

An EOQ Model for Time Dependent Backlogging Over Idle Time: A Step Order Fuzzy Approach

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Abstract The present article deals with a backorder Economic Order Quantity (EOQ) model for natural leisure/closing time system where the demand rate depends upon the total shortage period and the seasonal effect. A cost minimization problem is developed by trading off setup cost, inventory cost, cost for idle time and shortage cost. The intuitionistic step order fuzzy number for optimization has been developed, assuming all parameters as fuzzy numbers. Ranking is done by employing score function, accuracy value on the centre of gravity and Euclidean distance function over the objective function. Finally numerical examples are considered to justify the model.

Keywords Inventory · Shortage · Natural idle time · Intuitionistic fuzzy set · Centre of gravity · Euclidean distance · Optimization

Introduction

It is common to all enterprises that the inventory models determine optimal order quantity, shortage quantity and the duration of the opening and closing time of the shop/industry/inventory itself. In inventory literature, several research articles have been published along these directions except the use of natural opening or closing time duration

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in each day. Among those, some noteworthy works [1–6] are mentioned here. De [7] used first the concept of natural idle time. In this article, he developed a model measuring an error on customers' demand on spot for without backorder model under intuitionistic fuzzy environment. In the early stage of the invention of Step Order Fuzzy (SOF), Zadeh [8] used the concept of linguistic variable in fuzzy set. A book on module of SOF was developed by Kosinski [9]. Fortin et al. [10] studied an article over gradual numbers in fuzzy systems. A defuzzification method for ordered fuzzy numbers was analyzed by Kosinski and Wilezyska-Sztyma [11]. Also, Kosinski and Wegrzyn-Wolska [12] developed a neural network under defuzzification functional approximation method using step fuzzy numbers. On the other hand, several extensions in fuzzy concept as well as methodology of defuzzifications were discussed extensively. In intuitionistic fuzzy environment, Chakraborty et al. [13] considered a model for manufacturing inventory with shortages using Pareto-optimality method. Scoring of fuzzy numbers was nicely introduced by Chen et al. [14]. Recently, [15] and [16] published two novel articles of backlogging EOQ model using score functions and interpolating by pass to Pareto optimality over intuitionistic fuzzy technique respectively. The centroid in different approach was developed by [17, 18] and [19]. A numerous research articles over the ranking of fuzzy numbers were developed by Adamo [20], Chen and Lu [21] and Fortemps and Roubens [22]. A fast method for ranking fuzzy numbers was studied by Buckley and Chanas [23]. Jain [24, 25], Kerre [26], and Nakamura [27] studied a lot over the fuzzy decision making process. In general fuzzy system, Mabuchi [28] studied the index value where the index itself was α -cut dependent. Tran and Duckstein [29] investigated fuzzy numbers extensively, comparing among different types of fuzzy numbers. Recently, De and Sana [30] introduced fuzzy promotional effect on customers' demand in more realistic and concrete way in a fuzzy backorder EOQ model.

In the present article, we investigate a backorder EOQ model for natural idle time in which the demand rate in shortage time decreases exponentially with backorder time. Moreover, the demand rate may vary for season to season. The commodities like vegetables, fruits, woollen shawl, rain coats/ umbrella etc. are fall into this category. For this reason, we are to incorporate fuzzy environment in which the membership value may vary with domain set of the demand rate. Moreover, introducing a varying membership grade, we have had a step-up and step-down fuzzy numbers as well. Based on the center of gravity of the fuzzy objective function, the Ranking index and Euclidean Distance method under the score values and accuracy values of the minimization problem for both the cases of strong and weak fuzzy numbers are used in the model. Finally, a decision is made using the ranking values.

Assumptions and Notations

The following notations and assumptions are used to develop the model.

Assumptions

1. A day splits into two parts—one is inventory run time duration and other is closing time duration. In practice, it is observed that shops/enterprises are closed and opened at fixed times which may vary with location of enterprise. For instance, the closing and opening times of the enterprises are 8:00 p.m. to 8:00 a.m. respectively. Here, the duration (8:00 p.m. to 8:00 a.m.) is an unavoidable pause time.
2. Replenishments are instantaneous.
3. The time horizon is infinite (days).
4. The sum of opening and closing time period is unity.

5. Shortages are allowed.
6. Demand rate per unit time is constant for stock inventory and it decreases exponentially for stock-out period viz. $d' = de^{-\lambda(m+1)}$, where m is an positive integer, $0 \leq \lambda \leq 1$ and $d(> 0)$ is demand per unit time during inventory period. This is a realistic assumption, as the demand rate decreases gradually with the shortage period.
7. Holding cost and shortage cost are uniform over the cycle time.
8. Average natural idle time (leisure/pause) cost is constant per unit idle time.
9. Stock inventory period is $(n + 1)$ days and backlogging period is $(m + 1)$ days.
10. The security charge, telephone charge, transportation cost (if any) etc. beyond the working hours of inventory run time may be treated as cost for the idle time.

Notation

- (i) q : The order quantity per cycle
- (ii) t_1 : Duration of opening time (day)
- (iii) t_2 : Duration of closing /natural idle time (day)
- (iv) d : Demand rate per unit time in $(i - 1, t_1), i = 1, 2, 3, \dots, n + 1$.
- (v) c_1 : Inventory holding cost(\$) per unit quantity per day.
- (vi) c_2 : shortage cost(\$) per unit quantity per day.
- (vii) c_3 : Set up cost(\$) per cycle.
- (viii) i_c : Average natural idle time cost(\$) per unit idle time.
- (ix) $(n + m + 1)$: The number of days per cycle, n and m are positive integers and $n > m$.
- (x) T : Cycle length (days).
- (xi) t : Time horizon (days).
- (xii) z : Average cost of the system per cycle (\$).

