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RESEARCH

Quantitative Control of Solutions to the Axisymmetric Navier-Stokes Equations in Terms of the Weak L^3 Norm

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Abstract

We are concerned with strong axisymmetric solutions to the 3D incompressible Navier-Stokes equations. We show that if the weak L^3 norm of a strong solution u on the time interval $[0, T]$ is bounded by $A \gg 1$ then for each $k \geq 0$ there exists $C_k > 1$ such that $\| D^k u(t)\|_{L^\infty(\mathbb{R}^3)} \leq t^{-(1+k)/2} \exp\exp A^{C_k}$ for all $t \in (0, T]$.

Keywords Navier-Stokes equations · Type I blow up · Critical spaces · Axisymmetry · Quantitative regularity · Weak L^3 space

1 Introduction

We are concerned with the 3D incompressible Navier-Stokes equations,

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = 0, \\ \operatorname{div} u = 0 \qquad \text{in } \mathbb{R}^3 \end{cases} \tag{1}$$

for $t \in [0, T)$. While the question of global well-posedness of the equations remains open, it is well-known that the unique strong solution on a time interval $[0, T)$ can be continued past T provided a regularity criterion holds, such as $\int_0^T \| \operatorname{curl} u\|_\infty dt < \infty$ (the Beale-Kato-Majda [3] criterion), Lipschitz continuity up to $t = T$ of the direction of vorticity (the Constantin-Fefferman [11] criterion), or if $\int_0^T \|u\|_p^q dt < \infty$ for any $p \in [3, \infty]$, $q \in [2, \infty]$ such that $2/q + 3/p \leq 1$ (the Ladyzhenskaya-Prodi-Serrin condition), among many others. The non-endpoint case $q < \infty$ of the latter condition

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was settled in the 1960s [17, 34, 41], while the endpoint case $L_t^\infty L_x^3$ was only settled many years later by Escauriaza, Seregin, and Šverák [12]. The main difficulty of the endpoint case is related to the fact that L^3 is a critical space for 3D Navier–Stokes, and [12] settled it with an argument by contradiction using a blow-up procedure and new unique continuation results. This result implies that if $T_0 > 0$ is a putative blow-up time of (1), then $\|u(t)\|_3$ must blow-up at least along a sequence of times $t_k \to T_0^-$. While Seregin [38] showed that the L^3 norm must blow-up along any sequence of times converging to T_0^-, the question of quantitative control of the strong solution u in terms of the L^3 norm remained open until the recent breakthrough work by Tao [44], who showed that

$$|\nabla^j u(x,t)| \leq \exp\exp\exp(A^{O(1)}) t^{-\frac{j+1}{2}} \tag{2}$$

for all $t \in [0,T]$, $j = 0, 1$, $x \in \mathbb{R}^3$, whenever

$$\|u\|_{L^\infty([0,T];L^3(\mathbb{R}^3))} \leq A$$

for some $A \gg 1$. This result implies in particular a lower bound

$$\limsup_{t\to T_0^-} \frac{\|u(t)\|_3}{\left(\log\log\log(T_0 - t)^{-1})\right)^c} = \infty,$$

where $c > 0$ and $T_0 > 0$ is the putative blow-up time, and has subsequently been improved in some settings. For example, Barker and Prange [2] and Barker [1] provided remarkable local quantitative estimates, and the second author [31] proved that, in the case of axisymmetric solutions,

$$|\nabla^j u(x,t)| \leq \exp\exp(A^{O(1)}) t^{-\frac{j+1}{2}}$$

for all $t \in [0,T]$, $j = 0, 1$, $x \in \mathbb{R}^3$, whenever

$$\left\| r^{1-\frac{3}{p}} u \right\|_{L^\infty([0,T];L^p(\mathbb{R}^3))} \leq A$$

for some $A \gg 1$, $p \in (2,3]$. In another work [32] he generalized (2) to higher dimensions ($d \geq 4$), where, due to an issue related to the lack of Leray's intervals of regularity, one obtains an analogue of (2) with four exponential functions. Recently Feng, He, and Wang [13] extended (2) to the non-endpoint Lorentz spaces $L^{3,q}$ for $q < \infty$. We emphasize that all these generalizations rely on the same stacking argument by Tao [44]. In particular, the argument breaks down for the endpoint case $q = \infty$.

1.1 Tao's Stacking Argument and Type I Blow-Up

In order to illustrate the issue at the endpoint space $L^{3,\infty}$, let us recall that the main strategy of Tao [44] is to show that if u concentrates at a particular time, then there exists a widely separated sequence of length scales $(R_k)_{k=1}^K$ and $\alpha = \alpha(A) > 0$ such that $\|u\|_{L^3(\{|x|\sim R_k\})} \geq \alpha$ for all k, which implies that

$$\|u\|_3^3 = \int_{\mathbb{R}^3} |u|^3 \geq \sum_k \int_{|x|\sim R_k} |u|^3 \geq \alpha^3 K. \tag{3}$$

The more singularly u concentrates at the origin, the larger one can take K; thus the L^3 norm controls the regularity of u. More precisely, if $\|u\|_3 \leq A$ and u concentrates at a large frequency N at time T, then one can take $\alpha = \exp(-\exp(A^{O(1)}))$ and $K \sim \log(NT^{\frac{1}{2}})$, which, by (3), implies that $N \leq T^{-\frac{1}{2}} \exp\exp\exp(A^{O(1)})$. This controls the solution in the sense that higher frequencies do not admit concentrations, and so a simple argument [44, Section 6] implies the conclusion (2).

Let us contrast this L^3 situation with that of general Lorentz spaces $L^{3,q}$ with interpolation exponent $q \geq 3$. In that case, $\|u\|_{L^{3,q}(\{|x|\sim R_k\})} \geq \alpha$ implies

$$\|u\|_{L^{3,q}(\mathbb{R}^3)} \gtrsim \left\| \|u\|_{L^{3,q}(\{|x|\sim R_k\})} \right\|_{\ell_k^q} \geq \alpha K^{\frac{1}{q}},$$

and so one should expect the bounds from the stacking argument (3) used in the Lorentz space $L^{3,q}$ extension [13] to degenerate as $q \to \infty$. Indeed, if $|u(x)| = |x|^{-1}$ then, for some constant $\alpha > 0$, we have $\|u\|_{L^{3,\infty}(\{|x|\sim R\})} \geq \alpha$ for all $R > 0$, yet $\|u\|_{L^{3,\infty}(\mathbb{R}^3)} \sim 1$ which shows that the first inequality in (3) fails for the $L^{3,\infty}$ norm. For this reason, the approach of Tao [44] (and, for related reasons, of Escauriaza-Seregin-Šverák) to the L^3 problem cannot be extended to $L^{3,\infty}$.

This issue is in fact closely related to the study of Type 1 blow-ups and approximately self-similar solutions to (1). Leray famously conjectured the existence of backwards self-similar solutions that blow up in finite time, a possibility later ruled out by Nečas, Růžička, and Šverák [26] for finite-energy solutions and by Tsai [45] for locally-finite energy solutions. The latter reference identifies the following as a very natural ansatz for blow-up:

$$\begin{aligned} u(t,x) &= \frac{1}{(T_0-t)^{\frac{1}{2}}} U\left(\frac{x}{(T_0-t)^{\frac{1}{2}}}\right), \\ U(y) &= a\left(\frac{y}{|y|}\right)\frac{1}{|y|} + o\left(\frac{1}{|y|}\right) \text{ as } |y| \to \infty, \end{aligned} \tag{4}$$

where $a : S^2 \to \mathbb{R}^3$ is smooth. While Tsai [45] shows that there are no solutions *exactly* of this form, solutions that approximate this profile or attain it in a discretely self-similar way are promising candidates for singularity formation, as demonstrated by, for example, the Scheffer constructions [27, 28, 36, 37], and the recent numerical evidence of an approximately self-similar singularity for the axisymmetric system due to Hou [15]. Unfortunately, criteria pertaining to L^3 such as those in [12, 31, 44] are less effective at controlling such solutions because $|x|^{-1} \notin L^3(\mathbb{R}^3)$, which shows the relevance of the weak norm $L^{3,\infty}$.

Specializing to the case of axial symmetry, it is known, for instance, that certain critical pointwise estimates of u with respect to the distance from the axis imply regularity [6, 7, 33]. Moreover, Koch, Nadirashvili, Seregin, and Šverák [16] proved a Liouville-type theorem for ancient axisymmetric solutions. Furthermore, Seregin [39] proved that finite-time blow-up cannot be of Type I. Thus, roughly speaking, no axisymmetric solution can approximate the profile (4) all the way up to a putative blow-up time T_0. However, this regularity is only qualitative (indeed, the proof uses an argument by contradiction based on a "zooming in" procedure), and so explicit bounds on the solution have not been available.

The main purpose of this work is to make this regularity quantitative, in a similar sense in which Tao [44] quantified the Escauriaza-Seregin-Šverák theorem [12]. This allows us to not only to rule out Type I singularies, but also to control how singular they can possibly become. For example, it lets us estimate the length scale up to which a solution can be approximated by a self-similar profile, see Corollary 1.3 for details.

1.2 The Main Regularity Theorem

We suppose that a strong solution to (1) on the time interval $[0, T]$ is axisymmetric, namely that

$$\partial_\theta u_r = \partial_\theta u_3 = \partial_\theta u_\theta = 0, \tag{5}$$

where u_r, u_θ, u_3 denote (respectively) the radial, angular, and vertical components of u, so that

$$u = u_r e_r + u_\theta e_\theta + u_3 e_3$$

in cylindrical coordinates, where e_r, e_θ, e_3 denote the cylindrical basis vectors. We assume further that u remains bounded in $L^{3,\infty}$,

$$\|u\|_{L^\infty([0,T];L^{3,\infty}(\mathbb{R}^3))} \leq A \tag{6}$$

for some $A \gg 1$. We prove the following.

Theorem 1.1 (Main result) *Suppose u is a classical axisymmetric solution of (1) on $[0, T] \times \mathbb{R}^3$ obeying (6). Then*

$$\|\nabla^j u(t)\|_{L^\infty_x(\mathbb{R}^3)} \leq t^{-\frac{1+j}{2}} \exp\exp(A^{O_j(1)})$$

for all $j \geq 0$, $t \in [0, T]$.

We note that, although our proof of the above theorem does use some of the basic a priori estimates (see Section 4.2) pointed out by Tao [44], it follows a completely different scheme. Our main ingredients are parabolic methods applied to the swirl $\Theta := r u_\theta$ near the axis, as well as localized energy estimates on

$$\Phi := \frac{\omega_r}{r} \quad \text{and} \quad \Gamma := \frac{\omega_\theta}{r}. \tag{7}$$

In a sense, we use those estimates to replace the Carleman inequalities appearing in Tao's [44] approach.

To be more precise, our proof builds on the work of Chen, Fang, and Zhang [8], who showed that the energy norm of Φ, Γ,

$$\|\Phi\|_{L_t^\infty L_x^2} + \|\Gamma\|_{L_t^\infty L_x^2} + \|\nabla\Phi\|_{L_t^2 L_x^2} + \|\nabla\Gamma\|_{L_t^2 L_x^2}, \tag{8}$$

controls u via an estimate on $\|u_\theta^2/r\|_{L^2}$ (see [8, Lemma 3.1]). They also observed that one can indeed estimate this energy norm as long as the angular velocity u_θ remains small in any neighbourhood of the axis, namely if

$$\|r^d u_\theta\|_{L_t^\infty([0,T];L^{3/(1-d)}(\{r\le\alpha\}))} \text{ is sufficiently small for some } \alpha > 0 \text{ and } d \in (0,1). \tag{9}$$

In fact, this can be observed from the PDEs satisfied by Φ, Γ,

$$\begin{aligned} &\left(\partial_t + u\cdot\nabla - \Delta - \frac{2}{r}\partial_r\right)\Gamma + \frac{2}{r^2}u_\theta\omega_r = 0,\\ &\left(\partial_t + u\cdot\nabla - \Delta - \frac{2}{r}\partial_r\right)\Phi - (\omega_r\partial_r + \omega_3\partial_3)\frac{u_r}{r} = 0, \end{aligned} \tag{10}$$

which show that, in order to control the energy of Γ, Φ one needs to control u_r/r, ω_r, ω_3 and u_θ. However, u_r/r can be controlled by Γ in the sense that

$$\frac{u_r}{r} = \Delta^{-1}\partial_3\Gamma - 2\frac{\partial_r}{r}\Delta^{-2}\partial_3\Gamma \tag{11}$$

(see [8, p. 1929] for details), which is one of the main properties of function Γ. In particular, (11) lets us use the Calderón-Zygmund inequality to obtain that

$$\left\|D^2\frac{u_r}{r}\right\|_{L^q} \le \|\partial_3\Gamma\|_{L^q} \tag{12}$$

for $q \in (1,\infty)$ (see [8, Lemma 2.3] for details). Moreover $\omega_r = r\Phi$, and $\omega_3 = \partial_r(ru_\theta)/r$, which shows that the L^2 estimate of Φ, Γ relies only on control of u_θ. In fact, away from from the axis, one can easily control u_θ, while near the axis the smallness condition (9) is required in an absorption argument by the dissipative part of the energy, see [8, (3.11)–(3.14)] for details.

In this work we obtain such control of u_θ thanks to the weak-L^3 bound (6), by utilizing parabolic theory developped by Nazarov and Ural'tseva [24] in the spirit of the Harnack inequality. Namely, noting that the swirl $\Theta := ru_\theta$ satisfies the autonomous PDE

$$\left(\partial_t + \left(u + \frac{2}{r}e_r\right)\cdot\nabla - \Delta\right)\Theta = 0 \tag{13}$$

everywhere except for the axis, one can deduce (as observed in [24, Section 4]) Hölder continuity of Θ near the axis. A similar observation, but in a case of limited regularity of u was used by Seregin [39] in his proof of no Type I blow-ups for axisymmetric solutions. We quantify this approach (see Proposition 5.1 below) to obtain an estimate on the Hölder exponent in terms of the weak-L^3 norm, and hence we obtain sufficient control of the swirl Θ in a very small neighbourhood of the axis. As for the outside of the neighbourhood, we obtain pointwise estimates on u and all its derivatives, which are quantified with respect to A, and which improve the second author's estimates [31, Proposition 8]. This would enable one to close the energy estimates for the quantities in (8) if there exist sufficiently many starting times where the energy norms are finite. Indeed, given a weak L^3 bound (6) and short time control of the dynamics of the energy (8), control of $\|\Phi(T)\|_{L^2} + \|\Gamma(T)\|_{L^2}$ can be propagated from an initial time very close to $t = T$. Unfortunately, there are no times when we can explicitly control these energies in terms of A due to lack of quantitative decay in the x_3 direction. The standard approach of propagating L^2 control of Φ, Γ from the initial data at $t = 0$ (for instance, as in [8]) would lead to additional exponentials in Theorem 1.1.

To avoid this issue and prove efficient bounds, we replace (8) with L^2 norms that measure Φ and Γ uniformly-locally in x_3: namely, we consider

$$\|\Phi\|_{L^\infty_t L^2_{3-\mathrm{uloc}}} + \|\Gamma\|_{L^\infty_t L^2_{3-\mathrm{uloc}}} + \|\nabla\Phi\|_{L^2_t L^2_{3-\mathrm{uloc}}} + \|\nabla\Gamma\|_{L^2_t L^2_{3-\mathrm{uloc}}}, \tag{14}$$

where $\|\cdot\|_{L^2_{3-\mathrm{uloc}}} := \sup_{z\in\mathbb{R}} \|\cdot\|_{L^2(\mathbb{R}^2\times[z-1,z+1])}$. See Proposition 6.1 below for an estimate of such energy norm. This approach gives rise to two further challenges.

One of them is the x_3-uloc control of the solution u itself in terms of (14). We address this difficulty by an x_3-uloc generalization of the L^4 estimate on $u_\theta/r^{1/2}$ introduced by [8, Lemma 3.1], together with a x_3-uloc bootstrapping via $\|u\|_{L^\infty_t L^6_{3-\mathrm{uloc}}}$, as well as an inductive argument for the norms $\|u\|_{L^\infty_t W^{k-1,6}_{\mathrm{uloc}}}$ with respect to $k \geq 1$, where "uloc" refers to the uniformly locally integrable spaces (in all variables, not only x_3). We refer the reader to Steps 2–4 in Section 7 for details.

Another challenge is an x_3-uloc estimate on u_r in terms of Γ. To be more precise, instead of the global estimate (12), we require $L^2_{3-\mathrm{uloc}}$ control of u_r/r, which is much more challenging, particularly considering the bilaplacian term in (11) above. To this end we develop a bilaplacian Poisson-type estimate in $L^2_{3-\mathrm{uloc}}$ (see Lemma 5.5), which enables us to show that

$$\left\|\nabla\partial_r \frac{u_r}{r}\right\|_{L^2_{3-\mathrm{uloc}}} + \left\|\nabla\partial_3 \frac{u_r}{r}\right\|_{L^2_{3-\mathrm{uloc}}} \lesssim \|\Gamma\|_{L^2_{3-\mathrm{uloc}}} + \|\nabla\Gamma\|_{L^2_{3-\mathrm{uloc}}}, \tag{15}$$

see Lemma 5.3. Note that this is a x_3-uloc generalization of (12), and also requires the whole gradient on the right-hand side, rather than $\partial_3\Gamma$ only. Such an estimate lets us close the bound in (14), and thus control all subcritical norms of u in terms of $\|u\|_{L^{3,\infty}}$ (see Section 7 for details).

Having overcome the two difficulties of controlling the energy (14), we deduce (in (73)) that $\|\Gamma(t)\|_{L^2_{3-\mathrm{uloc}}} \leq \exp\exp A^{O(1)}$ for all $t \in [1/2, 1]$, whenever a solution u satisfies $\|u\|_{L^\infty([0,1];L^{3,\infty})} \leq A$; see Figure 1(supposing that $T = 1$). This suffices for

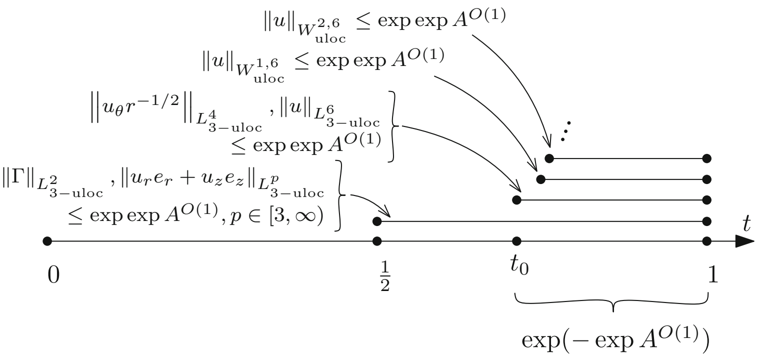

Fig. 1 A sketch of the proof of Theorem 1.1

iteratively improving the quantitative control of u until $t = 1$. Indeed, we first deduce a subcritical bound on the swirl-free part of the velocity on the same time interval, namely that $\|u_r e_r + u_z e_z\|_{L^p_{3-\mathrm{uloc}}} \lesssim_p \exp\exp A^{O(1)}$ for $p \ge 3$ and $t \in [1/2, 1]$. We can then control (in (74)) the time evolution of $\|u_\theta r^{-1/2}\|_{L^4_{3-\mathrm{uloc}}}$ over short time intervals, and so, choosing $t_0 \in [0, 1]$ sufficiently close to 1 (by picking a time of regularity, see Lemma 4.2) we then obtain (in (75)) that $\|u_\theta r^{-1/2}\|_{L^4_{3-\mathrm{uloc}}}$ and $\|u\|_{L^6_{3-\mathrm{uloc}}}$ are bounded by $\exp\exp A^{O(1)}$ for all $t \in [t_0, 1]$, see Figure 1. This subcritical bound allows one to also estimate $\|u\|_{W^{k,6}_{\mathrm{uloc}}} \le \exp\exp A^{C_k}$ for every k, on a time interval of the same size (see Step 4 in Section 7), which yields the claim of Theorem 1.1.

1.3 A Comparison of the Blow-Up Rate

We note that Theorem 1.1, together with the well-known blow-up criterion $\|u(t)\|_\infty \ge c/(T_0 - t)^{1/2}$ (see [30, Corollary 6.25], for example), where $T_0 > 0$ is a putative blow-up time, immediately implies the following lower bound on the blow-up rate of $\|u(t)\|_{L^{3,\infty}}$.

Corollary 1.2 (Blow-up rate of the weak-L^3 norm) *If u is a classical axisymmetric solution of (1) that blows up at T_0, then*

$$\limsup_{t\to T_0^-} \frac{\|u(t)\|_{L^{3,\infty}(\mathbb{R}^3)}}{(\log\log(T_0 - t)^{-1})^c} = +\infty. \tag{16}$$

This corollary is also a consequence of a recent theorem of Chen, Tsai, and Zhang [9], who prove[1]

$$\limsup_{t\to T_0^-} \frac{\|b(t)\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)}}{\left(\log\log \frac{100}{T_0-t}\right)^{\frac{1}{48}-}} = +\infty,$$

[1] Let us note the existence of a substantial misprint in the published version of [9]: in their Theorem 1.4, as in our Corollary 1.2, the blow-up rate is *double*-logarithmic.

where $b := u_r e_r + u_3 e_3$ denotes the swirl-less part of the velocity field u (see [22, Section 3.3] for the relevant definition of $\dot{B}^{-1}_{\infty,\infty}$). Thus, since $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3) \supset L^{3,\infty}$, the above blow-up rate implies (16). We conjecture that a variant of Theorem 1.1 holds with the weak-L^3 norm replaced by such a critical Besov norm and can be proved using the ideas presented here.

