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MANUSCRIPT

On convergence of approximate solutions to the compressible Euler system

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Abstract

We consider a sequence of approximate solutions to the compressible Euler system admitting uniform energy bounds and/or satisfying the relevant field equations modulo an error vanishing in the asymptotic limit. We show that such a sequence either **(i)** converges strongly in the energy norm, or **(ii)** the limit is not a weak solution of the associated Euler system. This is in sharp contrast to the incompressible case, where (oscillatory) approximate solutions may converge weakly to solutions of the Euler system. Our approach leans on identifying a system of differential equations satisfied by the associated turbulent defect measure and showing that it only has a trivial solution.

Keywords Compressible Euler system · Convergence · Weak solution · Defect measure

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1 Introduction

In [28, Section 4], Greengard and Thomann constructed a sequence $\{\mathbf{v}_n\}_{n=1}^{\infty}$ of exact solutions to the *incompressible* Euler system in R^2, compactly supported in the space variable, and converging *weakly* to the velocity field $\mathbf{v} = 0$. As $\mathbf{v} = 0$ is obviously a solution of the Euler system, this is an example of a sequence of solutions to the incompressible Euler system defined on the whole space R^2 and converging weakly to another solution of the same problem. We show that such a scenario is impossible in the context of *compressible* fluid flows.

We consider consider a sequence of *approximate* solutions to the compressible Euler system. Motivated by the numerical terminology we distinguish **(i)** *stable approximation*, where the approximate solutions satisfy the relevant uniform bounds, and **(ii)** *consistent approximation*, where the field equations of the Euler system are satisfied modulo an error vanishing in the asymptotic limit. A prominent example of consistent approximation is the *vanishing viscosity* limit, where the approximate solutions satisfy the Navier–Stokes system. In the light of the recent results [10–13] indicating essential ill–posedness of the compressible Euler system, the vanishing viscosity limit might be seen as a sound selection criterion to identify the physically relevant solutions of systems describing inviscid fluids, although this can be still arguable in view of the examples collected in the recent survey by Buckmaster and Vicol [6] and Constantin and Vicol [14]. The principal difficulties of this process, caused in particular by the presence of kinematic boundaries, are well understood in the case of incompressible fluids, see e.g. the survey of E [32]. However, much less is known in the compressible case. Leaving apart the boundary layer issue, Sueur [31] proved unconditional convergence in the barotropic case provided the Euler system admits a smooth solution. A similar result was obtained for the full Navier–Stokes/Euler systems in [20]. However, as many solutions of the Euler system are known to develop discontinuities in finite time, it is of essential interest to understand the inviscid limit provided the target solution is not smooth. Very recently, Basarić [3] identified the vanishing viscosity limit with a measure–valued solution to the Euler system on general, possibly unbounded, spatial domains, which can be seen as a "compressible" counterpart of the pioneering work of DiPerna and Majda [19] in the incompressible case. The incompressible setting was further studied in space dimension two and for vortex sheet initial data by DiPerna and Majda [17,18] and Greengard and Thomann [28]. Their results show that the set, where the approximate solutions do not converge strongly is either empty or its projection on the time axis is of positive measure.

As the name suggests, numerous consistent approximations can be identified with sequences of numerical solutions, see e.g. [21,22]. There is a strong piece of evidence, see e.g. Fjordholm et al. [24–26], that the numerical solutions to the compressible Euler system develop fast oscillations (wiggles) in the asymptotic limit. The resulting object is described by the associated Young measure and it is therefore of interest to know in which sense the limit Euler system is satisfied. In accordance with the seminal paper by DiPerna and Majda [19], the limit should be identified with a generalized *measure–valued* solution of the Euler system. The concept of measure–valued solution used also more recently in Basarić [3], however, follows the philosophy: the more general the better, while preserving a suitable weak (measure–valued)/strong uniqueness principle. Such an approach is typically beneficial for a number of applications in numerical analysis. As a matter of fact, a more refined description of the asymptotic limit can be obtained via Alibert–Bouchitté's [1] framework employed by Gwiazda, Świerczewska–Gwiazda, and Wiedemann [29]. Here, similarly to the work by Chen and Glimm [9], the measure–valued solutions are defined for the density ϱ and the weighted velocity $\sqrt{\varrho}\mathbf{u}$ yielding a rather awkward definition of a solution.