Crisp Model Formulation

The inventory starts at the beginning of the opening time duration t_1 and meets the demand quantity d per unit time. Then, it remains steady up to the closing time period t_2 . Again, it starts from the next day opening time and hence it continues up to $(n + 1)$ days. After that shortage continue up to $(m + 1)$ days (see Fig. 1). Now, the objective function can be developed as follows:

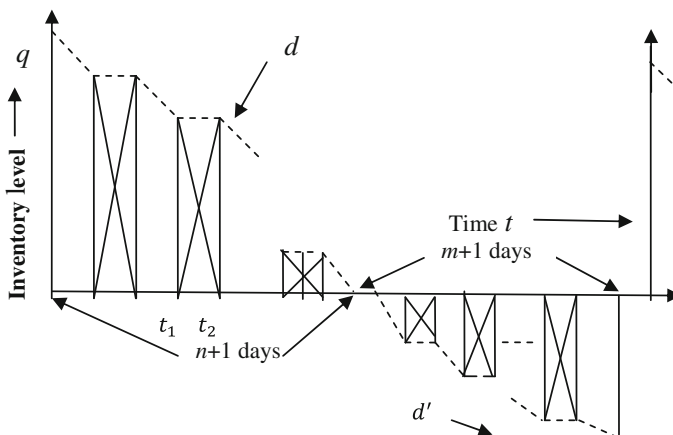


Fig. 1 Backorder EOQ model

The inventory holding cost (*HC*) can be stated as

$$\begin{aligned}
 HC &= c_1 \left[\frac{1}{2} (2q - dt_1) t_1 + \frac{1}{2} (2q - 3dt_1) t_1 + \frac{1}{2} (2q - 5dt_1) t_1 \right. \\
 &\quad \left. + \dots (n + 1) \text{ times } \frac{1}{2} (2q - ndt_1) t_1 \right] \\
 &\quad + c_1 [(q - dt_1) t_1 + (q - 2dt_1) t_1 + (q - 3dt_1) t_1 + \dots n \text{ times}] \\
 &= \frac{1}{2} (n + 1) dc_1 t_1 (n + t_1)
 \end{aligned} \tag{1}$$

where

$$q = (n + 1) dt_1 \tag{2}$$

and

$$t_1 + t_2 = 1 \tag{3}$$

The shortage cost (*SC*) is given by

$$\begin{aligned}
 SC &= c_2 \left[\frac{1}{2} d' t_1 + \frac{1}{2} t_1 (d' t_1 + 2d' t_1) + \frac{1}{2} t_1 (2d' t_1 + 3d' t_1) + \frac{1}{2} t_1 (3d' t_1 + 4d' t_1) \right. \\
 &\quad \left. + \dots + \frac{1}{2} t_1 (md' t_1 + (m + 1)d' t_1) \right] \\
 &= \frac{1}{2} d' c_2 t_1^2 (m + 1)^2 \\
 &= \frac{1}{2} c_2 t_1^2 (m + 1)^2 de^{-\lambda(m+1)}
 \end{aligned} \tag{4}$$

using

$$d' = de^{-\lambda(m+1)} \tag{5}$$

The idle time cost (*IC*) is given by

$$\begin{aligned}
 IC &= i_c \{ (n + 1) t_2 + (m + 1) t_2 \} \\
 &= i_c (m + n + 2) t_2 \\
 &= i_c (m + n + 2) (1 - t_1)
 \end{aligned} \tag{6}$$

The cycle time,

$$\begin{aligned}
 T &= (m + n + 2) (t_1 + t_2) \\
 &= (m + n + 2)
 \end{aligned} \tag{7}$$

Using equations (1), (4) and (5) the total average cost of the system by trading off holding cost, shortage cost, idle time cost and setup cost per cycle is given by

$$\begin{aligned}
 Z &= [\text{Holding Cost} + \text{Shortage Cost} + \text{Idle Time Cost} + \text{Setup cost}] \times \text{number of cycle} \\
 &= [HC + SC + IC + c_3] \frac{1}{T} \\
 &= \frac{1}{m + n + 2} \left[\frac{1}{2} (n + 1) dc_1 t_1 (n + t_1) + \frac{1}{2} c_2 d t_1^2 (m + 1)^2 e^{-\lambda(m+1)} \right. \\
 &\quad \left. + i_c (m + n + 2) (1 - t_1) + c_3 \right]
 \end{aligned} \tag{8}$$

Now, our objective is to minimize the objective function (Z) stated as follows:

$$\begin{aligned} \text{Minimize } Z = \frac{1}{m+n+2} & \left[\frac{1}{2} (n+1) dc_1 t_1 (n+t_1) + \frac{1}{2} c_2 dt_1^2 (m+1)^2 e^{-\lambda(m+1)} \right. \\ & \left. + i_c (m+n+2) (1-t_1) + c_3 \right] \end{aligned} \tag{9}$$

subject to the conditions

$$q = (n+1) dt_1; \quad t_1 + t_2 = 1; \quad n \geq 0 \quad \text{and} \quad m \geq 0.$$

Special cases If we take $t_2 \rightarrow 0$ then $i_c = 0$. Consequently, the equation (9) reduces to

$$z = \left[\frac{c_1 q}{2} + \frac{c_3 d}{q} \right] \tag{10}$$

which is the classical EOQ model without backorder.

Preliminaries

Step Order Fuzzy Numbers

It is worthwhile to point out that a class of order fuzzy number (OFNs) represents the whole class of convex fuzzy numbers that possess continuous membership functions. To include all OFN (with discontinuous membership function) some generalization of function f and g is needed, developed by Kosinski [31] from function of bounded variation (BV). Then, all convex fuzzy numbers are contained in this new space $\mathcal{R}_{BV} \supset \mathcal{R}$ of OFN. Then, operations are defined by \mathcal{R}_{BV} in the similar way. The norm, however, will change into the norm of the Cartesian product of the space of function of bounded variation. All convex fuzzy numbers are contained in the new space \mathcal{R}_{BV} of OFN. Notice that functions of BV (Lojasiewicz [32]) are continuous except for a countable numbers of points.