In order to describe the relation of Corollary 1.2 to [9], we note that the argument in [9] proceeds by proving a pointwise estimate of the form

$$|ru_\theta| \le C\exp(-c|\log r|^\tau), \tag{17}$$

where $c, C > 0$, $\tau \in (0,1)$, for axisymmetric solutions obeying the slightly supercritical bound

$$\frac{1}{R^{\frac{1}{2}}}\|u\|_{L^\infty((-R^2,0);L^2(B_R))} \le K\left(\log\log\frac{100}{R}\right)^\beta \quad \text{for all } R \in (0,1/4]$$

for some $\beta \in (0, \frac{1}{8})$ and $K > 0$. This is yet another application of Harnack inequality methods to axisymmetric Navier-Stokes equations. Rather than proving Hölder continuity of Θ under a global control of a critical norm as we do in Proposition 5.1, [9] obtains (17) by an “almost Hölder continuity,”

$$\operatorname*{osc}_{Q_\rho}\Theta \le \exp\left(-c\left(\left(\log\frac{100}{\rho}\right)^\tau - \left(\log\frac{100}{R}\right)^\tau\right)\right)\operatorname*{osc}_{Q_R}\Theta \tag{18}$$

for $0 < \rho < R \le 1/4$, $\tau \in (0,1)$; see [9, Proposition 1.2]. A similar result in the case of $\tau = 1/4$ has been obtained independently by Seregin [40, Proposition 1.3]. Note that the case of $\tau = 1$ corresponds to Hölder continuity.

We emphasize that the main point of our work is not to improve the blow-up rate but to give an explicit bound on u and its derivatives in terms of only the critical norm—this is a strictly stronger result in the sense that it pertains to *all* axisymmetric classical solutions, even those not blowing up. A naïve attempt to prove a similar quantitative theorem (e.g., using ideas of estimating axisymmetric vector fields from [21]) would lead to a bound which, compared to Theorem 1.1, would contain more iterated exponentials as well as severe dependence on the time t and subcritical norms of the initial data. Instead, Theorem 1.1 parallels the results in [44] and improves on those in [31] in the sense that the final bound depends only on $\|u\|_{L^\infty_t L^{3,\infty}_x}$ and a dimensional factor in t. This also leads to additional interesting corollaries: for instance, an explicit rate of convergence for $u(t) \to 0$ as $t \to +\infty$, and the non-existence of nontrivial ancient axisymmetric solutions in $L^\infty_t L^{3,\infty}_x$.

A comparison of these results with the work of Chen, Tsai, and Zhang [9] raises the following question: Is it possible to efficiently control (in the sense of Theorem 1.1) u and its derivatives in terms of only b measured in some critical norm? In fact, in our proof of Hölder continuity of Θ near the axis (Proposition 5.1) one can easily replace (6) with boundedness of $\|b(t)\|_{L^{3,\infty}}$ in time, since “u” in (13) can be replaced by “b”, due to axisymmetry. However, we do require $L^{3,\infty}$ control of all components of u for other quantitative estimates leading to Theorem 1.1. These include the basic estimates (Lemmas 4.2–4.4), quantitative decay away from the axis (Proposition 5.2), as well as energy estimates on Γ and Φ (Proposition 6.1) and their implementation in the main argument (Section 7).

A related open problem is to explicitly control u in terms of u_θ only. In fact, despite a number of works [8, 18, 20, 25, 40, 46] on the properties of the swirl ru_θ, its role in the regularity problem of axisymmetric solutions remains unclear.

1.4 An Estimate on the Self-similar Length Scale

One of the remarkable consequences of the quantitative estimate provided by Theorem 1.1 above is that it provides an estimate on the length scale up to which an axisymmetric solution to the NSE (1) can be approximated by a self-similar profile as in (4).

In order to make this precise, we will say that a vector field $b \in L^\infty(\mathbb{R}^3;\mathbb{R}^3)$ is *nearly-spherical* if there exists $\delta \in (0,1/2)$ such that for every $R>0$, there exists $x_0 \in \mathbb{R}^3$ with $|x_0| = R$ such that

$$|b(x_0)| \geq \frac{\|b\|_\infty}{2} \quad \text{and} \quad |b(x)-b(x_0)| \leq \frac{\|b\|_\infty}{4} \quad \text{for all } x \in B(x_0,\delta|x_0|). \tag{19}$$

Clearly any spherical profile $b(x) = a(x/|x|)$ is nearly-spherical for every $a \in C(\partial B(0,1))$ (in which case the choice of δ for (19) to hold can be made by a simple continuity argument). Let $\psi \in C_c^\infty(\mathbb{R}^3;[0,1])$ be such that $\int \psi = 1$, and let $\psi_l(x) := l^{-3}\psi(x/l)$ denote a mollifier at a given length scale $l>0$. We also set $\widetilde{\psi}_l := \psi_l * \psi_l$.

We note that, letting $R := 2l/\delta$, we can find $x_0 \in \mathbb{R}^3$ with $|x_0| = 2l/\delta$ and satisfying (19). In particular

$$\left|\left(\widetilde{\psi}_l * \frac{b(\cdot)}{|\cdot|}\right)(x_0)\right| = \left|\int_{B(x_0,2l)} \widetilde{\psi}_l(x_0-y)\frac{b(y)}{|y|}\mathrm{d}y\right| \gtrsim \frac{|b(x_0)| - \|b\|_\infty/4}{(1+\delta)|x_0|} \geq \frac{\delta\|b\|_\infty}{16l},$$

which shows that

$$\left\|\widetilde{\psi}_l * \frac{b(\cdot)}{|\cdot|}\right\|_\infty \geq \frac{\delta\|b\|_\infty}{16l} \tag{20}$$

for every length scale $l>0$. This simple fact lets us deduce from Theorem 1.1 that, if an axisymmetric solution approximates a self-similar profile $b(t,x)/|x|$ up to length scale $l(t)$, where b is nearly-spherical uniformly on $[0,t]$, then $l(t)$ cannot be smaller than a particular quantitative threshold.

Corollary 1.3 *If u is a strong axisymmetric solution u of (1) on* $[0,T]$,

$$\left\|u(t) - \psi_{l(t)} * \frac{b(t,x)}{|x|}\right\|_{L^{3,\infty}} \leq \sigma\|b(t)\|_\infty \tag{21}$$

for $t \in [0, T]$, and $\sigma < c\delta$, where $c > 0$ is a sufficiently small constant and $b(T)$ is nearly-spherical with constant δ, then

$$l(T) \gtrsim \delta T^{\frac{1}{2}} \|b(T)\|_\infty \exp\left(-\exp\left(\|b\|^{O(1)}_{L^\infty_{t,x}([0,T]\times\mathbb{R}^3)}\right)\right).$$

Proof We note that, at time T,

$$\begin{aligned}\|u\|_\infty &\gtrsim \|\psi_l * u\|_\infty \\ &\geq \left\|\widetilde{\psi}_l * \frac{b(\cdot)}{|\cdot|}\right\|_\infty - \left\|\psi_l * \left(u - \psi_l * \frac{b(\cdot)}{|\cdot|}\right)\right\|_\infty \\ &\geq \frac{\delta\|b\|_\infty}{16l} - Cl^{-1}\left\|u - \psi_l * \frac{b(\cdot)}{|\cdot|}\right\|_{L^{3,\infty}} \\ &\geq \left(\frac{\delta}{16} - C\sigma\right)\frac{\|b\|_\infty}{l}.\end{aligned}$$

Thus $\|u(T)\|_\infty \geq \delta\|b(T)\|_\infty/32l$ if $\sigma \in (0, \delta/32C)$. Since also

$$\|u(t)\|_{L^{3,\infty}} \leq \left\|\widetilde{\psi}_{l(t)} * \frac{b(t,\cdot)}{|\cdot|}\right\|_{L^{3,\infty}} + \left\|u(t) - \psi_{l(t)} * \frac{b(t,\cdot)}{|\cdot|}\right\|_{L^{3,\infty}} \leq C\|b(t,\cdot)\|_\infty$$

for *all* $t \in [0, T]$, Theorem 1.1 implies that

$$\frac{\delta\|b(T)\|_\infty}{32\, l(T)} \leq \|u(T)\|_\infty \lesssim T^{-1/2}\exp\exp\left(\|b\|^{O(1)}_{L^\infty([0,T]\times\mathbb{R}^3)}\right),$$

from which the claim follows. □

1.5 Organization of the Paper

The structure of the paper is as follows. In the following Section 2 we discuss preliminary concepts related to the Lorentz spaces $L^{p,q}$, the Bogovskiĭ operator, a simple Poisson-type tail estimate that we will later (in Section 5.3) expand to obtain our Poisson-type estimate (15) above, as well as some properties of cylindrical coordinates. In Section 3 we discuss some properties of axisymmetric functions, including an axisymmetric Bernstein inequality (Section 3.1) and a quantified version of Hardy's inequality (Section 3.2). In Section 4 we present some quantitative estimates of the 3D Navier–Stokes equations, including the Picard iterates (Section 4.1), times of regularity, bounded total speed, and second derivatives estimates (Section 4.2), all of which remain valid without the assumption of axisymmetry. The following section, Section 5, is dedicated to quantitative estimates that are specific to the axisymmetric setting (5) of the equations (1). These include the statement of the Hölder estimate of the swirl Θ mentioned above (Section 5.1), pointwise estimates away from the axis (Section 5.2), as well as the Poisson-type x_3-uloc estimate on u_r/r (15) (Section 5.3). In Section 6 we prove the energy estimate (14) for Γ and Φ mentioned above, and

Section 7 combines the developed methods to prove the main theorem, Theorem 1.1. Finally, Appendix A includes a detailed verification of the Hölder estimate of Θ.

2 Preliminaries

Given $f : \Omega \to \mathbb{R}$ we let

$$\operatorname*{osc}_{\Omega} f := \sup_{\Omega} f - \inf_{\Omega} f$$

denote the oscillation of f over Ω. We also denote by $⨍_{\Omega} := \frac{1}{|\Omega|}\int_{\Omega}$ the average over Ω.

We use standard definitions of Lebesgue spaces $L^p(\Omega)$, Sobolev spaces $W^{k,p}(\Omega)$, spaces of continuous functions $C(\Omega)$, spaces $C_c(\Omega)$ of continuous functions with compact support. For brevity of notation we often omit "Ω" in the notation if $\Omega = \mathbb{R}^3$; for example $W^{1,\infty} \equiv W^{1,\infty}(\mathbb{R}^3)$. We use the convention $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{R}^3)}$, and we reserve the notation $\|\cdot\| := \|\cdot\|_2$ for the $L^2(\mathbb{R}^3)$ norm. We also write $\int := \int_{\mathbb{R}^3}$. Given $p \in [1,\infty]$, we define the uniformly local L^p norms,

$$\|u\|_{L^p_{\mathrm{uloc}}} := \sup_{x\in\mathbb{R}^3} \|u\|_{L^p_x(B(x,1))} \quad \text{and} \quad \|u\|_{L^p_{t,x-\mathrm{uloc}}} := \Big\| \|u\|_{L^p_{\mathrm{uloc}}} \Big\|_{L^p_t}, \tag{22}$$

as well as the norms that are uniformly local in x_3 only,

$$\|f\|_{L^p_{3-\mathrm{uloc}}(\mathbb{R}^3)} := \sup_{z\in\mathbb{R}} \|f\|_{L^p(\mathbb{R}^2\times[z-1,z+1])}. \tag{23}$$

We let $\Psi(x,t) := (4\pi t)^{-3/2}\mathrm{e}^{-x^2/4t}$ denote the heat kernel, which satisfies

$$\|\nabla^k \Psi(t)\|_p = C_{k,p} t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{k}{2}}. \tag{24}$$

We often use the notation $\mathrm{e}^{t\Delta} f := \Psi(t) * f$.

Given $N \in \{2^k : k \in \mathbb{N}\}$ we let P_N denote the N-th Littlewood-Paley projection. We recall a localized version of the Bernstein inequality

$$\|P_N f\|_{L^q(\Omega)} \lesssim_k N^{\frac{3}{p_1}-\frac{3}{q}} \|P_N f\|_{L^{p_1}(\Omega_R)} + (RN)^{-k} N^{\frac{3}{p_2}-\frac{3}{q}} \|P_N f\|_{L^{p_2}}, \tag{25}$$

where $\Omega \subset \mathbb{R}^3$ is an open set, $k \geq 1$, $\Omega_R := \{x \in \mathbb{R}^3 : \mathrm{dist}(x,\Omega) < R\}$, $q \in [1,\infty]$ and $p_1, p_2 \in [1,q]$; see [44, Lemma 2.1] for a proof.

2.1 Lorentz Spaces

We recall the Lorentz spaces, defined by

$$\|f\|_{L^{p,q}} := p^{1/q} \|\lambda |\{|f| \geq \lambda\}|^{1/p}\|_{L^q(\mathbb{R}_+, \frac{\mathrm{d}\lambda}{\lambda})} \tag{26}$$

for $q < \infty$ and

$$\|f\|_{L^{p,\infty}} := \|\lambda |\{|f| \geq \lambda\}|^{1/p}\|_{L^\infty(\mathbb{R}_+, \frac{\mathrm{d}\lambda}{\lambda})}.$$

We recall the Hölder inequality for Lorentz spaces,

$$\|fg\|_{L^{p,q}} \leq C_{p_1,p_2,q_1,q_2} \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}, \tag{27}$$

whenever $1/p = 1/p_1 + 1/p_2$, $1/q = 1/q_1 + 1/q_2$, $p_1, p_2, p \in (0,\infty)$, $q_1, q_2, q \in (0,\infty]$. We refer the reader to [43, Theorem 6.9] for a proof of (27). The Hölder inequality can be very useful when estimating some localized integrals in terms of the $L^{p,\infty}$ norm. For example, if $\phi \in C_0^\infty(\Omega)$ is a smooth cutoff function then we have the simple estimate

$$\|\phi\|_{L^{p,1}} = p \int_0^\infty |\{|\phi| \geq \lambda\}|^{1/p} \mathrm{d}\lambda \leq p \int_0^{\|\phi\|_\infty} |\{|\phi| \geq \lambda\}|^{1/p} \mathrm{d}\lambda \leq p |\Omega|^{1/p} \|\phi\|_\infty,$$

which shows that, for example

$$\int_\Omega fg \leq \|f\|_{L^{3,\infty}} \|g\|_2 |\Omega|^{1/6}.$$

This simple method allows us to use the weak L^3 space to estimate some integrals over a region close to the axis of symmetry.

We also note two Young's inequalities involving weak L^p spaces

$$\|f * g\|_{L^{p,\infty}} \lesssim \|f\|_1 \|g\|_{L^{p,\infty}} \qquad \text{for } p \in (1,\infty), \tag{28}$$

$$\|f * g\|_p \lesssim \|f\|_r \|g\|_{L^{q,\infty}} \quad \text{for } p,q,r \in (1,\infty) \text{ with } \frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}, \tag{29}$$

see [22, Proposition 2.4(a)] and [35, Theorem A.16] for details (respectively).

2.2 The Bogovski Operator

We recall that, given $p \in (1,\infty)$, an open ball $B \subset \mathbb{R}^3$, $b \in W^{1,p}(B)$ such that $\operatorname{div} b = 0$, and $\phi \in C_0^\infty(B; [0,1])$ such that $\phi = 1$ on $B/2$ there exists $\overline{b} \in W^{1,p}(\mathbb{R}^3)$ such that $\overline{b} = 0$ outside B and inside $B/2$,

$$\operatorname{div} \overline{b} = \operatorname{div}(\phi b) \quad \text{and} \quad \|\overline{b}\|_{W^{1,p}} \lesssim_B \|b\|_{W^{1,p}(B)}, \tag{30}$$

due to the Bogovskiĭ lemma (see [4, 5] or [14, Lemma III.3.1], for example). Here we use the non-homogeneous $W^{1,p}$ norm and so the implicit constant in (30) may depend of the size of B. We note that the Bogovskiĭ lemma often assumes that the domain is star-shaped (which is not the case for $B \setminus B/2$), but it can be overcome in this particular setting by applying the partition of identity to ϕ; see [29, Section 2.3] for example.

2.3 A Poisson-Type Tail Estimate

Here we are concerned with a Poisson equation of the form $-\Delta f = D^2 g$, and we show that any $W^{k,\infty}(B(0,1))$ norm of ∇f can be bounded by the L^1_{uloc} norm of g, if $g = 0$ on $B(0,2)$.

To be more precise, we let $\psi \in C_c^\infty(B(0,1);[0,1])$ be such that $\psi = 1$ on $B(0,1/2)$. Given $y \in \mathbb{R}^3$ we set

$$\psi_y(x) := \psi(x-y). \tag{31}$$

and

$$\widetilde{\psi} := \sum_{\substack{j\in\mathbb{Z}^3\\ |j|\le 10}} \psi_j.$$

Lemma 2.1 *Suppose that* $f = D^2(-\Delta)^{-1}(g(1-\widetilde{\psi}))$ *for some* $g \in L^2$. *Then*

$$\|\psi \nabla f\|_{W^{k,\infty}} \lesssim_k \|g\|_{L^1_{\mathrm{uloc}}} \qquad \textit{for } k \ge 0.$$

Proof We note that

$$\partial_i f(x) = \int \frac{(x_i - y_i) g(y)(1-\tilde{\phi}(y))}{|x-y|^5} \mathrm{d}y$$

for $x \in \operatorname{supp} \phi$, and so

$$\begin{aligned}
|\nabla f(x)| &\le \int_{\{|x-y|\ge 5\}} \frac{|g(y)|}{|x-y|^4} \mathrm{d}y \\
&\le \sum_{\substack{j\in\mathbb{Z}^3\\ |j|\ge 2}} \int_{x_1+j_1}^{x_1+j_1+1} \int_{x_2+j_2}^{x_2+j_2+1} \int_{x_3+j_3}^{x_3+j_3+1} \frac{|g(y)|}{|x-y|^4} \mathrm{d}y_3\, \mathrm{d}y_2\, \mathrm{d}y_1 \\
&\lesssim \|g\|_{L^1_{\mathrm{uloc}}} \sum_{\substack{j\in\mathbb{Z}^3\\ |j|\ge 2}} |j|^{-4} \lesssim \|g\|_{L^1_{\mathrm{uloc}}},
\end{aligned}$$

as required. An analogous argument applies to higher derivatives of f. □

The above proof demonstrates a simple method of tail estimation which we will later use to obtain a $L^2_{3-\mathrm{uloc}}$ estimate of u_r/r in terms of Γ, mentioned in the introduction (recall (15)). In fact, to this end, a similar strategy can be applied in the x_3 direction only, and can be extended to the more challenging biLaplacian Poisson equation (see Lemma 5.5 below).

2.4 Cylindrical Coordinates

Given $x \in \mathbb{R}^3$ we denote by $x' := (x_1, x_2)$ the horizontal variables, and $r := (x_1^2 + x_2^2)^{1/2}$ denotes the radius in the cylindrical coordinates. We often use the notation

$$\{r < r_0\} := \{x \in \mathbb{R}^3 : r < r_0\}$$

for a given $r_0 > 0$.

We recall a version of the Hardy inequality

$$\|r^{-1} f\|_{L^q(\Omega)} \lesssim C(\Omega)\|f\|_{L^q(\Omega)} + \|\nabla f\|_{L^q(\Omega)}, \tag{32}$$

where Ω is a bounded domain and $q \in (1, 2]$; see [8, Lemma 2.4] for a proof.

We recall the divergence operator in cylindrical coordinates: if $v = v_r e_r + v_\theta e_\theta + v_3 e_3$ then

$$\operatorname{div} v = \frac{1}{r}\partial_r(r v_r) + \frac{1}{r}\partial_\theta v_\theta + \partial_3 v_3. \tag{33}$$

We say that a vector field v is axisymmetric if (5) holds. In such case we have

$$|\nabla' v|^2 = (\partial_r v_r)^2 + (\partial_r v_\theta)^2 + (\partial_r v_3)^2 + \frac{1}{r^2}(v_r^2 + v_\theta^2), \tag{34}$$

which implies the pointwise bounds

$$\frac{|v_r|}{r}, \ \frac{|v_\theta|}{r} \le |\nabla' v|.$$

Here ∇' refers to the gradient with respect to the horizontal variables x' only.

Moreover,

$$|\partial_{rr} f| \lesssim |D^2 f|. \tag{35}$$

Indeed, since

$$\partial_r = \cos\theta\, \partial_1 + \sin\theta\, \partial_2 = \frac{x_1}{|x'|}\partial_1 + \frac{x_2}{|x'|}\partial_2,$$

where $x' := (x_1, x_2)$ refers to the horizontal variables, we can compute that

$$\partial_{rr} = \frac{x_1^2}{|x'|^2}\partial_{11} + 2\frac{x_1 x_2}{|x'|^2}\partial_1\partial_2 + \frac{x_2^2}{|x'|^2}\partial_{22},$$

from which (35) follows. More generally,

$$\partial_{rrr} = \frac{x_1^3}{|x'|^3}\partial_{111} + \frac{3x_1^2x_2}{|x'|^3}\partial_{11}\partial_2 + \frac{3x_1x_2^2}{|x'|^3}\partial_1\partial_{22} + \frac{x_2^3}{|x'|^3}\partial_{222},$$
$$\partial_{rrrr} = \frac{x_1^4}{|x'|^4}\partial_{1111} + \frac{4x_1^3x_2}{|x'|^4}\partial_{111}\partial_2 + \frac{6x_1^2x_2^2}{|x'|^4}\partial_{11}\partial_{22} + \frac{4x_1x_2^3}{|x'|^4}\partial_1\partial_{222} + \frac{x_2^4}{|x'|^4}\partial_{2222}.$$

This shows that

$$|D^3_{r,x_3} f| \lesssim |D^3 f| \qquad \text{and} \qquad |D^4_{r,x_3} f| \lesssim |D^4 f| \tag{36}$$

for any axisymmetric f (here, for example, D^4 refers to all fourth order derivatives with respect to x_1, x_2, x_3).