Our approach is based on estimating the distance between an approximate sequence and its limits by means of the so–called *Bregman divergence*

$$\mathcal{E}\left(\mathbf{U}\,\middle|\mathbf{V}\right)=\int_{\Omega}\Big[E(\mathbf{U})-\xi\cdot(\mathbf{U}-\mathbf{V})-E(\mathbf{V})\Big]\,\mathrm{d}x,\ \xi\in\partial E(\mathbf{V}), \tag{1.1}$$

where $\mathbf{U}$, $\mathbf{V}$ are measurable functions on the fluid domain $\Omega\subset R^d$ ranging in R^m, and $E:R^m\to[0,\infty]$ is a strictly convex function, see e.g. Sprung [30]. In the context of the Euler system, the function E is the total energy; whence $\mathcal{E}$ may be see as *relative energy* in the sense of Dafermos [15]. Strict convexity of E is then nothing other than a formulation of the principle of thermodynamic stability, where the relevant phase variables are the density ϱ, the momentum $\mathbf{m}$, and the total entropy S, cf. Bechtel, Rooney, and Forrest [4].

We consider both the full Euler system and its isentropic variant. In the former case, we show that any *stable* approximation either converges pointwise or its limit is not a weak solution of the Euler system. The proof is based mainly on the fact that the total energy is a conserved quantity for the limit system. The isentropic case is more delicate, as the energy conservation is in general violated by the weak solutions. Here, we consider *consistent* approximation and show that the energy defect, expressed through the asymptotic limit of the Bregman distance is intimately related to *turbulent defect measure* in the momentum equation. In fact, the defect in the momentum equation directly controls the defect in the energy. (The converse, meaning the defect in the energy controls the defect in the momentum equation, is also true and indispensable but not of direct use in the present setting). Furthermore, the turbulent defect measure $\mathbb{D}(t)$ is for a.e. time given by a (symmetric) positive semidefinite matrix–valued finite Borel measure on the physical space $\Omega\subset R^d$ in the sense that

$$\mathbb{D}(t):(\xi\otimes\xi)\text{ is a non–negative finite measure on }\Omega\text{ for any }\xi\in R^d,$$

and it can be identified along with a system of differential equations it obeys. In particular, we show below that the problem of convergence towards a weak solution reduces to solving a system of differential equations

$$\mathrm{div}_x\mathbb{D}(t)=0. \tag{1.2}$$

The paper is organized as follows. In Section 2, we recall the concept of weak solution for both the complete Euler system and its isentropic variant. We introduce the notion of stable and consistent approximations and state the main results. In Section 3, we study convergence of stable approximations to the full Euler system.

Section 4 is devoted to the same problem for the isentropic Euler system. Possible extensions of the results are discussed in Section 5.

2 Approximate solutions to the Euler system, main results

The complete Euler system governing the time evolution of the density $\varrho = \varrho(t, x)$, the momentum $\mathbf{m} = \mathbf{m}(t, x)$, and the energy $E = E(t, x)$ of a compressible perfect fluid reads:

$$\begin{aligned} \partial_t \varrho + \mathrm{div}_x \mathbf{m} &= 0, \\ \partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p &= 0, \\ \partial_t E + \mathrm{div}_x \left[(E + p) \frac{\mathbf{m}}{\varrho} \right] &= 0. \end{aligned} \tag{2.1}$$

We suppose the fluid is confined to a domain $\Omega \subset R^d$ with impermeable boundary,

$$\mathbf{m} \cdot \mathbf{n}|_{\partial \Omega} = 0. \tag{2.2}$$

Mostly we deal with the *admissible* weak solutions satisfying the Euler system (2.1) in the sense of distributions, together with the (renormalized) entropy inequality

$$\partial_t (\varrho Z(s)) + \mathrm{div}_x \left(Z(s) \mathbf{m} \right) \geq 0 \tag{2.3}$$

for any $Z \in BC(R)$, $Z' \geq 0$, cf. e.g. Chen and Frid [8]. Here p is the pressure and s is the entropy related to the internal energy e through Gibbs' equation

$$\vartheta Ds = De + pD \left(\frac{1}{\varrho} \right), \ \text{where } \vartheta \text{ is the absolute temperature.} \tag{2.4}$$

Introducing the total entropy $S = \varrho s$, we write all thermodynamic functions in terms of the basic *phase variables* $[\varrho, \mathbf{m}, S]$,

$$E = \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + \varrho e(\varrho, S), \ \ p = p(\varrho, S).$$

The cornerstone of the forthcoming analysis is the *thermodynamic stability hypothesis*:

$$\text{The total energy } [\varrho, \mathbf{m}, S] \in R^{d+2} \mapsto E(\varrho, \mathbf{m}, S) \equiv \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + \varrho e(\varrho, S) \in [0, \infty]$$

is a strictly convex l.s.c. function, where we set

$$E(\varrho, \mathbf{m}, S) = \infty \text{ whenever } \varrho < 0, \ \ E(0, \mathbf{m}, S) = \lim_{\varrho \to 0+} E(\varrho, \mathbf{m}, S), \tag{2.5}$$

cf. Bechtel, Rooney, Forrest [4]. To avoid further technicalities, we suppose the polytropic relation between the pressure and the internal energy

$$p = (\gamma - 1)\varrho e,\ \gamma > 1,\ \text{and set } e = c_v \vartheta,\ c_v = \frac{1}{\gamma - 1}.$$

