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ORIGINAL PAPER

Doubling Transformations and Definite Integrals

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Abstract A doubling transformation $\psi(x) = x - \lambda - \frac{b}{x-\mu}$, $b, \lambda, \mu \in \mathbb{R}$, $b > 0$, has the property that for any absolutely integral function $F(x)$ on $\mathbb{R}$ we have $\int_{-\infty}^{\infty} F(\psi(x))\, dx = \int_{-\infty}^{\infty} F(x)\, dx$. Compositions of doubling transformations also satisfy this integral invariance property. In this paper we give criteria for determining when a given rational function is a composition of two or more doubling transformations, and use this criteria for giving explicit families of such transformations such as $\frac{(x-a)(x+a^2)(x-a^3)(x+a^4)}{x(x-a^2)(x+a^3)}$, for $a > 1$; $\frac{(x^2-a^2)(x^2-a^8)}{x(x+a^2)(x-a^3)}$, for $a > 1$; and $\frac{(x^2-a^2)(x^2-b^2)}{x(x^2-ab)}$, for $0 < a < b$.

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Introduction

Transformations on the real number line of the type

$$\phi(x) = x - \lambda - \frac{b}{x-\mu}, \tag{1}$$

with λ, μ, b real numbers with $b > 0$, called *doubling transformations*, have the interesting property that for any absolutely integrable function $F(x)$ on $\mathbb{R}$, we have

$$\int_{-\infty}^{\infty} F(\phi(x))\, dx = \int_{-\infty}^{\infty} F(x)\, dx. \tag{2}$$

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This result can be found for example in Wilson's *Advanced Calculus* [4, p. 386]; a proof is provided in Section "Proof of the Integral Formula" for the reader's convenience. The name *doubling* comes from the fact that $\phi(x)$ is a two-to-one function on $\mathbb{R} - \{\mu\}$, and thus the graph of $F(\phi(x))$ is a *doubling* of the graph of $F(x)$. By taking compositions of doubling functions we generate new rational functions satisfying (2). Our interest in this paper is to characterize all such rational functions. This extends work started by Hagler in [1], [2] and [3] where doubling transformation were used in the study of orthogonal systems of polynomials such as those of Jacobi, Hermite and Laguerre.

We shall establish criteria for recognizing when a given rational function is a composition of n doubling transformations. For example, the following are all examples of compositions of two doubling transformations,

$$\frac{(x-a)(x+a^2)(x-a^3)(x+a^4)}{x(x-a^2)(x+a^3)}, \quad (a>1), \tag{3}$$

$$\frac{(x^2-a^2)(x^2-a^8)}{x(x+a^2)(x-a^3)}, \quad (a>1), \tag{4}$$

$$\frac{(x^2-a^2)(x^2-b^2)}{x(x^2-ab)}, \quad (0<a<b); \tag{5}$$

see Examples 9, 10 and 11. Thus, by (2) we obtain integral formulae such as

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \int_{-\infty}^{\infty} e^{-\left[\frac{(x-a)(x+a^2)(x-a^3)(x+a^4)}{x(x-a^2)(x+a^3)}\right]^2} dx,$$

for $a>1$ and

$$\pi = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{\infty} \frac{x^2(x^2-ab)^2\, dx}{x^2(x^2-ab)^2+(x^2-a^2)^2(x^2-b^2)^2},$$

for $0<a<b$.

In general, for compositions of two doubling transformations we establish the following characterization.

Theorem 1 *A rational function is a composition of two doubling transformations if and only if it is of the form* $f_2(x-\mu)$ *for some real number* μ *and rational function*

$$f_2(x) = \frac{\prod_{i=1}^{4}(x-\theta_i^{(2)})}{x\prod_{i=1}^{2}(x-\theta_i^{(1)})},$$

where the $\theta_i^{(j)}$ *are real numbers satisfying*

$$\theta_4^{(2)} < \theta_2^{(1)} < \theta_2^{(2)} < 0 < \theta_3^{(2)} < \theta_1^{(1)} < \theta_1^{(2)},$$
$$\theta_1^{(1)}\theta_2^{(1)} = \theta_1^{(2)}\theta_2^{(2)} = \theta_3^{(2)}\theta_4^{(2)}.$$

Theorem 1 is an immediate consequence of Theorem 7, which establishes both existence and uniqueness criteria for such decompositions. Similar criteria are established for $n=3$ in Section "The case $n=3$" and for general n in Section "The General Case". For $n=3$ we give an explicit family of such functions in Example 12. Writing down explicit families for $n>3$ is more challenging and we have not endeavored to do so at this point.

Compositions of Doubling Transformations

Suppose that for $1 \leq k \leq n$, ϕ_k is a doubling transformation,

$$\phi_k(x) = x - \lambda_k - \frac{b_k}{x - \mu_k},$$

for some $\lambda_k, b_k, \mu_k \in \mathbb{R}$, $b_k > 0$. For any real number μ, let t_μ denote the translation $t_\mu(x) := x - \mu$, and t_μ^{-1} denote its inverse, $t_\mu^{-1}(x) = x + \mu$. Then we have

$$\begin{aligned}\phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1 &= (\phi_n \circ t_{\mu_n}^{-1}) \circ (t_{\mu_n} \circ \phi_{n-1} \circ t_{\mu_{n-1}}^{-1}) \circ \cdots \circ \left(t_{\mu_2} \circ \phi_1 \circ t_{\mu_1}^{-1}\right) \circ t_{\mu_1} \\ &= \rho_n \circ \rho_{n-1} \circ \cdots \circ \rho_1 \circ t_{\mu_1}\end{aligned}$$

say, where ρ_k is a doubling transformation with pole at 0, $1 \leq k \leq n$. Thus we have the following lemma.