Important consequence of this generalization is the possibility of introducing a sub-space of OFN, composed of pair of step function. If we fix a natural number K and split $[0,1]$ into $(K-1)$ sub-intervals $[a_i, a_{i+1})$ i.e. $\bigcup_{i=1}^{K-1} [a_i, a_{i+1}) = [0, 1)$, where $0 = a_1 < a_2 < \dots < a_K = 1$ and define a step function f of resolution K by putting u_i on each sub-interval $[a_i, a_{i+1})$, then each such function f is identified with a K -dimensional vectors $f \sim u = (u_1, u_2, \dots, u_K)$, the K -th value of u_K corresponds to $s = 1$, i.e., $f(1) = u_K$. Taking a pair of such functions, we have an order fuzzy number from \mathcal{R}_{BV} . Graphical representations of SOF are shown in Figs. 2, 3, and 4 in which the horizontal and vertical axes represent demand rate and the membership grade value respectively.

Definition 1 By a step order fuzzy number A of resolution K we mean an order pair (f, g) of functions such that $\{f, g : [0, 1] \rightarrow \mathbb{R}\}$ are K -step function.

We use \mathcal{R}_K for denotation the set of elements satisfying Definition 1. The example of a step order fuzzy number and its membership function are shown in Fig. 1. The set $\mathcal{R}_K \subset \mathcal{R}_{BV}$ has been extensively elaborated by Gruszczynska and Krajewska [33]. We can identify \mathcal{R}_K with the Cartesian product of $\mathbb{R}^K \times \mathbb{R}^K$ since each K -step function is represented by its K values. It is obvious that each elements of the space \mathcal{R}_K may be regarded as an approximation of elements from \mathcal{R}_{BV} . By increasing the number K of steps, we have the better approximation. The norm of \mathcal{R}_K is assumed to be the Euclidean one of \mathbb{R}^{2K} which provides us an inner product structure for our disposal.

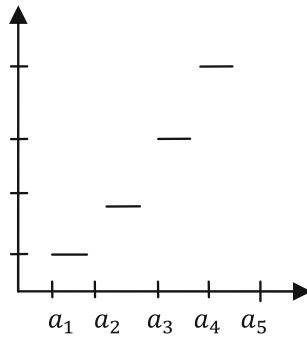


Fig. 2 Step up fuzzy number

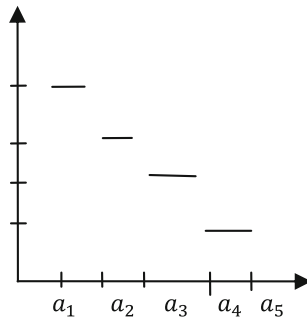


Fig. 3 Step down fuzzy number

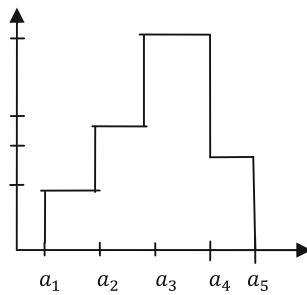


Fig. 4 Step function fuzzy number

Lattice Structure on \mathcal{R}_K

Let the set \mathcal{R}_K of step ordered fuzzy numbers with operations \vee and \wedge such that, for

$$A = (f_A, g_A) \text{ and } B = (f_B, g_B), A \vee B = (\sup \{f_A, f_B\}, \sup \{g_A, g_B\}),$$

$$A \wedge B = (\inf \{f_A, f_B\}, \inf \{g_A, g_B\}).$$

In the article (Kacprzak and Kosinski [34]), the algebra $(\mathcal{R}_K, \vee, \wedge)$ defines a lattice structure and it is proved in the following theorem.

Theorem 1 *The algebra $(\mathcal{R}_K, \vee, \wedge)$ is a lattice.*

Let \mathcal{B} be the set of two binary values: $\{0, 1\}$ and let us introduce the particular subset \mathcal{N} of \mathcal{R}_K where $\mathcal{N} = \{A = (\underline{u}, \underline{v}) \in \mathcal{R}_K : \underline{u} \in \mathcal{B}^K, \underline{v} \in \mathcal{B}^K\}$. It means such that each component of the vector \underline{u} as well as \underline{v} have value 1 and 0. It is easy to observe that all subsets of \mathcal{N} have both a join and a meet in \mathcal{N} . In fact, for pair of numbers form the set $\{0,1\}$ we can determine max and min and it is always constituted by 0 or 1. Therefore, \mathcal{N} creates a complete lattice. In such a lattice, we can distinguish the greatest element $\underline{1}$ represents by $(1,1,\dots,1)$ and the least element $\underline{0}$ represent by $(0,0,\dots,0)$.

Theorem 2 *The algebra $(\mathcal{N}, \vee, \wedge)$ is a complete lattice.*

We say that two elements A and B are compliments of each other if and only if $A \vee B = \underline{1}$ and $A \wedge B = \underline{0}$. The complement of a number is marked with A^c and is defined as follows:

Definition 2 Let $A \in \mathcal{N}$ be a step order fuzzy number represented by a binary vector $(a_1, a_2, \dots, a_{2K})$. By the complement of A , we understand

$A^c = (1 - a_1, 1 - a_2, \dots, 1 - a_{2K})$. A bounded lattice for which every element has a complement which is called a complemented lattice. Moreover, the structure of step order fuzzy numbers $(\mathcal{N}, \vee, \wedge)$ forms a complete and complemented lattice in which complements are unique. In fact it is a Boolean algebra. In the example with $K = 2$ a set of universe is created by vectors $\mathcal{N} = \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4 : a_i \in \{0, 1\}, \text{ for } i = 1, 2, 3, 4\}$. The complements of elements are $(0, 1, 0, 0)^c = (1, 0, 1, 1)$, $(1, 0, 0, 1)^c = (0, 1, 1, 0)$ etc. Now, we can rewrite the definition of the complements in terms of a new mapping.