Accordingly, the total energy takes the form

$$E(\varrho, \mathbf{m}, S) = \begin{cases} \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + \varrho^\gamma \exp\left(\frac{S}{c_v \varrho}\right) \text{ if } \varrho > 0, \\ 0 \text{ for } \varrho = 0,\ \mathbf{m} = 0,\ S \leq 0, \\ \infty,\ \text{otherwise} \end{cases} \tag{2.6}$$

for which the desired convexity has been verified in [5].

Remark 2.1 In what follows, we consider two particular settings, namely, the complete Euler system and the isentropic Euler system. We study the convergence of stable approximations of the complete Euler system and consistent approximations for the isentropic one. For both systems we are interested in fluids confined in a domain $\Omega \subset R^d$. In addition, it turns out that for the isentropic Euler system a much stronger result can be obtained on the full space, while the counterpart on a bounded domain requires additional assumption concerning the behavior close to the boundary, cf. Theorem 2.7 and Theorem 5.1. For the complete Euler system this additional assumption is not necessary but on the other hand we require a stronger assumption on the approximate initial data, cf. Theorem 2.6.

2.1 Weak solutions to the complete Euler system

Definition 2.2 *(Admissible weak solution to complete Euler system)* Let $\Omega \subset R^d$, $d = 1, 2, 3$, be a domain with Lipschitz boundary.

We say that $[\varrho, \mathbf{m}, S]$ is an *admissible weak solution* to the Euler system (2.1)–(2.3) in $(0, T) \times \Omega$ with the initial data $[\varrho_0, \mathbf{m}_0, S_0]$, if

- $$\varrho \geq 0 \text{ a.a. in } (0, T) \times \Omega,\ S = 0 \text{ a.a. in the set } \{\varrho = 0\};$$
- $$\left[\int_\Omega \varrho\varphi \ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi\right] \ \mathrm{d}x \, \mathrm{d}t,\ \varrho(0, \cdot) = \varrho_0, \tag{2.7}$$
- for any $0 \leq \tau < T$, $\varphi \in C^1_c([0, T) \times \overline{\Omega})$;

$$\left[\int_\Omega \mathbf{m} \cdot \boldsymbol{\varphi} \ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \mathbf{1}_{\varrho > 0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho, S) \mathrm{div}_x \boldsymbol{\varphi}\right] \ \mathrm{d}x \, \mathrm{d}t, \tag{2.8}$$
$$\mathbf{m}(0, \cdot) = \mathbf{m}_0,$$

- for any $0 \leq \tau < T$, $\boldsymbol{\varphi} \in C^1_c([0,T) \times \overline{\Omega})$, $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$;

$$\begin{aligned}
&\left[\int_\Omega E(\varrho, \mathbf{m}, S) \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} \\
&= \int_0^\tau \int_\Omega \left[E(\varrho, \mathbf{m}, S)\partial_t \varphi + \mathbf{1}_{\varrho>0} \left[(E(\varrho, \mathbf{m}, S) + p(\varrho, S)) \frac{\mathbf{m}}{\varrho} \right] \cdot \nabla_x \varphi \right] \ \mathrm{d}x \, \mathrm{d}t, \\
&E(\varrho, \mathbf{m}, S)(0, \cdot) = E(\varrho_0, \mathbf{m}_0, S_0),
\end{aligned} \tag{2.9}$$

  for any $0 \leq \tau < T$, $\varphi \in C^1_c([0,T) \times \overline{\Omega})$;

- $$\begin{aligned}
&\left[\int_\Omega \varrho Z\left(\frac{S}{\varrho}\right) \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} \geq \int_0^\tau \int_\Omega \left[\varrho Z\left(\frac{S}{\varrho}\right) \partial_t \varphi + Z\left(\frac{S}{\varrho}\right) \mathbf{m} \cdot \nabla_x \varphi \right] \ \mathrm{d}x \, \mathrm{d}t, \\
&\varrho Z\left(\frac{S}{\varrho}\right)(0, \cdot) = \varrho_0 Z\left(\frac{S_0}{\varrho_0}\right),
\end{aligned} \tag{2.10}$$

  for a.a. $0 \leq \tau < T$, and any $\varphi \in C^1_c([0,T) \times \overline{\Omega})$, $\varphi \geq 0$, and $Z \in BC(R) \cap C^1(R)$, $Z' \geq 0$.

