state.

We first introduce the following change of variables $x = z, u = zy = xy$ into Eq. (2.1), and it follows that

$$\frac{d^2u}{dz^2} + z^{1-\alpha}u^\alpha = 0$$
$$u(0) = 0, \quad \left.\frac{du}{dz}\right|_{z=0} = 1 \tag{2.2}$$

The Eq. (2.2) can be rewritten in an operator form as [6,7,12]

$$Lu + z^{1-\alpha}Nu = 0, \tag{2.3}$$

where L is the differential operator $\frac{d^2}{dz^2}$ and $N(u)$ represents the non-linear term u^α. Clearly, L is invertible and L^{-1} means a two-fold integral with respect to z. Generally, this choice of the highest order derivative for L is the most desirable, because the integrations are the simplest [6,7].

Operating with L^{-1} in (2.3), we get

$$u = u_0 - L^{-1}[z^{1-\alpha}Nu], \tag{2.4}$$

where u_0 is the solution of the equation $Lu = 0$ with the given initial conditions.

Assuming the non-linear term Nu is analytic, this is decomposed as

$$Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, ...u_n), \tag{2.5}$$

where A_n's are special set of polynomials, called Adomian's polynomials, and can be calculated for all types of non-linearity according to specific algorithms constructed in [6,7].

The Adomian decomposition method (ADM) assumes a series solution for

$$u(z) = \sum_{n=0}^{\infty} u_n(z) \tag{2.6}$$

The decomposition series (2.6) can be obtained from (2.4) and (2.5) as

$$u_0 = z,$$
$$u_{n+1} = -L^{-1}[z^{1-\alpha}\, A_n], \quad n \geq 0, \tag{2.7}$$

where A_n's are the Adomian polynomials.

We now calculate few A_n's for the non-linear term $Nu = u^{\alpha}$. It follows from [6.7] that

$$\begin{aligned} A_0 &= z^{\alpha} \\ A_1 &= -\frac{\alpha}{3!} z^{\alpha+2} \\ A_2 &= \frac{\alpha}{5!} z^{\alpha+4} + \frac{\alpha(\alpha-1)}{2!(3!)^2} z^{\alpha+4} \\ A_3 &= -\left\{ \frac{\alpha^3}{7!} + \frac{\alpha^2(\alpha-1)}{3!5!} + \frac{\alpha^2(\alpha-1)}{(3!)\cdot 14} + \frac{\alpha(\alpha-1)(\alpha-2)}{(3!)^4} \right\} z^{\alpha+6} \\ &\ldots \end{aligned} \tag{2.8}$$

Substituting Eq. (2.8) into (2.7) and performing the necessary integrations, we get

$$\begin{aligned} u_0 &= z \\ u_1 &= -\frac{z^3}{3!} \\ u_2 &= \frac{\alpha}{5!} z^5 \\ u_3 &= -\frac{1}{7!} \left\{ \alpha^2 + \frac{5\alpha(\alpha-1)}{3} \right\} z^7 \\ &\ldots \end{aligned} \tag{2.9}$$

All components are now determinable and substituting these components in Eq. (2.6), u is obtained.
We now obtain

$$u = z - \frac{1}{3!} z^3 + \frac{\alpha}{5!} z^5 - \frac{1}{7!} \left\{ \alpha^2 + \frac{5\alpha(\alpha-1)}{3} \right\} z^7 + \ldots \tag{2.10}$$

Therefore, in terms of original variable, we finally get the solution as

$$y = 1 - \frac{1}{3!} x^2 + \frac{\alpha}{5!} x^4 - \frac{1}{7!} \left\{ \alpha^2 + \frac{5\alpha(\alpha-1)}{3} \right\} x^6 + \ldots \tag{2.11}$$

2.1.1 Verification of the Solution

Verification of the solution for a particular type of non-linearity (taking $\alpha = 5$) in Eq. (2.1) and comparison with the decomposition solution.

