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RESEARCH

A Fully Nonlinear Degenerate Free Transmission Problem

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Abstract

We study a free transmission problem driven by degenerate fully nonlinear operators. Our first result concerns the existence of a viscosity solution to the associated Dirichlet problem. By framing the equation in the context of viscosity inequalities, we prove regularity results for the constructed viscosity solution to the problem. Our findings include regularity in $C^{1,\alpha}$ spaces, and an explicit characterization of α in terms of the degeneracy rates. We argue by perturbation methods, relating our problem to a homogeneous, fully nonlinear uniformly elliptic equation.

Keywords Free transmission problems · Optimal regularity of solutions · Existence of solutions · Viscosity inequalities

Mathematics Subject Classification 35B65 · 35J60 · 35J70 · 35R35

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1 Introduction

We examine viscosity solutions $u \in C(\overline{\Omega})$ to the free transmission problem

$$\begin{aligned} |Du|^{\theta_1} F(D^2u) &= f(x) \quad \text{in} \quad \Omega^+(u) \cap \Omega \\ |Du|^{\theta_2} F(D^2u) &= f(x) \quad \text{in} \quad \Omega^-(u) \cap \Omega, \end{aligned} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^d$, $\Omega^+(u) := \{u > 0\}$, and $\Omega^-(u) := \{u < 0\}$. In addition, $F : S(d) \to \mathbb{R}$ is a (λ, Λ)-elliptic operator, $\theta_i > 0, i = 1, 2$, are fixed constants, the source term $f : \Omega \to \mathbb{R}$ is a bounded function and $S(d)$ stands for the space of $d \times d$ symmetric matrices. We prove the existence of viscosity solutions to the Dirichlet problem associated with (1) and establish optimal regularity in $C^{1,\alpha}$-spaces, with appropriate estimates.

The model in (1) accounts for a diffusion process degenerating as a power of the gradient. The degeneracy law depends on the sign of the solution, which introduces discontinuities along $\partial\{u > 0\}$ and $\partial\{u < 0\}$. Since the subregions where distinct degeneracy regimes take place are *unknown a priori*, the transmission interface is understood as a free boundary. Also, there is no a priori reason for $|\{u = 0\}| = 0$; as a consequence, (1) prescribes a PDE only in a subregion of Ω.

Transmission problems account for diffusion processes in heterogeneous media, with applications to thermal and electromagnetic conductivity and composite materials, such as fiber-reinforced structures. A typical formulation can be described as follows. Given a domain $\Omega \subset \mathbb{R}^d$, we choose $k-1$ open, mutually disjoint subregions $\Omega_i \Subset \Omega$, for $i = 1, \ldots, k-1$, and set

$$\Omega_k := \Omega \setminus \bigcup_{i=1}^{k-1} \overline{\Omega_i}.$$

Inside each Ω_i a different equation is prescribed. For example, let $A : \Omega \to \mathbb{R}^{d^2}$ be a matrix-valued mapping and consider

$$\operatorname{div}(A(x)Du) = 0 \qquad \text{in} \quad \Omega,$$

where

$$A(x) = A_i,$$

for $x \in \Omega_i$, where A_i are constant matrices and $i = 1, \ldots, k$. Within each subregion, u solves a divergence-form equation governed by constant coefficients. However, across the transmission interface $\partial\Omega_i$, the diffusion process may be discontinuous.

Those discontinuities introduce difficulties in the study of the problem, affecting the understanding of properties such as existence and uniqueness of solutions and their regularity. Here, an important aspect of the analysis is the geometry of the transmission surfaces.

The first treatment of this class of problems appeared in [34], and was followed by a number of developments [13, 20–22, 26, 31, 33, 38, 40, 41]. The findings reported in these papers concern the well-posedness of transmission problems in distinct settings. For a comprehensive account of these results, we refer the reader to [14].

The regularity of solutions to transmission problems has also been investigated in the literature. In [30] the authors consider a bounded domain Ω in the presence of a finite number of subregions $(\Omega_i)_{i=1}^k$ which are known a priori. In the interior of each sub-region an equation in the divergence-form holds. Under regularity assumptions on the diffusion coefficients and the geometry of the transmission interface, the authors prove that solutions are $C^{1,\alpha}$-regular, locally; their estimates do not depend on the proximity of the sub-regions (compare with [12]). A vectorial counterpart of those results is reported in [29]. In that paper, the authors also derive bounds on higher derivatives of the solutions, under additional conditions on the data of the problem.

In [4, 5] the authors examine a transmission problem related to the theory of conductivity. The model in [4] consists of a bounded domain with two compactly contained sub-regions. These are ε-apart, where $\varepsilon > 0$ is a parameter. Inside each sub-region, the divergence-form equation governing the problem has constant coefficient $k > 0$. In the complementary region the diffusion coefficient is taken to be equal to 1. The authors consider the case of perfect conductivity ($k = +\infty$) and examine the behavior of gradient bounds for the solutions as $\varepsilon \to 0$. Though it is known that such bounds deteriorate as both subregions approach each other, the findings of [4] produce a blow-up rate for those estimates. In [5] the authors examine the case of multiple subregions – standing for multiple inclusions – and consider also the case of insulation ($k = 0$). We also mention [15].

A typical bottleneck in the regularity theory for solutions to transmission problems is the geometry of the transmission interface. In [19], the authors consider a domain $\Omega \subset \mathbb{R}^d$ and a subregion $\Omega_1 \Subset \Omega$. By prescribing Ω_1, they also define $\Omega_2 := \Omega \setminus \Omega_1$. The main contribution of that paper is in the fact that $\partial\Omega_1$ is supposed to be merely of class $C^{1,\alpha}$. Under this assumption, and a balance condition relating the normal derivatives of the solutions across the transmission interface, the authors prove $C^{1,\alpha}$-regularity of solutions to a problem driven by the Laplace operator. Their arguments rely on a new stability result, connecting the transmission problem under analysis with an auxiliary model, with flat interfaces.

The developments mentioned so far concern problems with transmission interfaces known a priori. However, a natural generalization regards the case where those sub-regions depend on the solution and, as a consequence, are endogenously determined. In this case, transmission problems can be framed in the context of free boundary analysis. This is precisely the context in the present work. Owing to the fact that the transmission interface behaves as a free boundary, this class of models is referred to as *free transmission problems*.

In [1], the authors consider a free transmission problem governed by the minimization of the functional

$$I(v) := \int_\Omega \left(\frac{1}{2}\langle A(x,v)Dv,\, Dv\rangle + \Lambda(v) + fv\right) dx, \tag{2}$$

where

$$\begin{aligned} A(x,u) &:= A_+(x)\chi_{\{u>0\}} + A_-(x)\chi_{\{u\leq 0\}},\\ \Lambda(u) &:= \lambda_+(x)\chi_{\{u>0\}} + \lambda_-(x)\chi_{\{u\leq 0\}},\\ f &:= f_+(x)\chi_{\{u>0\}} + f_-(x)\chi_{\{u\leq 0\}}, \end{aligned}$$

and $A_\pm : \Omega \to \mathbb{R}^{d^2}$ are matrix-valued mappings satisfying suitable ellipticity conditions, $f_\pm : \Omega \to \mathbb{R}$ are source terms in appropriate Lebesgue spaces and $\lambda_\pm : \Omega \to \mathbb{R}$ encode balance conditions of the model. The critical points of (2) satisfy the divergence form equations

$$\begin{cases} \operatorname{div}(A_+(x)Du(x)) = f_+ & \text{in } \{u>0\}\cap\Omega \\ \operatorname{div}(A_-(x)Du(x)) = f_- & \text{in } \{u\leq 0\}^\circ\cap\Omega, \end{cases} \tag{3}$$

equipped with a flux condition across the free transmission interface $\partial\{u > 0\} \cap \Omega$; the latter is derived through a Hadamard-type argument. The authors establish the existence of minimizers for (2), with uniform estimates in $L^\infty(\Omega)$. Notice that the functional under analysis lacks convexity, which entails further difficulties in the analysis. Moreover, the authors resort to a perturbation argument and suppose A_+ and A_- to be close, in a suitable topology. Under those conditions, solutions to (3) are proved to be asymptotically Lipschitz.

Free transmission problems in the fully nonlinear setting have also been studied in the literature. A uniformly elliptic problem is the subject of [35]. In that paper, the authors consider the model

$$F_1(D^2u)\chi_{\{u>0\}} + F_2(D^2u)\chi_{\{u<0\}} = f \qquad \text{in } \Omega, \tag{4}$$

where $F_i : S(d) \to \mathbb{R}^d$ are uniformly elliptic fully nonlinear operators, and $f \in L^\infty(\Omega)$. Under distinct assumptions on F_i, the authors prove regularity estimates in Hölder and in $W^{2,\mathrm{BMO}}$-spaces. Further density conditions on the negative phase of the problem unlock the proof of quadratic growth for the solutions at branch points of the free boundary. In [36], the authors prove the existence of L^p-viscosity solutions to (4); their argument relies on a combination of the comparison principle for an approximate problem, combined with Perron's method and a fixed point theorem. Once again, further conditions on F_i produce more refined information on the solutions. In this concrete case, leading to the existence of L^p-strong solutions.

The model under analysis in the present paper is a free transmission problem. Indeed, the regions where distinct degeneracy laws hold depend on the solution itself. In addition, the problem has a fully nonlinear, *non-variational*, structure. Finally, the equation is allowed to degenerate and does so as the gradient of the solutions vanishes.