Definition 3 For any $A \in \mathcal{N}$, we define its negation as $\mathcal{N}(A) = (1 - a_1, 1 - a_2, \dots, 1 - a_{2K})$, if $A = (a_1, a_2, \dots, a_{2K})$. It is obvious, from Definition 1 and 2, that the negation of given number A is its complement. Moreover, the operator \mathcal{N} is a strong negation, because it is involutive, i.e. $\mathcal{N}(\mathcal{N}(A)) = A$ for any $A \in \mathcal{N}$. One can refer [35] to define the strong negation N in terms of the standard strong negation \mathcal{N}_I on the unit interval $I = [0, 1]$ defined by $\mathcal{N}_I(x) = 1 - x, x \in I$, namely $((a_1, a_2, \dots, a_{2K})) = (\mathcal{N}_I(a_1), \mathcal{N}_I(a_2), \dots, \mathcal{N}_I(a_{2K}))$. In the classical Zadeh’s [8] fuzzy logic, the definition of a fuzzy implication on an abstract lattice $\mathcal{L} = (L, \leq_L)$ is based on the notation from the fuzzy set theory introduced in [35].

Implication of SOF in Inventory Systems

Generally, the features of SOF of an element appear whenever we can able to characterize it into clustered several subintervals. Also, the most vital factor of inventory model lies in its demand rate. Thus clustering is possible when the demand rate has exclusively sensitive on several seasons in a year. The commodities like medicine, umbrella, woollen shawl, rain coat, special types of vegetable etc. fall under this considerations. In each subinterval the membership values of fuzzy number is constant and are of step up and step down in nature which are to be determined by the decision maker. That is, within several interval values of demand rate, how much amount of membership grade will be added to the fuzzy information so that the total average inventory cost be minimum. Thus for any seasonal product, it is quite natural to incorporate SOF under consideration.

4-Tuples SOF

Let us define SOF for cost vector $\tilde{A} = \langle a_1, a_2, a_3, a_4 \rangle$ and for demand rate and other parameters as $\tilde{B} = \langle b_1, b_2, b_3, b_4 \rangle$ respectively and they can be expressed in Tables 1 and 2 respectively as follows:

Now, the member and non-membership functions for \tilde{A} may be stated as follows:

$$\mu(x) = \begin{cases} \mu_1 & \text{for } a_1 \leq x \leq a_2 \\ \mu_2 & \text{for } a_2 \leq x \leq a_3 \\ \mu_3 & \text{for } a_3 \leq x \leq a_4 \end{cases} \tag{11}$$

and

$$\gamma(x) = \begin{cases} (1 - \mu_1)w_1 & \text{for } a_1 \leq x \leq a_2 \\ (1 - \mu_2)w_2 & \text{for } a_2 \leq x \leq a_3 \\ (1 - \mu_3)w_3 & \text{for } a_3 \leq x \leq a_4 \end{cases} \tag{12}$$

where $w_1, w_2, w_3 < 1$ and they are called weights iff

$$w_1 + w_2 + w_3 = 1. \tag{13}$$

The schematic diagram of membership and non membership function of SOF are shown in Figs. 5 and 6 respectively.

Table 1 SOF for cost vector

Cost limit	Literal terms	Membership grade
$[a_1, a_2]$	Cheap	μ_1
$[a_2, a_3]$	High	μ_2
$[a_3, a_4]$	Very high	μ_3

Table 2 SOF for demand rate and other

Value limit	Literal terms	Membership grade
$[b_1, b_2]$	Poor	γ_1
$[b_2, b_3]$	Average	γ_2
$[b_3, b_4]$	High	γ_3

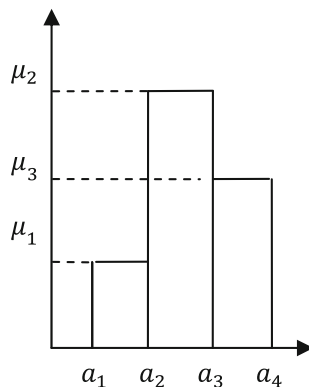


Fig. 5 Membership function of SOF

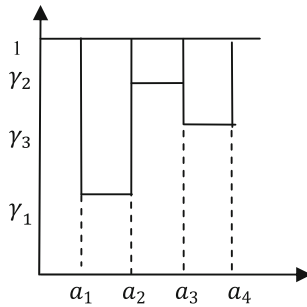


Fig. 6 Non-membership function of SOF

Basic Arithmetic Operations on Ordered SOF

Let $A = \langle a_1, a_2, a_3, a_4 \rangle$ and $B = \langle b_1, b_2, b_3, b_4 \rangle$ are two ordered SOF, then for usual Arithmetic operations $\{+, -, \times, \div\}$, namely addition, subtraction, multiplication between A and B are given below:

- (i) $A + B = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4 \rangle$,
- (ii) $A - B = \langle a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1 \rangle$
- (iii) $A \times B = \langle \text{Min}(a_i b_j), \text{Max}(a_i b_j) \rangle \forall i, j = 1, 2, 3, 4$
- (iv) $A/B = \langle \text{Min}(a_i/b_j), \text{Max}(a_i/b_j) \rangle$ for $b_j \neq 0, \forall i, j = 1, 2, 3, 4$
- (v) $\delta A = \langle \delta a_1, \delta a_2, \delta a_3, \delta a_4 \rangle$ if $\delta \geq 0$ and $\delta A = \langle \delta a_4, \delta a_3, \delta a_2, \delta a_1 \rangle$ if $\delta < 0$.