The above decomposition scheme will now be verified by taking a particular type of non-linearity u^5.
For this purpose we make the substitution

$$\begin{aligned} x &= \exp(-t) \\ y &= \frac{1}{\sqrt{2}} u(t) \exp(t/2) \end{aligned} \tag{2.12}$$

to eliminate the first-order term in (2.1) and obtain

$$4\frac{d^2u}{dt^2} - u + u^5 = 0. \tag{2.13}$$

Now, by writing $v = \frac{du}{dt}$, so that $\frac{d^2u}{dt^2} = dv/du$, the Eq. (2.13) reduces to the form

$$4v\frac{dv}{du} - u + u^5 = 0. \tag{2.14}$$

This equation can immediately be integrated to give

$$2v^2 - \frac{1}{2}u^2 + \frac{1}{6}u^6 = c, \tag{2.15}$$

where c is an arbitrary constant which can be set equal to zero by applying the given boundary conditions. For example, Eq. (2.12), for $x = 0$ corresponds to $t \to \infty$ and hence u and $\frac{du}{dt}$ must vanish as $t \to \infty$, implying that $u = v = 0$, as $t \to \infty$. Thus (2.15) becomes

$$v^2 = \frac{1}{4}u^2 - \frac{1}{12}u^{6,}$$
$$\text{or, } v \equiv \frac{du}{dt} = \frac{1}{2}u\left(1 - \frac{1}{3}u^4\right)^{1/2},$$

which leads to

$$2\int \frac{du}{u\left\{1 - \left(\frac{1}{3}\right)u^4\right\}^{1/2}} = \int dt + A \tag{2.16}$$

where A is a constant. To evaluate the integral (2.16), we substitute $u^4 = 3\cos^2\theta$ and obtain, $\sec\theta + \tan\theta = B\exp(-t)$, where $B = \exp(-A)$.

Further, using $\cos\theta = \frac{u^2}{\sqrt{3}}$ and x = exp(−t), we get $Bx = \frac{\sqrt{3}}{u^2} + \{\frac{3}{u^4} - 1\}^{1/2}$, which can be solved for u to give $u = [\frac{2\sqrt{3}Bx}{(1+B^2x^2)}]^{1/2}$.

Finally, using (2.12), the solution of (2.1), for $\alpha = 5$ becomes

$$y = \left[\frac{\sqrt{3}B}{(1 + B^2x^2)}\right]^{1/2}, \tag{2.17}$$

where $B = \frac{1}{\sqrt{3}}$ can be set after using the boundary condition $y = 1$, when $x = 0$. The solution of (2.1) with these boundary conditions has been used extensively in Astrophysics.

$$Bx = \frac{\sqrt{3}}{u^2} + \left\{\frac{3}{u^4} - 1\right\}^{1/2},$$
$$= u' + \left(u'^2 - 1\right)^{1/2}, \quad \text{where} \quad u' = \frac{\sqrt{3}}{u^2}$$

Now

$$\left(u'^2 - 1\right)^{1/2} = Bx - u' \tag{2.18}$$

Simplify the above Eq. (2.18), we have $u^2 = \frac{2\sqrt{3}Bx}{B^2x^2+1}$
Therefore,

$$y = \left[\frac{1}{1 + \frac{1}{3}x^2}\right]^{1/2} \tag{2.19}$$

Table 1 Comparison of numerical and exact results for case 1

x	y (ADM)	y (EXACT)	% ERROR
0.1	0.998337489	0.998337488	0.000001
0.2	0.993399894	0.993399267	0.0000627
0.3	0.985336295	0.985329278	0.0007017
0.4	0.974393227	0.974354703	0.0038524
0.5	0.960911665	0.960768922	0.0142743
0.6	0.945322858	0.944911182	0.0411676
0.7	0.928142974	0.92714554	0.0997434
0.8	0.909966560	0.907841299	0.2125261
0.9	0.891458795	0.887356509	0.4102286
1.0	0.87334656	0.866025403	0.732115767

For comparison with the decomposition solution (2.11), and to stress the point that the decomposition series is very rapidly convergent and only a few terms of the series solution (2.11) are sufficient for most purpose, taking $\alpha = 5$, it follows from (2.11) that

$$y = 1 - \frac{1}{6}x^2 + \frac{1}{24}\quad x^4 - \frac{5}{3024}x^6 + \dots. \tag{2.20}$$

The comparison is recorded in the Table 1.