In Definition 2.2, we tacitly assume that all quantities under integrals are at least locally integrable in $[0, T) \times \overline{\Omega}$.

2.2 Weak solutions to the isentropic Euler system

The *isentropic* Euler system is formally obtained from (2.1) by requiring the entropy $s = \overline{s}$ to be constant. The total energy given by (2.6) simplifies to

$$E = E(\varrho, \mathbf{m}) = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho), \ \ P(\varrho) \equiv \frac{a}{\gamma - 1}\varrho^\gamma, \ \ p = p(\varrho) = a\varrho^\gamma, \ \ a > 0. \tag{2.11}$$

We consider the isentropic Euler system on the whole space R^d, with the far field boundary conditions

$$\varrho \to \varrho_\infty \geq 0, \ \ \mathbf{m} \to \mathbf{m}_\infty = \varrho_\infty \mathbf{u}_\infty \text{ as } |x| \to \infty, \tag{2.12}$$

where ϱ_∞ and $\mathbf{u}_\infty$ are give constant fields. Consequently, it is more convenient to replace E by the relative energy

$$\begin{aligned}
&E\left(\varrho, \mathbf{m} \,\middle|\, \varrho_\infty, \mathbf{m}_\infty\right) \\
&= \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} - \mathbf{m} \cdot \mathbf{u}_\infty + \frac{1}{2}\varrho|\mathbf{u}_\infty|^2 + P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty) \\
&= \frac{1}{2}\varrho\left|\frac{\mathbf{m}}{\varrho} - \mathbf{u}_\infty\right|^2 + P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty).
\end{aligned}$$

As pointed out in the introductory part, the relative energy is nothing other than the Bregman divergence associated to the convex function E, cf. (1.1).

Definition 2.3 *(Weak solution to isentropic Euler system)* We say that $[\varrho, \mathbf{m}]$ is a *weak solution* to the Euler system in $(0, T) \times R^d$, with the initial data $[\varrho_0, \mathbf{m}_0]$ and the far field conditions (2.12), if

- $$\varrho \geq 0 \text{ a.a. in } (0, T) \times R^d;$$
- $$\left[\int_{R^d} \varrho \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{R^d} \left[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \right] \ \mathrm{d}x \, \mathrm{d}t, \ \ \varrho(0, \cdot) = \varrho_0, \tag{2.13}$$

  for any $0 \leq \tau < T$, $\varphi \in C^1_c([0, T) \times R^d)$;
- $$\left[\int_{R^d} \mathbf{m} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{R^d} \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \mathbf{1}_{\varrho > 0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x \, \mathrm{d}t, \tag{2.14}$$
  $$\mathbf{m}(0, \cdot) = \mathbf{m}_0,$$

  for any $0 \leq \tau < T$, $\boldsymbol{\varphi} \in C^1_c([0, T) \times R^d, R^d)$.

We say that a weak solution is *admissible*, if, in addition, the energy inequality

$$\int_{R^d} E \left(\varrho, \mathbf{m} \, \Big| \, \varrho_\infty, \mathbf{m}_\infty \right) (\tau, \cdot) \ \mathrm{d}x \leq \int_{R^d} E \left(\varrho_0, \mathbf{m}_0 \, \Big| \, \varrho_\infty, \mathbf{m}_\infty \right) \ \mathrm{d}x \tag{2.15}$$

holds for any $0 \leq \tau < T$.

Note that the total energy balance (2.9) that is an *integral part* of the weak formulation for the complete Euler system has been replaced by the integrated energy inequality (2.15) that plays the role of *admissibility condition* similar to the entropy inequality (2.10). In (2.15), we tacitly assume that the initial (relative) energy is finite, meaning that the initial data satisfy the far field conditions (2.12).

2.3 Stable and consistent approximations

The following two definitions are motivated by the terminology used in the *numerical analysis*.

Definition 2.4 *(Stable approximation of the full Euler system)* We say that a sequence

$$\{\varrho_n, \mathbf{m}_n, S_n\}_{n=1}^\infty$$

is a *stable approximation* of the full Euler system in $(0, T) \times \Omega$, with the initial data $[\varrho_0, \mathbf{m}_0, S_0]$, if:

$$\begin{aligned} \varrho_n \geq 0, \ \operatorname*{ess\,sup}_{\tau \in (0,T)} \int_\Omega \varrho_n(\tau, \cdot) \ \mathrm{d}x \leq M, \\ \operatorname*{ess\,inf}_{\tau \in (0,T)} \int_\Omega S_n(\tau, \cdot) \ \mathrm{d}x \geq \underline{S} \end{aligned} \tag{2.16}$$

uniformly for $n \to \infty$;

$$\operatorname*{ess\ sup}_{\tau \in (0,T)} \int_{\Omega} E(\varrho_n, \mathbf{m}_n, S_n) \ \mathrm{d}x \leq \int_{\Omega} E(\varrho_0, \mathbf{m}_0, S_0) \ \mathrm{d}x + e_n \text{ for all } n = 1, 2, \ldots \tag{2.17}$$

where $e_n \to 0$ as $n \to \infty$.