Centroid of SOF

From the perspectives of geometry, if a closed region R formed by a variable point (x, y) where y is a function of x , then the centroid of the enclosed region is defined by (\bar{x}_0, \bar{y}_0) and is defined by

$$\bar{x}_0 = \frac{\iint_R x dx dy}{\iint_R dx dy} \tag{14}$$

and

$$\bar{y}_0 = \frac{\iint_R y dx dy}{\iint_R dx dy} \tag{15}$$

Now, using (7) and (8) we get,

$$\begin{aligned} \bar{x}_0 &= \frac{(\iint_{R_1} + \iint_{R_2} + \iint_{R_3} + \iint_{R_4} \dots) x dx dy}{(\iint_{R_1} + \iint_{R_2} + \iint_{R_3} + \iint_{R_4} \dots) dx dy} \\ &= \frac{\frac{\mu_1}{2} (c_2^2 - c_1^2) + \frac{\mu_2}{2} (c_3^2 - c_2^2) + \frac{\mu_3}{2} (c_4^2 - c_3^2)}{\mu_1 (c_2 - c_1) + \mu_2 (c_3 - c_2) + \mu_3 (c_4 - c_3)} \end{aligned} \tag{16}$$

and

$$\bar{y}_0 = \frac{(\iint_{R_1} + \iint_{R_2} + \iint_{R_3} + \iint_{R_4} \dots) y dx dy}{(\iint_{R_1} + \iint_{R_2} + \iint_{R_3} + \iint_{R_4} \dots) dx dy}$$

$$= \frac{\frac{\mu_1^2}{2}(c_2 - c_1) + \frac{\mu_2^2}{2}(c_3 - c_2) + \frac{\mu_3^2}{2}(c_4 - c_3)}{\mu_1(c_2 - c_1) + \mu_2(c_3 - c_2) + \mu_3(c_4 - c_3)} \tag{17}$$

Ranking Index

According to the article proposed by Chen [36], the ranking index value of a given element is given by

$$I(x) = \sqrt{(\bar{x}_0)^2 + (\bar{y}_0)^2} \tag{18}$$

Maximum index value gives the value of strong fuzzy number, otherwise it is a weak fuzzy number.

Euclidean Distance

Let $\delta_{max} = \text{Max} \{x, x \in \text{Domain} (A_1, A_2, \dots, A_{n-1}, A_n)\}$ where A_i are fuzzy sets for $i = 1, 2, \dots, n$. The distance between the two points (\bar{x}_0, \bar{y}_0) and $(\delta_{max}, 0)$ is called Euclidean Distance and is defined by

$$d(x) = \sqrt{(\bar{x}_0 - \delta_{max})^2 + (\bar{y}_0)^2} \tag{19}$$

Now, minimizing the distance, we get the higher index value, which is generated from strong fuzzy number (degree of membership >0.5) otherwise it is a weak fuzzy number.

Fuzzy Mathematical Model

In reality, we know every parametric value portending to any inventory system have a characteristic of flexibility. That is why, in our model, the cost component vectors, demand rate and time fraction of the exponential demand rate are fuzzified occasionally. Therefore, the fuzzy mathematical problem can be stated as

$$\text{Minimize } \tilde{z} = \frac{(n+1)(n+t_1)t_1}{2(m+n+2)} \tilde{d}\tilde{c}_1 + \frac{(m+1)^2 t_1^2 e^{-\tilde{\lambda}(m+1)}}{2(m+n+2)} \tilde{d}\tilde{c}_2 + (1-t_1) \tilde{i}_c + \frac{\tilde{c}_3}{m+n+2} \tag{20}$$

$$= f_1 \tilde{A}_1 + f_2 \tilde{A}_2 + f_3 \tilde{A}_3 + f_4 \tilde{A}_4 \tag{21}$$

where

$$\begin{cases} f_1 = \frac{(n+1)(n+t_1)t_1}{2(m+n+2)} \\ f_2 = \frac{(m+1)^2 t_1^2}{2(m+n+2)} \\ f_3 = (1-t_1) \\ f_4 = \frac{1}{m+n+2} \end{cases} \tag{22}$$

and

$$\begin{cases} \tilde{A}_1 = \tilde{d}\tilde{c}_1 \\ \tilde{A}_2 = \tilde{d}\tilde{c}_2 e^{-\tilde{\lambda}(m+1)} \\ \tilde{A}_3 = \tilde{i}_c \\ \tilde{A}_4 = \tilde{c}_3 \end{cases} \tag{23}$$

Considering all the parameters as SOF, we have

$$\begin{cases} \widetilde{A}_1 = \langle d_1c_{11}, d_2c_{12}, d_3c_{14}c_{14} \rangle \\ \widetilde{A}_2 = \langle d_1c_{21}e^{-\lambda_1(m+1)}, d_2c_{22}e^{-\lambda_2(m+1)}, d_3c_{23}e^{-\lambda_3(m+1)}, d_4c_{24}e^{-\lambda_4(m+1)} \rangle \\ \widetilde{A}_3 = \langle i_{c1}, i_{c2}, i_{c3}, i_{c4} \rangle \\ \widetilde{A}_4 = \langle c_{31}, c_{32}, c_{33}, c_{34} \rangle \end{cases} \tag{24}$$