3 Properties of Axisymmetric Functions

Here we discuss some properties of axisymmetric functions, including an axisymmetric Bernstein inequality and a quantified Hardy's inequality.

3.1 Bernstein Inequalities

Here we discuss a version of the axisymmetric Bernstein inequality provided by [31, Proposition 1] that involves the weak L^3 space.

Lemma 3.1 *Let T_m be a Fourier multiplier whose symbol m is supported on $B(0, N)$ with $|\nabla^j m| \le MN^{-j}$ and $1 < q < p \le \infty$. If either $-\frac{2}{p} < \alpha < \frac{1}{q} - \frac{1}{p}$ or $p = \infty$ and $\alpha = 0$, we have*

$$\|r^\alpha T_m u\|_{L^p} \lesssim MN^{\frac{3}{q} - \frac{3}{p} - \alpha} \|u\|_{L^{q,\infty}}$$

for all axisymmetric scalar- or vector-valued functions u.

Proof We normalize $M = N = 1$. Under these assumptions on p, α, Proposition 1 in [31] implies

$$\|r^\alpha T_m u\|_{L^p} \lesssim \|P_{\le 10} u\|_{L^{q+\epsilon}}$$

for $T_m P_{\le 10} = T_m$, since an $\epsilon > 0$ sufficiently small depending on p, q, α. Let ψ be the kernel such that $P_{\le 10} = \psi *$. Then by the weak Young inequality (29),

$$\|P_{\le 10} u\|_{L^{q+\epsilon}} \lesssim \|\psi\|_{L^{1+O(\epsilon)}} \|u\|_{L^{q,\infty}} \lesssim \|u\|_{L^{q,\infty}}.$$

□

A useful consequence of the above lemma is the following heat kernel estimate

$$
\begin{aligned}
\|r^\alpha e^\Delta \nabla^j f\|_{L^p} &\le \|r^\alpha e^\Delta \nabla^j P_{\le 1} f\|_{L^p} + \sum_{N>1} \|r^\alpha e^\Delta \nabla^j P_N f\|_{L^p} \\
&\lesssim_{\alpha,p,q,j} \|f\|_{L^{q,\infty}} (1 + \sum_{N>1} e^{-N^2/100} N^{j+\frac{3}{q}-\frac{3}{p}}) \\
&\lesssim_{p,q,j} \|f\|_{L^{q,\infty}} \qquad (37)
\end{aligned}
$$

under the same assumptions on the parameters as in Lemma 3.1.

3.2 A Quantified Version of the Hardy Inequality

By the classical Hardy inequality

$$
\|r^{-\frac{3}{p}+\frac{1}{2}} f\|_p \lesssim_p (\|f\|_2 + \|\nabla f\|_2)
$$

for any axisymmetric f, and $p \in (2,6)$ (see [8, Lemma 2.6], for example). Here we prove a version of this inequality, which is localized in the horizontal variables, "uloc" in x_3, and which has a quantified divergence of the constant near $p = 2$. Namely we prove the following.

Lemma 3.2 (Quantified Hardy inequality) *For* $p \in (2, 6-\epsilon)$,

$$
\|r^{-\frac{3}{p}+\frac{1}{2}} f\|_{L^p_{3-\mathrm{uloc}}(\{r\le 1\})} \lesssim_\epsilon (p-2)^{-O(1)} \left(\|f\|_{L^2_{3-\mathrm{uloc}}(\{r\le 1\})} + \|\nabla f\|_{L^2_{3-\mathrm{uloc}}(\{r\le 1\})} \right).
$$

Proof From the Sobolev embedding

$$
\|u\|_{L^{2p/(2-p)}(\mathbb{R}^2)} \lesssim (2-p)^{-O(1)} \|\nabla u\|_{L^p(\mathbb{R}^2)}
$$

for $p < 2$, (see, e.g., [42] where the sharp constant is computed), one can prove the two-dimensional Gagliardo-Nirenberg inequality

$$
\|f\|_{L^q(B(1))} \lesssim q \left(\|f\|_{L^6(B(1))}^{\frac{6}{q}} \|\nabla f\|_{L^2(B(1))}^{1-\frac{6}{q}} + \|f\|_{L^p(B(1))} \right) \qquad (38)
$$

for $q > 6$. Fix $\epsilon > 0$ to be specified. Then

$$
\begin{aligned}
\left\| \frac{f(\cdot, x_3)}{r^{\frac{3}{q}-\frac{1}{2}}} \right\|_{L^q_{x'}(r\ge\epsilon)} &\le \|r^{-\frac{3}{q}+\frac{1}{2}}\|_{L^{6q/(6-q)}_{x'}(\{r\ge\epsilon\})} \|f(\cdot,x_3)\|_{L^6_{x'}(\mathbb{R}^2)} \\
&\lesssim \epsilon^{-\frac{1}{q}+\frac{1}{6}} \|f(\cdot,x_3)\|_{L^6_{x'}(\mathbb{R}^2)}.
\end{aligned}
$$

Inside, for any $\frac{1}{s} \in (\frac{3}{2p} - \frac{1}{4}, \frac{1}{p})$, by (38),

$$\begin{aligned}\left\| \frac{f(\cdot, x_3)}{r^{\frac{3}{p}-\frac{1}{2}}} \right\|_{L^p_{x'}(r \le \min(1,\epsilon))} &\le \|r^{-\frac{3}{p}+\frac{1}{2}}\|_{L^s_{x'}(r<\min(1,\epsilon))} \|f(\cdot, x_3)\|_{L^{ps/(s-p)}_{x'}(B(1))} \\ &\lesssim \left(\frac{1}{s} - \frac{3}{2p} + \frac{1}{4}\right)^{-\frac{1}{s}} \left(\frac{1}{p} - \frac{1}{s}\right)^{-1} \\ &\quad \times \left(\epsilon^{-\frac{3}{p}+\frac{1}{2}+\frac{2}{s}} \|f(\cdot, x_3)\|^{\frac{6}{p}-\frac{6}{s}}_{L^6_{x'}(B(1))} \|\nabla f(\cdot, x_3)\|^{1-\frac{6}{p}+\frac{6}{s}}_{L^2_{x'}(B(1))} \right. \\ &\quad \left. + \|f(\cdot, x_3)\|_{L^p_{x'}(B(1))} \right).\end{aligned}$$

Upon taking $\epsilon = \|f\|_6^3 / \|\nabla f\|_2^3$ and $\frac{1}{s} = \frac{4}{3p} - \frac{1}{6}$,

$$\begin{aligned}\left\| \frac{f(\cdot, x_3)}{r^{\frac{3}{p}-\frac{1}{2}}} \right\|_{L^p_{x'}(B(1))} \lesssim (p-2)^{-O(1)} &\left(\|f(\cdot, x_3)\|^{\frac{3}{2}-\frac{3}{p}}_{L^6_{x'}(B(1))} \|\nabla f(\cdot, x_3)\|^{-\frac{1}{2}+\frac{3}{p}}_{L^2_{x'}(B(1))} \right. \\ &\left. + \|f(\cdot, x_3)\|_{L^p_{x'}(B(1))} \right).\end{aligned}$$

Finally by Hölder's inequality, Sobolev embedding, and Gagliardo-Nirenberg interpolation, we find

$$\left\| \frac{f}{r^{\frac{3}{p}-\frac{1}{2}}} \right\|_{L^p_x(B_{\mathbb{R}^2}(1) \times B_{\mathbb{R}}(z,1))} \lesssim (p-2)^{-O(1)} \|f\|_{H^1_x(B_{\mathbb{R}^2}(1) \times B_{\mathbb{R}}(z,1))},$$

as required. □

4 Basic Estimates for the Navier-Stokes Solutions

Here we discuss some estimates for the Navier-Stokes equations without the assumption of axisymmetry.

4.1 The Picard Estimates

We define the flat and sharp Picard iterates

$$u^\flat_n(t) := \mathrm{e}^{(t-t_n)\Delta} u(t_n) - \int_{t_n}^t \mathrm{e}^{(t-t')\Delta} \mathbb{P}\mathrm{div}(u^\flat_{n-1} \otimes u^\flat_{n-1}(t'))\mathrm{d}t', \quad u^\sharp_n := u - u^\flat_n \tag{39}$$

for all $n = 1, 2, \ldots$ and $t \ge t_n$, where $t_n \in [0, \frac{1}{2})$ is an increasing sequence of times, and $u^\flat_0 := 0$, $u^\sharp_0 := u$. We have the following.

Lemma 4.1 (Basic Picard estimates) *Assume u solves (1) on $[0,1]\times\mathbb{R}^3$ with the bound (6). If $p\in(3,\infty]$ and $-\frac{2}{p}<\alpha<\frac{1}{3}-\frac{1}{p}$ or $p=\infty$ and $\alpha=0$, we have*

$$\|r^\alpha \nabla^j u_n^\flat\|_{L_t^\infty L_x^p([\frac12,1]\times\mathbb{R}^3)} \le A^{O_{n,j,p}(1)}, \tag{40}$$

$$\|u_n^\sharp\|_{L_t^\infty L_x^q([\frac12,1]\times\mathbb{R}^3)} \le A^{O_{n,q}(1)} \qquad \textit{for all } q\in(1,3), \tag{41}$$

$$\|\nabla^j P_N u_n^\flat\|_{L_{t,x}^\infty([\frac12,1]\times\mathbb{R}^3)} \le \mathrm{e}^{-N^2/O_{n,j}(1)} A^{O_{n,j}(1)}, \tag{42}$$

as well as the energy estimate

$$\|\nabla u_n^\sharp\|_{L_{t,x}^2([\frac12,1]\times\mathbb{R}^3)} \le A^{O_n(1)}. \tag{43}$$

In particular,

$$\|\nabla u\|_{L_{t,x-\mathrm{uloc}}^2([\frac12,1]\times\mathbb{R}^3)} \le A^{O(1)}. \tag{44}$$

The proof of (40)–(42) above relies only on the definition (39) as well as basic heat estimates (24), which, together with the weak Young's inequality (29), can be used in the same way as [44, (3.11)–(3.13)] and [32, Proposition 2.5] to obtain the estimates with $\|u\|_{L^\infty([0,1];L^{3,\infty})}\le A$ on the right-hand side.

4.2 Basic Estimates

Here we assume that u satisfies (1) with the weak $L^{3,\infty}$ bound (6) on the time interval $[0,T]$.

Lemma 4.2 (Choice of time of regularity) *If u solves (1) on a time interval I and satisfies $\|u\|_{L_t^\infty L_x^{3,\infty}(I\times\mathbb{R}^3)}\le A$, then there exists $t_*\in I$ such that*

$$\|\nabla^j u(t_*)\|_{L_x^\infty(\mathbb{R}^3)} \le |I|^{-\frac{1+j}{2}} A^{O(1)}$$

for all $j=0,1,2,\dots,10$.

Lemma 4.3 (Bounded total speed) *We have the bounded total speed estimate*

$$\|u\|_{L_t^1 L_x^\infty(I/2\times\mathbb{R}^3)} \le |I|^{\frac12} A^{O(1)}.$$

The two lemmas above follow by the same arguments in [44, Lemma 3.1] and [13, Propositions 3.1–2] using the estimates in Lemma 4.1. In particular, it is straightforward to check that the proofs of Propositions 3.1 and 3.2 in [13] are still valid in Lorentz spaces $L^{p,q}$ with $q=\infty$. Furthermore, we estimate $\nabla^2 u$ in terms of A.

Lemma 4.4 (2nd order derivatives estimates) *If u solves (1) on $[0, T]$ and obeys (6), then*

$$\|\nabla^2 u\|_{L^p_{t,x-\mathrm{uloc}}([\frac{T}{2},T]\times\mathbb{R}^3)} \lesssim_p A^{O(1)} T^{\frac{5}{2p}-\frac{3}{2}}$$

*for $p \in [1, \frac{4}{3})$, where the "*uloc*" norm is considered as the supremum of the L^p norms over $B(T^{1/2}) \subset \mathbb{R}^3$ (instead of $B(1)$, recall (22)).*

Proof We use an approach due to Constantin [10]. First rescale to make $T = 1$. For every $\epsilon \in (0, \frac{1}{2})$, we define the approximation to the function $\langle x\rangle := (1 + |x|^2)^{\frac{1}{2}}$,

$$q_\epsilon(x) := \langle x\rangle - \frac{1}{2(1-\epsilon)}\langle x\rangle^{1-\epsilon}$$

which satisfies the properties

$$|\nabla q_\epsilon| \le 1, \tag{45}$$

$$\xi^T \nabla^2 q_\epsilon(x)\xi > \frac{\epsilon}{2}\langle x\rangle^{-(1+\epsilon)}|\xi|^2, \tag{46}$$

$$\frac{1-2\epsilon}{2-2\epsilon}\langle x\rangle \le q_\epsilon(x) \le \langle x\rangle. \tag{47}$$

With τ a time scale to be specified, we define $w := q_\epsilon(\tau\omega)$ which obeys the equation

$$(\partial_t + u\cdot\nabla - \Delta)w = \tau\nabla q_\epsilon(\tau\omega)\cdot(\omega\cdot\nabla u) - \tau^2\,\mathrm{tr}(\nabla\omega^T\nabla^2 q_\epsilon\nabla\omega).$$

Recall that $\omega := \mathrm{curl}\, u$ denotes the vorticity vector. Multiplying by a spatial cutoff at length scale R and integrating over $\mathbb{R}^d$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} w\psi \le \int_{\mathbb{R}^3}(u\cdot\nabla\psi + \Delta\psi)w + O(\tau|\nabla u|^2)\psi - \frac{\epsilon}{2}\tau^2\langle\tau\omega\rangle^{-(1+\epsilon)}|\nabla\omega|^2\psi.$$

Let $\tilde\psi$ be an enlarged cutoff function so that $R|\nabla\psi| + R^2|\Delta\psi| \le 10\tilde\psi$. We set

$$\|f\|_{L^p_{\mathrm{uloc},R}} := \sup_{B(R)\subset\mathbb{R}^3}\|f\|_{L^p(B(R))}.$$

Integrating in time starting from a t_0 to be specified and taking a supremum over the balls,

$$\begin{aligned}&\|w\psi(t)\|_{L^1_{\mathrm{uloc},R}} + \frac{\epsilon}{2}\tau^2\int_{t_0}^t\int_{\mathbb{R}^3}\langle\tau\omega\rangle^{-(1+\epsilon)}|\nabla\omega|^2\psi\, dx\mathrm{d}t\\ &\quad\lesssim \|w(t_0)\|_{L^1_{\mathrm{uloc},R}} + \int_{t_0}^t(R^{-2}+R^{-1}\|u\|_\infty)\|w(t')\|_{L^1_{\mathrm{uloc},R}}\mathrm{d}t' + \tau\|\nabla u\|^2_{L^2_{t,x-\mathrm{uloc},R}}.\end{aligned}$$

Grönwall's inequality

$$\|w(t)\|_{L^1_{\mathrm{uloc},R}} \lesssim \left(\|w(t_0)\|_{L^1_{\mathrm{uloc},R}} + \tau R A^{O(1)}\right) \exp(R^{-2}|t-t_0| + R^{-1}A^{O(1)}|t-t_0|^{\frac{1}{2}}),$$

where $|t-t_0|^{1/2}$ comes from applying the Cauchy-Schwarz inequality in the time integral and by using the energy bound (44). Setting $R = A^{C_1}$ and $\tau = A^{-2C_1}$ for a sufficiently large C_1, we find

$$\|\langle \tau\omega(t)\rangle\|_{L^1_{\mathrm{uloc},R}} \lesssim \|\langle \tau\omega(t_0)\rangle\|_{L^1_{\mathrm{uloc},R}}.$$

By (44) and Hölder's inequality, we can find a $t_0 \in [1/4, 1/2]$ where the right-hand side is bounded by $A^{O(1)}$. Therefore

$$\int_{t_0}^t \int_{\mathbb{R}^3} \langle \tau\omega\rangle^{-(1+\epsilon)} |\nabla\omega|^2 \psi \,\mathrm{d}x\mathrm{d}t \le \epsilon^{-1} A^{O(1)}.$$

We use Hölder's inequality with the decomposition

$$|\nabla\omega|^{\frac{4}{3+\epsilon}} = \left(|\nabla\omega|^{\frac{4}{3+\epsilon}} \langle\tau\omega\rangle^{-2\frac{1+\epsilon}{3+\epsilon}}\right)\langle\tau\omega\rangle^{2\frac{1+\epsilon}{3+\epsilon}}$$

to conclude

$$\|\nabla\omega\|_{L^{4/(3+\epsilon)}_{t,x-\mathrm{uloc}}([t_0,t]\times\mathbb{R}^3)} \le \epsilon^{-O(1)} A^{O(1)}.$$

To convert this into a bound on $\nabla^2 u$, fix a unit ball $B \subset \mathbb{R}^3$ and a cutoff function $\varphi \in C_c^\infty(3B)$ with $\varphi \equiv 1$ in $2B$. We decompose $\nabla^2 u = a + b$ where $a = \nabla^2\Delta^{-1}\operatorname{curl}(\varphi\omega)$. Note that $b = \nabla f$ where $f = \nabla\Delta^{-1}\operatorname{curl}((1-\varphi)\omega)$ is harmonic in $2B$ so for any $p \in [1, \frac{4}{3})$,

$$\|a\|_{L^p_{t,x}([t_0,t]\times B)} \lesssim \|\nabla\omega\|_{L^p_{t,x}([t_0,t]\times 3B)} + \|\nabla\varphi\|_{L^\infty}\|\omega\|_{L^2_{t,x-\mathrm{uloc}}([t_0,t]\times\mathbb{R}^3)} \le \epsilon^{-O(1)}A^{O(1)}$$

and

$$\begin{aligned}\|b\|_{L^p_{t,x}([t_0,t]\times B)} &\lesssim \|\nabla\Delta^{-1}\operatorname{curl}((1-\varphi)\omega)\|_{L^2_{t,x}([t_0,t]\times 2B)} \\ &\lesssim \|\omega^\sharp\|_{L^2_{t,x}([t_0,t]\times\mathbb{R}^3)} + \|\omega^\flat\|_{L^\infty_{t,x}([t_0,t]\times\mathbb{R}^3)} \le A^{O(1)},\end{aligned}$$

where we have used (44), Hölder's inequality, (43), and (40). □

5 Estimates for Axisymmetric Navier-Stokes Solutions

Here we provide some estimates of classical solutions of (1) that are specific to the axisymmetric assumption on the solutions.

We first note that u_θ satisfies

$$\left(\partial_t + u\cdot\nabla - \Delta + \frac{1}{r^2}\right)u_\theta + \frac{u_r}{r}u_\theta = 0, \tag{48}$$

which in particular gives that the swirl $\Theta := ru_\theta$ satisfies

$$\left(\partial_t + \left(u + \frac{2}{r}e_r\right)\cdot\nabla - \Delta\right)\Theta = 0 \tag{49}$$

in $(\mathbb{R}^3\setminus\{r=0\})\times(0,T)$. It then follows that, at each time, $(r,x_3)\mapsto u_\theta(r,x_3,t)$ is a continuous function on $\overline{\mathbb{R}_+}\times\mathbb{R}$ with $u_\theta(0,x_3)=0$ for all x_3 (see [23, Lemma 1] for details). In particular

$$\Theta(0,x_3,t) = 0 \qquad \text{for all } x_3\in\mathbb{R}, t\in(0,T). \tag{50}$$

Moreover, since $\omega = \omega_r e_r + \omega_\theta e_\theta + \omega_3 e_3$ is a smooth vector field we see (also by [23, Lemma 1]) that $\Phi = \frac{\omega_r}{r}$, $\Gamma := \frac{\omega_\theta}{r}$ (recall (7)) satisfy

$$|\Phi(r,x_3,t)|, |\Gamma(r,x_3,t)| \lesssim C(x_3,t) \tag{51}$$

for $r\in[0,1]$.

5.1 Hölder Continuity Near the Axis

Here we consider the parabolic equation

$$\mathcal{M}V := \partial_t V - \Delta V + b\cdot\nabla V = 0 \tag{52}$$

in a space-time cylinder

$$Q_R(x_0,t_0) := B(x_0,R)\times(t_0-R^2,t_0).$$

We assume that at each point of $Q_R := Q_R(0,0)$

$$\text{either } \operatorname{div} b = 0 \quad \text{or} \quad V = 0. \tag{53}$$

We also assume that

$$\mathcal{N}(R) := 2 + \sup_{R'\le R}(R')^{-\alpha}\|b\|_{L_t^\ell L_x^q(Q_{R'})} < \infty, \tag{54}$$

where $\alpha := \frac{3}{q}+\frac{2}{\ell}-1\in[0,1)$. In such setting [24, Corollary 3.6] observed that V must be Hölder continuous in the interior of Q_R, and in the proposition below we state a version of their result in which we quantify the dependence of the Hölder exponent in terms of $\mathcal{N}$.