2.2 Case 2 ($f(y) = e^y$)

The Lane–Emden equation has the form [16]

$$y'' + \left(\frac{2}{x}\right)y' + e^y = 0, \quad 0 < x \le 1$$
$$y(0) = y'(0) = 0 \tag{2.21}$$

Equation (2.21) is the Lane–Emden equation that models the non-dimensional density distribution $y(x)$ in an isothermal gas sphere [16].
We first introduce the following change of variables $x = z, u = zy = xy$ into Eq. (2.21), and it follows that

$$\frac{d^2u}{dz^2} + ze^{u/z} = 0$$
$$u(0) = 0, \quad \left.\frac{du}{dz}\right|_{z=0} = 0 \tag{2.22}$$

The Eq. (2.22) can be rewritten in an operator form as [6,7,12]

$$Lu + zNu = 0, \tag{2.23}$$

where L is the differential operator $\frac{d^2}{dz^2}$ and $N(u)$ represents the non-linear term $e^{u/z}$.
Operating with L^{-1} in (2.23), we get

$$u = u_0 - L^{-1}[zNu], \tag{2.24}$$

where u_0 is the solution of the equation $Lu = 0$ with the given initial conditions.

Assuming the non-linear term Nu is analytic, this is decomposed as

$$Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, ...u_n), \tag{2.25}$$

where A_n's are special set of polynomials, called Adomian's polynomials, and can be calculated for all types of non-linearity according to specific algorithms constructed in [6,7].
The Adomian decomposition method (ADM) assumes a series solution for

$$u(z) = \sum_{n=0}^{\infty} u_n(z) \tag{2.26}$$

The decomposition series (2.26) can be obtained from (2.24) and (2.25) as

$$\begin{aligned} u_0 &= 0, \\ u_{n+1} &= -L^{-1}[zA_n], \quad n \geq 0, \end{aligned} \tag{2.27}$$

where A_n's are the Adomian polynomials.
We now calculate few A_n's for the non-linear term $Nu = e^{u/z}$. It follows from [6.7] that

$$\begin{aligned} A_0 &= 1 \\ A_1 &= -\frac{1}{6}z^2 \\ A_2 &= \frac{1}{45}z^4 \\ A_3 &= -\frac{61}{22680}z^6 \\ &... \end{aligned} \tag{2.28}$$

Substituting Eq. (2.28) into (2.27) and performing the necessary integrations, we get

$$\begin{aligned} u_0 &= 0 \\ u_1 &= -\frac{z^3}{6} \\ u_2 &= \frac{1}{120}z^5 \\ u_3 &= -\frac{1}{1890}z^7 \\ &... \end{aligned} \tag{2.29}$$

All components are now determinable and substituting these components in Eq. (2.26), u is obtained.
We now obtain

$$u = -\frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{1890}z^7 + ... \tag{2.30}$$

Therefore, in terms of original variable, we finally get the solution as

$$y = -\frac{1}{6}x^2 + \frac{1}{120}x^4 - \frac{1}{1890}x^6 + ... \tag{2.31}$$

Verifying the results obtained by Adomian decomposition method in Eq. (2.31), we can compare with the results obtained by Variational iteration method (VIM) [17] and it is clear that both results are same.