Note that both (2.16) and (2.17) obviously hold for any admissible weak solution in the sense of Definition 2.2.

Next, we introduce the concept of consistent approximation of the isentropic Euler system in $(0, T) \times R^d$ supplemented with the far field conditions (2.12).

Definition 2.5 *(Consistent approximation of isentropic Euler system)* We say that a sequence $\{\varrho_n, \mathbf{m}_n\}_{n=1}^{\infty}$ is a *consistent approximation* of the isentropic Euler system in $(0, T) \times R^d$, with the far field conditions (2.12) if:

- $$\varrho_n \geq 0 \text{ a.a. in } (0, T) \times R^d;$$
- $$\int_0^{\tau} \int_{R^d} \left[\varrho_n \partial_t \varphi + \mathbf{m}_n \cdot \nabla_x \varphi \right] \ \mathrm{d}x = e_n^1(\tau, \varphi) \tag{2.18}$$

  for any $\varphi \in C_c^{\infty}((0, T) \times R^d)$, $0 \leq \tau \leq T$;
- $$\int_0^{\tau} \int_{R^d} \left[\mathbf{m}_n \cdot \partial_t \boldsymbol{\varphi} + \mathbf{1}_{\varrho_n > 0} \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \boldsymbol{\varphi} + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x \, \mathrm{d}t = e_n^2(\tau, \boldsymbol{\varphi}) \tag{2.19}$$

  for any $\boldsymbol{\varphi} \in C_c^{\infty}((0, T) \times R^d, R^d)$, $0 \leq \tau \leq T$;
- $$\int_{R^d} E\left(\varrho_n, \mathbf{m}_n \ \middle| \ \varrho_{\infty}, \mathbf{m}_{\infty} \right) (\tau, \cdot) \ \mathrm{d}x \leq c \tag{2.20}$$

  uniformly for $0 \leq \tau \leq T$, $n = 1, 2, \ldots$;
- $$e_n^1(\tau, \varphi) \to 0, \ e_n^2(\tau, \boldsymbol{\varphi}) \to 0 \text{ as } n \to \infty \tag{2.21}$$

  for any fixed $0 \leq \tau \leq T$, $\varphi \in C_c^{\infty}((0, T) \times R^d)$, $\boldsymbol{\varphi} \in C_c^{\infty}((0, T) \times R^d; R^d)$.

Note carefully the difference between stable and consistent approximation. Stable approximation only satisfies the relevant *a priori* bounds and approaches the energy of the initial data in the asymptotic limit. Consistent approximation satisfies the weak formulation of the field equations modulo a small error vanishing in the asymptotic limit.

2.4 Main results

We start by the result concerning stable approximation to the complete Euler system. Recall that the only uniform bounds available result from the hypothesis (2.16), and the energy inequality (2.17). In particular, as we shall see below, the uniform bounds (2.16), (2.17) guarantee only L^1-integrability of the phase variables $(\varrho_n, \mathbf{m}_n, S_n)$ with respect to the x-variable. Accordingly, we consider the concept of *biting limit* in the sense of Ball and Murat [2] to describe the asymptotic behavior of a stable approximation to the complete Euler system. The result reads as follows.

Theorem 2.6 (Asymptotic limit of stable approximation) *Let $\Omega \subset R^d$ be a bounded Lipschitz domain. Let $\{\varrho_n, \mathbf{m}_n, S_n\}_{n=1}^{\infty}$ be a stable approximation of the complete Euler system in the sense of Definition 2.4, with the initial data*

$$\varrho_0 > 0, \ \mathbf{m}_0, \ S_0 \geq \varrho_0 \underline{s}, \ \textit{where } \underline{s} \in R. \tag{2.22}$$

Then there exists a subsequence (not relabeled for simplicity) enjoying the following properties:

$$\operatorname*{ess\ sup}_{\tau\in(0,T)} \left[\|\varrho_n(\tau,\cdot)\|_{L^1(\Omega)} + \|\mathbf{m}_n(\tau,\cdot)\|_{L^1(\Omega;R^d)} + \|S_n(\tau,\cdot)\|_{L^1(\Omega)} \right] \leq c; \tag{2.23}$$

the sequence $\{\varrho_n, \mathbf{m}_n, S_n\}_{n=1}^{\infty}$ admits a biting limit $[\varrho, \mathbf{m}, S]$,

$$[\varrho, \mathbf{m}, S] \in L^\infty(0,T; L^1(\Omega; R^{d+2})).$$

If, moreover, $[\varrho, \mathbf{m}, S]$ is an admissible weak solution to the complete Euler system specified in Definition 2.2, then

$$\varrho_n \to \varrho, \ \mathbf{m}_n \to \mathbf{m}, \ S_n \to S \textit{ a.a. in } (0,T) \times \Omega.$$

Our second result concerns the asymptotic behavior of a consistent approximation to the isentropic Euler system on R^d. The corresponding result on a bounded domain $\Omega \subset R^d$ is formulated in Theorem 5.1.