Proposition 5.1 *If V is a Lipschitz solution of (52) on Q_{2R} then*

$$\operatorname*{osc}_{B(r)} V(0) \lesssim \left(\frac{r}{R}\right)^{\gamma} \operatorname*{osc}_{Q_R} V$$

for all $r \le R$, where $\gamma = \exp(-\mathcal{N}^{O(1)})$.

Proof See Appendix 1. □

We note that the swirl Θ satisfies (52) with $b := u + 2e_r/r$ (recall (49) above). Moreover $\operatorname{div} b = 0$ everywhere except for the axis, since $\operatorname{div} u = 0$, $\operatorname{div}(e_r/r) = 0$ (recall (33)) there. Furthermore, $\Theta = 0$ on the axis (recall (50)), and so the assumption (53) holds. Thus Proposition 5.1 shows that Θ is Hölder continuous in a neighborhood of the axis. We explore this in more detail in the proof of Theorem 1.1 below, where we quantify $\mathcal{N}$ in terms of the weak-L^3 bound A (see Step 1 in Section 6 below).

5.2 Pointwise Estimates Away from the Axis

The following is a more precise version of Proposition 8 in [31].

Proposition 5.2 (Pointwise bounds away from the axis) *Let u solve (1) on* $[0, 1]$ *satisfying (5) and (6). Then for every $\epsilon \in (0, 4/15)$, we have*

$$|\nabla^j u| \le \left(r^{-1-j} + r^{-\frac{1}{3}+\epsilon}\right) A^{O_{\epsilon,j}(1)}$$

for each $t \in [1/2, 1]$. We also have

$$\|u\|_{L^p(\{r\ge 1\})} \le A^{O_p(1)}$$

for each such t, and $p \in (3, \infty]$.

Proof We first pick any $\alpha \in (1/3 - \epsilon/2, 1/3)$ and $c = c(j) > 0$ sufficiently small so that

$$(1-\alpha+j)c < \epsilon/2 \quad \text{and} \quad c < \alpha/(1-\alpha). \tag{55}$$

We also pick $n = n(j) \in \mathbb{N}$ sufficiently large so that

$$n \ge (2+j)\left(1+\frac{1}{c}\right). \tag{56}$$

We set $t_k := 1/2 - (1/2)^k$ and we define a sequence of regions $\{x \in \mathbb{R}^3 : r \ge R/2\} = \Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n = \{x \in \mathbb{R}^3 : r \ge R\}$ such that $\operatorname{dist}(\Omega_i^c, \Omega_{i+1}) \ge R/2n$.

Given such a sequence of times we now consider the corresponding Picard iterates $u_k^\flat, u_k^\sharp$, for $k \in \{0, 1, \dots, n\}$.

Step 1. We show that

$$\| P_N u_k^\flat(t)\|_{L^\infty(\{r\geq R/2\})}, \| P_N u_k^\sharp(t)\|_{L^\infty(\{r\geq R/2\})} \lesssim R^{-\alpha} N^{1-\alpha} A^{O_k(1)} \tag{57}$$

for all $\alpha \in [0, \frac{1}{3})$, $R > 0$ and $t \in [t_k, 1]$, $k \geq 0$.

In fact, we first observe that Lemma 3.1 gives that

$$\| r^\alpha P_N u(t)\|_\infty \lesssim N^{1-\alpha} \|u(t)\|_{L^{3,\infty}} \lesssim N^{1-\alpha} A^{O(1)}. \tag{58}$$

Thus, since the first inequality above is valid for any axisymmetric function, it remains to note that the second inequality is also valid for each $u_k^\flat, u_k^\sharp$, on $[t_k, 1]$, $k \geq 0$. Indeed, the case $k = 0$ follows trivially, while the inductive step follows by applying Young's inequality (28) for weak L^p spaces, and Hölder's inequality (27) for Lorentz spaces

$$\begin{aligned} \|u_k^\flat(t)\|_{L^{3,\infty}} &\lesssim \|\Psi(t-t_k)\|_1 \|u(t_k)\|_{L^{3,\infty}} \\ &\quad + \int_{t_k}^t \|\nabla \Psi(t-t')\|_1 \|(u_{k-1}^\flat \otimes u_{k-1}^\flat)(t')\|_{L^{3/2,\infty}} \mathrm{d}t' \\ &\leq C_k A + C_k \|u_{k-1}^\flat\|^2_{L^\infty([t_{k-1},1]; L^{3,\infty})} \int_{t_k}^t (t-t')^{-\frac{1}{2}} \mathrm{d}t' \leq A^{O_k(1)} \end{aligned}$$

for $t \in [t_k, 1]$, as required, where we also used the heat kernel bounds (24).

Step 2. We show that the inequality from Step 1 can be improved for $u_k^\sharp$ for large k, namely

$$\| P_N u_k^\sharp \|_{L^\infty([\frac{1}{2},1]\times\{r\geq R\})} \leq N A^{O_k(1)} ((RN)^{-(k-1)\alpha} + N^{-(k-1)}) \tag{59}$$

for every $k \geq 1$ and $N \in 2^{\mathbb{N}} \cap [100^k \max(1, R^{-1}), \infty)$.

We will show that,

$$X_{k,N} \leq N^{-\frac{4}{5}} A^{O_k(1)} ((RN)^{-(k-1)\alpha} + N^{-(k-1)}), \tag{60}$$

for $k \geq 1$ and $N \geq 100^k \max(1, R^{-1})$, using induction with respect to k, where

$$X_{k,N} := \| P_N u_k^\sharp \|_{L^\infty([t_{k+1},1]; L^{5/3}(\Omega_k))}.$$

Then (59) follows by the local Bernstein inequality (25).

As for the base case $k = 1$ we note that (37) gives that

$$\begin{aligned} \|P_N u_1^\sharp(t)\|_{5/3} &\lesssim \int_{t_1}^t \|P_N \mathrm{e}^{(t-t')\Delta} \mathbb{P} \mathrm{div}(u\otimes u)(t')\|_{5/3} \mathrm{d}t' \\ &\lesssim \int_{t_1}^t \mathrm{e}^{-(t-t')N^2/O(1)} N^{\frac{6}{5}} \|(u\otimes u)(t')\|_{L^{\frac{3}{2},\infty}} \mathrm{d}t' \\ &\lesssim N^{\frac{6}{5}} \|\mathrm{e}^{-tN^2/O(1)}\|_{L^1(t_1,1)} \|u\|^2_{L^{3,\infty}} \end{aligned}$$

for $t \in [t_1, 1]$. Thus

$$X_{1,N} \le \| P_N u_1^\sharp \|_{L^\infty([t_2,1];L^{5/3})} \le N^{-\frac{4}{5}} A^{O(1)}, \tag{61}$$

due to Hölder's inequality for Lorentz spaces (27).

As for the inductive step, we use the Duhamel formula for $u_k^\sharp$ (recall (39)), and the local Bernstein inequality (25) to obtain

$$\begin{aligned}
&\| P_N u_k^\sharp(t) \|_{L^{5/3}(\Omega_k)} \\
&\lesssim \int_{t_k}^t \| P_N \mathrm{e}^{(t-t')\Delta} \mathbb{P}\mathrm{div}(u \otimes u - u_{k-1}^\flat \otimes u_{k-1}^\flat) \|_{L^{5/3}(\Omega_k)} \mathrm{d}t' \\
&\le \int_{t_k}^t N \mathrm{e}^{-(t-t')N^2/O(1)} \mathrm{d}t' \Big(\| P_N (u \otimes u - u_{k-1}^\flat \otimes u_{k-1}^\flat) \|_{L^\infty([t_k,1];L^{5/3}(\Omega_{k-1}))} \\
&\quad + (NR)^{-(k-1)\alpha} \| P_N (u \otimes u - u_{k-1}^\flat \otimes u_{k-1}^\flat) \|_{L^\infty([t_k,1];L^{5/3})} \Big) \\
&\lesssim N^{-1} \Big(\| P_N (u \otimes u - u_{k-1}^\flat \otimes u_{k-1}^\flat) \|_{L^\infty([t_k,1];L^{5/3}(\Omega_{k-1}))} \\
&\quad + N^{\frac{1}{5}} (NR)^{-(k-1)\alpha} A^{O(1)} \Big),
\end{aligned}$$

where we used the weak L^3 bound (6) and Lemma 3.1 for the $u \otimes u$ term and (40) for the $u_{k-1}^\flat \otimes u_{k-1}^\flat$ term. Thus we can use the paraproduct decomposition in the first term on the right-hand side to obtain

$$X_{k,N} \lesssim N^{-1} \| Y_1 + \cdots + Y_5 \|_{L^\infty([t_k,1];L^{5/3}(\Omega_{k-1}))} + N^{-\frac{4}{5}} (NR)^{-(k-1)\alpha} A^{O_k(1)}, \tag{62}$$

where

$$\begin{aligned}
Y_1 &:= 2 \sum_{N' \sim N} P_{N'} u_{k-1}^\sharp \odot P_{\le N/100} u_{k-1}^\sharp, \\
Y_2 &:= \sum_{N_1 \sim N_2 \gtrsim N} P_{N_1} u_{k-1}^\sharp \otimes P_{N_2} u_{k-1}^\sharp, \\
Y_3 &:= \sum_{N_1 \sim N_2 \gtrsim N} P_{N_1} u_{k-1}^\flat \otimes P_{N_2} u_{k-1}^\sharp, \\
Y_4 &:= 2 \sum_{N' \sim N} P_{N'} u_{k-1}^\flat \odot P_{\le N/100} u_{k-1}^\sharp, \\
Y_5 &:= 2 \sum_{N' \sim N} P_{\le N/100} u_{k-1}^\flat \odot P_{N'} u_{k-1}^\sharp,
\end{aligned}$$

where we use the notation $a \odot b := a \otimes b + b \otimes a$. Using (57),

$$\| Y_1 \|_{L^\infty([t_k,1];L^{5/3}(\Omega_{k-1}))} \lesssim \sum_{N' \sim N} X_{k-1,N'} \sum_{N' \lesssim N} R^{-\alpha} (N')^{1-\alpha} A^{O_k(1)}$$

$$\lesssim R^{-\alpha} N^{1-\alpha} A^{O_k(1)} \sum_{N' \sim N} X_{k-1,N'}$$

and

$$\|Y_2\|_{L^\infty([t_k,1];L^{5/3}(\Omega_{k-1}))} \lesssim R^{-\alpha} A^{O_k(1)} \sum_{N' \gtrsim N} (N')^{1-\alpha} X_{k-1,N'}.$$

Moreover, the frequency-localized bounds (42) for $u^\flat_{k-1}$ give that

$$\|Y_3\|_{L^\infty([t_k,1];L^{5/3}(\Omega_{k-1}))} \lesssim A^{O_k(1)} \sum_{N' \gtrsim N} \mathrm{e}^{-(N')^2/O_k(1)} N' X_{k-1,N'},$$

and (41), as well as boundedness of $P_{\leq N/100}$ on $L^{5/3}$ give that

$$\|Y_4\|_{L^\infty([t_k,1];L^{5/3}(\Omega_{k-1}))} \lesssim A^{O_k(1)} \sum_{N' \sim N} \mathrm{e}^{-(N')^2/O_k(1)} N' \lesssim \mathrm{e}^{-N^2/O_k(1)} A^{O_k(1)}.$$

Finally, using boundedness of $P_{\leq N/100}$ on L^∞ and (40) we obtain

$$\|Y_5\|_{L^\infty([t_k,1];L^{5/3}(\Omega_{k-1}))} \lesssim A^{O_k(1)} \sum_{N' \sim N} X_{k-1,N'}.$$

Combining these estimates into (62), we have shown

$$\begin{aligned} X_{k,N} \leq A^{O_k(1)} \Bigg(&((RN)^{-\alpha} + N^{-1}) \sum_{N' \sim N} X_{k-1,N'} + N^{-1} R^{-\alpha} \sum_{N' \gtrsim N} (N')^{1-\alpha} X_{k-1,N'} \\ &+ N^{-1} \sum_{N' \gtrsim N} \mathrm{e}^{-(N')^2/O_k(1)} N' X_{k-1,N'} + N^{-\frac{4}{5}} (NR)^{-(k-1)\alpha} + N^{-1} \mathrm{e}^{-N^2/O_k(1)} \Bigg). \end{aligned} \tag{63}$$

Since the upper bounds on $X_{k-1,N'}$ provided by the inductive assumption (60) are comparable for all $N' \sim N$, up to constants depending only on k, we thus obtain that

$$\begin{aligned} \sum_{N' \sim N} X_{k-1,N'} &\leq A^{O_k(1)} N^{-\frac{4}{5}} \left((RN)^{-\alpha(k-2)} + N^{-k-2} \right), \\ R^{-\alpha} &\sum_{N' \gtrsim N} (N')^{1-\alpha} X_{k-1,N'} \\ &\leq A^{O_k(1)} R^{-\alpha} \sum_{N' \gtrsim N} (N')^{1-\alpha-\frac{4}{5}} \left((RN')^{-\alpha(k-2)} + (N')^{-(k-2)} \right) \\ &\leq A^{O_k(1)} N^{\frac{1}{5}} \left((RN)^{-\alpha(k-1)} + N^{-(k-1)} \right), \end{aligned}$$

where, in the last line we used the fact that $(k-1)(1-\alpha)-4/5<0$ for any $k\geq 2$. A similar estimate for $\sum_{N'\gtrsim N} \mathrm{e}^{-(N')^2/O_k(1)} N' X_{k-1,N'}$ now allows us to deduce from (63) that

$$X_{k,N} \leq N^{-\frac{4}{5}} A^{O_k(1)} ((RN)^{-(k-1)\alpha} + N^{-(k-1)}),$$

as required.

Step 3. We prove the claim.

We first consider the case $R\geq 100^{n/c}$, and we note that, by (57)

$$\begin{aligned}\| P_{N\leq R^c} \nabla^j u_n^\sharp \|_{L^\infty_{t,x}([\frac{1}{2},1]\times\{r\geq R\})} &\leq \sum_{N\leq R^c} A^{O_n(1)} N^{1-\alpha+j} R^{-\alpha}\\ &\leq A^{O_n(1)} R^{-\alpha+(1-\alpha+j)c} \leq A^{O_n(1)} R^{-\frac{1}{3}+\varepsilon},\end{aligned}$$

where we used the choice of $\alpha>1/3-\epsilon/2$ and the first property of our choice (55) of c in the last inequality. On the other hand for $N>R^c$ we can use (59) with $k=n$ to obtain arbitrarily fast decay in N. Comparing the terms on the right-hand side of (59) we see that $N^{-(n-2)}$ dominates $(RN)^{-(n-2)\alpha}$ if and only if $N\leq R^{\alpha/(1-\alpha)}$, which allows us to apply the decomposition

$$\begin{aligned}\| P_{N> R^c} \nabla^j u_n^\sharp \|_{L^\infty_{t,x}([\frac{1}{2},1]\times\{r\geq R\})} &\leq \sum_{R^c<N\leq R^{\alpha/(1-\alpha)}} A^{O_n(1)} N^{-n+2+j}\\ &\quad + \sum_{N>R^{\alpha/(1-\alpha)}} A^{O_n(1)} N^{1+j} (RN)^{-(n-1)\alpha}\\ &\leq A^{O_n(1)} R^{c(-n+2+j)}\\ &\leq A^{O_n(1)} R^{-1-j},\end{aligned}$$

where we used the second property of our choice (55) of c in the second inequality, and the choice (56) of n in the last inequality.

We now suppose that $R\leq 100^{n/c}$. The low frequencies can be estimated directly from the weak L^3 bound (6),

$$\| P_{\leq 100^{2n/c}R^{-1}} \nabla^j u \|_{L^\infty_{t,x}([\frac{1}{2},1]\times\{r\geq R\})} \lesssim_{n,c} A^{O(1)} R^{-1-j}.$$

On the other hand, for $N>100^{2n/c}R^{-1}$ we have in particular $N>R^{\alpha/(1-\alpha)}$, which shows that the dominant term on the right-hand side of (59) is $(RN)^{-(n-2)\alpha}$, and so

$$\begin{aligned}&\| P_{>100^{2n/c}R^{-1}} \nabla^j u_n^\sharp(t) \|_{L^\infty(\{r\geq R\})}\\ &\leq \sum_{N>100^{2n/c}R^{-1}} N^{1+j} A^{O_n(1)} (RN)^{-(n-1)\alpha} \leq A^{O_n(1)} R^{-1-j}\end{aligned}$$

for every $t \in [1/2, 1]$, as desired. As for the estimate for $u^\flat$ we use (40) to obtain

$$\|\nabla^j u_n^\flat\|_{L^\infty(\{r\geq R\})} \leq R^{-1/3+\epsilon}\|r^{1/3-\epsilon}\nabla^j u_n^\flat\|_\infty \lesssim_\epsilon R^{-1/3+\epsilon} A^{O_{\epsilon,j}(1)},$$

as needed.

The estimate for $\|u\|_{L^p(\{r\geq 1\})}$ follows by an L^p analogue of Step 1, as well as applying the $X_{k,N}$ estimates (60) in the L^p variant of Step 3. □

5.3 A Poisson-Type Estimate on u_r/r

Here we discuss how derivatives of u_r/r can be controlled by Γ using the representation (11),

$$\frac{u_r}{r} = \Delta^{-1}\partial_3\Gamma - 2\frac{\partial_r}{r}\Delta^{-2}\partial_3\Gamma, \tag{64}$$

see [8, p. 1929], which will be an essential part of our x_3-uloc energy estimates for Φ and Γ (see Proposition 6.1 below).

Lemma 5.3 (The $L^2_{3-\mathrm{uloc}}$ estimate on u_r/r)

$$\left\|\nabla\partial_r\frac{u_r}{r}\right\|_{L^2_{3-\mathrm{uloc}}} + \left\|\nabla\partial_3\frac{u_r}{r}\right\|_{L^2_{3-\mathrm{uloc}}} \lesssim \|\Gamma\|_{L^2_{3-\mathrm{uloc}}} + \|\nabla\Gamma\|_{L^2_{3-\mathrm{uloc}}}. \tag{65}$$

A version of the above estimate without the localization in x_3 has appeared in [8, Lemma 2.3]. As mentioned in the introduction, the localization makes the estimate much more challenging, particularly due to the bilaplacian term in (64).

In order to prove Lemma 5.3 we note that, since

$$\frac{\partial_r}{r} = \Delta' - \partial_{rr},$$

(64) gives that

$$\frac{u_r}{r} = -\Delta^{-1}\partial_3\Gamma + 2(\partial_{rr} - \Delta')\Delta^{-2}\partial_3\Gamma. \tag{66}$$

Thus, since $|\nabla\partial_3\frac{u_r}{r}| = |(\partial_r\partial_3\frac{u_r}{r}, \partial_3\partial_3\frac{u_r}{r})|$ (and similarly for $|\nabla\partial_r\frac{u_r}{r}|$), we can use (35) and (36) to observe that

$$\begin{aligned}\left|\nabla\partial_3\frac{u_r}{r}\right| + \left|\nabla\partial_r\frac{u_r}{r}\right| &\lesssim |D^2_{r,x_3}\Delta^{-1}\partial_3\Gamma| + |D^2_{r,x_3}(\partial_{rr} - \Delta')\Delta^{-2}\partial_3\Gamma| \\ &\lesssim |\nabla\Gamma| + |D^2\Delta^{-1}\nabla'\Gamma| + |D^4\Delta^{-2}\nabla'\Gamma|,\end{aligned}$$

where we used $\partial_{33} = \Delta - \Delta'$ in the last line. In particular, each of the terms on the right-hand side involves at least one derivative in the horizontal variables. Thus, in order to estimate the left-hand side of (65) it suffices to find suitable bounds on the last two terms, which we achieve in Lemmas 5.4–5.5 below. Their claims give us (65), as required.