Fully nonlinear equations degenerating as a power of the gradient have been examined in various contexts; the work-horse of the theory takes the form

$$|Du|^\theta F(D^2u) = f(x) \qquad \text{in } \Omega, \tag{5}$$

where F is a fully nonlinear uniformly elliptic operator, $\theta > -1$ is a constant, and $f \in L^\infty(\Omega) \cap C(\Omega)$. This is modeled as a non-variational, fully nonlinear, variant of the p-Poisson equation. Among the results available for the solutions to (5), we mention comparison and maximum principles, well-posedness for the Dirichlet problem, and an Aleksandroff-Bakelman-Pucci estimate; see [6], [7], [8], [9], [10], [24].

The regularity of solutions to (5) is the subject of [27], [11] and [3]. If $u \in C(\Omega)$ is a viscosity solution to (5), then $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$, with the appropriate estimates. In particular, the Hölder-exponent α satisfies

$$\alpha \in (0, \alpha_0), \qquad \alpha \leq \frac{1}{1+\theta},$$

where α_0 stands for the exponent in the regularity theory associated with the homogeneous equation $F = 0$; see Proposition 3 and, for instance, [17, Section 5.3]. If F is convex, then $\alpha_0 = 1$ and solutions are precisely $C^{1,\frac{1}{1+\theta}}$-regular, locally.

The case of variable exponents $\theta = \theta(x)$ is the topic of [16]. In that paper, the authors prove optimal regularity estimates for viscosity solutions in $C^{1,\alpha}$-spaces. Here, $\alpha \in (0, 1)$ depends on the regularity of solutions to equation $F = 0$ and on lower and upper bounds for the exponent $\theta = \theta(x)$. Although the authors work under the assumption that $\theta(\cdot)$ is a continuous function, their estimates do no depend on the modulus of continuity of θ. Because the continuity of $\theta(\cdot)$ plays no role in the estimates in [16], the findings of that paper are very related to the present work and most of the proofs in Section 4 follow by adjusting techniques and methods of [16]. See also [27, 28, 32].

We also mention the findings recently reported in [2]. In that paper, the authors consider viscosity solutions to equations of the form

$$\sigma(|Du|)\, F(D^2u) = f \quad \text{in} \quad \Omega, \tag{6}$$

under the assumption that σ^{-1} is a Dini-continuous modulus of continuity. In this context, they prove that solutions are locally of class C^1, with estimates.

Our first main result, concerning the existence of viscosity solutions, reads as follows.

Theorem 1 *(Existence of solutions) Let $\Omega \subset \mathbb{R}^d$ be a bounded domain which satisfies a uniform exterior sphere condition. Suppose Assumptions A1 and A2 (see below) hold, $f \in C(\overline{\Omega})$ and $g \in C(\partial\Omega)$. Then there exists a viscosity solution $u \in C(\overline{\Omega})$ to*

$$\begin{cases} |Du|^{\theta_1}\, F(D^2u) = f(x) & \text{in} \quad \Omega^+(u) \cap \Omega \\ |Du|^{\theta_2}\, F(D^2u) = f(x) & \text{in} \quad \Omega^-(u) \cap \Omega \\ u = g & \text{on} \quad \partial\Omega. \end{cases} \tag{7}$$

The solution u obtained is a viscosity subsolution to

$$\min\left\{ |Du|^{\max(\theta_1,\theta_2)} F(D^2u),\ F(D^2u) \right\} = C_0 \quad \text{in} \quad \Omega \tag{8}$$

and a viscosity supersolution to

$$\max\left\{|Du|^{\max(\theta_1,\theta_2)}F(D^2u),\ F(D^2u)\right\} = -C_0 \quad in \quad \Omega, \tag{9}$$

where $C_0 = \|f\|_{L^\infty(\Omega)}$.

Compare Theorem 1 with the findings in [36]. We remark that our arguments do not imply a comparison principle for the equations in (7), and we do not know if solutions to the Dirichlet problem (7) are unique.

Once the existence of solutions is addressed, we examine their regularity. From this point on we assume for simplicity that $\Omega = B_1$. We observe that the $C^{1,\alpha}$-regularity only relies on the two differential inequalities (8)-(9). The $C^{1,\alpha}$ result is based on a perturbation argument which relates (8)-(9) with equation $\overline{F} = 0$ for some $\overline{F}$ having the same structure as F. Our second main theorem is the following.

Theorem 2 *(Hölder-regularity of the gradient) Assume* $\Omega = B_1$. *Let* $u \in C(B_1)$ *be a viscosity subsolution to* (8) *and a viscosity supersolution to* (9) *for some* $C_0 \geq 0$. *Suppose Assumptions A1 and A2 (see below) hold. Let*

$$\alpha \in (0, \alpha_0), \ \alpha \leq \frac{1}{1 + \max(\theta_1, \theta_2)}. \tag{10}$$

Then $u \in C^{1,\alpha}_{\text{loc}}(B_1)$ *and, for every* $0 < \tau < 1$, *there exists* $C > 0$ *such that*

$$\|u\|_{C^{1,\alpha}(B_\tau)} \leq C\left(\|u\|_{L^\infty(B_1)} + \max\left\{C_0, C_0^{\frac{1}{1+\max(\theta_1,\theta_2)}}\right\}\right), \tag{11}$$

where $C = C(d, \lambda, \Lambda, \theta_2, \alpha, \tau)$.

If F is a convex operator, the Evans-Krylov theory for equation $F = 0$ becomes available giving $\alpha_0 = 1$. Then, Theorem 2 produces an *optimal regularity result* and viscosity solutions to a pair of differential inequalities (8)-(9) are of class $C^{1,\alpha^*}_{\text{loc}}(B_1)$, where

$$\alpha^* := \frac{1}{1 + \max(\theta_1, \theta_2)}, \tag{12}$$

with appropriate estimates. We notice, however, the convexity assumption on F can be substantially relaxed. Indeed, one may require solutions to the equation $F = 0$ to have uniform $C^{1,1}$-estimates, as in [17, Chapter 7]. Alternatively, one may impose the convexity condition on the *recession profile* associated with F; see [37, 39]. Indeed, to impose a convexity condition on the operator F^*, defined by

$$F^*(M) := \lim_{\mu\to 0} \mu F\left(\mu^{-1}M\right),$$

the modulus of continuity driven by the exponent in (12) follows. In addition, we believe the methods in the present paper may be adjusted to free transmission problems driven by degeneracies of the form (6), yielding differentiability of the solutions in that case as well.

Remark 1 For the sake of simplicity, suppose $0 < \theta_1 \leq \theta_2$. Consider the problem

$$|Du|^{\theta_1 \chi_{\{u>0\}} + \theta_2 \chi_{\{u<0\}}} F(D^2u) = f \quad \text{in} \quad \Omega. \tag{13}$$

It differs from (1) in the sense that it prescribes the equation in the entire domain. The proof of Theorem 1 also yields the existence of a viscosity solution u to (13) in the standard sense on $\Omega^+(u) \cup \Omega^-(u)$, however on the set $\{u = 0\}$, u is only a viscosity subsolution to

$$\min \left\{ |Du|^{\theta_2} F(D^2u),\, F(D^2u) \right\} = f(x)$$

and a viscosity supersolution to

$$\max \left\{ |Du|^{\theta_2} F(D^2u),\, F(D^2u) \right\} = f(x).$$

Hence, the conclusion of Theorem 2 is also available in this setting.

The remainder of this paper is organized as follows. In Section 2.1 we describe the main assumptions of the paper. In Section 2.2 we recall some elementary notions and show preliminary results, whereas in Section 2.3 we discuss scaling properties of the model. The proof of Theorem 1 is in Section 3. Section 4 is devoted to the proof of Theorem 2.

2 Preliminaries

In this section we collect a few notions, known and preliminary results, and the assumptions under which we will work in this article.

2.1 Main Assumptions

Our main assumptions concern the uniform ellipticity of the fully nonlinear operator governing (1) and the degeneracy degree constants θ_1 and θ_2.

A 1 (Uniform ellipticity) The function $F : S(d) \to \mathbb{R}$ is such that $F(0) = 0$ and is (λ, Λ)-uniformly elliptic; that is, for some fixed constants $0 < \lambda \leq \Lambda$,

$$\lambda \, \|N\| \;\leq\; F(M) \;-\; F(M \;+\; N) \;\leq\; \Lambda \, \|N\|$$

for every $M, N \in S(d)$, with $N \geq 0$.

A 2 (Degeneracy rates) The constants $\theta_1, \theta_2 \in \mathbb{R}$ satisfy

$$0 \;<\; \theta_1 \;\leq\; \theta_2.$$

We notice that Assumption 1 in particular implies that F is degenerate elliptic.

2.2 Preliminary Notions and Results

We begin with the definition of the Pucci extremal operators.

Definition 1 (Extremal operators) Let $0 < \lambda \le \Lambda$ be as in Assumption 1. We define the extremal Pucci operators $\mathcal{P}^{\pm}_{\lambda,\Lambda} : S(d) \to \mathbb{R}$ as follows:

$$\mathcal{P}^{+}_{\lambda,\Lambda}(M) := -\Lambda \sum_{e_i<0} e_i - \lambda \sum_{e_i>0} e_i$$

and

$$\mathcal{P}^{-}_{\lambda,\Lambda}(M) := -\Lambda \sum_{e_i>0} e_i - \lambda \sum_{e_i<0} e_i,$$

where $\{e_1, e_2, \ldots, e_d\}$ are the eigenvalues of M.