Theorem 2.7 (Asymptotic limit of consistent approximation) *Let $\{\varrho_n, \mathbf{m}_n\}_{n=1}^{\infty}$ be a consistent approximation of the isentropic Euler system in $(0,T) \times R^d$ in the sense of Definition 2.5.*

Then there exists a subsequence (not relabeled for simplicity) enjoying the following properties:

$$\begin{aligned} (\varrho_n - \varrho) &\to 0 \textit{ weakly-(*) in } L^\infty(0,T; L^\gamma + L^2(R^d)), \\ (\mathbf{m}_n - \mathbf{m}) &\to 0 \textit{ weakly-(*) in } L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}} + L^2(R^d; R^d)), \end{aligned} \tag{2.24}$$

where

$$\operatorname*{ess\ sup}_{\tau\in(0,T)} E\left(\varrho, \mathbf{m} \,\middle|\, \varrho_\infty, \mathbf{m}_\infty\right) < \infty.$$

If, moreover, $[\varrho, \mathbf{m}]$ *is a weak solution to the isentropic Euler system specified in Definition 2.3, then*

$$\int_0^T \int_K E\left(\varrho_n, \mathbf{m}_n \Big| \varrho, \mathbf{m}\right) \, \mathrm{d}x \, \mathrm{d}t \to 0, \text{ for any compact } K \subset R^d,$$

in particular,

$$\varrho_n \to \varrho \text{ in } L^q(0, T; L^{\gamma}_{\mathrm{loc}}(R^d)),$$
$$\mathbf{m}_n \to \mathbf{m} \text{ in } L^q(0, T; L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(R^d; R^d)),$$

for any $1 \le q < \infty$. *Thus, for a suitable subsequence,*

$$\varrho_n \to \varrho, \ \mathbf{m}_n \to \mathbf{m} \text{ a.a. in } (0, T) \times R^d.$$

We point out that the results stated in Theorems 2.6, 2.7 require extracting a suitable subsequence. In both cases, the convergence is necessarily strong (pointwise a.a.) as soon as the limit is an admissible weak solution to the system.

3 Convergence of stable approximations to the full Euler system

Our goal is to prove Theorem 2.6. We start by establishing uniform bounds for the stable approximation.

3.1 Uniform bounds

We establish the uniform bounds claimed in (2.23). To see this, we choose an arbitrary point $[\tilde{\varrho}, 0, \widetilde{S}] \in R^{d+2}$, $\tilde{\varrho} > 0$, and consider the quantity

$$0 \le E(\varrho_n, \mathbf{m}_n, S_n) - \frac{\partial E(\tilde{\varrho}, 0, \widetilde{S})}{\partial \varrho}(\varrho_n - \tilde{\varrho}) - \frac{\partial E(\tilde{\varrho}, 0, \widetilde{S})}{\partial \mathbf{m}} \cdot (\mathbf{m}_n - \tilde{\mathbf{m}}) - \frac{\partial E(\tilde{\varrho}, 0, \widetilde{S})}{\partial S}(S_n - \widetilde{S})$$
$$- E(\tilde{\varrho}, 0, \widetilde{S}) = \frac{1}{2}\frac{|\mathbf{m}_n|^2}{\varrho_n} + \varrho_n e(\varrho_n, S_n) - \frac{\partial(\varrho e)(\tilde{\varrho}, \widetilde{S})}{\partial \varrho}(\varrho - \tilde{\varrho}) - \frac{\partial(\varrho e)(\tilde{\varrho}, \widetilde{S})}{\partial S}(S - \widetilde{S}) - \tilde{\varrho} e(\tilde{\varrho}, \widetilde{S}).$$