Lemma 2 *Any function that can be expressed as a composition of doubling transformations can be expressed in the form $f(x - \mu)$ for some $\mu \in \mathbb{R}$, where f is a composition of doubling transformations having poles at 0.*

Henceforth, we shall assume that the poles μ_k in all of our doubling transformations are zero. The ϕ_k then take the form

$$\phi_k(x) := x - \lambda_k - \frac{b_k}{x} = \frac{(x - \alpha_k)(x - \beta_k)}{x}, \tag{6}$$

where $\alpha_k > 0$, $\beta_k < 0$ are real numbers given by

$$\alpha_k, \beta_k = \frac{\lambda_k \pm \sqrt{\lambda_k^2 + 4b_k}}{2},$$

and satisfy the basic relations

$$\alpha_k + \beta_k = \lambda_k, \tag{7}$$

$$\alpha_k \beta_k = -b_k. \tag{8}$$

Set

$$f_n(x) = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1(x). \tag{9}$$

Each ϕ_k is strictly increasing on $(-\infty, 0)$ and on $(0, \infty)$, and is a 2-to-1 mapping on the extended real number line $\mathbb{R} \cup \{\infty\}$, where we define $\phi_k(0) = \infty$, $\phi_k(\infty) = \infty$. Thus f_n is a 2^n-to-1 mapping on the extended real number line, strictly increasing on the open intervals where it is defined, having $2^n - 1$ distinct real simple poles, and can be expressed in reduced form as a rational function

$$f_n(x) = \frac{p_n(x)}{q_n(x)},$$

for some monic polynomials $p_n(x)$, $q_n(x)$ of degrees 2^n, $2^n - 1$ respectively. The graph of $f_n(x)$ consists of 2^n connected components $\mathcal{C}_r^{(n)}$, $1 \leq r \leq 2^n$, each containing a unique zero $\theta_r^{(n)}$ of $f_n(x)$. We call $\mathcal{C}_r^{(n)}$ the component of the graph corresponding to $\theta_r^{(n)}$. Thus we have,

$$p_n(x) = \prod_{r=1}^{2^n} (x - \theta_r^{(n)}).$$

Since

$$f_n(x) = \phi_n(f_{n-1}(x)) = f_{n-1}(x) - \lambda_n - b_n/f_{n-1}(x),$$

we see that the poles of f_n are just the zeros of f_{n-1} together with the poles of f_{n-1}. From this observation and the fact that $p_n(x), q_n(x)$ are monic, we see that $f_1(x) = \frac{p_1(x)}{x}$, $f_2(x) = \frac{p_2(x)}{xp_1(x)}$, and by induction that,

$$f_n(x) = \frac{p_n(x)}{xp_1(x)p_2(x)\cdots p_{n-1}(x)}.$$

The zeros of $f_n(x)$ can be put into positive-negative pairs inductively as follows. For $n = 1$ we already have the positive-negative pair $(\theta_1^{(1)}, \theta_2^{(1)}) = (\alpha_1, \beta_1)$. We will use odd subcripts for the positive zeros and even subscripts for the negative zeros. Suppose that we are given a pairing of the zeros of $f_{n-1}(x)$, say $(\theta_{2k-1}^{(n-1)}, \theta_{2k}^{(n-1)})$, $1 \le k \le 2^{n-2}$. Let $\mathcal{C}_{2k-1}^{(n-1)}, \mathcal{C}_{2k}^{(n-1)}$ be the corresponding components of the graph of $f_{n-1}(x)$. Note that $\mathcal{C}_{2k-1}^{(n-1)}$ lies in the right half-plane ($x > 0$), while $\mathcal{C}_{2k}^{(n-1)}$ lies in the left half-plane ($x < 0$). Since $f_n(x) = \phi_n(f_{n-1}(x))$, the zeros of $f_n(x)$ are just the solutions of the equations $f_{n-1}(x) = \alpha_n$, $f_{n-1}(x) = \beta_n$. Let $\theta_{2k-1}^{(n)}$ be the solution of $f_{n-1}(x) = \alpha_n$ with (x, α_n) on $\mathcal{C}_{2k-1}^{(n-1)}$ and $\theta_{2k}^{(n)}$ the solution of $f_{n-1}(x) = \alpha_n$ with (x, α_n) on $\mathcal{C}_{2k}^{(n-1)}$. Then $(\theta_{2k-1}^{(n)}, \theta_{2k}^{(n)})$ is a uniquely defined pair of zeros of $f_n(x)$. Similarly, let $(\theta_{2^{n-1}+2k-1}, \theta_{2^{n-1}+2k})$ be the positive negative pair corresponding to the solutions of $f_{n-1}(x) = \beta_n$.