We write $\mathcal{P}^{\pm}_{\lambda,\Lambda} = \mathcal{P}^{\pm}$, when ellipticity constants have been set. For properties of the extremal operators, we refer the reader to [17, Section 2.2] or [18]. For the sake of completeness, we recall the notion of viscosity solution, see [23]. It is the so called C-viscosity solution in the terminology of [18].

Definition 2 (Viscosity solution) Let $G : \Omega \times \mathbb{R} \times \mathbb{R}^d \times S(d) \to \mathbb{R}$ be a degenerate elliptic operator. We say that an upper semicontinuous function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution to

$$G(x, u, Du, D^2u) = 0 \tag{14}$$

in U if, whenever $\varphi \in C^2(\Omega)$ and $u - \varphi$ attains a local maximum at $x_0 \in \Omega$, we have

$$G(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \le 0.$$

Similarly, we say that a lower semicontinuous function $u : \Omega \to \mathbb{R}$ is a viscosity supersolution to (14) if, whenever $\varphi \in C^2(\Omega)$ and $u - \varphi$ attains a local minimum at $x_0 \in \Omega$, we have

$$G(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \ge 0.$$

If u is both a viscosity subsolution and supersolution to (14), we say u is a viscosity solution to (14).

We recall Perron's method, see e.g. Theorem 4.1 of [23].

Lemma 1 *(Perron's method) Let Ω be a bounded domain and $G \in C(\Omega \times \mathbb{R}^d \times S(d))$ be degenerate elliptic. Suppose the comparison principle holds for* (14) *with this G. Suppose further that there exist a viscosity subsolution $\underline{w} \in C(\overline{\Omega})$ and a viscosity supersolution $\overline{w} \in C(\overline{\Omega})$ of* (14) *such that $\underline{w} \le \overline{w}$ in Ω and $\underline{w} = \overline{w}$ on $\partial\Omega$. Then*

$$u(x) := \left\{u(x) \mid \underline{w} \le v \le \overline{w},\ v \text{ is a viscosity subsolution to } (14)\right\}$$

is a viscosity solution to (7).

We continue by stating the maximum principle for viscosity solutions, Theorem 3.2 of [23].

Proposition 1 *Let Ω be a bounded domain and $H, G \in C(B_1 \times \mathbb{R}^d \times S(d))$ be degenerate elliptic. Let u be a viscosity subsolution to $G(x, Du, D^2u) = 0$ and w be a viscosity supersolution to $H(x, Dw, D^2w) = 0$ in Ω. Let $\psi \in C^2(\Omega \times \Omega)$. Define $v : \Omega \times \Omega \to \mathbb{R}$ by*

$$v(x, y) := u(x) - w(y).$$

Suppose further that $(\overline{x}, \overline{y}) \in \Omega \times \Omega$ is a local maximum of $v - \psi$ in $\Omega \times \Omega$. Then, for each $\varepsilon > 0$, there exist matrices X and Y in $\mathcal{S}(d)$ such that

$$G\left(\overline{x}, D_x\psi(\overline{x}, \overline{y}), X\right) \leq 0 \leq H\left(\overline{y}, -D_y\psi(\overline{x}, \overline{y}), Y\right),$$

and the matrix inequality

$$-\left(\frac{1}{\varepsilon} + \|A\|\right) I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \varepsilon A^2$$

holds true, where $A = D^2\psi\,(\overline{x}, \overline{y})$.

When developing perturbation methods we will need compactness properties of the solutions. We will use [28, Theorem 1.1] which is stated below in a simplified form.

Proposition 2 *(Hölder-continuity) Let $u \in C(B_1)$ be a bounded viscosity supersolution to*

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u) = -C_0 \qquad in \qquad \{|Du| > \gamma\}$$

and a bounded viscosity subsolution to

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2u) = C_0 \qquad in \qquad \{|Du| > \gamma\},$$

for some fixed $\gamma > 0$, and $C_0 \geq 0$. Then $u \in C^{\beta}_{\text{loc}}(B_1)$ and for every $0 < \tau < 1$ there exists $C > 0$ such that

$$\|u\|_{C^\beta(B_\tau)} \leq C.$$

The constant β depends only on d, λ and Λ, and C depends only on d, λ, Λ, γ, $\|u\|_{L^\infty(B_1)}$, C_0 and τ.

The above proposition implies Hölder regularity of viscosity solutions to variants of (8)-(9). Indeed, consider $u \in C(B_1)$ which is a viscosity subsolution to

$$\min\left\{|q + Du|^{\theta_2}\, F(D^2u),\ F(D^2u)\right\} = 1 \tag{15}$$

and a viscosity supersolution to

$$\max\left\{|q+Du|^{\theta_2}\,F(D^2u),\ F(D^2u)\right\}\ =-1 \tag{16}$$

in the unit ball B_1, where $q\in\mathbb{R}^d$ is an arbitrary vector. Let $A_0>1$ be such that $|q|<A_0$. Then since $|q+p|>A_0>1$ if $|p|>2A_0$, it is easy to see that u is a viscosity subsolution to

$$F(D^2u)=0 \qquad \text{in}\ \ \{|Du|>2A_0\}.$$

As a consequence, in the set $\{|Du|>2A_0\}$, u is a viscosity subsolution to

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2u)=0.$$

Similarly we obtain that in the set $\{|Du|>2A_0\}$, u is a viscosity supersolution to

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u)=0.$$

A straightforward application of Proposition 2 thus leads to the following corollary.

Corollary 1 *Let $u\in C(B_1)$ be a viscosity subsolution to (15) and a viscosity supersolution to (16). Let Assumptions A1, A2 hold and let $\|u\|_{L^\infty(B_1)}\le 1$. Suppose further that $|q|<A_0$, for some fixed constant $A_0>1$. Then $u\in C^{\beta}_{\text{loc}}(B_1)$ for some $\beta\in(0,1)$ depending only on d, λ, Λ. In addition, for every $0<\tau<1$, there exists $C>0$ such that*

$$\|u\|_{C^\beta(B_\tau)}\ \le\ C \tag{17}$$

with $C=C\,(d,\lambda,\Lambda,A_0,\tau)$.

We recall the standard $C^{1,\alpha_0}_{\text{loc}}$-regularity result for solutions to $F=0$, see e.g. [17, Corollary 5.7].

Proposition 3 *Let F satisfy Assumption A1 and let $h\in C(B_1)$ be a viscosity solution to*

$$F(D^2h)\ =\ 0 \qquad in\ \ B_1.$$

Then $h\in C^{1,\alpha_0}_{\text{loc}}(B_1)$, for some universal constant $\alpha_0\in(0,1)$. Furthermore, there exists $C>0$ depending only on d, λ and Λ, such that

$$\|h\|_{C^{1,\alpha_0}(B_{1/2})}\ \le\ C\,\|h\|_{L^\infty(B_{3/4})}\,.$$

2.3 Scaling Properties

In this section we examine scaling properties of equations (8) and (9). Similar properties apply to (1). We only discuss scaling about the origin but the procedure can be obviously done about every point with obvious adjustments.

Suppose u is a viscosity subsolution to (8) and a viscosity supersolution to (9) in B_1 but in fact we only require that (8)-(9) be satisfied in B_r for some $r > 0$. We define for $K > 0$,

$$v(x) := \frac{u(rx)}{K}.$$

A straightforward computation implies that in particular v is a viscosity subsolution to

$$\min\left\{|Dv|^{\theta_2}\overline{F}(D^2v),\ \overline{F}(D^2v)\right\} = \overline{C}_0 \quad \text{in} \quad B_1$$

and a viscosity supersolution to

$$\max\left\{|Dv|^{\theta_2}\overline{F}(D^2v),\ \overline{F}(D^2v)\right\} = -\overline{C}_0 \quad \text{in} \quad B_1,$$

where

$$\overline{F}(M) := \frac{r^2}{K}F\left(\frac{K}{r^2}M\right)$$

and

$$\overline{C}_0 = C_0 \max\left(\frac{r^{2+\theta_2}}{K^{1+\theta_2}}, \frac{r^2}{K}\right).$$

Choosing

$$K := \left[\|u\|_{L^\infty(B_1)} + \max\left\{C_0, C_0^{\frac{1}{1+\theta_2}}\right\}\right]$$

and setting $r := \varepsilon < 1$ we obtain $\|v\|_{L^\infty(B_1)} \leq 1$ and $\overline{C}_0 \leq \varepsilon$. Thus by this kind of scaling we can always assume that viscosity subsolutions/supersolutions u of (8)/(9) satisfy $\|u\|_{L^\infty(B_1)} \leq 1$ and $C_0 \leq 1$ or C_0 is arbitrarily small.