Lemma 5.4 *If* $f=\Delta^{-1}\nabla'\Gamma$ *then*

$$\|D^2 f\|_{L^2_{3-\mathrm{uloc}}} \le \|\Gamma\|_{L^2_{3-\mathrm{uloc}}} + \|\nabla'\Gamma\|_{L^2_{3-\mathrm{uloc}}}.$$

Proof Let $I(x)$ denote the kernel matrix of $D^2(-\Delta)^{-1}$. We have that

$$|\nabla^j I(x)| \le \frac{C}{|x|^{3+j}} \qquad \text{for } j=0,1,$$

and

$$\begin{aligned} D^2 f(x) &= \mathrm{p.v.}\int_{\mathbb{R}^3} I(x-y)\nabla'\Gamma(y)\mathrm{d}y \\ &= \mathrm{p.v.}\int_{\mathbb{R}^3} \nabla'\Gamma(y)\tilde{\phi}(y_3)I(x-y)\mathrm{d}y + \mathrm{p.v.}\int_{\mathbb{R}^3} \Gamma(y)(1-\tilde{\phi}(y_3))\nabla' I(x-y)\mathrm{d}y \\ &=: f_1(x)+f_2(x). \end{aligned}$$

The Calderón-Zygmund inequality (see [35, Theorem B.5], for example) gives that

$$\|f_1\|_{L^2_{3-\mathrm{uloc}}} \le \|f_1\|_{L^2} \lesssim \|\nabla'\Gamma\,\tilde{\phi}\|_{L^2} \lesssim \|\nabla'\Gamma\|_{L^2_{3-\mathrm{uloc}}}.$$

Moreover, noting that $\int_{\mathbb{R}^2}\frac{\mathrm{d}x_1\,\mathrm{d}x_2}{(a^2+x_1^2+x_2^2)^2}=Ca^{-2}$, we can use Young's inequality for convolutions to obtain

$$\begin{aligned} \|f_2(\cdot,x_3)\|_{L^2} &\le \int_{\mathbb{R}} \frac{\|\Gamma(\cdot,y_3)\|_{L^2}(1-\tilde{\phi}(y_3))}{|x_3-y_3|^2}\mathrm{d}y_3 \\ &\le \sum_{j\ge 1}\int_{\{|x_3-y_3|\in(j,j+1)} \frac{\|\Gamma(\cdot,y_3)\|_{L^2}(1-\tilde{\phi}(y_3))}{|x_3-y_3|^2}\mathrm{d}y_3 \\ &\le \sum_{j\ge 1} j^{-2}\int_{\{|x_3-y_3|\in(j,j+1)} \|\Gamma(\cdot,y_3)\|_{L^2}\mathrm{d}y_3 \\ &\le \|\Gamma\|_{L^2_{3-\mathrm{uloc}}}. \end{aligned}$$

Integration in x_3 over supp ϕ finishes the proof. □

For the bilaplacian term in (66) one needs to work harder:

Lemma 5.5 *Let* $f=D^4\Delta^{-2}\nabla'\Gamma$. *Then*

$$\|f\|_{L^2_{3-\mathrm{uloc}}} \le \|\Gamma\|_{L^2_{3-\mathrm{uloc}}} + \|\nabla\Gamma\|_{L^2_{3-\mathrm{uloc}}}.$$

Proof We have that

$$f(x) = \text{p.v.} \int_{\mathbb{R}^3} \text{p.v.} \int_{\mathbb{R}^3} \nabla' \Gamma(z) I(x-y) I(y-z) \mathrm{d}z \, \mathrm{d}y.$$

Recalling that $\tilde{\phi} = \sum_{|j| \leq 10} \phi_j$, and $\tilde{\tilde{\phi}} = \sum_{|j| \leq 20} \phi_j$ we use the partition of unity,

$$\begin{aligned}
1 &= \tilde{\tilde{\phi}}(z_3) + (1 - \tilde{\tilde{\phi}}(z_3))\tilde{\phi}(y_3) + \sum_{\substack{|j|>10 \\ |k|>20}} \phi_j(y_3)\phi_k(z_3) \\
&= \tilde{\tilde{\phi}}(z_3) + (1 - \tilde{\tilde{\phi}}(z_3))\tilde{\phi}(y_3) \\
&\quad + \sum_{|j|>10} \phi_j(y_3) \left(\sum_{\substack{|k|>20 \\ |k-j| \leq 10}} \phi_k(z_3) + \sum_{\substack{|k|>20 \\ |k-j|>10 \\ k \leq j/2}} \phi_k(z_3) + \sum_{\substack{|k|>20 \\ |k-j|>10 \\ j/2<k \leq 2j}} \phi_k(z_3) + \sum_{\substack{|k|>20 \\ |k-j|>10 \\ k>2j}} \phi_k(z_3) \right),
\end{aligned}$$

to decompose f accordingly,

$$\begin{aligned}
f(x) &= \text{p.v.} \int_{\mathbb{R}^3} \text{p.v.} \int_{\mathbb{R}^3} \nabla' \Gamma(z) \tilde{\tilde{\phi}}(z_3) I(x-y) I(y-z) \mathrm{d}y \, \mathrm{d}z \\
&\quad + \text{p.v.} \int_{\mathbb{R}^3} I(x-y)\tilde{\phi}(y_3) \text{p.v.} \int_{\mathbb{R}^3} \nabla' \Gamma(z)(1 - \tilde{\tilde{\phi}}(z_3)) I(y-z) \mathrm{d}z \, \mathrm{d}y \\
&\quad + \text{p.v.} \int_{\mathbb{R}^3} I(x-y) \sum_{|j|>10} \phi_j(y_3) \text{p.v.} \int_{\mathbb{R}^3} \nabla' \Gamma(z) \sum_{\substack{|k|>20 \\ |k-j| \leq 10}} \phi_k(z_3) I(y-z) \mathrm{d}z \, \mathrm{d}y \\
&\quad + \text{p.v.} \int_{\mathbb{R}^3} I(x-y) \sum_{|j|>10} \phi_j(y_3) \text{p.v.} \int_{\mathbb{R}^3} \nabla' \Gamma(z) \sum_{\substack{|k|>20 \\ |k-j|>10 \\ k \leq j/2}} \phi_k(z_3) I(y-z) \mathrm{d}z \, \mathrm{d}y \\
&\quad + \text{p.v.} \int_{\mathbb{R}^3} I(x-y) \sum_{|j|>10} \phi_j(y_3) \text{p.v.} \int_{\mathbb{R}^3} \nabla' \Gamma(z) \sum_{\substack{|k|>20 \\ |k-j|>10 \\ j/2<k \leq 2j}} \phi_k(z_3) I(y-z) \mathrm{d}z \, \mathrm{d}y \\
&\quad + \text{p.v.} \int_{\mathbb{R}^3} I(x-y) \sum_{|j|>10} \phi_j(y_3) \text{p.v.} \int_{\mathbb{R}^3} \nabla' \Gamma(z) \sum_{\substack{|k|>20 \\ |k-j|>10 \\ k>2j}} \phi_k(z_3) I(y-z) \mathrm{d}z \, \mathrm{d}y \\
&=: f_1(x) + f_2(x) + f_3(x) + f_4(x) + f_5(x) + f_6(x).
\end{aligned}$$

Clearly f_1 involves localization of $\nabla' \Gamma$ in z_3, and so we can use the Calderón-Zygmund inequality twice to obtain

$$\|f_1\|_{L^2} \lesssim \|\nabla \Gamma\|_{L^2_{3-\mathrm{uloc}}}.$$

As for f_2 we integrate by parts in the z-integral (note that this does not conflict with the principal value, as the singularity has been cut off, and the far field has sufficient decay) and apply the Calderón-Zygmund estimate in x to obtain

$$\begin{aligned}\|f_2\|_{L^2} &\lesssim \left\| \tilde\phi(y_3) \int_{\mathbb{R}^3} \frac{|\Gamma(z)|(1-\tilde{\tilde\phi}(z_3))}{|y-z|^4} \mathrm{d}z \right\|_{L^2} \lesssim \sup_{y_3\in \operatorname{supp}\tilde\phi} \left\| \int_{\mathbb{R}^3} \frac{|\Gamma(z)|(1-\tilde{\tilde\phi}(z_3))}{|y-z|^4} \mathrm{d}z \right\|_{L^2_{y'}} \\ &\lesssim \sup_{y_3\in \operatorname{supp}\tilde\phi} \int_{\mathbb{R}} \frac{\|\Gamma(\cdot,z_3)\|_{L^2}(1-\tilde{\tilde\phi}(z_3))}{|y_3-z_3|^2} \mathrm{d}z_3 \\ &\lesssim \sup_{y_3\in \operatorname{supp}\tilde\phi} \sum_{j\geq 1} j^{-2} \int_{|z_3-y_3|\in(j,j+1)} \|\Gamma(\cdot,z_3)\|_{L^2_{z'}} \mathrm{d}z_3 \lesssim \|\Gamma\|_{L^2_{3-\mathrm{uloc}}},\end{aligned}$$

where we used Young's inequality in the second line (as in the lemma above).

As for f_3, we integrate by parts in z and then in y to obtain

$$|f_3(x)| \lesssim \sum_{|j|>10} \int_{\mathbb{R}^3} \frac{\phi_j(y_3)}{|x-y|^4} \left| \mathrm{p.v.} \int_{\mathbb{R}^3} \Gamma(z) \sum_{\substack{|k|>20\\ |k-j|\leq 10}} \phi_k(z_3) I(y-z) \mathrm{d}z \right| \mathrm{d}y.$$

We note that the integration by parts is justified as

$$f_3 = D^2(-\Delta)^{-1}\left((1-\sum_{|j|\leq 10}\phi_j(y_3)) D^2(-\Delta)^{-1} \left(\nabla'\Gamma(1-\sum_{k\in I}\phi_k(z_3)\right)\right),$$

where $I := \{-20,\ldots,20\}\cup\{j-10,\ldots,j+10\}$ is a finite index set. Thus, the operation of integration by parts above is equivalent to moving ∇' outside of the outer brackets, which in turn holds since the sums do not depend on x' and ∇' commutes with other differential symbols.

Thus, using Young's inequality in x'

$$\begin{aligned}\|f_3(\cdot,x_3)\|_{L^2_{x'}} &\lesssim \sum_{|j|>10} \int_{\mathbb{R}} \frac{\phi_j(y_3)}{|x_3-y_3|^2} \left\| \mathrm{p.v.} \int_{\mathbb{R}^3} \Gamma(z) \sum_{\substack{|k|>6\\ |k-j|\leq 2}} \phi_k(z_3) I(y-z) \mathrm{d}z \right\|_{L^2_{y'}} \mathrm{d}y_3 \\ &\lesssim \sum_{|j|>2} j^{-2} \left\| \mathrm{p.v.} \int_{\mathbb{R}^3} \Gamma(z) \sum_{\substack{|k|>20\\ |k-j|\leq 10}} \phi_k(z_3) I(y-z) \mathrm{d}z \right\|_{L^2_y} \\ &\lesssim \sum_{|j|>10} j^{-2} \left\| \Gamma(z) \sum_{\substack{|k|>20\\ |k-j|\leq 10}} \phi_k(z_3) \right\|_{L^2} \lesssim \|\Gamma\|_{L^2_{3-\mathrm{uloc}}}\end{aligned}$$

for each $x_3 \in \operatorname{supp} \phi$, where we applied the Cauchy-Schwarz inequality (in y_3) in the second line.

As for f_4 we note that

$$|y_3 - z_3| \geq |y_3| - |z_3| \geq (j-1) - (k+1) \geq \frac{j}{2}$$
$$-2 \geq (j+2)/4 \geq (|y_3|+1)/4 \geq |y_3 - x_3|/4.$$

Thus, we can integrate by parts in z to obtain

$$|f_4(x)| \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \cap \{|y_3 - z_3| \geq |x_3 - y_3|/4\}} \frac{|\Gamma(z)|(1-\tilde{\phi}(y_3))(1-\tilde{\phi}(y_3 - z_3))}{|x-y|^3 |y-z|^4} \mathrm{d}z \, \mathrm{d}y.$$

Hence, applying Young's inequality in x' and then in y' we obtain

$$\begin{aligned} \|f_4(\cdot, x_3)\|_{L^2} &\leq \int_{\mathbb{R}} \left\| \int_{\mathbb{R}^3 \cap \{|y_3 - z_3| \geq |x_3 - y_3|/4\}} \frac{\Gamma(z)(1-\tilde{\phi}(y_3))(1-\tilde{\phi}(y_3 - z_3))}{|y-z|^4} \mathrm{d}z \right\|_{L^2_{y'}} \\ &\quad \cdot \underbrace{\int_{\mathbb{R}^2} \frac{\mathrm{d}x_1 \, \mathrm{d}x_2}{\left(|x_3 - y_3|^2 + x_1^2 + x_2^2\right)^{3/2}}}_{= C|x_3 - y_3|^{-1}} \mathrm{d}y_3 \\ &\lesssim \int_{\mathbb{R}} \int_{\mathbb{R} \cap \{|y_3 - z_3| \geq |x_3 - y_3|/4\}} \\ &\quad \frac{\|\Gamma(\cdot, z_3)\|_{L^2}(1-\tilde{\phi}(y_3))(1-\tilde{\phi}(y_3 - z_3))}{|x_3 - y_3| \, |y_3 - z_3|^2} \mathrm{d}z_3 \, \mathrm{d}y_3. \end{aligned} \tag{67}$$

Hence

$$\begin{aligned} \|f_4(\cdot, x_3)\|_{L^2} &\leq \int_{\mathbb{R}} \frac{1-\tilde{\phi}(y_3)}{|x_3 - y_3|^{3/2}} \left(\sum_{j \geq 1} \int_{\{|y_3 - z_3| \in (j, j+1)\}} \frac{\|\Gamma(\cdot, z_3)\|_{L^2}}{|y_3 - z_3|^{3/2}} \mathrm{d}z_3 \right) \mathrm{d}y_3 \\ &\lesssim \|\Gamma\|_{L^2_{3-\mathrm{uloc}}} \int_{\mathbb{R}} \frac{1-\tilde{\phi}(y_3)}{|x_3 - y_3|^{3/2}} \mathrm{d}y_3 \lesssim \|\Gamma\|_{L^2_{3-\mathrm{uloc}}}. \end{aligned}$$

As for f_5 we have

$$\frac{1}{4} \leq \frac{|x_3 - y_3|}{|x_3 - z_3|} \leq 4,$$

since

$$|x_3 - y_3| \leq |y_3| + |x_3| \leq j + 2 \leq 2j - 8 \leq 4k - 8 \leq 4(|z_3| - |x_3|) \leq 4|x_3 - z_3|$$

and

$$|x_3 - z_3| \le |z_3| + |x_3| \le k + 2 \le 2j + 2 \le 4(j-2) \le 4(|y_3| - |x_3|) \le 4|x_3 - y_3|.$$

In particular, the triangle inequality gives that

$$|y_3 - z_3| \le 5|x_3 - z_3|.$$

Thus we can integrate by parts twice (in z and then in y, so that the derivative falls on $I(x-y)$), and then use Young's inequality twice (as in (67) above) and Tonelli's Theorem to obtain

$$\begin{aligned}
\|f_5(\cdot, x_3)\|_{L^2} \le& \int_{\mathbb{R}} \int_{\{|x_3-y_3|/4 \le |x_3-z_3| \le 4|x_3-y_3|\}} \\
&\frac{\|\Gamma(\cdot, z_3)\|_{L^2}(1-\tilde{\phi}(y_3-z_3))(1-\tilde{\tilde{\phi}}(z_3))}{|x_3-y_3|^2|y_3-z_3|} \mathrm{d}z_3\, \mathrm{d}y_3 \\
&\le \int_{\mathbb{R}} \frac{\|\Gamma(\cdot, z_3)\|_{L^2}(1-\tilde{\tilde{\phi}}(z_3))}{|x_3-z_3|^2} \int_{\{|y_3-z_3| \le 5|x_3-x_3|\}} \frac{1-\tilde{\phi}(y_3-z_3)}{|y_3-z_3|} \mathrm{d}y_3\, \mathrm{d}z_3 \\
&\lesssim \int_{\mathbb{R}} \frac{\|\Gamma(\cdot, z_3)\|_{L^2}(1-\tilde{\tilde{\phi}}(z_3))}{|x_3-z_3|^2} \log(5|x_3-z_3|)\mathrm{d}z_3 \\
&\lesssim \sum_{j\ge 1} \int_{|z_3-x_3|\in(j,j+1)} \frac{\|\Gamma(\cdot, z_3)\|_{L^2}}{|x_3-z_3|^2} \log(5|x_3-z_3|)\mathrm{d}z_3 \\
&\lesssim \sum_{j\ge 1} j^{-2}\log(5j)\|\Gamma\|_{L^2_{3-\mathrm{uloc}}} \lesssim \|\Gamma\|_{L^2_{3-\mathrm{uloc}}}.
\end{aligned}$$

Finally, for f_6 we observe that

$$\frac{1}{4} \le \frac{|x_3 - z_3|}{|y_3 - z_3|} \le 4,$$

since

$$|y_3 - z_3| \ge |z_3| - |y_3| \ge k - j - 2 > \frac{k-8}{2} \ge \frac{k+2}{4} \ge \frac{|x_3| + |z_3|}{4} \ge \frac{|x_3 - z_3|}{4}$$

and

$$\begin{aligned}
|y_3 - z_3| &\le |y_3| + |z_3| \le j + k + 2 \\
&\le \frac{3k+4}{2} \le 4(k-2) \le 4(|z_3| - |x_3|) \le 4|x_3 - z_3|.
\end{aligned}$$

In particular, the triangle inequality gives that

$$|x_3 - y_3| \le 5|x_3 - z_3|.$$

Thus, similarly to the case of f_5 (although without integrating by parts in y), we apply Young's inequality twice, and Tonelli's Theorem to obtain

$$\begin{aligned}
\|f_6(\cdot,x_3)\|_{L^2} &\le \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\|\Gamma(\cdot,z_3)\|_{L^2}(1-\tilde{\phi}(y_3))(1-\tilde{\tilde{\phi}}(z_3))}{|x_3-y_3||y_3-z_3|^2}\mathrm{d}z_3\,\mathrm{d}y_3\\
&\le \int_{\mathbb{R}} \frac{\|\Gamma(\cdot,z_3)\|_{L^2}(1-\tilde{\tilde{\phi}}(z_3))}{|x_3-z_3|^2}\\
&\qquad \int_{\{\frac14|z_3-x_3|\le|y_3-z_3|\le 4|z_3-x_3|\}} \frac{1-\tilde{\phi}(y_3)}{|x_3-y_3|}\mathrm{d}y_3\,\mathrm{d}z_3\\
&\le \int_{\mathbb{R}} \frac{\|\Gamma(\cdot,z_3)\|_{L^2}(1-\tilde{\tilde{\phi}}(z_3))}{|x_3-z_3|^2}\int_{\{1\le|x_3-y_3|\le 5|x_3-z_3|\}} \frac{1}{|x_3-y_3|}\mathrm{d}y_3\,\mathrm{d}z_3\\
&\lesssim \int_{\mathbb{R}} \frac{\|\Gamma(\cdot,z_3)\|_{L^2}(1-\tilde{\tilde{\phi}}(z_3))}{|x_3-z_3|^2}\log(5|x_3-z_3|)\mathrm{d}z_3\\
&\lesssim \sum_{j\ge1}\int_{|z_3-x_3|\in(j,j+1)} \frac{\|\Gamma(\cdot,z_3)\|_{L^2}\log(5|x_3-z_3|)}{|x_3-z_3|^2}\mathrm{d}z_3\\
&\lesssim \sum_{j\ge1}\log(5j)j^{-2}\|\Gamma\|_{L^2_{3-\mathrm{uloc}}} \lesssim \|\Gamma\|_{L^2_{3-\mathrm{uloc}}}
\end{aligned}$$

for $x_3 \in \operatorname{supp}\phi$. Integration of the squares of the above estimates for f_3, f_4, f_5, f_6 gives the claim. □

6 Energy Estimates for ω/r

In this section, we assume the weak L^3 bound (6) on the time interval [0, 1] and prove an energy bound for $\Phi^2+\Gamma^2$ at time 1, that is we prove the following.

Proposition 6.1 (An $L^2_{3-\mathrm{uloc}}$ energy estimate for Φ and Γ) *Let u be a classical solution of (1) satisfying the weak L^3 bound (6) on* [0, 1]. *Then*

$$\|\Phi(1)\|_{L^2_{3-\mathrm{uloc}}(\mathbb{R}^3)}+\|\Gamma(1)\|_{L^2_{3-\mathrm{uloc}}(\mathbb{R}^3)} \le \exp\exp A^{O(1)}. \tag{68}$$

Recall (23) that $\|\cdot\|_{L^p_{3-\mathrm{uloc}}} := \sup_{z\in\mathbb{R}}\|\cdot\|_{L^p(\mathbb{R}^2\times[z-1,z+1])}$. We note that we will only use (in (73) below) the bound on Γ.

Proof We fix a cutoff function $\phi \in C_c^\infty((-1,1);[0,1])$ such that $\phi \equiv 1$ in $[-1/2, 1/2]$, and we define the translate

$$\phi_z(y) := \phi(y-z).$$

Clearly, we have the pointwise inequality

$$\phi_z', \phi_z'' \lesssim \sum_{i=-2}^{2} \phi_{z+i}.$$

We will consider the energies

$$\begin{aligned} E(t) &:= \sup_{z\in\mathbb{R}} E_z(t), \quad E_z(t) := \frac{1}{2}\int_{\mathbb{R}^3} (\Phi(t,x)^2 + \Gamma(t,x)^2)\phi_z(x_3)\mathrm{d}x, \\ F(t) &:= \sup_{z\in\mathbb{R}} F_z(t), \quad F_z(t) := \int_{t_0}^{t}\int_{\mathbb{R}^3} (\nabla\Phi(s,x)^2 + \nabla\Gamma(s,x)^2)\phi_z(x_3)\mathrm{d}x\,\mathrm{d}s \end{aligned}$$

for $t \in [t_0, 1]$, where $t_0 \in [0, 1]$ will be chosen in Step 3 below. Given $z \in \mathbb{R}$, we multiply the equations (10) by $\phi_z\Gamma$ and $\phi_z\Phi$, respectively, and integrate to obtain, at a given time t,

$$\begin{aligned} E_z' &\leq \int_{\mathbb{R}^3} \Big(-(|\nabla\Phi|^2 + |\nabla\Gamma|^2)\phi_z + \frac{1}{2}(\Phi^2+\Gamma^2)(u_z\phi_z' + \phi_z'') \\ &\quad + (\omega_r\partial_r + \omega_3\partial_3)\frac{u_r}{r}\Phi\phi_z - 2r^{-1}u_\theta\Phi\Gamma\phi_z\Big)\mathrm{d}x \\ &=: -F_z'(t) + I_1 + I_2 + I_3. \end{aligned} \tag{69}$$

The second term on the right hand side can be bounded directly,

$$I_1 \lesssim (1 + \|u_z\|_{L^\infty_x(\mathbb{R}^3)})E(t). \tag{70}$$

The remaining terms I_2, I_3 are more challenging. In order to estimate them, as well as choose t_0 and deduce the claim (68), we follow the steps below.