In particular, we see that for $n = 2$,

$$\theta_4^{(2)} < \theta_2^{(1)} < \theta_2^{(2)} < 0 < \theta_3^{(2)} < \theta_1^{(1)} < \theta_1^{(2)}. \tag{10}$$

Thus, between any two consecutive zeros of $q_2(x) = xp_1(x)$ there is a zero of $p_2(x)$. Also, to the right of the largest zero of $q_2(x)$ and to the left of the smallest zero of $q_2(x)$ there is a zero of $p_2(x)$. Similarly, for $n = 3$ one obtains from the construction above that,

$$\begin{aligned} \theta_8^{(3)} < \theta_4^{(2)} < \theta_4^{(3)} < \theta_2^{(1)} < \theta_6^{(3)} < \theta_2^{(2)} < \theta_2^{(3)} < 0 \\ 0 < \theta_7^{(3)} < \theta_3^{(2)} < \theta_3^{(3)} < \theta_1^{(1)} < \theta_5^{(3)} < \theta_1^{(2)} < \theta_1^{(3)}, \end{aligned} \tag{11}$$

and we see the same type of splicing of zeros. Continuing this process we have the following lemma.

Lemma 3 *The Splicing Principle. For $n \ge 2$, between any two consecutive zeros of $q_n(x)$ (with respect to the standard ordering on $\mathbb{R}$), as well as to the right of the largest zero and to the left of the smallest zero of $q_n(x)$, there is a unique zero of $p_n(x)$. Moreover, we have the following consecutive triples of zeros of $q_n(x)$:*

$$\theta_{2^{n-1}+2k-1}^{(n)} < \theta_{2k-1}^{(n-1)} < \theta_{2k-1}^{(n)}, \quad \theta_{2^{n-1}+2k}^{(n)} < \theta_{2k}^{(n-1)} < \theta_{2k}^{(n)}, \quad 1 \le k \le 2^{n-2}. \tag{12}$$

The positive-negative pairs can be defined in a purely algebraic manner as follows. They correspond to the choice of positive-negative signs in the successive applications of the quadratic formula that one would use for calculating the zeros. For instance,

$$\theta_1^{(2)} = \left(\lambda_1 + \alpha_2 + \sqrt{(\lambda_1 + \alpha_2)^2 + 4b_1}\right)/2$$
$$\theta_2^{(2)} = \left(\lambda_1 + \alpha_2 - \sqrt{(\lambda_1 + \alpha_2)^2 + 4b_1}\right)/2$$
$$\theta_3^{(2)} = \left(\lambda_1 + \beta_2 + \sqrt{(\lambda_1 + \beta_2)^2 + 4b_1}\right)/2$$
$$\theta_4^{(2)} = \left(\lambda_1 + \beta_2 - \sqrt{(\lambda_1 + \beta_2)^2 + 4b_1}\right)/2.$$

Fundamental Relations

As noted above, the zeros of $f_n(x)$ are the solutions of the equations $f_{n-1}(x) = \alpha_n$, $f_{n-1}(x) = \beta_n$. We call the former the α zeros of $f_n(x)$ and the latter the β zeros of $f_n(x)$. Let

$$f_{n-1}^*(x) := \phi_{n-1}^* \circ \phi_{n-2} \circ \cdots \circ \phi_1(x),$$
$$f_{n-1}^{**}(x) := \phi_{n-1}^{**}(x) \circ \phi_{n-2} \circ \cdots \circ \phi_1(x),$$

where

$$\phi_{n-1}^*(x) := x - (\lambda_{n-1} + \alpha_n) - b_{n-1}/x,$$
$$\phi_{n-1}^{**}(x) := x - (\lambda_{n-1} + \beta_n) - b_{n-1}/x.$$

Let $\theta_j^{*(n-1)}, \theta_j^{**(n-1)}$ denote the zeros of f_{n-1}^*, f_{n-1}^{**} respectively. Then the alpha zeros of f_n are just the zeros of f_{n-1}^*, while the beta zeros of f_n are the zeros of f_{n-1}^{**}, that is,

$$\theta_{2k-1}^{(n)} = \theta_{2k-1}^{*(n-1)}, \quad \theta_{2k}^{(n)} = \theta_{2k}^{*(n-1)}, \tag{13}$$
$$\theta_{2^{n-1}+2k-1}^{(n)} = \theta_{2k-1}^{**(n-1)}, \quad \theta_{2^{n-1}+2k}^{(n)} = \theta_{2k}^{**(n-1)}. \tag{14}$$

Because $f_{n-1}^*(x)$ and $f_{n-1}^{**}(x)$ are functions of type (9), identical to $f_{n-1}(x)$ except for the value of λ_{n-1}, we have

Lemma 4 *The Correspondence Principle. Any relationship satisfied by the zeros of $f_{n-1}(x)$ that has no dependence on λ_{n-1}, will be satisfied by the corresponding α-zeros and β-zeros of $f_n(x)$.*

As an example of this phenomena we give the most basic such relationship.

Lemma 5 *The Basic Relationship. For any $n \geq 1$ and any positive-negative pair ($\theta_{2k-1}^{(n)}$, $\theta_{2k}^{(n)}$) of zeros of $f_n(x)$ we have*

$$\theta_{2k-1}^{(n)}\theta_{2k}^{(n)} = -b_1.$$

Proof The relationship holds for $n = 1$ by (8). Since it has no dependence on the λ_n the same relationship holds by induction and the Correspondence Principle for all n. □

The next lemma provides a more general class of relationships.