3 Existence of Solutions

Next we prove the existence of a viscosity solution to (7) with the required properties. We start by considering an approximating problem and establishing a comparison principle. Let $v \in C(\overline{\Omega})$. For $0 < \varepsilon < 1$, define the function g_ε^v as

$$g_\varepsilon^v := \max\left(\min\left(\frac{v+\varepsilon}{2\varepsilon}, 1\right), 0\right) \quad \text{on } \Omega$$

and $g_\varepsilon^v = 0$ on $\mathbb{R} \setminus \Omega$. Let $\eta_\varepsilon(\cdot)$ be the standard mollifier and consider $h_\varepsilon^v(x) := (g_\varepsilon^v * \eta_\varepsilon)(x)$, for $x \in \Omega$. Finally, we define the exponent function $\theta_\varepsilon^v : \Omega \to \mathbb{R}$ by setting

$$\theta_\varepsilon^v(x) := \theta_1 h_\varepsilon^v(x) + (1 - h_\varepsilon^v(x))\theta_2.$$

Notice that $\theta_1 \le \theta_\varepsilon^v \le \theta_2$. We consider the family of equations

$$(\varepsilon + |Du|)^{\theta_v^\varepsilon(x)}\left[\varepsilon u + F(D^2u)\right] = f(x) \qquad \text{in } \Omega, \tag{18}$$

and prove a comparison principle for its sub and supersolutions.

Proposition 4 *(Comparison principle) Let Ω be a bounded domain, F be degenerate elliptic and $f \in C(\overline{\Omega})$. Let $u \in USC(\overline{\Omega})$ be a viscosity subsolution to (18) and $w \in LSC(\overline{\Omega})$ be a viscosity supersolution to (18). Suppose $u \le w$ on $\partial\Omega$. Then, $u \le w$ in $\overline{\Omega}$.*

Proof If the statement is false, we have $\max_{x\in\overline{\Omega}}(u - w)(x) =: \tau > 0$. For $\delta > 0$ we define $\Phi_\delta : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ as

$$\Phi_\delta(x, y) := u(x) - w(y) - \frac{|x-y|^2}{2\delta}.$$

Let $(x_\delta, y_\delta) \in \overline{\Omega} \times \overline{\Omega}$ be such that

$$\max_{x,y\in\overline{\Omega}} \Phi_\delta(x, y) = \Phi_\delta(x_\delta, y_\delta) \ge \tau.$$

We know (see Lemma 3.1 of [23]) that

$$\lim_{\delta\to 0} \frac{|x_\delta - y_\delta|^2}{\delta} = 0 \tag{19}$$

and thus, for small δ, we have $x_\delta, y_\delta \in \Omega$. From Theorem 3.2 of [23] (see also Proposition 1), there exist $X, Y \in S(d)$ such that

$$\left(\frac{x_\delta - y_\delta}{\delta}, X\right) \in \overline{J}^{2,+}u(x_\delta) \quad \text{and} \quad \left(\frac{x_\delta - y_\delta}{\delta}, Y\right) \in \overline{J}^{2,-}w(y_\delta),$$

with

$$-\frac{3}{\delta}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{3}{\delta}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{20}$$

where I is the identity matrix. Inequality (20) implies $X \leq Y$ and, as a consequence of the degenerate ellipticity of F, we thus have for sufficiently small δ,

$$\frac{\varepsilon\tau}{2} \leq \varepsilon\left(u(x_\delta) - w(y_\delta)\right) \leq \frac{f(x_\delta)}{\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}\right)^{\theta_v^\varepsilon(x_\delta)}} - \frac{f(y_\delta)}{\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}\right)^{\theta_v^\varepsilon(y_\delta)}}. \tag{21}$$

Let $|f(x)| \leq C_1$ for all $x \in \Omega$ and let ω be a modulus of continuity of f on $\overline{\Omega}$. We notice that

$$\min\left(\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}\right)^{\theta_v^\varepsilon(x_\delta)}, \left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}\right)^{\theta_v^\varepsilon(y_\delta)}\right) \geq \varepsilon^{\theta_2},$$
$$\max\left(-\theta_v^\varepsilon(x_\delta)\ln\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}\right), -\theta_v^\varepsilon(y_\delta)\ln\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}\right)\right) \leq -\theta_2 \ln\varepsilon.$$

Let C_2 be the Lipschitz constant of the function $\theta_v^\varepsilon(x)$ and recall that $|\ln(\varepsilon + r)| \leq |\ln\varepsilon| + r$ for $r \geq 0$. Then

$$\begin{aligned}
&\frac{f(x_\delta)}{\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}\right)^{\theta_v^\varepsilon(x_\delta)}} - \frac{f(y_\delta)}{\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}\right)^{\theta_v^\varepsilon(y_\delta)}} \leq \frac{f(x_\delta) - f(y_\delta)}{\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}\right)^{\theta_v^\varepsilon(x_\delta)}} \\
&+ f(y_\delta)\left(\frac{1}{\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}\right)^{\theta_v^\varepsilon(x_\delta)}} - \frac{1}{\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}\right)^{\theta_v^\varepsilon(y_\delta)}}\right) \\
&\leq \omega(|x_\delta - y_\delta|)\varepsilon^{-\theta_2} + C_1\left|e^{-\theta_v^\varepsilon(x_\delta)\ln\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}\right)} - e^{-\theta_v^\varepsilon(y_\delta)\ln\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}\right)}\right| \\
&\leq \omega(|x_\delta - y_\delta|)\varepsilon^{-\theta_2} + C_1 e^{-\theta_2\ln\varepsilon}|\theta_v^\varepsilon(x_\delta) - \theta_v^\varepsilon(y_\delta)|\left|\ln\left(\varepsilon + \frac{|x_\delta - y_\delta|}{\delta}\right)\right| \\
&\leq \omega(|x_\delta - y_\delta|)\varepsilon^{-\theta_2} + C_1\varepsilon^{-\theta_2}C_2|x_\delta - y_\delta|\left(|\ln\varepsilon| + \frac{|x_\delta - y_\delta|}{\delta}\right).
\end{aligned}$$

Therefore, letting $\delta \to 0$ in (21) and using (19), we obtain $\frac{\varepsilon\tau}{2} \leq 0$, which is a contradiction. □

Once the comparison principle is available for (18), we examine the existence of viscosity solutions for this equation. To use Perron's method, we construct continuous viscosity sub and supersolutions to (18), agreeing with g on the boundary $\partial\Omega$.

Lemma 2 *(Existence of global sub and supersolutions) Let Ω be a bounded domain which satisfies a uniform exterior sphere condition. Let Assumptions A1 and A2 hold and let $f \in C(\overline{\Omega})$, $g \in C(\partial\Omega)$. Then there exist a viscosity subsolution $\underline{w} \in C(\overline{\Omega})$ to (18) and a viscosity supersolution $\overline{w} \in C(\overline{\Omega})$ to (18) for every $0 < \varepsilon < 1$ and $v \in C(\overline{\Omega})$, such that $\underline{w} = \overline{w} = g$ on $\partial\Omega$.*

Proof We construct a continuous viscosity supersolution $\overline{w}$ of (18) for every $0 < \varepsilon < 1$ and $v \in C(\overline{\Omega})$ such that $\overline{w} = g$ on $\partial\Omega$. We first construct a global supersolution to (18). Let $\|f\|_{L^\infty(\Omega)} =: K$. We choose a point x_0 such that $\mathrm{dist}(x_0, \Omega) \geq 1$. Denote $K_1 := \max(K, \lambda d)$ and let

$$w_1(x) := K_2 - \frac{K_1}{2\lambda d}|x - x_0|^2,$$

where K_2 is such that $w_1 > \|g\|_{L^\infty(\partial\Omega)}$ on $\partial\Omega$. Then for $x \in \Omega$,

$$(\varepsilon + |Dw_1(x)|)^{\theta_v^\varepsilon(x)}\left[\varepsilon w_1(x) + F(D^2 w_1(x))\right] \geq K_1 \geq f(x).$$

Let $R > 0$; for every $y \in \partial\Omega$, let x_y be such that $|y - x_y| = R$ and $\overline{B_R(x_y)} \cap \overline{\Omega} = \{y\}$. Denote $R_1 := R + \mathrm{diam}(\Omega)$. Define for $\alpha > 2$, $M > 0$, $w_y(x) := M(R^{-\alpha} - |x - x_y|^{-\alpha})$. Then $w_y(y) = 0$, $w_y(x) > 0$ in Ω and

$$Dw_y(x) = M\alpha\frac{x - x_y}{|x - x_y|^{\alpha+2}}$$

so

$$|Dw_y| \geq M\alpha\frac{1}{R_1^{1+\alpha}} \quad \text{in } \Omega.$$

Also

$$D^2 w_y(x) = M\alpha\frac{I}{|x - x_y|^{\alpha+2}} - M\alpha(\alpha + 2)\frac{(x - x_y) \otimes (x - x_y)}{|x - x_y|^{\alpha+4}}.$$

We notice that if $\lambda(\alpha + 2) - d\Lambda \geq 1$, then

$$F(D^2 w_y(x)) \geq M\alpha\frac{1}{|x - x_y|^{\alpha+2}}(\lambda(\alpha + 2) - d\Lambda) \geq M\alpha\frac{1}{|x - x_y|^{\alpha+2}}.$$

We now fix M such that

$$M\alpha\frac{1}{R_1^{1+\alpha}} \geq 1 \quad \text{and} \quad M\alpha\frac{1}{R_1^{2+\alpha}} \geq K + \|g\|_{L^\infty(\partial\Omega)}.$$