`Step 1.` We use the Hölder estimate (Proposition 5.1) to show that $|\Theta| \leq r^\gamma A^{O(1)}$ whenever $r \leq \frac{1}{2}$ and $t \in [3/4, 1]$, where $\gamma = \exp(-A^{O(1)})$.

To this end we note that, due to incompressibility, $\operatorname{div}(u + \frac{2}{r}e_r) = 4\pi\delta_{\{x'=0\}}$, which enables us to apply Proposition 5.1 to the equation for the swirl Θ (recall (13)).

Moreover, in the notation of Proposition 5.1, for every $R < \frac{1}{2}$, $t_0 \in [\frac{1}{2}, 1]$ and $x_0 \in (0,0)\times\mathbb{R}$ (i.e., on the x_3-axis),

$$R^{-\frac{4}{5}}\|u + \frac{e_r}{r}\|_{L^\infty_t L^{\frac{5}{3}}_x(Q((t_0,x_0),R))} \lesssim R^{-\frac{1}{2}}\|u\|_{L^\infty_t L^2_{\mathrm{uloc}}([t_0-R^2,t_0]\times\mathbb{R}^3)} + 1 \leq A^{O(1)},$$

by Hölder's inequality and (44) applied on the timescale R^2. (In particular note that each scale R leads to a different decomposition $u = u_n^\flat + u_n^\sharp$, but they all obey the same bounds up to being suitably rescaled.) Thus, for every $r \in (0, 1/2)$, $\operatorname{osc}_{B(x_0,r)} \Theta(t_0) \lesssim r^\gamma \operatorname{osc}_{Q(1/2)} \Theta$ for $r \in (0, 1/2)$, which implies the claim.

Step 2. We show that

$$\int_{t_0}^{t} |I_2 + I_3| \lesssim \frac{1}{2}F(t) + r_0^{-10} + \int_{t_0}^{t} GE$$

for each $t_0 \in [3/4, 1]$ and $t \in [t_0, 1]$, where

$$r_0 := \mathrm{e}^{-\gamma^{-2}}, \tag{71}$$

$\gamma = \exp(-A^{O(1)})$ is given by Step 1, and

$$G := r_0^{-3} + \|u\|_\infty + \|D^2 u\|_{L^{5/4}_{\mathrm{uloc}}} + \|\nabla u\|_{L^2_{\mathrm{uloc}}}$$

at each $t' \in [t_0, t]$.

To this end, we proceed similarly to [8]. Using integration by parts, we compute

$$\begin{aligned} I_2 &= 2\pi \int_{\mathbb{R}} \int_0^\infty (-\partial_3 u_\theta \partial_r \frac{u_r}{r} \Phi + \frac{\partial_r(r u_\theta)}{r} \partial_3 \frac{u_r}{r} \Phi) \phi_z(x_3) r \,\mathrm{d}r \,\mathrm{d}x_3 \\ &= \int_{\mathbb{R}^3} u_\theta (\partial_r \frac{u_r}{r} \partial_3 \Phi \phi_z - \partial_3 \frac{u_r}{r} \partial_r \Phi \phi_z + \partial_r \frac{u_r}{r} \Phi \phi_z') \\ &=: I_{2,1} + I_{2,2} + I_{2,3}. \end{aligned}$$

Let us further decompose $I_{2,i} = I_{2,i,\mathrm{in}} + I_{2,i,\mathrm{out}}$ $(i = 1, 2, 3)$ by writing

$$\int = \int_{\{r < r_0\}} + \int_{\{r \geq r_0\}}.$$

We decompose

$$I_{2,1,\mathrm{in}} = I_{2,1,\mathrm{in},1} + I_{2,1,\mathrm{in},2},$$

where

$$I_{2,1,\mathrm{in},1} := \int_{\{r<r_0\}} u_\theta \left(\fint_\Omega \partial_r \frac{u_r}{r} \right) \partial_3 \Phi \phi_z$$

and $\Omega := \{x' : r < 1\} \times \operatorname{supp} \phi_z$. We compute using Hölder's inequality and Sobolev embedding

$$\begin{aligned} \left| \int_\Omega \partial_r \frac{u_r}{r} \right| &\leq \|r^{-1} \partial_r u_r\|_{L^1(\Omega)} + \|r^{-2} u_r\|_{L^1(\Omega)} \\ &\lesssim \|r^{-1}\|_{L^{15/8}(\Omega)} \|\nabla u\|_{L^{15/7}(\Omega)} \lesssim \|\nabla^2 u\|_{L^{5/4}(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \lesssim G. \end{aligned}$$

Thus, integrating by parts, and applying Hölder's inequality in Lorentz spaces (27), and Young's inequality, we obtain

$$
\begin{aligned}
|I_{2,1,\mathrm{in},1}| &\le G \int_{B(r_0)\times\mathbb{R}} \left(r\Phi^2 \phi_z + |u_\theta \Phi \phi_z'| \right) \mathrm{d}x \\
&\lesssim G(r_0 E + \|u_\theta\|_{L^{3,\infty}_x(\mathbb{R}^3)} \|\Phi\|_{L^2_x(\Omega)} |\Omega|^{\frac{1}{6}}) \\
&\lesssim G(E + A^{O(1)}).
\end{aligned}
$$

As for $I_{2,1,\mathrm{in},2}$ we note that $p = 2(1-\gamma)/(1-2\gamma)$ is such that $p-2 = 2\gamma/(1-2\gamma) \ge \gamma$ and so we can use the quantified Hardy inequality (Lemma 3.2) to obtain, for we estimate for $t \in [\frac{1}{2}, 1]$,

$$
\begin{aligned}
|I_{2,1,\mathrm{in},2}| &\lesssim \|r^{\frac{3}{p}-\frac{1}{2}} u_\theta\|_{L^{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}}(\{r\le r_0\}\cap \operatorname{supp}\phi_z)} \\
&\quad \left\| r^{-\frac{3}{p}+\frac{1}{2}} (\partial_r \frac{u_r}{r} - \fint_\Omega \partial_r \frac{u_r}{r}) \phi_z^{\frac{1}{2}} \right\|_{L^p(\{r\le 1\})} \|\partial_3 \Phi \phi_z^{\frac{1}{2}}\|_2 \\
&\lesssim \gamma^{-O(1)} r_0^{\gamma/3} \left\| \nabla \partial_r \frac{u_r}{r} \right\|_{L^2_{3-\mathrm{uloc}}} \|\nabla \Phi \phi_z^{\frac{1}{2}}\|_2 \\
&\le \mathrm{e}^{-\gamma^{-1}/4} (\|\nabla \Gamma\|_{L^2_{3-\mathrm{uloc}}(\mathbb{R}^3)} + \|\Gamma\|_{L^2_{3-\mathrm{uloc}}(\mathbb{R}^3)}) \|\nabla \Phi \phi_z^{\frac{1}{2}}\|_2,
\end{aligned}
$$

where we have also applied Poincaré's inequality and our choice (71) of r_0. Thus

$$
\int_{t_0}^t I_{2,1,\mathrm{in},2} \le \frac{1}{20} F(t) + \int_{t_0}^t E.
$$

An analogous argument, in which "∂_r" and "∂_3" are switch, gives us the same bound for $I_{2,2,\mathrm{in},2}$. As for $I_{2,2,\mathrm{in},1}$, we integrate by parts, and apply Hölder's inequality for Lorentz spaces (27), and Young's inequality, to obtain

$$
\begin{aligned}
|I_{2,2,\mathrm{in},1}| &\le \left| \fint_\Omega \partial_3 \frac{u_r}{r} \phi_z \right| \int_{\{r\le r_0\}\cap \operatorname{supp}\phi_z} |u_\theta \partial_r \Phi| \\
&\lesssim \left| \fint_\Omega \frac{u_r}{r} \phi_z' \right| \|u_\theta\|_{L^{3,\infty}} \|\nabla \Phi\|_{L^2_x(\operatorname{supp}\phi_z)} r_0^{\frac{1}{3}} \\
&\lesssim \sum_{i=-2}^{2} \|\nabla u\|_{L^1(\Omega)} A (F_{z+i}')^{\frac{1}{2}} r_0^{\frac{1}{3}} \\
&\lesssim G A r_0^{1/3} \left(\sum_{i=-2}^{2} F_{z+i}' \right)^{\frac{1}{2}},
\end{aligned}
$$

which, thanks to the smallness of $r_0 = \exp(-\exp(A^{O(1)}))$ (recall (71)), gives that

$$\int_{t_0}^{t} |I_{2,2,\text{in},1}| \le \frac{1}{20} F(t) + (t - t_0).$$

We similarly decompose $I_{2,3,\text{in}} = I_{2,3,\text{in},1} + I_{2,3,\text{in},2}$ to find

$$\begin{aligned} |I_{2,3,\text{in},1}| = \left| \fint_{\Omega} \partial_r \frac{u_r}{r} \right| \left| \int_{\{r \le r_0\}} u_\theta \Phi \phi_z' \right| &\lesssim (\|\nabla u\|_{L^2(\Omega)} + \|\nabla^2 u\|_{L^{5/4}(\Omega)}) A E^{\frac{1}{2}} r_0^{\frac{1}{3}} \\ &\lesssim G(E+1), \end{aligned}$$

where we have used Lemma 3.2 and change of variables, the pointwise estimate $|u_r/r| \le |\nabla u|$, and Hölder's inequality to bound

$$\begin{aligned} \left| \fint_{\Omega} \partial_r \frac{u_r}{r} \right| &\lesssim \int_{z-10}^{z+10} \int_0^1 \left(|\partial_r u_r| + \frac{|u_r|}{r} \right) \mathrm{d}r\, \mathrm{d}z \\ &\lesssim \|r^{-1} \partial_r u_r\|_{L^1(\Omega)} + \|r^{-1} \nabla u\|_{L^1(\Omega)} \\ &\lesssim \|r^{-1} \nabla u\|_{L^{5/4}(\Omega)} \\ &\lesssim \|\nabla u\|_{L^2(\Omega)} + \|\nabla^2 u\|_{L^{5/4}(\Omega)}, \end{aligned}$$

where we used (34) in the third line, and the Hardy inequality (32) in the last line. Next

$$\begin{aligned} |I_{2,3,\text{in},2}| &= \left| \int_{\{r \le r_0\}} u_\theta \left(\partial_r \frac{u_r}{r} - \fint_{\Omega} \partial_r \frac{u_r}{r} \right) \Phi \phi_z' \right| \\ &\lesssim \|r u_\theta\|_{L^3(\{r \le r_0\})} \left\| r^{-\frac{1}{2}} \left(\partial_r \frac{u_r}{r} - \fint_{\Omega} \partial_r \frac{u_r}{r} \right) \right\|_{L^3(\mathbb{R}^2 \times \operatorname{supp} \phi_z)} \\ &\quad \|r^{-\frac{1}{2}} \Phi\|_{L^3(\mathbb{R}^2 \times \operatorname{supp} \phi_z)} \\ &\le A^{O(1)} r_0^{\frac{2}{3}} \left\| \nabla \partial_r \frac{u_r}{r} \right\|_{L^2_{3-\text{uloc}}} \|\nabla \Phi\|_{L^2_{3-\text{uloc}}}, \end{aligned}$$

where we have used the Hardy inequality (Lemma 3.2). Thus Lemma 5.3 and Young's inequality imply that

$$\int_{t_0}^{t} |I_{2,3,\text{in},2}| \le \frac{1}{20} F(t) + \int_{t_0}^{t} E.$$

Next let us consider the contributions to I_2 from outside $B(r_0)$. Using Hölder's inequality, we obtain that

$$|I_{2,1,\text{out}}| = \left| \int_{\{r > r_0\}} u_\theta \partial_r \frac{u_r}{r} \partial_3 \Phi \phi_z \, \mathrm{d}x \right|$$

$$\leq \|u_\theta\|_{L^6_{3-\mathrm{uloc}}(\{r>r_0\})}\|r^{-1}\partial_r u_r - r^{-2}u_r\|_{L^3_{3-\mathrm{uloc}}(\{r>r_0\})}\|\nabla\Phi\|_{L^2_{3-\mathrm{uloc}}(\mathbb{R}^3)}.$$

Hence, since Proposition 5.2 shows that $|u| \leq A^{O(1)}(r^{-1}+r^{-1/4})$ and $|\partial_r u_r| \leq A^{O(1)}(r^{-2}+r^{1/4})$, we see that the first two norms on the right hand side are finite and bounded by, say, r_0^{-10}. Thus, an application of Young's inequality gives that

$$\int_{t_0}^t |I_{2,1,\mathrm{out}}| \leq \frac{1}{20}F(t) + r_0^{-10}(t-t_0).$$

The remaining outer parts of I_2, i.e. $I_{2,2,\mathrm{out}}$ and $I_{2,3,\mathrm{out}}$ can be estimated in a similar way, with the latter bounded by, say, $E + r_0^{-10}$.

Finally let us consider I_3. Taking p such that, for example, $\frac{1}{p} = \frac{1}{2} - \frac{\gamma}{4}$, we have $p-2 = 2\gamma/(2-\gamma) \geq \gamma$, and so our quantified Hardy's inequality (Lemma 3.2) shows that

$$\begin{aligned}|I_{3,\mathrm{in}}| &\leq \left\| r^{-2+\frac{6}{p}}u_\theta \right\|_{L^{\left(1-\frac{2}{p}\right)^{-1}}(\{r\leq r_0\})} \|r^{-\frac{3}{p}+\frac{1}{2}}\Phi\|_{L^p_{3-\mathrm{uloc}}}\|r^{-\frac{3}{p}+\frac{1}{2}}\Gamma\|_{L^p_{3-\mathrm{uloc}}}\\ &\lesssim \gamma^{-O(1)}r_0^{\gamma/2}\left(\|\Phi\|_{L^2_{3-\mathrm{uloc}}}+\|\nabla\Phi\|_{L^2_{3-\mathrm{uloc}}}\right)\left(\|\Gamma\|_{L^2_{3-\mathrm{uloc}}}+\|\nabla\Gamma\|_{L^2_{3-\mathrm{uloc}}}\right),\end{aligned}$$

which gives that $\int_{t_0}^t |I_{3,\mathrm{in}}| \leq \frac{1}{20}F(t) + \int_{t_0}^t E$. On the other hand, for $r \geq r_0$ we have the simple bound

$$|I_{3,\mathrm{out}}| \leq 2\|r^{-1}u_\theta\|_{L^\infty_x(\{r\geq r_0\})}\|\Phi\|_{L^2_{3-\mathrm{uloc}}}\|\Gamma\|_{L^2_{3-\mathrm{uloc}}} \leq r_0^{-5/4}E,$$

as required.

`Step 3.` Given $\tau > 0$ we use the choice of time of regularity (Lemma 4.2) to find $t_0 \in [1-\tau, 1]$ such that $E(t_0) \lesssim A^{O(1)}\tau^{-3}$.

Indeed, Lemma 4.2 lets us choose $t_0 \in [1-\tau, 1]$ such that

$$\|\nabla^2 u(t_0)\|_\infty \leq A^{O(1)}\tau^{-\frac{3}{2}}.$$

It follows from the axial symmetry and (34) that $|\Phi| + |\Gamma| \leq |\nabla\omega|$, and so

$$\|\Phi(t_0)\phi_z^{1/2}\|_{L^2(\{r\leq 1\})} + \|\Gamma(t_0)\phi_z^{1/2}\|_{L^2(\{r\leq 1\})} \lesssim \|\nabla\omega(t_0)\|_{L^\infty(B(1)\times\mathbb{R})} \leq A^{O(1)}\tau^{-\frac{3}{2}} \tag{72}$$

for every $z \in \mathbb{R}$. Using the decomposition $\omega = \omega_1^\sharp + \omega_1^\flat$ on the interval $[0, 1]$, by (44), (40), and Hölder's inequality,

$$\begin{aligned}&\|\Phi(t_0)\phi_z^{1/2}\|_{L^2(\{r>1\})} + \|\Gamma(t_0)\phi_z^{1/2}\|_{L^2(\{r>1\})}\\ &\lesssim \|\omega_1^\sharp\|_{L^2(\mathbb{R}^3)} + \|r^{-1}\omega_1^\flat\|_{L^2(\{r>1\}\cap \mathrm{supp}\,\phi_z)}\\ &\lesssim \|\nabla u_1^\sharp\|_{L^2(\mathbb{R}^3)} + \|r^{-1}\|_{L^4_{x'}(B(1)^c)}\|\omega_1^\flat\|_{L^4(\mathbb{R}^3)}\end{aligned}$$

$$\leq A^{O(1)}.$$

This and (72) proves the claim of this step.

Step 4. We prove the claim.

Integration in time of the energy inequality (69) from initial time t_0 chosen in Step 3 above, taking $\sup_{z\in\mathbb{R}}$, and applying the estimate (70) for I_1 and Step 2 for I_2, I_3 we find that

$$E(t) + \frac{1}{2}F(t) \leq \underbrace{E(t_0)}_{\leq A^{O(1)}\tau^{-3}} + r_0^{-10}$$
$$+ \int_{t_0}^{t} O(r_0^{-3} + \|u\|_\infty + \|\nabla^2 u\|_{L^{5/4}_{\mathrm{uloc}}} + \|\nabla u\|_{L^2_{\mathrm{uloc}}})E(t')\mathrm{d}t'.$$

for $t \in [t_0, 1]$. Thus, by Grönwall's inequality,

$$E(1) \leq (A^{O(1)}\tau^{-3} + r_0^{-10}) \exp\left(O\left(r_0^{-3}(t-t_0) + A^{O(1)}(t-t_0)^{\frac{1}{5}}\right)\right).$$

Setting $\tau := r_0^4$, we see that the last exponential function is $O(1)$, and the prefactor gives the required estimate (68). □

7 Proof of Theorem 1.1

In this section we prove Theorem 1.1. Namely, given the $L^{3,\infty}$ bound (6) on the time interval [0, 1], we show that $|\nabla^j u| \leq \exp\exp A^{O_j(1)}$ at time 1.

Step 1. We show that $\|b\|_{L^p_{3-\mathrm{uloc}}(\mathbb{R}^3)} \leq C_p \exp\exp A^{O(1)}$ for each $p \in [3, \infty)$, $t \in [1/2, 1]$, where $b := u_r e_r + u_z e_z$ denotes the swirl-free part of the velocity field.

To this end we apply Proposition 6.1 to find

$$\|\Gamma\|_{L^\infty_t L^2_{3-\mathrm{uloc}}([\frac{1}{2},1]\times\mathbb{R}^3)} \leq \exp\exp A^{O(1)}. \tag{73}$$

On the other hand Proposition 5.2 shows that

$$\|r^2\omega\|_{L^\infty_x(\{r\leq 10\})} \leq A^{O(1)}.$$

Interpolating between this inequality and (73) we obtain

$$\|\omega_\theta\|_{L^p_{3-\mathrm{uloc}}(\{r\leq 10\})} = \|\Gamma^{\frac{2}{3}}(r^2\omega_\theta)^{\frac{1}{3}}\|_{L^p_{3-\mathrm{uloc}}(\{r\leq 10\})}$$
$$\lesssim \|\Gamma\|^{\frac{2}{3}}_{L^2_{3-\mathrm{uloc}}} \|r^2\omega_\theta\|^{\frac{1}{3}}_{L^\infty_x(\{r\leq 10\})} \leq \exp\exp A^{O(1)}$$

for all $p \leq 3$. Noting that

$$\mathrm{curl}\, b = \omega_\theta e_\theta, \quad \mathrm{div} b = 0$$

almost everywhere, and that $\operatorname{div} b = 0$ we now localize b to obtain an L^p estimate near the axis. Namely, for any unit ball $B \subset \{r \le 10\}$, let $\phi \in C_c^\infty(B)$ such that $\phi \equiv 1$ on $B/2$. Observe that for all $p \in [1,3)$ we can use Hölder's inequality for Lorentz spaces (27) to obtain

$$\|\operatorname{div}(\phi b)\|_{L^p(\mathbb{R}^3)} = \|b \cdot \nabla \phi\|_p \lesssim \|b\|_{L^{3,\infty}} \|\nabla \phi\|_{L^{3p/(3-p),1}} \lesssim A.$$

Applying the Bogovskiĭ operator (30) to $\operatorname{div}(\phi b)$ on the domain $B \setminus (B/2)$, we find $\tilde{b} \in W^{1,p}$ such that $\operatorname{div}\tilde{b} = 0$, $\|b - \tilde{b}\|_{W^{1,p}(B)} \le A^{O(1)}$, $\tilde{b} \equiv b$ in $B/2$, and $\tilde{b} \equiv 0$ outside B. Then for any $p \in (1,3)$,

$$\begin{aligned} \|b\|_{L^{3p/(3-p)}(B/2)} &\le \|\tilde{b}\|_{3p/(3-p)} \lesssim \|\nabla \tilde{b}\|_p \lesssim \|\operatorname{curl} \tilde{b}\|_{L^p(B)} \\ &\le \|\omega_\theta\|_{L^p(B)} + \|b - \tilde{b}\|_{W^{1,p}(B)} \\ &\le \exp\exp A^{O(1)}, \end{aligned}$$

which is our desired localized estimate. Here we have used the boundedness of the operator $\nabla f \mapsto \operatorname{curl} f$ in L^p (which is a consequence of the identity $\operatorname{curl}\operatorname{curl} f = \nabla(\operatorname{div} f) - \Delta f$, which in turn implies that $\nabla f = \nabla(-\Delta)^{-1}\operatorname{curl}(\operatorname{curl} f)$ for divergence-free f). Combining this with the pointwise estimates away from the axis (Proposition 5.2) gives the claim of this step.