Lemma 6 *For any positive integer n, we have*

$$(i)\quad \sum_{i=1}^{2^n} \theta_i^{(n)} = \lambda_n + 2\lambda_{n-1} + 2^2\lambda_{n-2} + \cdots + 2^{n-1}\lambda_1. \tag{15}$$

$$(ii)\quad \sum_{i=1}^{2^{n-1}} \theta_i^{(n)} - \sum_{i=1}^{2^{n-1}} \theta_i^{(n-1)} = \alpha_n \tag{16}$$

$$(iii)\quad \sum_{i=1}^{2^{n-1}} \theta_{2^{n-1}+i}^{(n)} - \sum_{i=1}^{2^{n-1}} \theta_i^{(n-1)} = \beta_n. \tag{17}$$

$$(iv)\quad -b_n = \left(\sum_{i=1}^{2^{n-1}} \theta_i^{(n)} - \sum_{i=1}^{2^{n-1}} \theta_i^{(n-1)}\right)\left(\sum_{i=1}^{2^{n-1}} \theta_{2^{n-1}+i}^{(n)} - \sum_{i=1}^{2^{n-1}} \theta_i^{(n-1)}\right). \tag{18}$$

When $n = 1$, the relationship in (iv) is just the basic relationship of Lemma 5.

Proof The proof of (i) is by induction on n the case $n = 1$ being the identity $\alpha_1 + \beta_1 = \lambda_1$. Suppose the statement is true for n. Then for $n+1$ the α-zeros of $f_{n+1}(x)$ are the solutions of $f_n(x) = \alpha_{n+1}$, that is, the zeros of $f_n^*(x)$ where $f_n^*(x)$ is the same as $f_n(x)$ with λ_n replaced by $\lambda_n + \alpha_{n+1}$. Thus by the induction assumption the sum of the α-zeros is $\lambda_n + \alpha_{n+1} + \sum_{i=1}^{n-1} 2^i\lambda_{n-i}$. Similarly, the sum of the β-zeros of $f_{n+1}(x)$ is $\lambda_n + \beta_{n+1} + \sum_{i=1}^{n-1} 2^i\lambda_{n-i}$. Thus the full sum of zeros of $f_{n+1}(x)$ is

$$\alpha_{n+1} + \beta_{n+1} + 2\lambda_n + 2\sum_{i=1}^{n-1} 2^i\lambda_{n-i} = \lambda_{n+1} + 2\lambda_n + 2^2\lambda_{n-1} + \cdots + 2^n\lambda_1.$$

The sum in (ii) is just the sum of the α-zeros of $f_n(x)$ minus the sum of all the zeros of $f_{n-1}(x)$, which by (i) equals $\sum_{i=0}^{n-1} 2^i\lambda_{n-i} + \alpha_n - \sum_{i=0}^{n-1} 2^i\lambda_{n-i} = \alpha_n$. The proof of (iii) is identical, using the β-zeros in place of the α-zeros. The identity in (iv) follows from (ii), (iii) and the fact that $\alpha_n\beta_n = -b_n$. □

Since the identity in part (iv) of the lemma has no dependence on the λ_i, one can invoke the correspondence lemma to produce further relations. We do this in Section "The General Case", but first we explore the cases $n = 2$ and $n = 3$ in detail.

The case $n = 2$

Compositions of two doubling transformations are *quadrupling* transformations, that is, 4-to-1 mappings on the extended real number line. In the notation of the previous section, any such transformation has the form

$$f_2(x) = \phi_2 \circ \phi_1(x) = \frac{p_2(x)}{xp_1(x)} = \frac{\prod_{i=1}^4 (x - \theta_i^{(2)})}{x\prod_{i=1}^2 (x - \theta_i^{(1)})}, \tag{19}$$

where, by the Splicing Principle (Lemma 3), Basic Relationship (Lemma 5) and Lemma 6 (iv) and (i), we have

$$\theta_4^{(2)} < \theta_2^{(1)} < \theta_2^{(2)} < 0 < \theta_3^{(2)} < \theta_1^{(1)} < \theta_1^{(2)}, \tag{20}$$

$$-b_1 = \theta_1^{(1)}\theta_2^{(1)} = \theta_1^{(2)}\theta_2^{(2)} = \theta_3^{(2)}\theta_4^{(2)}, \tag{21}$$

$$-b_2 = \left(\theta_1^{(2)} + \theta_2^{(2)} - \theta_1^{(1)} - \theta_2^{(1)}\right)\left(\theta_3^{(2)} + \theta_4^{(2)} - \theta_1^{(1)} - \theta_2^{(1)}\right), \tag{22}$$

$$\lambda_1 = \theta_1^{(1)} + \theta_2^{(1)}, \tag{23}$$

$$\lambda_2 = \theta_1^{(2)} + \theta_2^{(2)} + \theta_3^{(2)} + \theta_4^{(2)} - 2\theta_1^{(1)} - 2\theta_2^{(1)}. \tag{24}$$

Thus, the values $\lambda_1, \lambda_2, b_1, b_2$ are uniquely determined by $f_2(x)$, and we have established one direction of the following theorem.

Theorem 7 *Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function over $\mathbb{R}$, expressed in reduced form with $q(x)$ monic. Then $f(x)$ is a composition of two doubling transformations with poles at zero if and only if $p(x)$ is a monic fourth degree polynomial with distinct real zeros $\theta_i^{(2)}$, $1 \le i \le 4$, $q(x)$ is a third degree polynomial with distinct real zeros $0, \theta_1^{(1)}, \theta_2^{(1)}$, and the zeros of $p(x), q(x)$ can be ordered in such a manner that the relations in (20) and (21) hold for some $b_1 \in \mathbb{R}$.*

To establish the converse part of the theorem we need the following uniqueness lemma.