For $0 < \eta < 1$ we now define the functions

$$w_{y,\eta}(x) := g(y) + \eta + C_\eta w_y(x),$$

where the constants C_η are such that $C_\eta \geq 1$ and $w_{y,\eta} \geq g$ on $\partial\Omega$. We notice that the C_η only depend on the modulus of continuity of g and are independent of y. Then, for every $0 < \varepsilon, \eta < 1$ and $x \in \Omega$

$$\left(\varepsilon + |Dw_{y,\eta}(x)|\right)^{\theta_v^\varepsilon(x)} \left[\varepsilon w_{y,\eta}(x) + F(D^2 w_{y,\eta}(x))\right]$$
$$\geq -\|g\|_{L^\infty(\partial\Omega)} + C_\eta M\alpha \frac{1}{R_1^{2+\alpha}} \geq K \geq f(x).$$

Therefore, the functions $w_{y,\eta}$ are supersolutions of (18) for every $0 < \varepsilon, \eta < 1$ and $y \in \partial\Omega$. Thus the functions

$$\tilde{w}_{y,\eta}(x) := \min(w_{y,\eta}(x), w_1(x))$$

are viscosity supersolutions of (18). Finally the function

$$\overline{w}(x) := \inf\{\tilde{w}_{y,\eta}(x) : y \in \partial\Omega, 0 < \eta < 1\}$$

is the required viscosity supersolution of (18) and $\overline{w} = g$ on $\partial\Omega$. A viscosity subsolution $\underline{w}$ of (18) such that $\underline{w} = g$ on $\partial\Omega$ is constructed similarly. □

The existence of a unique viscosity solution to the approximating equations (18) follows from Lemma 2, together with the comparison principle in Proposition 4 and Perron's method.

Corollary 2 *Let Ω be a bounded domain which satisfies a uniform exterior sphere condition. Let Assumptions A1 and A2 hold and let $f \in C(\overline{\Omega})$, $g \in C(\partial\Omega)$. Then, for every $0 < \varepsilon < 1$ and $v \in C(\overline{\Omega})$, there exists a unique viscosity solution u_ε^v to (18) such that $\underline{w} \leq u_\varepsilon^v \leq \overline{w}$ in $\overline{\Omega}$. Moreover, there exists $\beta = \beta(d, \lambda, \Lambda) > 0$ such that for every $\Omega' \Subset \Omega$,*

$$\left\|u_\varepsilon^v\right\|_{C^\beta(\Omega')} \leq C, \tag{22}$$

for some $C = C\left(d, \lambda, \Lambda, \|u_\varepsilon^v\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(\Omega)}, \operatorname{dist}(\Omega', \partial\Omega)\right)$.

Proof We only need to show (22). To this end we notice that u_ε^v is a viscosity subsolution of

$$F(D^2 u_\varepsilon^v) = \|f\|_{L^\infty(\Omega)} + \|u_\varepsilon^v\|_{L^\infty(\Omega)} \qquad \text{in } \{|Du| > 1\},$$

and thus it is a viscosity subsolution to

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2 u_\varepsilon^v) = \|f\|_{L^\infty(\Omega)} + \|u_\varepsilon^v\|_{L^\infty(\Omega)} \qquad \text{in } \{|Du| > 1\}.$$

Similarly we obtain that in the set $\{|Du| > 1\}$, u_ε^v is a viscosity supersolution to

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2 u_\varepsilon^v) = -\|f\|_{L^\infty(\Omega)} - \|u_\varepsilon^v\|_{L^\infty(\Omega)} \qquad \text{in } \{|Du| > 1\}.$$

The result now follows by an easy application of Proposition 2. □

Now we introduce the set $B \subset C(\overline{\Omega})$, given by

$$B := \left\{ w \in C(\overline{\Omega}) \ \mid \ \underline{w} \leq w \leq \overline{w} \right\}, \tag{23}$$

where $\underline{w}, \overline{w} : \Omega \to \mathbb{R}$ are the sub and supersolution from Lemma 2, respectively. It is clear that B is a convex and closed subset of $C(\overline{\Omega})$. Define a map $T : B \to C(\overline{\Omega})$ as follows. Given $v \in B$, let u_ε^v be the unique solution to (18) such that $u_\varepsilon^v = g$ on $\partial\Omega$, whose existence is the subject of Corollary 2. Set

$$Tv := u_\varepsilon^v \tag{24}$$

The next lemma collects some properties of the map T.

Lemma 3 *(Properties of the map T) Let $B \subset C(\overline{\Omega})$ and $T : B \to C(\overline{\Omega})$ be defined as in (23) and (24), respectively. Then $T(B) \subset B$. In addition, $T(B)$ is precompact in B and the map T is continuous.*

Proof Let $v \in B$. Corollary 2 and the definition of T imply that $\underline{w} \leq Tv \leq \overline{w}$, hence $T(B) \subset B$. We emphasize that $\underline{w}$ and $\overline{w}$ are independent of $v \in C(\overline{\Omega})$ and $\varepsilon > 0$.

Now we observe that $T(B)$ is precompact. Let $(Tv_n)_{n\in\mathbb{N}}$ be a sequence in $T(B)$. Estimate (22), together with $\underline{w} \leq Tv_n \leq \overline{w}$, implies that the sequence $(Tv_n)_{n\in\mathbb{N}}$ is equibounded and equicontinuous in $C(\overline{\Omega})$. Hence, it has a subsequence which converges to some $w \in B$.

To complete the proof, we show that T is continuous. Suppose that $(v_n)_{n\in\mathbb{N}}$ is a sequence in B which converges in $C(\overline{\Omega})$ to $v \in B$. We need to verify that $Tv_n \to Tv$ in $C(\overline{\Omega})$. Since $T(B)$ is precompact, there exists $w \in B$ such that $Tv_n \to w$ in $C(\overline{\Omega})$, through a subsequence if necessary.

We notice that $h_\varepsilon^{v_n}$ converges uniformly to h_ε^v in $\overline{\Omega}$, since $v_n \to v$ uniformly. As a consequence, the sequence of operators $(G_\varepsilon^n)_{n\in\mathbb{N}}$ given by

$$G_\varepsilon^n(x, r, p, M) := (\varepsilon + |p|)^{\theta_\varepsilon^{v_n}(x)} (\varepsilon r + F(M))$$

converges locally uniformly to G_ε^∞, where

$$G_\varepsilon^\infty(x, r, p, M) := (\varepsilon + |p|)^{\theta_\varepsilon^{v}(x)} (\varepsilon r + F(M)) .$$

The stability of viscosity solutions and the uniqueness of viscosity solutions to our Dirichlet boundary value problem for (18) ensure that $w = Tv$. To complete the proof it remains to notice that this argument does not depend on the subsequence.

Suppose through a different subsequence $(Tv_{n_j})_{j\in\mathbb{N}}$, we obtain $Tv_{n_j} \to w'$. Once again, the stability of viscosity solutions yields $Tv = w'$. The uniqueness of viscosity solutions ensures $w = w'$ and the proof is complete. □

In the sequel we detail the proof of Theorem 1.

Proof of Theorem 1 Lemma 3 and properties of the set B allow us to apply the Schauder Fixed Point Theorem; see, for example, [25, Corollary 11.2]. For every $\varepsilon > 0$, we conclude that there exists a viscosity solution $u_\varepsilon \in C(\overline{\Omega})$ to

$$(\varepsilon + |Du_\varepsilon|)^{\theta_\varepsilon^{u_\varepsilon}(x)} \left(\varepsilon u_\varepsilon + F(D^2 u_\varepsilon)\right) = f(x) \quad \text{in } \Omega,$$

such that $u_\varepsilon = g$ on $\partial\Omega$. Again, estimate (22), together with $\underline{w} \le u_\varepsilon \le \overline{w}$, ensures the existence of a sequence $(u_{\varepsilon_n})_{n\in\mathbb{N}}$, with $\varepsilon_n < 1/n$, and a function $u \in B$, such that $u_{\varepsilon_n} \to u$ in $C(\overline{\Omega})$. Using the fact that $\theta_\varepsilon^{u_\varepsilon}$ converges to $\theta_1\chi_{\{u>0\}} + \theta_2\chi_{\{u<0\}}$ uniformly on compact subsets of $(\Omega^+(u) \cup \Omega^-(u)) \cap \Omega$, a standard consistency argument now allows us to conclude that u is a viscosity solution to (7) in $(\Omega^+(u) \cup \Omega^-(u)) \cap \Omega$. Moreover, since $0 \le \theta_\varepsilon^{u_\varepsilon} \le \theta_2$, u is also a viscosity subsolution to (8) and a viscosity supersolution to (9) in Ω.

Remark 2 Equation (18) is slightly different from the equations considered in [16] and no quotable result from [16] can be used to claim directly that the solutions u_ε^v from Corollary 2 are $C^{1,\alpha}$ with the $C^{1,\alpha}$ estimates independent of the modulus of continuity of the variable exponent function $\theta_\varepsilon^{u_\varepsilon}$. If this was done, we could then claim that the functions u_ε in the proof of Theorem 1 are $C^{1,\alpha}$ with estimates independent of ε, which then would give the $C^{1,\alpha}$-regularity of the solution u to (7) obtained in Theorem 1. We do not attempt to follow this path. Moreover, equation (18) is only auxiliary and we are interested in the regularity of all solutions of (7). For this reason we show in Section 4 the $C^{1,\alpha}$-regularity by working directly with equation (7) or more precisely with the differential inequalities (8) and (9). Overall, the proof of Theorem 2 is perhaps similar to the proof of a $C^{1,\alpha}$-regularity result in [16], however we provide the full proof in Section 4 to keep the manuscript self-contained and rigorous.