And, the SOF of the fuzzy objective function is

$$\tilde{z} = \langle z_1, z_2, z_3, z_4 \rangle \tag{25}$$

where

$$\begin{cases} z_1 = \frac{(n+1)(n+t_1)t_1}{2(m+n+2)}d_1c_{11} + \frac{(m+1)^2t_1^2e^{-\lambda_1(m+1)}}{2(m+n+2)}d_1c_{21} + (1-t_1)i_1 + \frac{c_{31}}{(m+n+2)} \\ z_2 = \frac{(n+1)(n+t_1)t_1}{2(m+n+2)}d_2c_{12} + \frac{(m+1)^2t_1^2e^{-\lambda_2(m+1)}}{2(m+n+2)}d_1c_{22} + (1-t_1)i_2 + \frac{c_{32}}{(m+n+2)} \\ z_3 = \frac{(n+1)(n+t_1)t_1}{2(m+n+2)}d_3c_{13} + \frac{(m+1)^2t_1^2e^{-\lambda_3(m+1)}}{2(m+n+2)}d_1c_{23} + (1-t_1)i_3 + \frac{c_{33}}{(m+n+2)} \\ z_4 = \frac{(n+1)(n+t_1)t_1}{2(m+n+2)}d_1c_{14} + \frac{(m+1)^2t_1^2e^{-\lambda_4(m+1)}}{2(m+n+2)}d_1c_{24} + (1-t_1)i_4 + \frac{c_{34}}{(m+n+2)} \end{cases} \tag{26}$$

The centroid of \tilde{z} is given by (\bar{x}_0, \bar{y}_0) where

$$\bar{x}_0 = \frac{\frac{\mu_1}{2}(z_2^2 - z_1^2) + \frac{\mu_2}{2}(z_3^2 - z_2^2) + \frac{\mu_3}{2}(z_4^2 - z_3^2)}{\mu_1(z_2 - z_1) + \mu_2(z_3 - z_2) + \mu_3(z_4 - z_3)} \tag{27}$$

and

$$\bar{y}_0 = \frac{\frac{\mu_1^2}{2}(z_2 - z_1) + \frac{\mu_2^2}{2}(z_3 - z_2) + \frac{\mu_3^2}{2}(z_4 - z_3)}{\mu_1(z_2 - z_1) + \mu_2(z_3 - z_2) + \mu_3(z_4 - z_3)} \tag{28}$$

We may rewrite the problem is as follows:

$$\text{Minimize } I(x) = \sqrt{(\bar{x}_0)^2 + (\bar{y}_0)^2} \tag{29}$$

and

$$\text{Minimize } d(x) = \sqrt{(\bar{x}_0 - \delta_{max})^2 + (\bar{y}_0)^2} \tag{30}$$

Cases of Optimality

Now, we shall study the cases where the cost component vectors and the demand parameters are of step up/down fuzzy numbers. For different seasons, we have assumed that the cost vectors and demand vectors follow the same membership grade values. We consider the cases of three seasons such as Summer, Rainy and Winter which are as follows:

Where we put $\gamma_1 = (1 - \mu_1)w_1, \gamma_2 = (1 - \mu_2)w_2, \gamma_3 = (1 - \mu_3)w_3$, and $w_1, w_2, w_3 < 1$ (Table 3).

Table 3 Assumptions on SOF

Cases of optimality	Summer	Rainy	Winter
Case-I: $\mu_1 < \mu_2 < \mu_3$	Down	Up	Up
Case-II: $\mu_1 > \mu_2 > \mu_3$	Up	Down	Down
Case-III: $\mu_1 < \mu_3 < \mu_2$	Down	Up	Down

Numerical Example-1

Let us consider the holding cost $c_1 = \$1.5$, the shortage cost $c_2 = \$1.2$, set up cost $c_3 = \$150$, idle time cost $i_c = \$4.5$, demand rate $d = 150$ units and time exponent multiplier $\lambda = 0.5$. Then, the optimal solution of the crisp optimal is

$$(q^* = 225.0, s^* = 55.182, t_1^* = t_2^* = 0.5, n^* = 2, m^* = 1, z^* = \$123.25).$$

Numerical Example-2

Let us assume SOF values of the different parameters $\lambda = (0.4, 0.6, 0.8, 0.9)$, $c_1 = (1.2, 1.4, 1.6, 1.8)$, $c_3 = (80, 100, 150, 160)$, $c_2 = (0.9, 1, 1.3, 1.4)$, $d = (130, 140, 150, 160)$, $i_c = (3.5, 4.5, 5, 5.5)$ and $w_1 = w_2 = w_3 = \frac{1}{3}$. The solutions of the three models for strong and weak fuzzy cases are provided in Table 4.

Now, considering only the expected values of the optimal solutions for strong and weak fuzzy numbers, we have the Table 5.

Moreover, we need to justify our result by the accuracy test, developed by Chen [36] and rank them accordingly. We shall do this only for case-II, as Table 5 shows model-II is optimum.

Table 4 Optimal solution under SOF

Cases	Fuzzy				Intuitionistic fuzzy				
	I(z)		d(z)		I(z)		d(z)		
	Max	Min	Max	Min	Max	Min	Max	Min	
Case-I	Z =	258.09	120.18	39.82	0.346	123.09	0.31	198.26	0.31
$\mu_1 < \mu_2 < \mu_3$	q =	441.31	220.658	220.66	132.69	222.72	128.86	433.91	128.86
$\mu_1 = 0.4$	S =	97.613	48.808	111.194	95.26	49.95	95.21	60.102	95.21
$\mu_2 = 0.7$	$t_1 =$	1	0.5	0.5	0.5	0.5	0.65	1	0.65
$\mu_3 = 0.8$	$t_2 =$	0	0.5	0.5	0.5	0.5	0.35	0	0.35
	n =	2	2	2	2	3	2	3	2
	m =	1	1	1	1	1	1	1	1
Case-II	Z =	235.66	110.40	75.66	0.333	309.55	107.04	237.69	0.3
$\mu_1 > \mu_2 > \mu_3$	q =	428.684	214.34	268.68	148.178	566.09	212.28	547.6	154.67
$\mu_1 = 0.8$	S =	85.531	42.766	74.47	98.51	79.08	39.54	80.92	101.39
$\mu_2 = 0.7$	$t_1 =$	1	0.5	1	0.72	1	0.5	1	0.74
$\mu_3 = 0.4$	$t_2 =$	0	0.5	0	0.28	0	0.5	0	0.26
	n =	2	2	2	2	3	2	4	2
	m =	1	1	1	1	1	1	1	1
Case-III	Z =	338.23	116.31	273.64	0.41	239.23	116.68	264.28	0.38
$\mu_1 < \mu_3 < \mu_2$	q =	581.67	218.12	567.08	138.43	582.42	218.41	561.97	142.42
$\mu_1 = 0.7$	S =	71.89	46.64	66.72	96.19	93.24	45.62	70.25	97.34
$\mu_2 = 0.8$	$t_1 =$	1	0.5	1	0.68	1	0.5	1	0.7
$\mu_3 = 0.9$	$t_2 =$	0	0.5	0	0.32	0	0.5	0	0.3
	n =	3	2	4	2	3	2	4	2
	m =	1	1	1	1	1	1	1	1