Lemma 8 *For a given set of values $b_1, b_2, \lambda_1, \lambda_2 \in \mathbb{R}$, with $b_1 > 0, b_2 > 0$, the system of equations (21), (22), (23), (24) has a unique solution $\theta_1^{(1)}, \theta_2^{(1)}, \theta_i^{(2)}$, $1 \le i \le 4$, satisfying (20).*

Proof The values $\theta_1^{(1)}, \theta_2^{(1)}$ are roots of the quadratic equation $x^2 - \lambda_1 x - b_1 = 0$, with $\theta_1^{(1)} > 0 > \theta_2^{(1)}$. Put $X = \theta_1^{(2)} + \theta_2^{(2)}$, $Y = \theta_3^{(2)} + \theta_4^{(2)}$. Then we have

$$(X - \lambda_1)(Y - \lambda_1) = -b_2, \quad X + Y - 2\lambda_1 = \lambda_2, \quad X > Y.$$

The first two equations give X, Y as zeros of a quadratic equation, and the inequality $X > Y$ then uniquely determines X, Y. Next, the system

$$\theta_1^{(2)}\theta_2^{(2)} = -b_1, \quad \theta_1^{(2)} + \theta_1^{(2)} = X, \quad \theta_1^{(2)} > \theta_2^{(2)}.$$

uniquely determines $\theta_1^{(2)}, \theta_2^{(2)}$, while the system

$$\theta_3^{(2)}\theta_4^{(2)} = -b_1, \quad \theta_3^{(2)} + \theta_4^{(2)} = Y, \quad \theta_3^{(2)} > \theta_4^{(2)},$$

uniquely determines $\theta_3^{(2)}, \theta_4^{(2)}$. □

Proof of Theorem 7 Suppose that $f(x) = \frac{p(x)}{q(x)}$ is a given rational function with $p(x)$ monic and having zeros $\theta_1^{(2)}, \theta_2^{(2)}, \theta_3^{(2)}, \theta_4^{(2)}$, and $q(x)$ monic with zeros $0, \theta_1^{(1)}, \theta_2^{(1)}$. Suppose also that the relations in (20) and (21) hold for some $b_1 \in \mathbb{R}$. Define b_2, λ_1 and λ_2 as in (22), (23) and (24), and set

$$\phi_1(x) := x - \lambda_1 - \frac{b_1}{x}, \qquad \phi_2(x) := x - \lambda_2 - \frac{b_2}{x}.$$

By (20) we see that $b_1 > 0$ and $b_2 > 0$ and so ϕ_1, ϕ_2 are doubling transformations. The zeros and poles of $\phi_2 \circ \phi_1$ satisfy the relations (20), (21), (22), (23) and (24). Thus, by Lemma 8, the zeros and poles of $\phi_2 \circ \phi_1$ must be the values $\theta_1^{(1)}, \theta_2^{(1)}, \theta_i^{(2)}$, $1 \le i \le 4$. Since $\phi_2 \circ \phi_1$ is a ratio of monic polynomials, we have $\phi_2 \circ \phi_1(x) = f(x)$. □

In the next three examples, we give families of quadrupling transformations having particularly nice sets of zeros and poles. The strategy for constructing such examples is to find a set of real numbers satisfying (20) and (21).

Example 9 Let $a > 1$. Observing that $-a^4 < -a^3 < -a^2 < 0 < a < a^2 < a^3$ we put

$$\theta_1^{(2)} = a^3, \quad \theta_2^{(2)} = -a^2, \quad \theta_3^{(2)} = a, \quad \theta_4^{(2)} = -a^4, \quad \theta_1^{(1)} = a^2, \quad \theta_2^{(1)} = -a^3.$$

Then, with $b_1 = a^5$, $b_2 = -2a^3(a-1)^2(a^2+1)$, relations (21) and (22) hold. Defining $\lambda_1 = a^2 - a^3$, $\lambda_2 = -a(a-1)^3$, we have

$$f_2(x) = \phi_2 \circ \phi_1(x) = \frac{(x-a)(x+a^2)(x-a^3)(x+a^4)}{x(x-a^2)(x+a^3)}.$$

Example 10 Let $a > 1$. Observing that $-a^4 < -a^2 < -a < 0 < a < a^3 < a^4$, we put

$$\theta_4^{(2)} = -a^4, \quad \theta_3^{(2)} = a, \quad \theta_2^{(2)} = -a, \quad \theta_1^{(2)} = a^4, \quad \theta_2^{(1)} = -a^2, \quad \theta_1^{(1)} = a^3.$$

Then we have $b_1 = a^5$, $b_2 = a^2(a^4-1)(a^2-1)$, $\lambda_1 = a^3 - a^2$, $\lambda_2 = -2(a^3 - a^2)$, and

$$f_2(x) = \phi_2 \circ \phi_1(x) = \frac{(x^2-a^2)(x^2-a^8)}{x(x+a^2)(x-a^3)}.$$

Example 11 Let $0 < a < b$, so that $-b < -\sqrt{ab} < -a < 0 < a < \sqrt{ab} < b$. Then with $b_1 = ab$, $b_2 = (a-b)^2$, $\lambda_1 = \lambda_2 = 0$ we have

$$f_2(x) = \phi_2 \circ \phi_1(x) = \frac{(x^2-a^2)(x^2-b^2)}{x(x^2-ab)}.$$

The Case $n = 3$.