4 Towards Improved Regularity

In this section we prove Theorem 2. We first establish Hölder continuity of viscosity solutions to differential inequalities (15)-(16), for arbitrary vector $q \in \mathbb{R}^d$. Proposition 5 below is a version of Lemma 3 of [27] and its proof follows the strategy of the proof of Lemma 3 of [27] (see also Section 4 of [16]). However we present the proof with all details. We emphasize that the Hölder-estimate in Proposition 5 does not depend on q. Then the proof of Theorem 2 is done in several steps and uses techniques and methods of [16].

Proposition 5 *(Hölder-continuity) Let Assumptions A1, A2 hold. Let $u \in C(B_1)$ be a viscosity subsolution to (15) and a viscosity supersolution to (16) in the unit ball B_1, where $q \in \mathbb{R}^d$ is arbitrary. Suppose that $\|u\|_{L^\infty(B_1)} \le 1$. Then $u \in C^\beta_{\text{loc}}(B_1)$, where β is from Corollary 1 and for every $0 < \tau < 1$, there exists a universal constant $C_\tau > 0$ such that*

$$\|u\|_{C^\beta(B_\tau)} \le C_\tau. \tag{25}$$

Proof Fix $0 < r < \frac{1-\tau}{2}$ and define

$$\omega(t) := t - \frac{t^2}{2}.$$

For constants $L_1,\ L_2 > 0$ and $x_0 \in B_\tau$, we set

$$L := \sup_{x,y \in B_r(x_0)} \left[u(x) - u(y) - L_1\omega(|x-y|) - L_2(|x-x_0|^2 + |y-x_0|^2) \right].$$

We aim at verifying that there are choices of L_1 and L_2 for which $L \le 0$ for every $x_0 \in B_\tau$. This will imply that u is Lipschitz continuous in B_τ by taking $x_0 = x$.

We argue by contradiction. Suppose there exists $x_0 \in B_\tau$ for which $L > 0$ regardless of the choices of L_1 and L_2. Consider the auxiliary functions $\psi,\ \phi : \overline{B}_1 \times \overline{B}_1 \to \mathbb{R}$ given by

$$\psi(x, y) := L_1\omega(|x - y|) + L_2 \left(|x - x_0|^2 + |y - x_0|^2 \right)$$

and

$$\phi(x, y) := u(x) - u(y) - \psi(x, y).$$

Let $(\overline{x}, \overline{y})$ be a point where ϕ attains its maximum. Then

$$\phi(\overline{x}, \overline{y}) = L > 0$$

and

$$L_1\omega(|\overline{x} - \overline{y}|) + L_2 \left(|\overline{x} - x_0|^2 + |\overline{y} - x_0|^2 \right) \le \sup_{x \in B_1} u(x) - \inf_{x \in B_1} u(x) \le 2.$$

Set

$$L_2 := \left(\frac{4\sqrt{2}}{r} \right)^2.$$

Then,

$$|\overline{x} - x_0| + |\overline{y} - x_0| \le \frac{r}{2}.$$

It follows that $\overline{x}, \overline{y} \in B_r(x_0)$. In addition, $\overline{x} \ne \overline{y}$; indeed, if this is not the case, we would conclude $L \le 0$.

At this point, we use Proposition 1 to obtain elements in the closures of subjets and superjets of u and produce a viscosity inequality relating those elements. We split the rest of the proof into four steps.

Step 1 - Proposition 1 ensures the existence of $(q_{\overline{x}}, X)$ in the closure of the superjet of u at $\overline{x}$ and of $(q_{\overline{y}}, Y)$ in the closure of the subjet of u at $\overline{y}$, with

$$q_{\bar{x}} := D_x\psi(\bar{x},\bar{y}) = L_1\omega'(|\bar{x}-\bar{y}|)\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|} + 2L_2(\bar{x}-x_0)$$

and

$$q_{\bar{y}} := -D_y\psi(\bar{x},\bar{y}) = L_1\omega'(|\bar{x}-\bar{y}|)\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|} - 2L_2(\bar{y}-x_0).$$

In addition, the matrices X and Y satisfy the inequality

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix} + (2L_2+\iota)I, \tag{26}$$

for

$$Z := L_1\omega''(|\bar{x}-\bar{y}|)\frac{(\bar{x}-\bar{y})\otimes(\bar{x}-\bar{y})}{|\bar{x}-\bar{y}|^2} + L_1\frac{\omega'(|\bar{x}-\bar{y}|)}{|\bar{x}-\bar{y}|}\left(I - \frac{(\bar{x}-\bar{y})\otimes(\bar{x}-\bar{y})}{|\bar{x}-\bar{y}|^2}\right)$$

where $0 < \iota \ll 1$ depends solely on the norm of Z.

Next, we apply the matrix inequality (26) to special vectors as to recover information about the eigenvalues of $X - Y$. First, apply (26) to vectors of the form $(z, z) \in \mathbb{R}^{2d}$ to get

$$\langle (X-Y)z, z\rangle \leq (4L_2 + 2\iota)|z|^2.$$

We then conclude that all the eigenvalues of $(X-Y)$ are less than or equal to $4L_2+2\iota$. Now, by applying (26) to

$$\bar{z} := \left(\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}, \frac{\bar{y}-\bar{x}}{|\bar{x}-\bar{y}|}\right),$$

we obtain

$$\begin{aligned}\left\langle (X-Y)\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}, \frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}\right\rangle &\leq (4L_2+2\iota)\left|\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}\right|^2 \\ &= 4L_1\omega''(|\bar{x}-\bar{y}|) = 4L_2 + 2\iota - 4L_1.\end{aligned}$$

We conclude that at least one eigenvalue of $(X-Y)$ is below

$$4L_2 + 2\iota - 4L_1.$$

We notice this quantity will be negative for large values of L_1. Evaluating the minimal Pucci operator on $X - Y$, we then get

$$\mathcal{P}^-_{\lambda,\Lambda}(X-Y) \geq 4\lambda L_1 - (\lambda + (d-1)\Lambda)(4L_2+2\iota). \tag{27}$$

At this point we evoke the differential inequalities (15) and (16) satisfied by u in the viscosity sense. They yield

$$\min\left\{|q+q_{\bar{x}}|^{\theta_2}F(X),\ F(X)\right\}\ \leq\ 1 \tag{28}$$

and

$$\max\left\{|q+q_{\bar{y}}|^{\theta_2}F(Y),\ F(Y)\right\}\ \geq\ -1. \tag{29}$$

Since F is (λ,Λ)-elliptic, we have

$$F(X)\geq F(Y)+\mathcal{P}^{-}_{\lambda,\Lambda}(X-Y). \tag{30}$$

Step 2 - In what follows, we relate (27), (30) and (28)-(29). Below, we consider all possible cases.

Case 1: Suppose

$$\min\left\{|q+q_{\bar{x}}|^{\theta_2}F(X),\ F(X)\right\}\ =\ F(X)$$

and

$$\max\left\{|q+q_{\bar{x}}|^{\theta_2}F(Y),\ F(Y)\right\}\ =\ F(Y).$$

In this case we get

$$4\lambda L_1\leq(\lambda+(d-1)\Lambda)(4L_2+2\iota)+2.$$

Case 2: Suppose

$$\min\left\{|q+q_{\bar{x}}|^{\theta_2}F(X),\ F(X)\right\}\ =\ |q\ +\ q_{\overline{x}}|^{\theta_2}\ F(X)$$

and

$$\max\left\{|q+q_{\bar{x}}|^{\theta_2}F(Y),\ F(Y)\right\}\ =\ \left|q\ +\ q_{\overline{y}}\right|^{\theta_2}\ F(Y).$$

Then,

$$4\lambda L_1\leq(\lambda+(d-1)\Lambda)(4L_2+2\iota)\ +\ \left|q\ +\ q_{\overline{y}}\right|^{-\theta_2}\ +\ |q\ +\ q_{\overline{x}}|^{-\theta_2}.$$

Case 3: Suppose

$$\min\left\{|q+q_{\bar{x}}|^{\theta_2}F(X),\ F(X)\right\}\ =\ F(X)$$

and

$$\max\left\{|q+q_{\bar{x}}|^{\theta_2}F(Y),\ F(Y)\right\}\ =\ \left|q\ +\ q_{\overline{y}}\right|^{\theta_2}\ F(Y).$$

In this case we produce

$$4\lambda L_1 \le (\lambda + (d-1)\Lambda)(4L_2 + 2\iota) + \left|q + q_{\bar{y}}\right|^{-\theta_2} + 1.$$

Case 4: Suppose

$$\min\left\{|q + q_{\bar{x}}|^{\theta_2} F(X),\ F(X)\right\} = |q + q_{\bar{x}}|^{\theta_2} F(X)$$

and

$$\max\left\{|q + q_{\bar{x}}|^{\theta_2} F(Y),\ F(Y)\right\} = F(Y).$$

Here, we obtain

$$4\lambda L_1 \le (\lambda + (d-1)\Lambda)(4L_2 + 2\iota) + 1 + |q + q_{\bar{x}}|^{-\theta_2}.$$

Step 3 - Next, we explore the fact that $q \in \mathbb{R}^d$ is arbitrary, in close connection with Cases 1-4. Observe that

$$|q_{\bar{x}}| \le L_1(1 + |\bar{x} - \bar{y}|) + 2L_2 \le aL_1,$$

for some $a > 0$, universal. Let $A_0 := 10aL_1$ and assume $|q| \ge A_0$. In this case, we ensure that $q \ne q_{\bar{x}}$. A similar reasoning leads to $q \ne q_{\bar{y}}$.