Table 5 General feasible solutions of the models

	Methods	IFS			FS		
		Case-I	Case-II	Case-III	Case-I	Case-II	Case-III
Strong fuzzy	Max $I(z)$	123.09	–	–	–	–	–
	Min $d(z)$	–	–	–	–	–	–
Weak fuzzy	Min $I(z)$	–	107.04	116.68	120.18	110.40	116.31
	Max $d(z)$	–	–	–	–	–	–

Table 6 Various $I(z)$ values for model-II

(μ_1, μ_2, μ_3)	$(\gamma_1, \gamma_2, \gamma_3)$	Score $(\mu - \gamma)$	Accuracy $(\mu + \gamma)$	Rank
(0.9, 0.8, 0.7)	(0.033, 0.067, 0.1)	$z^* = 112.63, q^* = 215.68, s^* = 44.89,$ $t_1^* = t_2^* = 0.5, n^* = 2, m^* = 1$	114.29	3
(0.7, 0.6, 0.5)	(0.1, 0.133, 0.167)	$z^* = 110.96, q^* = 214.64, s^* = 43.72,$ $t_1^* = t_2^* = 0.5, n^* = 2, m^* = 1$	114.07	2
(0.5, 0.4, 0.3)	(0.167, 0.2, 0.233)	$z^* = 104.69, q^* = 210.83, s^* = 38.35,$ $t_1^* = t_2^* = 0.5, n^* = 2, m^* = 1$	113.75	1

From Table 6, we see that the minimum membership grade for SOF in model-II provides optimum result under the considerations of standard score and accuracy function as well. Also, in Table 5, the optimality occurs when the demand rate is weak fuzzy number. The step down fuzzy number generates the minimum inventory cost due to seasonal variations (summer, rainy, winter). The required optimal solution is: the minimum average cost is $z^* = \$104.69$, the optimal order quantity is $q^* = 210.83$, the optimum shortage quantity is $s^* = 38.35$, the optimal opening and closing time are 12 h each, the inventory run time is 3 days and the shortage period is 2 days only. Also, from Table 5, the optimality occurs when the demand rate is considered as weak fuzzy number.

Conclusion

The present article develops a realistic EOQ model incorporating opening time and closing time of the inventory system. During stock out situation, demands of the customers follow an exponentially decaying with total time of shortage period. The basic insight lies in the consideration of SOF for several parameters of the model itself. In SOF model, the several membership grade values are considered but which one will give the minimum cost is the managerial part of an inventory system as a whole. However, introducing SOF in the intuitionistic fuzzy model, we have had a scope to utilize the laws of lattice algebra. Because, the negation or complementation of SOF numbers have been illustrated properly by max-min operators of a lattice. The larger step will give the stronger fuzzy number, but in our study, we have assumed only three steps; if it is possible to develop a model for 12 steps (the value of the parameter whose value can vary monthly in a year and perform 12 non-intersecting intervals) then the model under SOF would be viewed in real life problems. Moreover, in our study, we have shown that the step down fuzzy number provides the minimum inventory cost under weak fuzzy environment. The application of intuitionistic SOF in such EOQ model is quite new compared to the existing inventory literature. The present article may be extended in future in the light of α -levels set of both member and non-

membership functions of SOF, Hamming distance method and several Aggregation Operators methods.