Let

$$f_3(x) = \phi_3 \circ \phi_2 \circ \phi_1(x) = \frac{\prod_{i=1}^{8}(x-\theta_i^{(3)})}{x\prod_{i=1}^{2}(x-\theta_i^{(1)})\prod_{i=1}^{4}(x-\theta_i^{(2)})}. \tag{25}$$

By the Splicing principle,

$$\begin{aligned} \theta_8^{(3)} < \theta_4^{(2)} < \theta_4^{(3)} < \theta_2^{(1)} < \theta_6^{(3)} < \theta_2^{(2)} < \theta_2^{(3)} < 0 \\ 0 < \theta_7^{(3)} < \theta_3^{(2)} < \theta_3^{(3)} < \theta_1^{(1)} < \theta_5^{(3)} < \theta_1^{(2)} < \theta_1^{(3)}. \end{aligned} \tag{26}$$

By the basic relationship,

$$-b_1 = \theta_1^{(3)}\theta_2^{(3)} = \theta_3^{(3)}\theta_4^{(3)} = \theta_5^{(3)}\theta_6^{(3)} = \theta_7^{(3)}\theta_8^{(3)} = \theta_1^{(2)}\theta_2^{(2)} = \theta_3^{(2)}\theta_4^{(2)} = \theta_1^{(1)}\theta_2^{(1)}. \tag{27}$$

By Lemma 6 (iv) together with the Correspondence Principle,

$$\begin{aligned} -b_2 &= \left(\theta_1^{(3)} + \theta_2^{(3)} - \theta_1^{(1)} - \theta_2^{(1)}\right)\left(\theta_3^{(3)} + \theta_4^{(3)} - \theta_1^{(1)} - \theta_2^{(1)}\right) \\ &= \left(\theta_5^{(3)} + \theta_6^{(3)} - \theta_1^{(1)} - \theta_2^{(1)}\right)\left(\theta_7^{(3)} + \theta_8^{(3)} - \theta_1^{(1)} - \theta_2^{(1)}\right) \\ &= \left(\theta_1^{(2)} + \theta_2^{(2)} - \theta_1^{(1)} - \theta_2^{(1)}\right)\left(\theta_3^{(2)} + \theta_4^{(2)} - \theta_1^{(1)} - \theta_2^{(1)}\right), \end{aligned} \tag{28}$$

and again by Lemma 6 (iv),

$$-b_3 = \left(\theta_1^{(3)} + \theta_2^{(3)} + \theta_3^{(3)} + \theta_4^{(3)} - \theta_1^{(2)} - \theta_2^{(2)} - \theta_3^{(2)} - \theta_4^{(2)}\right) \cdot \left(\theta_5^{(3)} + \theta_6^{(3)} + \theta_7^{(3)} + \theta_8^{(3)} - \theta_1^{(2)} - \theta_2^{(2)} - \theta_3^{(2)} - \theta_4^{(2)}\right). \tag{29}$$

By Lemma 6 (i) the values $\lambda_1, \lambda_2, \lambda_3$ satisfy

$$\lambda_1 = \sum_{i=1}^{2} \theta_i^{(1)},$$
$$\lambda_2 = \sum_{i=1}^{4} \theta_i^{(2)} - 2\sum_{i=1}^{2} \theta_i^{(1)},$$
$$\lambda_3 = \sum_{i=1}^{8} \theta_i^{(3)} - 2\sum_{i=1}^{4} \theta_i^{(2)} - 4\sum_{i=1}^{2} \theta_i^{(1)}.$$

Again, given any $\theta_i^{(k)}$ satisfying the relations in (26), (27) and (28), by defining $b_1, b_2, b_3, \lambda_1, \lambda_2, \lambda_3$ as above we obtain an $f_3(x)$ of the form (25). Thus we have the analogue of Theorem 7 for the case $n = 3$; see Theorem 13.

Example 12 Let $a_0 = 1.839...$ be the real zero of $x^3 - x^2 - x - 1 = 0$. Let a be any real with $a > a_0$, and set $\theta_1^{(1)} = 1, \theta_2^{(1)} = -1$,

$$\theta_1^{(2)} = a, \quad \theta_2^{(2)} = -\frac{1}{a}, \quad \theta_3^{(2)} = \frac{a-1}{a+1}, \quad \theta_4^{(2)} = -\frac{a+1}{a-1}$$
$$\theta_1^{(3)} = a^2, \quad \theta_2^{(3)} = -\frac{1}{a^2}, \quad \theta_3^{(3)} = \frac{a^2-1}{a^2+1}, \quad \theta_4^{(3)} = -\frac{a^2+1}{a^2-1},$$
$$\theta_5^{(3)} = \frac{a^2+1}{a^2-1}, \quad \theta_6^{(3)} = -\frac{a^2-1}{a^2+1}, \quad \theta_7^{(3)} = \frac{1}{a^2}, \quad \theta_8^{(3)} = -a^2.$$