From the choice of A_0 we conclude that

$$|q + q_{\bar{x}}| \ge A_0 - \frac{A_0}{10} = \frac{9}{10}A_0;$$

also $|q + q_{\bar{y}}| \ge \frac{9}{10}A_0$. Hence, we get

$$|q + q_{\bar{x}}|^{-\theta_2} \le \left(\frac{9}{10}A_0\right)^{-\theta_2}.$$

Similarly, we obtain

$$|q + q_{\bar{y}}|^{-\theta_2} \le \left(\frac{9}{10}A_0\right)^{-\theta_2}.$$

Thus, the choice of A_0 ensures that in all Cases 1-4

$$4\lambda L_1 \le (\lambda + (d-1)\Lambda)(4L_2 + 2\iota) + \frac{C}{L_1^{\theta_2}} + C.$$

By choosing $L_1 \gg 1$ large enough, depending only on $d,\ \lambda,\ \Lambda,\ \theta_2$ and L_2, which in turn depends only on $0 < r \ll 1$ and τ, we produce a contradiction. Therefore, we cannot have $L > 0$.

As a result, in the case $|q| \geq A_0$, solutions to (15)-(16) are locally Lipschitz-continuous, with universal estimates.

Step 4 - If $|q| < A_0$ then Corollary 1, applied with $\gamma = 2A_0$, guarantees that $u \in C^{\beta}_{\text{loc}}(B_1)$ for some universal $\beta \in (0, 1)$ and u satisfies (25). □

Proposition 6 *(Approximation Lemma) Let Assumptions A1, A2 hold. Let $u \in C(B_1)$ be a viscosity subsolution to*

$$\min\left\{|q + Du|^{\theta_2} F(D^2u),\ F(D^2u)\right\} = c \tag{31}$$

and a viscosity supersolution to

$$\max\left\{|q + Du|^{\theta_2} F(D^2u),\ F(D^2u)\right\} = -c \tag{32}$$

in the unit ball B_1, where $c > 0$ and $q \in \mathbb{R}^d$ is arbitrary. For every $0 < \delta < 1$ there exists $0 < \varepsilon < 1$ such that, if $\|u\|_{L^\infty(B_1)} \leq 1$ and

$$c \leq \varepsilon,$$

then one can find $h \in C^{1,\alpha_0}_{\text{loc}}(B_1)$ satisfying $\overline{F}(D^2h) = 0$ in the viscosity sense in B_1 for some $\overline{F}$ satisfying Assumption A1, such that

$$\|u - h\|_{L^\infty(B_{3/4})} \leq \delta. \tag{33}$$

Such function h satisfies

$$\|h\|_{C^{1,\alpha_0}(B_{1/2})} \leq C\, \|h\|_{L^\infty(B_{3/4})}, \tag{34}$$

where $C = C(d, \lambda, \Lambda) > 0$ is from Proposition 3 and is independent of q.

Proof For ease of presentation, we split the proof into five steps. As before, we resort to a contradiction argument.

Step 1 - Suppose the statement of the proposition is false. If this is the case, there exist $0 < \delta_0 < 1$, a sequence of functions $(u_n)_{n\in\mathbb{N}} \subset C(B_1)$, $\|u_n\|_{L^\infty(B_1)} \leq 1$, a sequence of positive numbers $(c_n)_{n\in\mathbb{N}}$, $c_n \to 0$, a sequence of vectors $(q_n)_{n\in\mathbb{N}}$ and a sequence $(F_n)_{n\in\mathbb{N}}$ of operators satisfying A1, such that u_n is a viscosity subsolution to

$$\min\left\{|q_n + Du_n|^{\theta_2} F_n(D^2u_n),\ F_n(D^2u_n)\right\} = c_n \tag{35}$$

and a viscosity supersolution to

$$\max\left\{|q_n + Du_n|^{\theta_2} F_n(D^2u_n),\ F_n(D^2u_n)\right\} = -c_n \tag{36}$$

in the unit ball B_1 for every $n \in \mathbb{N}$ but

$$\|u_n - h\|_{L^\infty(B_{3/4})} \geq \delta_0,$$

for every $h \in C^{1,\alpha_0}_{\text{loc}}(B_1)$ satisfying $\overline{F}(D^2h) = 0$ in the viscosity sense in B_1 for some $\overline{F}$.

Step 2 - By Proposition 5, we know that $(u_n)_{n\in\mathbb{N}}$ is bounded in $C^\beta(B_\tau)$ for every $0 < \tau < 1$. Therefore, choosing a subsequence if necessary, it converges uniformly on every compact subset of B_1 to some function $u_\infty \in C^\beta_{\text{loc}}(B_1)$, where $\|u_\infty\|_{L^\infty(B_1)} \leq 1$. In addition, $(F_n)_{n\in\mathbb{N}}$ is uniformly Lipschitz continuous. Hence, there exists an operator F_∞ satisfying Assumption A1 such that F_n converges to F_∞, locally uniformly.

Our goal is to prove that u_∞ is a viscosity solution to

$$F_\infty(D^2u_\infty) = 0 \qquad \text{in} \quad B_1.$$

We will only show that u_∞ is a viscosity subsolution of $F_\infty(D^2u_\infty) = 0$ as the proof of the supersolution property is analogous. We consider two cases, depending on the behavior of the sequence $(q_n)_{n\in\mathbb{N}}$.

Step 3 - Firstly, suppose that the sequence $(q_n)_{n\in\mathbb{N}}$ does not admit a convergent subsequence. Then, $|q_n| \to \infty$ as $n \to \infty$. Let $\varphi \in C^2(B_1)$ and suppose that $u_\infty - \varphi$ attains a local maximum at $x_0 \in B_1$; we assume this maximum to be strict. Suppose that

$$F_\infty(D^2\varphi(x_0)) > 0.$$

Standard arguments yield a sequence $(x_n)_{n\in\mathbb{N}}$, converging to x_0, such that $u_n - \varphi$ attains a local maximum at x_n. Notice also that $D\varphi(x_n) \to D\varphi(x_0)$ and $D^2\varphi(x_n) \to D^2\varphi(x_0)$. We choose $M \in \mathbb{N}$ such that

$$F_n(D^2\varphi(x_n)) > c_n$$

and

$$|q_n + D\varphi(x_n)| > 1,$$

for $n > M$. Then, for $n > M$, we conclude

$$\min\left\{|q_n + D\varphi(x_n)|^{\theta_2} F_n(D^2\varphi(x_n)),\, F_n(D^2\varphi(x_n))\right\} > c_n$$

which is a contradiction. A similar argument, using (36), shows that u_∞ is a viscosity supersolution and so we get that u_∞ is a viscosity solution to

$$F_\infty(D^2u_\infty) = 0 \qquad \text{in} \quad B_1$$

in this case.

Step 4 - We now consider the complementary case. Namely, suppose $(q_n)_{n\in\mathbb{N}}$ admits a convergent subsequence, still denoted by $(q_n)_{n\in\mathbb{N}}$, such that $q_n \to q_\infty$. By resorting to standard stability results (see, for instance, [23, Section 6, Remarks 6.2 and 6.3]), we conclude that u_∞ is a viscosity subsolution to

$$\min\left\{|q_\infty + Du_\infty|^{\theta_2} F_\infty(D^2u_\infty),\, F_\infty(D^2u_\infty)\right\} = 0 \tag{37}$$

and a viscosity supersolution to

$$\max\left\{|q_\infty + Du_\infty|^{\theta_2} F_\infty(D^2u_\infty),\, F_\infty(D^2u_\infty)\right\} = 0 \tag{38}$$

in the unit ball B_1.

We now reduce the problem to the case $q_\infty \equiv 0$. Indeed, by considering $w_\infty := u_\infty + q_\infty \cdot x$ we get that w_∞ is a viscosity subsolution to

$$\min\left\{|Dw_\infty|^{\theta_2} F_\infty(D^2w_\infty),\, F_\infty(D^2w_\infty)\right\} = 0 \tag{39}$$

and a viscosity supersolution to

$$\max\left\{|Dw_\infty|^{\theta_2} F_\infty(D^2w_\infty),\, F_\infty(D^2w_\infty)\right\} = 0 \tag{40}$$

in B_1. Because $D^2w_\infty = D^2u_\infty$ in the viscosity sense, by verifying $F_\infty(D^2w_\infty) = 0$, we infer $F_\infty(D^2u_\infty) = 0$. We will only argue that w_∞ is a viscosity subsolution of $F_\infty(D^2w_\infty) = 0$.

Let $p(x)$ be a second order polynomial of the form

$$p(x) := b \cdot x + \frac{1}{2}x^T Ax,$$

for a vector $b \in \mathbb{R}^d$ and a matrix $A \in \mathbb{R}^{d^2}$, fixed. Suppose that $w_\infty - p$ attains its maximum at $x_0 \in B_1$. Without loss of generality we suppose $x_0 = 0$, $w_\infty(0) = 0$ and the maximum is strict. Our goal is to prove that $F_\infty(A) \leq 0$.