`Step 2.` We show that there exists $C_0 > 1$ such that

$$\left\| \frac{u_\theta(t)}{r^{\frac{1}{2}}} \right\|^4_{L^4_{3-\mathrm{uloc}}} \le \left\| \frac{u_\theta(t_0)}{r^{\frac{1}{2}}} \right\|^4_{L^4_{3-\mathrm{uloc}}} + 1 + \exp\exp A^{C_0} \int_{t_0}^t \left\| \frac{u_\theta}{r^{\frac{1}{2}}} \right\|^4_{L^4_{3-\mathrm{uloc}}} \tag{74}$$

for each $t_0 \in [1/2, 1]$ and $t \in [t_0, 1]$.

To this end we provide a localization of the estimate of $u_\theta / r^{1/2}$ in the spirit of [8, Lemma 3.1]. Indeed, one can calculate from the equation (48) for u_θ that for a smooth cutoff $\psi = \psi(x_3)$,

$$\begin{aligned} &\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \frac{u_\theta^4}{r^2}\psi + \frac{3}{4}\int_{\mathbb{R}^3} \left|\nabla \frac{u_\theta^2}{r}\right|^2 \psi + \frac{3}{4}\int_{\mathbb{R}^3} \frac{u_\theta^4}{r^4}\psi_z \\ &\quad = -\frac{3}{2}\int_{\mathbb{R}^3} \frac{1}{r^3} u_r u_\theta^4 \psi + \frac{1}{8}\int_{\mathbb{R}^3} \frac{1}{r^2} u_\theta^2 (2u_\theta^2 u_z - \partial_z(u_\theta^2))\psi' =: I_1 + I_2 + I_3. \end{aligned}$$

As before, we choose $\psi \in C_c^\infty((-2,2))$ with $\psi \equiv 1$ in $[-1,1]$ and define the translates $\psi_z(x) := \psi(x - z)$ for all $z \in \mathbb{R}$. We consider the energies

$$\begin{aligned} E_z(t) &:= \frac{1}{4}\int_{\mathbb{R}^3} \frac{u_\theta^4}{r^2}\psi_z, & F_z(t) &:= \frac{3}{4}\int_{t_0}^t \int_{\mathbb{R}^3} \left|\nabla \frac{u_\theta^2}{r}\right|^2 \psi_z, \\ E(t) &:= \sup_{z\in\mathbb{R}} E_z(t), & F(t) &:= \sup_{z\in\mathbb{R}} F_z(t). \end{aligned}$$

By Step 1 and Sobolev embedding,

$$|I_1| \lesssim \|u_r\|_{L^6_{3-\mathrm{uloc}}} \left\| r^{-\frac{1}{2}} \frac{u_\theta^2}{r} \right\|^2_{L^{12/5}(\Omega)}$$
$$\leq \exp\exp A^{O(1)} \left(\left\| \frac{u_\theta^2}{r} \right\|^{\frac{1}{2}}_{L^2(\Omega)} \left\| \nabla \frac{u_\theta^2}{r} \right\|^{\frac{3}{2}}_{L^2(\Omega)} + \left\| \frac{u_\theta^2}{r} \right\|^{\frac{1}{2}}_{L^2(\Omega)} \right),$$

where $\Omega := \mathbb{R}^2 \times \operatorname{supp} \psi$. It follows that

$$\int_{t_0}^t |I_1| \leq \frac{1}{20} F(t) + \exp\exp A^{O(1)} \int_{t_0}^t E + (t - t_0).$$

Similarly,

$$|I_2| \lesssim \|u_z\|_{L^6_{3-\mathrm{uloc}}} \left\| \frac{u_\theta^2}{r} \right\|_{L^2_{3-\mathrm{uloc}}} \left\| \frac{u_\theta^2}{r} \right\|_{L^3(\Omega)}$$
$$\leq \exp\exp A^{O(1)} E^{\frac{1}{2}} \left(\left\| \frac{u_\theta^2}{r} \right\|^{\frac{1}{2}}_{L^2(\Omega)} \left\| \nabla \frac{u_\theta^2}{r} \right\|^{\frac{1}{2}}_{L^2(\Omega)} + \left\| \frac{u_\theta^2}{r} \right\|_{L^2(\Omega)} \right),$$

which yields the same bound as I_1. Finally,

$$|I_3| = \frac{1}{8} \left| \int_{\mathbb{R}^3} \frac{u_\theta^2}{r} \partial_3 \frac{u_\theta^2}{r} \psi' \right| \lesssim \left\| \frac{u_\theta^2}{r} \right\|_{L^2_{3-\mathrm{uloc}}} \left\| \nabla \frac{u_\theta^2}{r} \right\|_{L^2(\Omega)},$$

so we have

$$\int_{t_0}^t |I_3| \leq \frac{1}{20} F(t) + \int_{t_0}^t O(E).$$

Summing and taking the supremum over $z \in \mathbb{R}$ gives the claim of this step.

`Step 3.` We deduce that

$$\|u\|_{L^\infty_t L^6_{3-\mathrm{uloc}}([t_0,1]\times\mathbb{R}^3)} \leq \exp\exp A^{O(1)}, \tag{75}$$

where

$$t_0 := 1 - \exp(-\exp A^{O(1)}).$$

Indeed, Lemma 4.2 and Proposition 5.2 give a $t_0 \in [1 - \exp(-\exp A^{C_0}), 1]$ such that $\|r^{-\frac{1}{2}} u_\theta(t_0)\|_{L^4_x(\mathbb{R}^3)} \leq \exp\exp A^{2C_0}$. Therefore, applying Grönwall's inequality to the claim of the previous step,

$$\left\| \frac{u_\theta}{r^{\frac{1}{2}}} \right\|_{L^\infty_t L^4_{3-\mathrm{uloc}}([t_0,1]\times\mathbb{R}^3)} \leq \exp\exp A^{O(1)}.$$

Combining this with Proposition 5.2 and Hölder's inequality,

$$\begin{aligned}&\|u_\theta\|_{L^\infty_t L^6_{3-\mathrm{uloc}}([t_0,1]\times\mathbb{R}^3)}\\&\le \|ru_\theta\|^{\frac13}_{L^\infty_x(\{r\le 1\})}\|r^{-\frac12}u_\theta\|^{\frac23}_{L^\infty_t L^4_{3-\mathrm{uloc}}([t_0,1]\times\mathbb{R}^3)}+\|u\|_{L^\infty_t L^6_x([t_0,1]\times\{r>1\})}\\&\le \exp\exp A^{O(1)},\end{aligned}$$

which, together with Step 1, implies (75).

We note that Step 3 already provides a subcritical local regularity condition of the type of Ladyzhenskaya-Prodi-Serrin, which guarantees local boundedness of all spatial derivatives of u, and can be proved by employing the vorticity equation for example (see [35, Theorem 13.7]). In the last step below we use a robust tail estimate of the pressure function (recall Lemma 2.1) to provide a simpler justification of pointwise bounds by $\exp\exp A^{O(1)}$.

Step 4. We prove that, if $\|u\|_{L^\infty([1-t_1,1];W^{k-1,6}_{\mathrm{uloc}})}\lesssim \exp\exp A^{O(1)}$ for some $k\ge 1$ and $t_1=\exp(-\exp A^{O(1)})$, then the same is true for k (with some other t_1 of the same order).

Let $I=[a,b]\subset[t_1,1]$, and let $\chi\in C^\infty(\mathbb{R})$ be such that $\chi(t)=0$ for $t<a+(b-a)/8$ and $\chi(t)=1$ for $t>(a+b)/2$. We set $\phi\in C^\infty_c(B(0,2);[0,1])$ such that $\phi=1$ on $B(0,1/2)$ and $\sum_{j\in\mathbb{Z}^3}\phi_j=1$, where $\phi_j:=\phi(\cdot-j)$ for each $j\in\mathbb{R}^3$.

Letting $v:=\chi\phi\nabla^k u$ we see that $v(t_1)=0$, and

$$\begin{aligned}v_t-\Delta v&=\underbrace{-\chi'\phi\nabla^k u-2\chi\nabla\phi\cdot\nabla(\nabla^k u)-\chi\Delta\phi(\nabla^k u)}_{=:f_1}-\chi\phi\,\mathrm{div}(1+T)\nabla^k(u\otimes u)\\&=f_1-\phi\,\mathrm{div}(1+T)((\chi\nabla^k u\otimes u+u\otimes\chi\nabla^k u)\tilde\phi)\\&\quad-\chi\phi\,\mathrm{div}(1+T)\sum_{\substack{|\alpha|+|\beta|+|\gamma|=k\\|\alpha|,|\beta|<k}}C_{\alpha,\beta,\gamma}(D^\alpha u\otimes D^\beta u D^\gamma\tilde\phi)\\&\quad-\chi\phi\,\mathrm{div}T\nabla^k(u\otimes u(1-\tilde\phi))\\&=:f_1+f_2+f_3+f_4.\end{aligned}$$

We can now estimate $\|v(t)\|_6$, by extracting the same norm on the right-hand side and ensuring that the length of the interval is sufficiently small, so that the norm can be absorbed. Namely,

$$\begin{aligned}\|v(t)\|_6&=\left\|\int_a^t e^{(t-t')\Delta}f_1(t')dt'+\int_a^t e^{(t-t')\Delta}f_2(t')dt'+\int_a^t e^{(t-t')\Delta}f_3(t')dt'\right.\\&\quad\left.+\int_a^t e^{(t-t')\Delta}f_4(t')dt'\right\|_6\\&\le\left(\|\chi\nabla^k u\tilde\phi\|_{L^\infty([a,t];L^6)}+\|\chi'\nabla^{k-1}u\tilde\phi\|_{L^\infty([a,1];L^6)}\right)\int_a^t\|\Psi(t-t')\|_{W^{1,1}}dt'\\&\quad+\|\chi\nabla^k u\tilde\phi^{1/2}\|_{L^\infty([a,t];L^6)}\|u\tilde\phi^{1/2}\|_{L^\infty([a,t];L^6)}\int_a^t\|\Psi(t-t')\|_{W^{1,6/5}}dt'\end{aligned}$$

$$
\begin{aligned}
&+ \|u\|^2_{L^\infty([a,1];W^{k-1,6}_{\mathrm{uloc}})} \int_a^t \|\Psi(t-t')\|_{W^{1,6/5}} \mathrm{d}t' \\
&+ \|\operatorname{div} T(u \otimes u(1-\tilde{\phi}))\|_{L^\infty([a,1];W^{k,6}(B(0,2)))} \int_a^t \|\Psi(t-t')\|_1 \mathrm{d}t' \\
&\leq \|\chi \nabla^k u\|_{L^\infty([a,t];L^6_{\mathrm{uloc}})} \left((b-a)^{1/2} + \exp\exp A^{O(1)} (b-a)^{1/4}\right) + \exp\exp A^{O(1)}
\end{aligned}
$$

for each $t \in (a,b)$, where we used Young's inequality, heat estimates (24) and the Calderón-Zygmund inequality. By replacing ϕ (in the definition of v) by ϕ_z for any $z \in \mathbb{R}^3$, we obtain the same bound, and so

$$
\begin{aligned}
&\|\chi \nabla^k u\|_{L^\infty([a,b];L^6_{\mathrm{uloc}})} \\
&\quad \leq \|\chi \nabla^k u\|_{L^\infty([a,b];L^6_{\mathrm{uloc}})} (b-a)^{1/4} \exp\exp A^{O(1)} + \exp\exp A^{O(1)}.
\end{aligned}
$$

Thus, for any b, a such that $t_1 \leq a < b \leq 1$ and $(b-a)^{1/4} \leq \exp\exp A^{O(1)}/2$ we can absorb the first term on the right-hand side by the left-hand side to obtain

$$
\|\nabla^k u\|_{L^\infty([(a+b)/2,b];L^6_{\mathrm{uloc}})} \leq \exp\exp A^{O(1)}.
$$

Since the upper bound is independent of the location of $[a,b] \subset [t_1, 1]$, we obtain the claim.

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Appendix A: Quantitative Parabolic Theory

Here we prove Proposition 5.1. Namely, we consider parabolic cylinders

$$
Q_R^{\lambda,\theta}(t_0, x_0) := [t_0 - \theta R^2, t_0] \times B(x_0, \lambda R), \quad Q_R^{\lambda,\theta} := Q_R^{\lambda,\theta}(0,0), \quad Q_R := Q_R^{1,1}
$$

and we consider Lipschitz solutions V of $\mathcal{M}V = 0$ on $Q_R^{\lambda,\theta}$, namely we suppose that

$$
\int_{\mathbb{R}} \int (\partial_t V \phi + \nabla V \cdot \nabla \phi + b \cdot \nabla V \phi) = 0 \tag{76}
$$

for all $\phi \in C_c^\infty(Q_R^{\lambda,\theta})$, where the (distributional) supports of div$b$ and V are disjoint. Moreover we assume that (54) holds, namely

$$\mathcal{N}(R) := 2 + \sup_{R' \leq 2R} (R')^{-\alpha} \|b\|_{L_t^\ell L_x^q(Q_{R'})} < \infty,$$

where $\alpha := \frac{n}{q} + \frac{2}{\ell} - 1 \in [0,1)$. We also say that V is a *subsolution* (or *supersolution*) of $\mathcal{M}V = 0$, i.e. $\mathcal{M}V \leq 0$ (or $\mathcal{M}V \geq 0$), if (76) holds with "=" replaced by "$\leq$" (or "$\geq$") for all nonnegative test functions.

We will show that

$$\operatorname*{osc}_{B(r)} V(0) \lesssim \left(\frac{r}{R}\right)^\gamma \operatorname*{osc}_{Q(R)} V \tag{77}$$

for all $r \leq R$, where $\gamma = \exp(-\mathcal{N}^{O(1)})$.

To this end we first prove the Harnack inequality for Lipschitz subsolutions of $\mathcal{M}V = 0$.

Lemma A.1 (based on Lemma 3.1 in [24]) *Let V be a Lipschitz solution of $\mathcal{M}\mathcal{V} \leq 0$ in $Q_R^{\lambda,\theta}$ where $\lambda \in (1,2]$ and $\theta \in (0,1]$. Then*

$$\sup_{Q_R^{1,\theta/2}} V_+ \leq (\mathcal{N}/\theta)^C \left(⨍_{Q_R^{\lambda,\theta}} V_+^2 \right)^{\frac{1}{2}}.$$

Proof We first note that, for any r, a satisfying

$$\frac{3}{r} + \frac{2}{a} \in \left[\frac{3}{2}, \frac{5}{2}\right],$$

we have the interpolation inequality

$$\|\zeta U\|_{L_t^a L_x^r(Q_R^{\lambda,\theta})} \lesssim_{\lambda,\theta} R^{\frac{3}{r}+\frac{2}{a}-\frac{3}{2}} \|\zeta U\|_{\mathcal{V}(Q_R^{\lambda,\theta})}, \tag{78}$$

by [19, (3.4) in Chapter II], where $\mathcal{V}$ is the energy space $L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$.

Since V is a subsolution, we have, for a non-negative test function η,

$$\int_{Q_R^{\lambda,\theta}} (\partial_t V\eta + \nabla V \cdot \nabla \eta + b \cdot \nabla V \eta) \leq 0.$$

We let $\eta := \varphi'(V)\xi$ where ξ is a cutoff function vanishing on a neighborhood of the boundary of $Q_R^{\lambda,\theta}$, and φ is a convex function vanishing on $\mathbb{R}_-$. Taking $U := \varphi(V)$ we obtain

$$\int_{Q_R^{\lambda,\theta} \cap \{V>0\}} \left(\partial_t U\xi + \nabla U \cdot \nabla \xi + \frac{\varphi''(V)}{\varphi'(V)^2} |\nabla U|^2 \xi + b \cdot \nabla U \xi \right) \leq 0.$$

We now take

$$\varphi(\tau) := \tau_+^p \quad (p > 1) \qquad \text{and} \qquad \xi := \chi_{\{t<\bar{t}\}} U \zeta^2,$$

where ζ is a smooth cutoff function in $Q_R^{\lambda,\theta}$ and $\bar{t} \in (-\theta R^2, 0)$,

$$\begin{aligned}\int_{B_{\lambda R}} (\zeta U)^2(\bar{t}) dx + \int_{Q_R^{\lambda,\theta} \cap \{t<\bar{t}\}} &(2 - p^{-1})|\nabla U|^2 \zeta^2 + U \nabla U \cdot \nabla(\zeta^2) \\ &+ \frac{1}{2} b \cdot \nabla(U^2)\zeta^2 - \partial_t(\zeta^2) U^2 \leq 0. \end{aligned} \tag{79}$$

Using integration by parts and recalling the assumption $\operatorname{div} b \geq 0$, we can apply Hölder's inequality to obtain

$$\begin{aligned} \int_{Q_R^{\lambda,\theta} \cap \{t<\bar{t}\}} b \cdot \nabla(U^2)\zeta^2 &\geq - \int_{Q_R^{\lambda,\theta} \cap -\{t<\bar{t}\}} b \cdot \nabla(\zeta^2) U^2 \\ &\geq -\|b\|_{L_t^\ell L_x^q(Q_R^{\lambda,\theta})} \| |U|^{\frac{1}{s}} \zeta^{\frac{1}{s}-1} \nabla \zeta \|_{L^{2s}_{t,x}(Q)} \| (\zeta |U|)^{2-\frac{1}{s}} \|_{L_t^{(1-\frac{1}{2s}-\frac{1}{\ell})^{-1}} L_x^{(1-\frac{1}{2s}-\frac{1}{q})^{-1}}(Q)} \\ &= -\|b\|_{L_t^\ell L_x^q(Q_R^{\lambda,\theta})} \| U \zeta^{1-s} |\nabla \zeta|^s \|^{\frac{1}{s}}_{L^2_{t,x}(Q_R^{\lambda,\theta})} \| \zeta U \|^{2-\frac{1}{s}}_{L_t^a L_x^r(Q_R^{\lambda,\theta})}, \end{aligned}$$

where $s > 2$ and r and a are defined by

$$\frac{1}{2s} + \frac{1}{q} + \frac{1}{r}\Big(2 - \frac{1}{s}\Big) = 1, \quad \frac{1}{2s} + \frac{1}{\ell} + \frac{1}{a}\Big(2 - \frac{1}{s}\Big) = 1.$$

Applying Young's inequality to separate the last term, and utilizing the interpolation inequality (78) (which is valid since

$$\frac{3}{r} + \frac{2}{a} = \frac{3}{2} + 1 - 2\left(1 + 2/\left(\frac{3}{q} + \frac{2}{\ell}\right)\right)^{-1} \in (3/2, 11/6),$$

as needed) we obtain, after plugging into the local energy inequality (79),

$$\begin{aligned} \sup_{t \in [-\theta R^2, 0]} \int_{B_{\lambda R}} (\zeta U)^2 dx + \int_{Q_R^{\lambda,\theta} \cap \{t<\bar{t}\}} &(2 - p^{-1})|\nabla U|^2 \zeta^2 + U \nabla U \cdot \nabla(\zeta^2) - \partial_t(\zeta^2) U^2 \\ &- O\left(R^2 \|b\|^{2s}_{L_t^\ell L_x^q(Q_R^{\lambda,\theta})} \| U \zeta^{1-s} |\nabla \zeta|^s \|^2_{L^2_{t,x}(Q_R^{\lambda,\theta})}\right) - \frac{1}{10} \| \zeta U \|^2_{\mathcal{V}(Q_R^{\lambda,\theta})} \leq 0. \end{aligned}$$

Absorbing ∇U from the term on the third term on the left-hand side by the second term we obtain

$$\| \zeta U \|^2_{\mathcal{V}(Q_R^{\lambda,\theta})} \lesssim \int_{Q_R^{\lambda,\theta}} \left(|\nabla \zeta|^2 + \zeta |\partial_t \zeta| + R^2 \|b\|^{2s}_{L_t^\ell L_x^q(Q_R^{\lambda,\theta})} \zeta^{2-2s} |\nabla \zeta|^{2s} \right) U^2.$$

We now set

$$\lambda_m := 1 + 2^{-m}(\lambda - 1) \qquad \text{and} \qquad \theta_m := \frac{1}{2}\theta(1 + 4^{-m}),$$

and we substitute ζ with ζ_m such that

$$\zeta_m \equiv 1 \text{ in } Q_R^{\lambda_{m+1},\theta_{m+1}}, \quad \zeta_m \equiv 0 \text{ outside } Q_R^{\lambda_m,\theta_m}, \quad |\partial_t \zeta_m| \leq \frac{4^m C}{\theta R^2}, \quad \frac{|\nabla \zeta_m|}{\zeta_m^{1-\frac{1}{s}}} \leq \frac{2^m C}{R},$$

where C may depend on λ. Then the energy estimate and (78), taken with $r = l = 10/3$, yield

$$\|\zeta_m U\|_{L^{10/3}_{t,x}(Q_R^{\lambda,\theta})} \lesssim \|\zeta_m U\|_{\mathcal{V}(Q_R^{\lambda,\theta})} \leq C R^{-1}(\theta^{-\frac{1}{2}} + 2^m + \mathcal{N}^s)2^{ms}\|U\|_{L^2_{t,x}(Q_R^{\lambda,\theta})}.$$

Recalling the definition of U and replacing p with $p_m := (5/3)^m$, Hölder's inequality implies

$$\begin{aligned} \left(\fint_{Q_R^{\lambda_{m+1},\theta_{m+1}}} u_+^{2p_{m+1}} \right)^{\frac{1}{2p_{m+1}}} &\leq \left(C \fint_{Q_R^{\lambda_m,\theta_m}} (\zeta_m U)^{10/3} \right)^{\frac{1}{rp_m}} \\ &\leq \left(C\theta_m^{-1}\mathcal{N}^{2s}4^{m(s+1)} \fint_{Q_R^{\lambda_m,\theta_m}} u_+^{2p_m} \right)^{\frac{1}{2p_m}}. \end{aligned}$$

Iterating, we have

$$\left(\fint_{Q_R^{\lambda_m,\theta_m}} u_+^{2p_m} \right)^{\frac{1}{2p_m}} \leq \prod_{k=0}^{m-1} \left(\frac{C}{\theta} 4^{k(s+1)} \mathcal{N}^s \right)^{\frac{1}{2p_k}} \left(\fint_{Q_R^{\lambda,\theta}} u_+^2 \right)^{\frac{1}{2}},$$

and we conclude by taking $m \to \infty$. □

In the next three lemmas we focus on nonnegative solutions to $\mathcal{M}V \leq 0$ and we find lower bounds on the mass distribution of such solutions. We first show that if $V \geq k$ in Q_R, except for a small (quantified) "portion of Q_R", then in fact $V \geq k/2$ everywhere in a smaller cylinder.