2.3 Case 3 ($f(y) = (y^2 - C)^{3/2}$)

The Lane–Emden equation has the form [18]

$$y'' + \left(\frac{2}{x}\right) y' + (y^2 - C)^{3/2} = 0, \;\; 0 < x \le 1$$
$$y(0) = 1, \quad y'(0) = 0 \tag{2.32}$$

Equation (2.32) is considered as "white-dwarf" equation which is introduced by Davis [19] and Chandrasekhar [11] in his study of the gravitational potential of the degenerate white-dwarf stars.
We first introduce the following change of variables $x = z, u = zy = xy$ into Eq. (2.32), and it follows that

$$\frac{d^2u}{dz^2} + z^{-2}\left(u^2 - Cz^2\right)^{3/2} = 0$$
$$u(0) = 0, \quad \left.\frac{du}{dz}\right|_{z=0} = 1 \tag{2.33}$$

The Eq. (2.33) can be rewritten in an operator form as [6,7,12]

$$Lu + z^{-2}Nu = 0, \tag{2.34}$$

where L is the differential operator $\frac{d^2}{dz^2}$ and $N(u)$ represents the non-linear term $(u^2 - Cz^2)^{3/2}$. Applying the same procedure, the decomposition series $u(z) = \sum_{n=0}^{\infty} u_n(z)$ can be obtained as

$$u_0 = z,$$
$$u_{n+1} = -L^{-1}[z^{-2}A_n], \quad n \ge 0, \tag{2.35}$$

where A_n's are the Adomian polynomials.
We now calculate few A_n's for the non-linear term $Nu = (u^2 - Cz^2)^{3/2}$. It follows from [6,7] that

$$A_0 = \left[(1 - C)z^2\right]^{3/2}$$
$$A_1 = \frac{1}{2}(1 - C)^2 z^5$$
$$A_2 = \frac{(1 - C)^3(19 - 5C)}{120\sqrt{1 - C}} z^7$$
$$A_3 = -\frac{(1 - C)^3(619 - 339C)}{15120} z^9$$
$$\ldots \tag{2.36}$$

and

$$\begin{aligned}
u_0 &= z \\
u_1 &= -\frac{1}{6}\left[(1-C)z^2\right]^{3/2} \\
u_2 &= \frac{1}{40}(1-C)^2 z^5 \\
u_3 &= -\frac{(1-C)^3(19-5C)}{5040\sqrt{1-C}} z^7 \\
&\ldots
\end{aligned} \tag{2.37}$$

All components are now determinable and substituting these components in decomposition series $u(z) = \sum_{n=0}^{\infty} u_n(z)$.
We now obtain

$$u = z - \frac{1}{6}(1-C)^{3/2} z^3 + \frac{1}{40}(1-C)^2 z^5 - \frac{(1-C)^3(19-5C)}{5040\sqrt{1-C}} z^7 + \ldots \tag{2.38}$$

Therefore, in terms of original variable, we finally get the solution as

$$y = 1 - \frac{1}{6}(1-C)^{3/2} x^2 + \frac{1}{40}(1-C)^2 x^4 - \frac{(1-C)^3(19-5C)}{5040\sqrt{1-C}} x^6 + \ldots \tag{2.39}$$

It is more interesting to point out that if $C = 0$, then the white-dwarf equation change to Lane–Emden equation of index 3.

3 Conclusion

The advantage of this global methodology lies on the fact that it not only leads to an analytical continuous approximation which is very rapidly convergent [2,4,6,20], but also shows the dependence, giving insight into the character and behavior of the solution just as in a closed form solution [3,4,6]. The present analysis exhibits the applicability of the decomposition method to solve Lane–Emden equation in the form of a power series. Furthermore, this method does not require any transformation technique for linearization, and discretization of the variables and it does not make closure approximation of a smallness assumptions. Finally, we point out that, if the conditions on one variable are better known than the others, we consider the appropriate operator equation which can yield the solution without suffering transitional difficulty. This technique may be applied to the non-linear partial differential equations such as KdV equation, Burgers equation, and many other important nonlinear equations which will be considered in subsequent papers.

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