References

1. Cárdenas-Barrón, L.E., Taleizadeh, A.A., Treviño-Garza, G.: An improved solution to replenishment lot size problem with discontinuous issuing policy and rework, and the multi-delivery policy into economic production lot size problem with partial rework. *Expert Syst. Appl.* **39**, 3888–3895 (2012)
2. Ghosh, S.K., Chaudhuri, K.S.: An EOQ model with a quadratic demand, time proportional deterioration and shortages in all cycles. *Int. J. Syst. Sci.* **37**, 663–672 (2006)
3. Ghosh, S.K., Khanra, S., Chaudhuri, K.S.: Optimal price and lot size determination for a perishable product under conditions of finite production, partial backordering and lost sale. *Appl. Math. Comput.* **217**, 6047–6053 (2011)
4. Sarkar, B.: An EOQ model with delay-in-payments and time-varying deterioration rate. *Math. Comput. Model.* **55**, 367–377 (2012)
5. Sarkar, B., Sarkar, S.: An improved inventory model with partial backlogging, time varying deterioration and stock-dependent demand. *Econ. Model.* **30**, 924–932 (2013)
6. Taleizadeh, A.A., Niaki, S.T.A., Aryanezhad, M.B., Nima, S.: A hybrid method of fuzzy simulation and genetic algorithm to optimize constrained inventory control systems with stochastic replenishments and fuzzy demand. *Inf. Sci.* **220**, 425–441 (2013)
7. De, S.K.: EOQ model with natural idle time and wrongly measured demand rate. *Int. J. Inventory Control Manag.* **3**, 329–354 (2013)
8. Zadeh, L.A.: The concept of a linguistic variable and its application to approximate reasoning, Part I. *Inf. Sci.* **8**, 199–249 (1975)
9. Kosinski, K.: Module of Step Ordered Fuzzy Numbers in Control of Material Point Motion. PJWSTK, Warszawa (2010)
10. Fortin, J., Dubois, D., Fargier, H.: Gradual numbers and their application to fuzzy interval analysis. *IEEE Trans. Fuzzy Syst.* **16**, 388–402 (2008)
11. Kosinski, W., Wilezyska-Sztyma, D.: Defuzzification and implication within ordered fuzzy numbers. In: WCCI 2010 IEEE World Congress on Computational Intelligence-CCIB, pp. 1073–1079, Barcelona, Spain, 18–23 July 2010
12. Kosinski, W., Wegrzyn-Wolska, K.: Step fuzzy numbers and neural networks in defuzzification functional approximation. *Adv. Intell. Model. Simul. Stud. Comput. Intell.* **422**, 25–41 (2012)
13. Chakraborty, S., Pal, M., Nayak, P.K.: Intuitionistic fuzzy optimization technique for Pareto-optimal solution of manufacturing inventory models with shortages. *Eur. J. Oper. Res.* **228**, 381–387 (2013)
14. Chen, S.J., Hwang, C.L.: Fuzzy scoring of fuzzy number—a direct comparison index. In: Chen, S.J., Hwang, C.L., Hwang, F.P. (eds.) *Fuzzy multiple attribute decision making: methods and applications*, pp. 256–258. Springer, Berlin (1989)
15. De, S.K., Sana, S.S.: Backlogging EOQ model for promotional effort and selling price sensitive demand—an intuitionistic fuzzy approach. *Ann. Oper. Res.* (2013). doi:10.1007/s10479-013-1476-3. ISSN-0254-5330
16. De, S.K., Goswami, A., Sana, S.S.: An interpolating by pass to Pareto optimality in intuitionistic fuzzy technique for an EOQ model with time sensitive backlogging. *Appl. Math. Comput.* **230**, 664–674 (2014)
17. Murakami, M., Maeda, S.S., Inamura, S.: Fuzzy decision analysis on the development of centralized regional energy control system. In: IFAC Symposium on Fuzzy Information, Knowledge Representation and Decision Analysis, pp. 363–368 (1983)
18. Yager, R.R.: A procedure of ordering fuzzy subsets of the unit interval. *Inf. Sci.* **24**, 143–161 (1981)
19. Yong, D., Qi, L.: A TOPSIS-based centroid-index ranking method of fuzzy numbers and its application to decision making. *Cybern. Syst.: An Int. J.* **36**, 581–595 (2005)
20. Adamo, J.M.: Fuzzy decision trees. *Fuzzy Set Syst.* **4**, 207–220 (1980)
21. Chen, L.H., Lu, H.W.: An approximate approach for ranking fuzzy numbers based on left and right dominance. *Int. J. Comput. Math. Appl.* **41**, 183–224 (1983)
22. Fortemps, P., Roubens, M.: Ranking of defuzzification method based on area compensation. *Fuzzy Set Syst.* **82**, 319–330 (1996)
23. Buckley, J.J., Chanas, S.: A fast method of ranking alternatives using fuzzy numbers. *Fuzzy Set Syst.* **30**, 337–339 (1989)
24. Jain, R.: Decision making in the presence of fuzzy variables. *IEEE Trans. Syst. Man Cybern.* **6**, 698–703 (1976)

25. Jain, R.: A procedure for multi aspect decision making using fuzzy sets. *Int. J. Syst.* **8**, 1–7 (1977)
26. Kerre, E.E.: The use of fuzzy set theory in electrocardiological diagnostics. In: Gupta, M.M., Sanchez, E. (eds.) *Approximate Reasoning in Decision Analysis*, pp. 277–282. North-Holland, Amsterdam (1982)
27. Nakamura, K.: Preference relation on a set of fuzzy utilities as a basis for decision making. *Fuzzy Sets Syst.* **20**, 147–162 (1986)
28. Mabuchi, S.: An approach to the comparison of fuzzy subsets with an α -cut dependent index. *IEEE Trans. Syst. Man Cybern.* **18**, 264–272 (1988)
29. Tran, L., Duckstein, L.: Comparison of fuzzy numbers using distance measure. *Fuzzy Set Syst.* **130**, 331–341 (2002)
30. De, S.K., Sana, S.S.: Fuzzy order quantity inventory model with fuzzy shortage quantity and fuzzy promotional index. *Econ. Model.* **31**, 351–358 (2013)
31. Kosinski, W.: On fuzzy number Calculus. *Int. J. Appl. Math. Comput. Sci.* **16**, 51–57 (2006)
32. Lojasiewicz, S.: *Introduction to the Theory of Real Functions*, Biblioteka Matematyczna, Tom 46 (in Polish) PWN, Warszawa (1973)
33. Gruszczynska, A., Krajewska, I.: Fuzzy Calculator on Step Ordered Fuzzy Numbers. UKW, Bydgoszcz (2008) (in Polish)
34. Kacprzak, M., Kosinski, W.: On lattice structure and implications on ordered fuzzy numbers, Submitted to Conference: EUSFLAT (2011)
35. Fodor, J.C., Roubens, M.: *Fuzzy Preference Modeling and Multi-criteria Decision Support*. Kluwer Academic Publishers, Dordrecht (1994)
36. Chen, S.H.: Ranking fuzzy numbers with maximizing sets and minimizing sets. *Fuzzy Set Syst.* **17**, 113–129 (1985)