Lemma A.2 (based on part 2 of Corollary 3.1 in [24]) *If V is a non-negative solution of $\mathcal{M}\mathcal{V} \geq 0$ in $Q_R^{\lambda,\theta}$ and*

$$|\{V < k\} \cap Q_R^{\lambda,\theta}| \leq (\mathcal{N}/\theta)^{-5C}|Q_R^{\lambda,\theta}|,$$

then

$$V \geq \frac{k}{2} \quad \text{in } Q_R^{1,\theta/2}.$$

Proof We apply Lemma A.1 to $k - V$ to find

$$\sup_{Q_1^{1,\theta/2}} (k-V)_+ \leq (\mathcal{N}/\theta)^C \left(\fint_{Q_R^{\lambda,\theta}} (k-V)_+^2 \right)^{\frac{1}{2}} \leq \mathcal{N}^{-1} k,$$

which implies the result. □

We now show that, if the cylinder $Q_R^{1,\theta}$ is flat enough, then a lower bound on the bottom lid of $Q_R^{1,\theta}$ (i.e. at $t = -\theta R^2$) implies a similar lower bound at every t.

Lemma A.3 (based on Lemma 3.2 in [24]) *Suppose V is non-negative with $\mathcal{M}\mathcal{V} \geq 0$ in a neighbuorhood of Q_R^{1,θ_0} and*

$$|\{V(-\theta_0 R^2) \geq k\} \cap B_R| \geq \delta_0 |B_R|$$

for some $\delta_0 > 0$ and $\theta_0 \leq C^{-1}\delta_0^6 \mathcal{N}^{-1}$. Then

$$|\{V(\bar{t}) \geq \frac{1}{3}\delta_0 k\} \cap B_R| \geq \frac{1}{3}\delta_0 |B_R|$$

for all $\bar{t} \in [-\theta_0 R^2, 0]$.

Proof By the calculations in [24], with ζ a smooth cutoff function supported in B_R,

$$\int_{B_R} (V(\bar{t}) - k)_-^2 \zeta^2 + \int_{Q_R^{1,\theta_0}} \chi_{\{t<\bar{t}\}} |\nabla (V-k)_-|^2 \zeta^2 \leq \int_{B_R} (V(-\theta_0 R^2) - K)_-^2 \zeta^2 \tag{80}$$

$$+ \int_{Q_R^{1,\theta_0}} \chi_{\{t<\bar{t}\}} (V-k)_-^2 \big(O(|\nabla \zeta|^2) + b \cdot \nabla(\zeta^2) + (\mathrm{div} b)\zeta^2 \big). \tag{81}$$

We choose ζ such that $\zeta \equiv 1$ in $B_{(1-\sigma)R}$ and $|\nabla \zeta| \leq \frac{2}{\sigma R}$ where $\sigma < 1$ is to be specified. Note that, due to (53),

$$\begin{aligned} \int_{Q_R^{1,\theta_0}} \chi_{\{t<\bar{t}\}} (V-k)_-^2 (\mathrm{div} b) \zeta^2 &\leq k^2 \int_{Q_R^{1,\theta_0}} \chi_{\{t<\bar{t}\}} (\mathrm{div} b) \zeta^2 \\ &= -k^2 \int_{Q_R^{1,\theta_0}} \chi_{\{t<\bar{t}\}} b \cdot \nabla(\zeta^2). \end{aligned}$$

Then the right-hand side of (80) is bounded by

$$k^2 \Big((1-\delta_0)|B_R| + O(\theta_0 \sigma^{-2} |B_R|) + \frac{4}{\sigma R} \|b\|_{L_t^\ell L_x^q(Q_R)} \|1\|_{L_t^{\ell'} L_x^{q'}(Q_R^{1,\theta_0})} \Big).$$

From here one can proceed with the argument exactly as in [24] to arrive at

$$\left|\left\{V(\bar{t}) < \frac{1}{3}\delta_0 k\right\} \cap B_R\right| \leq \left(1 - \frac{1}{3}\delta_0\right)^{-2} (1 - \delta_0 + O(\sigma + \sigma^{-2}\theta_0 + \sigma^{-1}\theta_0^{2/\ell'}\mathcal{N})).$$

Setting $\sigma = C^{-1/5}\delta_0^2$ and θ_0 as above proves the claimed bound. □

We now show that for any given "portion of $Q_R^{1,\theta}$" (in the sense of a set with the measure arbitrarily close to $|Q^{1,\theta}|$) V is greater or equal a constant multiple of some lower bound, if, for each t, the lower bound occurs at least on some "portion of B_R". Although this enables us to obtain a lower bound on almost the entire cylinder, we lose an exponential in the process.

Lemma A.4 (based on Lemma 3.3 in [24]) *Let $V \geq 0$ be a solution of $\mathcal{MV} \geq 0$ in $Q_R^{\lambda,\theta}$ satisfying*

$$|\{V(t) \geq k_0\} \cap B_R| \geq \delta_1 |B_R| \textit{ for all } t \in [-\theta R^2, 0]$$

for some $k_0 > 0$, $\delta_1 > 0$. Then for any $\mu > 0$ and $s > C(\mathcal{N} + \theta^{-1})/(\delta_1\mu)^2$,

$$|\{V < 2^{-s}k_0\} \cap Q_R^{1,\theta}| \leq \mu |Q_R^{1,\theta}|.$$

Proof With $k_m = 2^{-m}k_0$, we define

$$\mathcal{E}_m(t) := \{x \in B_R : k_{m+1} \leq V(x,t) < k_m\}; \quad \mathcal{E}_m := \{(t,x) \in Q_R^{1,\theta} : x \in \mathcal{E}_m(t)\}.$$

Integrating the inequality $\mathcal{MV} \geq 0$ against the test function $\eta = (V - k_m)_- \xi(x)^2$ where ξ is a smooth cutoff vanishing in a neighborhood of $\partial B_{\lambda R}$ and satisfying $\xi \equiv 1$ in B_R,

$$\begin{aligned}\int_{Q_R^{\lambda,\theta} \cap \{V < k_m\}} |\nabla V|^2 \xi^2 &\leq \int_{Q_R^{\lambda,\theta}} |\nabla (V - k_m)_-|^2 \xi^2 \lesssim \int_{B_{\lambda R} \cap \{V < k_m\}} (V - k_m)_-^2 \xi^2 \Big|_{t = -\theta R^2} \\ &\quad + \int_{-\theta R^2}^0 \int_{B_{\lambda R} \cap \{V < k_m\}} (V - k_m)_-^2 |\nabla \xi|^2 + 2(V - k_m)_-^2 \xi b \cdot \nabla \xi \\ &\lesssim k_m^2 R^n (1 + \theta \mathcal{N}), \end{aligned} \tag{82}$$

by Hölder's inequality and the trivial bound $0 \leq (V - k_m)_- \leq k_m$. From De Giorgi's inequality [19, (5.6) in Chapter II],

$$(k_m - k_{m+1})|\{V(t) < k_{m+1}\} \cap B_R| \lesssim \frac{R}{\delta_1} \int_{\mathcal{E}_m(t)} |\nabla V(t)|$$

for all $t \in [-\theta R^2, 0]$. Integrating in time, squaring, and applying Cauchy-Schwarz gives

$$k_{m+1}^2 \left|\{V < k_{m+1}\} \cap Q_R^{1,\theta}\right|^2 \lesssim \frac{R^2}{\delta_1^2} \int_{\mathcal{E}_m} |\nabla V|^2 \mathrm{d}x\mathrm{d}t |\mathcal{E}_m|.$$

Combined with (82), this gives

$$\left|\{V < k_{m+1}\} \cap Q_R^{1,\theta}\right|^2 \lesssim \delta_1^{-2} R^{n+2}(1+\theta\mathcal{N})|\mathcal{E}_m|.$$

We conclude

$$\begin{aligned} s\left|\{V < k_s\} \cap Q_R^{1,\theta}\right|^2 &\le \sum_{m=0}^{s-1} \left|\{V < k_{m+1}\} \cap Q_R^{1,\theta}\right|^2 \\ &\lesssim \delta_1^{-2} R^{n+2}(1+\theta\mathcal{N}) \sum_{m=0}^{s-1} |\mathcal{E}_m| \\ &\lesssim \delta_1^{-2}(\theta^{-1}+\mathcal{N})|Q_R^{1,\theta}|^2. \end{aligned}$$

□

We can now combine Lemmas A.2–A.4 to obtain a pointwise lower bound for V in the interior of a cylinder, with an exponential dependence on $\mathcal{N}$.

Lemma A.5 (based on part 1 of Corollary 3.2 in [24]) *If V is a non-negative solution of $\mathcal{MV} \ge 0$ in $Q_R^{2,1}$ and*

$$|\{V(-\Theta R^2) \ge k\} \cap B_R| \ge \delta|B_R|$$

for some $k > 0$ and $\Theta \le C^{-1}\delta^6\mathcal{N}^{-1}$, then

$$V \ge \exp(-\delta^{-2}(\mathcal{N}/\Theta)^{20C})k \quad \text{in } Q_R^{1,\Theta/2}.$$

Proof This is a straightforward application of Lemmas A.3, A.4, and A.2 in sequence, with the latter two applied with $R \to \frac{3}{2}R$ to compensate for the shrinking domain in Lemma A.2. □

By considering $V - \inf V$ and $\sup V - V$ the above lemma now allows us to estimate oscillations of solutions to $\mathcal{M}V = 0$ with no sign restrictions.

Lemma A.6 (based on Lemma 3.5 of [24]) *If V solves $\mathcal{M}V = 0$ in $Q_R^{2,1}$ then*

$$\operatorname*{osc}_{Q^{(1)}} V \le (1 - \exp(-\mathcal{N}^{50C})) \operatorname*{osc}_{Q^{(2)}} V$$

where $Q^{(1)} = Q_R^{1,\Theta/2}$, $Q^{(2)} = Q_R^{2,1}$, and $\Theta = C^{-2}\mathcal{N}^{-1}$.

Proof Consider the positive supersolutions $V_1 = V - \inf_{Q^{(2)}} V$ and $V_2 = \sup_{Q^{(2)}} V - V$. With $k = \operatorname*{osc}_{Q^{(2)}} V$, clearly we must have $|\{V_i(-\Theta R^2) \ge k\} \cap B_{2R}| \ge |B_{2R}|/2$ for either $i = 1$ or $i = 2$. Fix this i, so V_i obeys the hypotheses of Lemma A.5. Let us assume for concreteness that $i = 1$; the other case is analogous. Then by the lemma,

$$\inf_{Q^{(2)}} V + \exp(-\mathcal{N}^{50C}) \operatorname*{osc}_{Q^{(2)}} V \le V \le \sup_{Q^{(2)}} V$$

for all $(t, x) \in Q^{(1)}$, which immediately implies the result. □

Finally, iterating Lemma A.6 we obtain the required Hölder continuity (77), i.e. we can prove Proposition 5.1.

Proof of Proposition 5.1 Iterating Lemma A.6, we have

$$\operatorname*{osc}_{Q^{2,1}_{(\Theta/2)^{k/2}R/2}} V \leq (1-\exp(-\mathcal{N}^{50C}))^k \operatorname*{osc}_{Q^{2,1}_{R/2}} V.$$

We conclude upon taking $k = \lfloor \log \frac{R}{r} (\log \frac{2}{\Theta})^{-1} \rfloor$. □

References

1. Barker, T.: Localized quantitative estimates and potential blow-up rates for the Navier-Stokes equations. arXiv preprint arXiv:2209.15627, (2022)
2. Barker, T., Prange, C.: Quantitative regularity for the Navier-Stokes equations via spatial concentration. Comm. Math. Phys. **385**(2), 717–792 (2021)
3. Beale, J.T., Kato, T., Majda, A.: Remarks on the breakdown of smooth solutions for the 3-D Euler equations. Comm. Math. Phys. **94**(1), 61–66 (1984)
4. Bogovskiĭ, M.E.: Solution of the first boundary value problem for an equation of continuity of an incompressible medium. Dokl. Akad. Nauk SSSR **248**(5), 1037–1040 (1979)
5. Bogovskiĭ, M. E.: Solutions of some problems of vector analysis, associated with the operators div and grad. In Theory of cubature formulas and the application of functional analysis to problems of mathematical physics, volume 1980 of Trudy Sem. S. L. Soboleva, No. 1, pp 5–40, 149. Akad. Nauk SSSR Sibirsk. Otdel., Inst. Mat., Novosibirsk, (1980)
6. Chen, C., Strain, R. M., Tsai, T., Yau, H.: Lower bound on the blow-up rate of the axisymmetric Navier-Stokes equations. Int. Math. Res. Not. IMRN, (9):Art. ID rnn016, 31, (2008)
7. Chen, C., Strain, R.M., Tsai, T., Yau, H.: Lower bounds on the blow-up rate of the axisymmetric Navier-Stokes equations. II. Comm. Partial Differential Equations **34**(1–3), 203–232 (2009)
8. Chen, H., Fang, D., Zhang, T.: Regularity of 3D axisymmetric Navier-Stokes equations. Discrete Contin. Dyn. Syst. **37**(4), 1923–1939 (2017)
9. Chen, H., Tsai, T.-P., Zhang, T.: Remarks on local regularity of axisymmetric solutions to the 3D Navier-Stokes equations. Commun. Partial. Differ. Equ. **47**(2), 1680–1699 (2022)
10. Constantin, P.: Navier-Stokes equations and area of interfaces. Communications in mathematical physics **129**(2), 241–266 (1990)
11. Constantin, P., Fefferman, C.: Direction of vorticity and the problem of global regularity for the Navier-Stokes equations. Indiana Univ. Math. J. **42**(3), 775–789 (1993)
12. Escauriaza, L., Serëgin, G.A., Šverák, V.: $L_{3,\infty}$-solutions of Navier-Stokes equations and backward uniqueness. Uspekhi Mat. Nauk **58**(2(350)), 3–44 (2003)
13. Feng, W., He, J., Wang, W.: Quantitative bounds for critically bounded solutions to the three-dimensional Navier-Stokes equations in Lorentz spaces. arXiv preprint arXiv:2201.04656, (2022)
14. Galdi, G. P.: An introduction to the mathematical theory of the Navier-Stokes equations. Springer Monographs in Mathematics. Springer, New York, second edition, (2011). Steady-state problems
15. Hou, Thomas Y.: Potentially singular behavior of the 3D Navier–Stokes equations. Foundations of Computational Mathematics, pp. 1–49, (2022)
16. Koch, G., Nadirashvili, N., Seregin, G.A., Šverák, V.: Liouville theorems for the Navier-Stokes equations and applications. Acta Math. **203**(1), 83–105 (2009)
17. Ladyženskaja, O.A.: Uniqueness and smoothness of generalized solutions of Navier-Stokes equations. Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **5**, 169–185 (1967)
18. Ladyženskaja, O.A.: Unique global solvability of the three-dimensional Cauchy problem for the Navier-Stokes equations in the presence of axial symmetry. Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **7**, 155–177 (1968)

19. Ladyženskaja, O. A., Solonnikov, V. A., Ural'tseva, N. N.: Linear and quasilinear equations of parabolic type. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., (1968). Translated from the Russian by S. Smith
20. Lei, Z., Zhang, Q.S.: A Liouville theorem for the axially-symmetric Navier-Stokes equations. J. Funct. Anal. **261**(8), 2323–2345 (2011)
21. Lei, Zhen, Zhang, Qi.: Criticality of the axially symmetric Navier-Stokes equations. Pacific Journal of Mathematics **289**(1), 169–187 (2017)
22. Lemarié-Rieusset, P.G.: Recent developments in the Navier-Stokes problem. Chapman & Hall/CRC Research Notes in Mathematics, vol. 431. Chapman & Hall/CRC, Boca Raton, FL (2002)
23. Liu, J.-G., Wang, W.-C.: Characterization and regularity for axisymmetric solenoidal vector fields with application to Navier-Stokes equation. SIAM J. Math. Anal. **41**(5), 1825–1850 (2009)
24. Nazarov, A.I., Ural'tseva, N.N.: The Harnack inequality and related properties of solutions of elliptic and parabolic equations with divergence-free lower-order coefficients. Algebra i Analiz **23**(1), 136–168 (2011)
25. Neustupa, J., Pokorný, M.: Axisymmetric flow of Navier-Stokes fluid in the whole space with non-zero angular velocity component. In Proceedings of Partial Differential Equations and Applications (Olomouc, 1999) **126**, 469–481 (2001)
26. Nečas, J., Růžička, M., Šverák, V.: On Leray's self-similar solutions of the Navier-Stokes equations. Acta Math. **176**(2), 283–294 (1996)
27. Ożański, W.S.: The partial regularity theory of Caffarelli, Kohn, and Nirenberg and its sharpness. Lecture Notes in Mathematical Fluid Mechanics. Birkhäuser/Springer, Cham (2019)
28. Ożański, W.S.: Weak solutions to the Navier-Stokes inequality with arbitrary energy profiles. Comm. Math. Phys. **374**(1), 33–62 (2020)
29. Ożański, W.S.: Quantitative transfer of regularity of the incompressible Navier-Stokes equations from $\mathbb{R}^3$ to the case of a bounded domain. J. Math. Fluid Mech., **23**(4), Paper No. 98, 14, (2021)
30. Ożański, W. S., Pooley, B. C.: Leray's fundamental work on the Navier-Stokes equations: a modern review of "sur le mouvement d'un liquide visqueux emplissant l'espace". In Partial differential equations in fluid mechanics, vol. 452 of London Math. Soc. Lecture Note Ser., pp. 113–203. Cambridge Univ. Press, Cambridge
31. Palasek, S.: Improved quantitative regularity for the Navier-Stokes equations in a scale of critical spaces. Arch. Ration. Mech. Anal. **242**(3), 1479–1531 (2021)
32. Palasek, S.: A minimum critical blowup rate for the high-dimensional Navier-Stokes equations. Journal of Mathematical Fluid Mechanics **24**(4), 1–28 (2022)
33. Pan, X.: Regularity of solutions to axisymmetric Navier-Stokes equations with a slightly supercritical condition. J. Differential Equations **260**(12), 8485–8529 (2016)
34. Prodi, G.: Un teorema di unicità per le equazioni di Navier-Stokes. Ann. Mat. Pura Appl. **4**(48), 173–182 (1959)
35. Robinson, J. C., Rodrigo, J. L., Sadowski, W.: The three-dimensional Navier-Stokes equations, volume 157 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, (2016). Classical theory
36. Scheffer, V.: A solution to the Navier-Stokes inequality with an internal singularity. Comm. Math. Phys. **101**(1), 47–85 (1985)
37. Scheffer, V.: Nearly one-dimensional singularities of solutions to the Navier-Stokes inequality. Comm. Math. Phys. **110**(4), 525–551 (1987)
38. Seregin, G.: A note on necessary conditions for blow-up of energy solutions to the Navier-Stokes equations. In Parabolic problems, vol. 80 of Progr. Nonlinear Differential Equations Appl., pp. 631–645. Birkhäuser/Springer Basel AG, Basel, (2011)
39. Seregin, G.: Local regularity of axisymmetric solutions to the Navier-Stokes equations. Anal. Math. Phys. **10**(4), (2020)
40. Seregin, G.: A note on local regularity of axisymmetric solutions to the Navier-Stokes equations. J. Math. Fluid Mech. **24**(1), (2022)
41. Serrin, J.: The initial value problem for the Navier-Stokes equations. In Nonlinear Problems (Proc. Sympos., Madison, Wis., 1962), pp. 69–98. Univ. Wisconsin Press, Madison, Wis., (1963)
42. Talenti, G.: Best constant in Sobolev inequality. Annali di Matematica pura ed Applicata **110**(1), 353–372 (1976)
43. Tao, T.: Lecture notes 1 for 247A. available at https://www.math.ucla.edu/~tao/247a.1.06f/notes1.pdf

44. Tao, T.: Quantitative bounds for critically bounded solutions to the Navier-Stokes equations. In Nine mathematical challenges, vol. 104 of Proc. Sympos. Pure Math., pp. 149–193. Amer. Math. Soc., Providence, RI, (2021)
45. Tsai, T.P.: On Leray's self-similar solutions of the Navier-Stokes equations satisfying local energy estimates. Arch. Ration. Mech. Anal. **143**(1), 29–51 (1998)
46. Ukhovskii, M.R., Iudovich, V.I.: Axially symmetric flows of ideal and viscous fluids filling the whole space. J. Appl. Math. Mech. **32**, 52–61 (1968)

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