From (39) we infer one of the following inequalities:

$$|b|^{\theta_1} F_\infty(A) \leq 0, \qquad |b|^{\theta_2} F_\infty(A) \leq 0 \quad \text{or} \quad F_\infty(A) \leq 0.$$

In case $b \neq 0$, $F_\infty(A) \leq 0$ and we are done. If $b = 0$ the argument is exactly the same as that in the proof of [27, Lemma 6].

Step 5 - Since u_∞ is a viscosity solution to $F_\infty(D^2u_\infty) = 0$ in B_1, which satisfies Assumption A1, Proposition 3 guarantees that $u_\infty \in C^{1,\alpha_0}_{\text{loc}}(B_1)$. Proposition 3 implies

$$\|u_\infty\|_{C^{1,\alpha_0}(B_{1/2})} \leq C\,\|u_\infty\|_{L^\infty(B_{3/4})}.$$

Thus, taking $h = u_\infty$, $\overline{F} = F_\infty$, we reach a contradiction. This completes the proof. □

Proposition 7 *(Oscillation control) Let Assumptions A1, A2 hold. Let $u \in C(B_1)$ and $\|u\|_{L^\infty(B_1)} \leq 1$. Let $\alpha \in (0, \alpha_0)$. Set*

$$\rho := \min\left\{\frac{1}{2}, \left(\frac{1}{2C}\right)^{\frac{1}{\alpha_0-\alpha}}\right\} \qquad \text{and} \qquad \delta := \frac{\rho^{1+\alpha}}{2},$$

where C is from Proposition 6. If u is a viscosity subsolution to (31) and a viscosity supersolution to (32) for any q and for $c = \varepsilon$, where ε is from Proposition 6 for the δ above, then there exists an affine function $\ell(x) = a + b \cdot x$ such that $|a| \leq C$, $|b| \leq C$ and

$$\sup_{x \in B_\rho} |u(x) - \ell(x)| \leq \rho^{1+\alpha}.$$

Proof Let h be the approximating function whose existence is ensured by Proposition 6. Set $\ell(x) := h(0) + Dh(0) \cdot x$. The triangle inequality yields

$$\begin{aligned}\sup_{x \in B_\rho} |u(x) - \ell(x)| &\leq \sup_{x \in B_\rho} |u(x) - h(x)| + \sup_{x \in B_\rho} |h(x) - h(0) - Dh(0) \cdot x| \\ &\leq \delta + C\rho^{1+\alpha_0} \leq \rho^{1+\alpha}.\end{aligned}$$

□

Proposition 8 *(Oscillation control in discrete scales) Let the assumptions of Proposition 7 be satisfied, however let $q = 0$ now and let in addition*

$$\alpha \leq \frac{1}{1 + \theta_2}.$$

There exists a sequence of affine functions $(\ell_n)_{n \in \mathbb{N}}$, of the form

$$\ell_n(x) := a_n + b_n \cdot x,$$

satisfying

$$\sup_{x \in B_{\rho^n}} |u(x) - \ell_n(x)| \leq \rho^{(1+\alpha)n} \tag{41}$$

and

$$|a_{n+1} - a_n| + \rho^n |b_{n+1} - b_n| \leq C\rho^{(1+\alpha)n}, \tag{42}$$

for every $n \in \mathbb{N}$ and some universal constant $C > 0$.

Proof We argue by induction.

Step 1 - The basis case follows from Proposition 7. In fact, for h as in that proposition, set $\ell_0 \equiv 0$ and define $\ell_1(x) := h(0) + Dh(0) \cdot x$. It is clear that

$$\sup_{x \in B_\rho} |u(x) - \ell_1(x)| \leq \rho^{1+\alpha}$$

and

$$|h(0)| + |Dh(0)| \le C,$$

where $C > 0$ is a universal constant.
Step 2 - Now, suppose the case $n = k$ has been verified. Next we examine the case $n = k + 1$. To that end, we introduce the auxiliary function

$$v_k(x) := \frac{u(\rho^k x) - \ell_k(\rho^k x)}{\rho^{k(1+\alpha)}}.$$

It is easy to see that (see Section 2.3) v_k is a viscosity subsolution to

$$\min\left\{\left|\rho^{-k\alpha} b_k + Dv_k\right|^{\theta_2} F_k(D^2 v_k), F_k(D^2 v_k)\right\} = c_k$$

and a viscosity supersolution to

$$\max\left\{\left|\rho^{-k\alpha} b_k + Dv_k\right|^{\theta_2} F_k(D^2 v_k), F_k(D^2 v_k)\right\} = -c_k,$$

where

$$F_k(M) := \rho^{k(1-\alpha)} F\left(\frac{1}{\rho^{k(1-\alpha)}} M\right)$$

and

$$c_k := \varepsilon\left(\max\left\{\rho^{k(1-\alpha(1+\theta_2))}, \rho^{k(1-\alpha)}\right\}\right).$$

Notice that F_k satisfies Assumption A1 and the choice of the exponent α, together with $\rho \le 1/2$, yields $c_k \le \varepsilon$. The induction hypothesis ensures $\|v_k\|_{L^\infty(B_1)} \le 1$.
Step 3 - Proposition 7 now implies that there exists an affine function $\overline{\ell}_k$ satisfying

$$\sup_{x \in B_\rho} \left|v_k(x) - \overline{\ell}_k(x)\right| \le \rho^{1+\alpha}, \tag{43}$$

where

$$\overline{\ell}_k(x) := \overline{a}_k + \overline{b}_k \cdot x$$

is such that $|\overline{a}_k|$ and $|\overline{b}_k|$ are bounded by a constant $C > 0$, depending solely on d, λ and Λ. Using the definition of v_k, estimate (43) becomes

$$\sup_{x \in B_\rho} \left|\frac{u(\rho^k x) - a_k - b_k \cdot (\rho^k x) - \rho^{k(1+\alpha)}\left(\overline{a}_k + \overline{b}_k \cdot x\right)}{\rho^{k(1+\alpha)}}\right| \le \rho^{1+\alpha}.$$

It implies

$$\sup_{x \in B_{\rho^{k+1}}} |u(x) - \ell_{k+1}(x)| \leq \rho^{(k+1)(1+\alpha)}$$

for

$$\ell_{k+1} = a_{k+1} + b_{k+1} \cdot x := \left(a_k + \rho^{k(1+\alpha)}\overline{a}_k\right) + \left(b_k + \rho^{k\alpha}\overline{b}_k\right) \cdot x.$$

To complete the proof, we notice that

$$|a_{k+1} - a_k| \leq |\overline{a}_k| \, \rho^{k(1+\alpha)} \leq C\rho^{k(1+\alpha)}$$

and

$$\rho^k \, |b_{k+1} - b_k| \leq \left|\overline{b}_k\right| \rho^{k(1+\alpha)} \leq C\rho^{k(1+\alpha)}.$$

□

Proof of Theorem 2 Let δ, ε be as in Proposition 7. Denote

$$K = \|u\|_{L^\infty(B_1)} + \max\left\{\|f\|_{L^\infty(B_1)}, \|f\|_{L^\infty(B_1)}^{\frac{1}{1+\theta_2}}\right\}$$

and set

$$v(x) = \frac{u(\varepsilon x)}{K}.$$

Then $\|v\|_{L^\infty(B_1)} \leq 1$ and (see Section 2.3) v is a viscosity subsolution to

$$\min\left\{|Dv|^{\theta_2}\overline{F}(D^2v), \ \overline{F}(D^2v)\right\} = \varepsilon \quad \text{in} \quad B_1$$

and a viscosity supersolution to

$$\max\left\{|Dv|^{\theta_2}\overline{F}(D^2v), \ \overline{F}(D^2v)\right\} = -\varepsilon \quad \text{in} \quad B_1,$$

where

$$\overline{F}(X) = \frac{\varepsilon^2}{K} F\left(\frac{K}{\varepsilon^2} X\right).$$

Thus we are in the framework of Propositions 6-8. From (41)-(42) we infer

$$a_n \longrightarrow v(0) \qquad \text{and} \qquad b_n \longrightarrow Dv(0),$$

with convergence rates of the order

$$|a_n - v(0)| \sim \rho^{n(1+\alpha)} \qquad \text{and} \qquad |b_n - Dv(0)| \sim \rho^{n\alpha}.$$

Therefore

$$\begin{aligned}\sup_{x\in B_{\rho^n}} |v(x) - v(0) - Dv(0)\cdot x| &\le \sup_{x\in B_{\rho^n}} |v(x) - a_n - b_n \cdot x| + C\rho^{n(1+\alpha)} \\ &\le C\rho^{n(1+\alpha)},\end{aligned}$$

where the second inequality comes from (41). Now if $\rho^{m+1} < r \le \rho^m$ for some $m \in \mathbb{N}$, then

$$\begin{aligned}\sup_{x\in B_r} |v(x) - v(0) - Dv(0)\cdot x| &\le \sup_{x\in B_{\rho^m}} |v(x) - v(0) - Dv(0)\cdot x| \\ &\le C\rho^{m(1+\alpha)} \\ &\le \frac{C}{\rho^{1+\alpha}}\rho^{(m+1)(1+\alpha)} \\ &\le Cr^{1+\alpha}.\end{aligned}$$

This concludes the proof since the same argument can be done for every point $x_0 \in B_\tau$ provided we also take $\varepsilon < 1 - \tau$. □

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