It is easy to see that (26) holds since, by assumption, $a^3 - a^2 - a - 1 > 0$, and that (27) holds with $b_1 = 1$. Next, we have

$$\left(\theta_1^{(3)} + \theta_2^{(3)} - \theta_1^{(1)} - \theta_2^{(1)}\right)\left(\theta_3^{(3)} + \theta_4^{(3)} - \theta_1^{(1)} - \theta_2^{(1)}\right)$$
$$= \left(a^2 - \frac{1}{a^2}\right)\left(\frac{a^2-1}{a^2+1} - \frac{a^2+1}{a^2-1}\right) = -4,$$
$$\left(\theta_5^{(3)} + \theta_6^{(3)} - \theta_1^{(1)} - \theta_2^{(1)}\right)\left(\theta_7^{(3)} + \theta_8^{(3)} - \theta_1^{(1)} - \theta_2^{(1)}\right)$$
$$= \left(\frac{a^2+1}{a^2-1} - \frac{a^2-1}{a^2+1}\right)\left(\frac{1}{a^2} - a^2\right) = -4,$$
$$\left(\theta_1^{(2)} + \theta_2^{(2)} - \theta_1^{(1)} - \theta_2^{(1)}\right)\left(\theta_3^{(2)} + \theta_4^{(2)} - \theta_1^{(1)} - \theta_2^{(1)}\right)$$
$$= \left(a - \frac{1}{a}\right)\left(\frac{a-1}{a+1} - \frac{a+1}{a-1}\right) = -4,$$

and so (28) holds with $b_2 = 4$. Thus there exist ϕ_1, ϕ_2, ϕ_3 such that

$$f_3(x) := \phi_3 \circ \phi_2 \circ \phi_1(x) = \frac{\left(x^2 - \frac{1}{a^4}\right)\left(x^2 - a^4\right)\left(x^2 - \left(\frac{a^2+1}{a^2-1}\right)^2\right)\left(x^2 - \left(\frac{a^2-1}{a^2-1}\right)^2\right)}{x(x^2-1)\left(x + \frac{1}{a}\right)(x-a)\left(x + \frac{a+1}{a-1}\right)\left(x - \frac{a-1}{a+1}\right)}.$$

The General Case

For general n, we start with the relationship of Lemma 6,

$$-b_m = \left(\sum_{i=1}^{2^{m-1}} \theta_i^{(m)} - \sum_{i=1}^{2^{m-1}} \theta_i^{(m-1)}\right)\left(\sum_{i=1}^{2^{m-1}} \theta_{2^{m-1}+i}^{(m)} - \sum_{i=1}^{2^{m-1}} \theta_i^{(m-1)}\right),$$

for $m \le n$. For a fixed m we apply the Correspondence Principle to obtain $2^{n-m+1} - 1$ companion relationships:

$$-b_m = \left(\sum_{i=1}^{2^{m-1}} \theta_{2^m\ell+i}^{(m+j)} - \sum_{i=1}^{2^{m-1}} \theta_i^{(m-1)}\right)\left(\sum_{i=1}^{2^{m-1}} \theta_{2^{m-1}(2\ell+1)+i}^{(m+j)} - \sum_{i=1}^{2^{m-1}} \theta_i^{(m-1)}\right), \tag{30}$$

$$0 \le j \le n-m, \quad 0 \le \ell \le 2^j - 1, \tag{31}$$

where $\theta_1^{(0)} := 0$. Thus for $m = 1, 2, 3$ and j, ℓ satisfying (31), we have

$$-b_1 = \theta_{2\ell+1}^{(1+j)}\theta_{2\ell+2}^{(1+j)}, \tag{32}$$

$$-b_2 = \left(\theta_{4\ell+1}^{(2+j)} + \theta_{4\ell+2}^{(2+j)} - \theta_1^{(1)} - \theta_2^{(1)}\right)\left(\theta_{4\ell+3}^{(2+j)} + \theta_{4\ell+4}^{(2+j)} - \theta_1^{(1)} - \theta_2^{(1)}\right), \tag{33}$$

$$\begin{aligned}-b_3 = &\left(\theta_{8\ell+1}^{(3+j)} + \theta_{8\ell+2}^{(3+j)} + \theta_{8\ell+3}^{(3+j)} + \theta_{8\ell+4}^{(3+j)} - \theta_1^{(2)} - \theta_2^{(2)} - \theta_3^{(2)} - \theta_4^{(2)}\right)\\ &\cdot \left(\theta_{8\ell+5}^{(3+j)} + \theta_{8\ell+6}^{(3+j)} + \theta_{8\ell+7}^{(3+j)} + \theta_{8\ell+8}^{(3+j)} - \theta_1^{(2)} - \theta_2^{(2)} - \theta_3^{(2)} - \theta_4^{(2)}\right).\end{aligned} \tag{34}$$

There are $2^n - 1$, $2^{n-1} - 1$ and $2^{n-2} - 1$ equations in (32), (33) and (34) respectively. For $m = n$ we have a single relationship,

$$\begin{aligned}-b_n = &\left(\theta_1^{(n)} + \cdots + \theta_{2^{n-1}}^{(n)} - \theta_1^{(n-1)} - \cdots - \theta_{2^{n-1}}^{(n-1)}\right)\\ &\cdot \left(\theta_{2^{n-1}+1}^{(n)} + \cdots + \theta_{2^n}^{(n)} - \theta_1^{(n-1)} - \cdots - \theta_{2^{n-1}}^{(n-1)}\right).\end{aligned} \tag{35}$$

Altogether, there are $\sum_{m=1}^{n}(2^{n-m+1} - 1) = 2^{n+1} - 2 - n$ equations in $2^{n+1} - 2$ variables $\theta_i^{(j)}$. Insisting that the b_i be positive gives n more soft conditions.