Seeing that $\frac{\partial E}{\partial S} = \vartheta > 0$, we conclude

$$\int_\Omega \Big[E(\varrho_n, \mathbf{m}_n, S_n) - \frac{\partial E(\tilde{\varrho}, 0, \widetilde{S})}{\partial \varrho}(\varrho_n - \tilde{\varrho}) - \frac{\partial E(\tilde{\varrho}, 0, \widetilde{S})}{\partial \mathbf{m}} \cdot (\mathbf{m}_n - \tilde{\mathbf{m}}) - \frac{\partial E(\tilde{\varrho}, 0, \widetilde{S})}{\partial S}(S_n - \widetilde{S})$$
$$- E(\tilde{\varrho}, 0, \widetilde{S}) \Big] \mathrm{d}x \le c(\tilde{\varrho}, \widetilde{S}) \left(1 + \int_\Omega E(\varrho_n, \mathbf{m}_n, S_n) \, \mathrm{d}x + \int_\Omega \varrho_n \, \mathrm{d}x - \int_\Omega S_n \, \mathrm{d}x \right)$$
$$\le c(\tilde{\varrho}, \widetilde{S}) \left(1 + \int_\Omega E(\varrho_0, \mathbf{m}_0, S_0) \, \mathrm{d}x + M - \underline{S} + e_n \right)$$

As E is strictly convex at $[\tilde{\varrho}, 0, \widetilde{S}]$, we have

$$E(\varrho_n, \mathbf{m}_n, S_n) - \frac{\partial E(\tilde{\varrho}, 0, \widetilde{S})}{\partial \varrho}(\varrho_n - \tilde{\varrho}) - \frac{\partial E(\tilde{\varrho}, 0, \widetilde{S})}{\partial \mathbf{m}} \cdot (\mathbf{m}_n - \tilde{\mathbf{m}}) - \frac{\partial E(\tilde{\varrho}, 0, \widetilde{S})}{\partial S}(S_n - \widetilde{S})$$
$$- E(\tilde{\varrho}, 0, \widetilde{S}) \gtrsim |\varrho_n - \tilde{\varrho}| + |\mathbf{m}_n| + |S_n - \widetilde{S}|$$

as soon as

$$|\varrho_n - \tilde{\varrho}| + |\mathbf{m}_n| + |S_n - \widetilde{S}| \geq 1.$$

Since Ω is bounded, the estimates (2.23) follow.

3.2 Strong convergence

We shall systematically extract various subsequence keeping the labeling of the original sequence. In view of (2.17), (2.23), the sequence $\{\varrho_n, \mathbf{m}_n, S_n\}_{n=1}^{\infty}$ generates a Young measure

$$\mathcal{V} \in L^{\infty}_{\text{weak}-(*)}((0, T) \times \Omega; \mathcal{P}(R^{d+2})), \ \ R^{d+2} = \left\{(\tilde{\varrho}, \tilde{\mathbf{m}}, \widetilde{S}) \in R^{d+2}\right\}.$$

Moreover, $\mathcal{V}_{t,x}$ possesses finite first moments for a.a. (t, x) and we can set

$$\varrho(t, x) = \langle \mathcal{V}_{t,x}; \tilde{\varrho} \rangle, \ \ \mathbf{m}(t, x) = \langle \mathcal{V}_{t,x}; \tilde{\mathbf{m}} \rangle, \ \ S(t, x) = \langle \mathcal{V}_{t,x}; \widetilde{S} \rangle.$$

As observed by Ball and Murat [2], the trio $[\varrho, \mathbf{m}, S]$ corresponds to the biting limit of the sequence $\{\varrho_n, \mathbf{m}_n, S_n\}_{n=1}^{\infty}$. Finally, in view of the energy bound (2.17), we have

$$E(\varrho_n, \mathbf{m}_n, S_n) \to \overline{E(\varrho, \mathbf{m}, S)} \text{ weakly-}(*) \text{ in } L^{\infty}_{w^*}(0, T; \mathcal{M}^+(\overline{\Omega})),$$

where the symbol $\mathcal{M}^+$ denotes the set of non–negative Borel measures. In view of the hypothesis (2.17),

$$\int_{\Omega} E(\varrho_0, \mathbf{m}_0, S_0) \ \mathrm{d}x \geq \int_{\overline{\Omega}} \mathrm{d}\overline{E(\varrho, \mathbf{m}, S)}(\tau, \cdot), \tag{3.1}$$

and

$$\overline{E(\varrho, \mathbf{m}, S)}(\tau, \cdot) \geq \langle \mathcal{V}_{\tau,\cdot}; E(\tilde{\varrho}, \tilde{\mathbf{m}}, \widetilde{S}) \rangle \geq E(\varrho, \mathbf{m}, S)(\tau, \cdot) \text{ for a.a. } \tau \in (0, T) \tag{3.2}$$

in the sense of non–negative measures on $\overline{\Omega}$. Note that the first inequality in (3.2) follows from lower semi–continuity of the energy, while the second one follows from its convexity, see e.g. [23, Section 3.2]. In particular, the biting limit $[\varrho, \mathbf{m}, S]$ belongs to the class

$$[\varrho, \mathbf{m}, S] \in L^{\infty}(0, T; L^1(\Omega; R^{d+2})).$$

Finally, suppose that $[\varrho, \mathbf{m}, S]$ is an admissible weak solution of the Euler system in the sense of Definition 2.2. In particular, the total energy balance (2.9) holds; whence

$$\int_\Omega E(\varrho, \mathbf{m}, S)(\tau, \cdot) \ \mathrm{d}x = \int_\Omega E(\varrho_0, \mathbf{m}_0, S_0) \ \mathrm{d}x \text{ for any } 0 \leq \tau \leq T. \tag{3.3}$$