Theorem 13 *Existence-Uniqueness property. Let n be a positive integer. Given a collection of $2^{n+1}-2$ real numbers $\theta_i^{(j)}$, $1 \le j \le n$, $1 \le i \le 2^j$ satisfying the $2^{n+1}-2-n$ equations in* (30) *for some positive real numbers b_m, $m \le n$, and ordered in accordance with the Splicing principle* (12)*, there exists a unique sequence of doubling transformations $\phi_i(x)$, $1 \le i \le n$, with poles at zero, such that with $f_n(x) := \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1(x)$, we have*

$$f_n(x) = \frac{\prod_{i=1}^{2^n}(x - \theta_i^{(n)})}{x\prod_{j=1}^{n-1}\prod_{i=1}^{2^j}(x - \theta_i^{(j)})}.$$

Proof The proof is an extension of the proof of Theorem 7 and so we just give a sketch. To define the ϕ_i, we simply define the b_i as in (30), and the λ_i by the relations in Lemma 6 (i), to wit,

$$\lambda_1 = \sum_{i=1}^{2} \theta_i^{(1)}, \tag{36}$$

$$\lambda_2 = \sum_{i=1}^{4} \theta_i^{(2)} - 2\sum_{i=1}^{2} \theta_i^{(1)} \tag{37}$$

$$\lambda_3 = \sum_{i=1}^{8} \theta_i^{(3)} - 2\sum_{i=1}^{4} \theta_i^{(2)} - 4\sum_{i=1}^{2} \theta_i^{(1)} \tag{38}$$

$$\vdots$$

$$\lambda_n = \sum_{i=1}^{2^n} \theta_i^{(n)} - 2\sum_{i=1}^{2^{n-1}} \theta_i^{(n-1)} - \cdots - 2^{n-1}\sum_{i=1}^{2} \theta_i^{(1)}. \tag{39}$$

The composition $\phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1$ then has zeros and poles satisfying (30) and the λ_i equations (36), (37), ..., (39). The theorem then follows by the analogue of the uniqueness lemma, Lemma 8, which can be proven by induction following the line of argument in the proof of Lemma 8. □

Proof of the Integral Formula

Theorem 14 *Let $\phi(x) = x - \lambda - \frac{b}{x-\mu}$ with $\lambda, b \in \mathbb{R}$, $b > 0$. Then for any continuous, absolutely integrable function $F(x)$ on $(-\infty, \infty)$, we have*

$$\int_{-\infty}^{\infty} F(\phi(x))\, dx = \int_{-\infty}^{\infty} F(x)\, dx.$$

Proof Let $I = \int_{-\infty}^{\infty} F(x)\, dx$. Replacing x by $x + \mu$, we may assume that $\mu = 0$ and

$$\phi(x) = x - \lambda - \frac{b}{x}.$$

Since $\phi(x)$ is continuously differentiable on $(0, \infty)$ with image $(-\infty, \infty)$, and $\phi'(x) = 1 + \frac{b}{x^2}$, we have, substituting $x = \phi(u)$,

$$I = \int_0^{\infty} F(\phi(u))\phi'(u)\, du = \int_0^{\infty} F(\phi(u)) + F(\phi(u))\frac{b}{u^2}\, du. \tag{40}$$

Now, since $|F(\phi(u))| \leq |F(\phi(u))||\phi'(u)|$, and $|F(\phi(u))\frac{b}{u^2}| \leq |F(\phi(u))||\phi'(u)|$ for $u \in (0, \infty)$, and $F(\phi(u))\phi'(u)$ is absolutely integrable on $(0, \infty)$, the functions $F(\phi(u))$ and $F(\phi(u))\frac{b}{u^2}$ are integrable on $(0, \infty)$, and so we can break up the integral in (40) to get,

$$I = \int_0^{\infty} F(\phi(u))\, du + \int_0^{\infty} F(\phi(u))\frac{b}{u^2}\, du.$$

Next, we substitute $u = -b/t$, and note that $\phi\left(\frac{-b}{t}\right) = \phi(t)$, $dt = \frac{b}{u^2}\,du$, to obtain

$$\begin{aligned} I &= \int_0^\infty F(\phi(u))\,du + \int_{-\infty}^0 F(\phi(-b/t))\,dt \\ &= \int_0^\infty F(\phi(u))\,du + \int_{-\infty}^0 F(\phi(t))\,dt \\ &= \int_{-\infty}^\infty F(\phi(u))\,du. \end{aligned}$$

□

References

1. Hagler, B.A.: A transformation of orthogonal polynomial sequences into orthogonal Laurent polynomial sequences, Ph.D. thesis, University of Colorado at Boulder, Boulder, 1–124 (1997)
2. Hagler, B.A.: Formulas for the moments of some strong moment distributions. Orthogonal functions, moment theory, and continued fractions (Campinas, 1996), Lecture Notes in Pure and Applied Mathematics, 199, Dekker, New York, 179–186 (1998)
3. Hagler, B.A., Jones, W.B., Thron, W.J.: Orthogonal Laurent polynomials of Jacobi, Hermite, and Laguerre types. Orthogonal functions, moment theory, and continued fractions, (Campinas, 1996). Lecture Notes in Pure and Applied Mathematics, 199, Dekker, New York, 187-208 (1998)
4. Wilson, E.B.: Advanced calculus. A Text Upon Select Parts of Differential Calculus, Differential Equations, Integral Calculus, Theory of Functions, with Numerous Exercises. Dover Publications Inc, New York (1959)