Moreover, as the entropy equation (2.10) is satisfied in the renormalized sense, we can deduce from the hypothesis (2.22) the entropy minimum principle,

$$S(t,x) \geq \varrho(t,x)\underline{s} \text{ for a.a. } (t,x), \tag{3.4}$$

see [7].

Going back to (3.2) we conclude

$$\begin{aligned} \int_\Omega E(\varrho_0, \mathbf{m}_0, S_0) \ \mathrm{d}x &= \int_{\overline{\Omega}} \mathrm{d}\overline{E(\varrho, \mathbf{m}, S)}(\tau, \cdot), \\ \overline{E(\varrho, \mathbf{m}, S)} &= \langle \mathcal{V}; E(\tilde{\varrho}, \tilde{\mathbf{m}}, \widetilde{S}) \rangle = E(\varrho, \mathbf{m}, S). \end{aligned} \tag{3.5}$$

The second equality, specifically,

$$\overline{E(\varrho, \mathbf{m}, S)} = \langle \mathcal{V}; E(\tilde{\varrho}, \tilde{\mathbf{m}}, \widetilde{S}) \rangle$$

means that the concentration defect associated to the sequence $\{E(\varrho_n, \mathbf{m}_n, S_n)\}_{n=1}^\infty$ vanishes, specifically,

$$E(\varrho_n, \mathbf{m}_n, S_n) \to \langle \mathcal{V}; E(\tilde{\varrho}, \tilde{\mathbf{m}}, \widetilde{S}) \rangle = E(\varrho, \mathbf{m}, S) \text{ weakly in } L^1((0,T) \times \Omega),$$

cf. [23].

The third equality, together with (3.4), implies the desired pointwise convergence. To see this, we need the following result that may be of independent interest.

Lemma 3.1 (Sharp form of Jensen's inequality) *Suppose that $E : R^m \to [0, \infty]$ is an l.s.c. convex function satisfying:*

- *E is strictly convex on its domain of positivity, meaning for any $y_1, y_2 \in R^m$ such that $0 < E(y_1) < \infty$, $E(y_2) < \infty$, $y_1 \neq y_2$, we have*

$$E\left(\frac{y_1 + y_2}{2}\right) < \frac{1}{2}E(y_1) + \frac{1}{2}E(y_2).$$

- *If $y \in \partial \mathrm{Dom}[E]$, then either $E(y) = \infty$ or $E(y) = 0$, in other words,*

$$E(y) = 0 \textit{ whenever } y \in \mathrm{Dom}[E] \cap \partial\mathrm{Dom}[E]. \tag{3.6}$$

Let $\nu \in \mathcal{P}[R^m]$ be a (Borel) probability measure with finite first moment satisfying

$$E(\langle \nu; \widetilde{y} \rangle) = \langle \nu; E(\widetilde{y}) \rangle < \infty. \tag{3.7}$$

Then **(i)** *either*

$$\nu = \delta_Y, \ Y = \langle \nu; \widetilde{y} \rangle \in \mathrm{Dom}[E], \ E(Y) > 0,$$

(ii) *or*

$$\mathrm{supp}[\nu] \subset \left\{ y \in R^m \ \middle| \ E(y) = 0 \right\}.$$

Proof First observe that, obviously, $\langle \nu; \widetilde{u} \rangle \in \mathrm{Dom}[E]$, and, by virtue of (3.7) and positivity of E,

$$\nu \left\{ R^m \setminus \mathrm{Dom}[E] \right\} = 0.$$

(i) Suppose first that $Y \equiv \langle \nu; \widetilde{y} \rangle \in \mathrm{int}[\mathrm{Dom}[E]]$, $E(Y) > 0$. Then there exists

$$\Lambda \in \partial E(Y)$$

such that

$$E(y) \geq E(Y) + \Lambda \cdot (y - Y) \text{ for any } y \in R^m.$$

As E is strictly convex in $\mathrm{Dom}[E] \cap \{E > 0\}$, however, we claim that the above inequality must be sharp:

$$E(y) - E(Y) - \Lambda \cdot (y - Y) > 0 \text{ for all } y \in R^d, \ y \neq Y.$$

Now it follows from (3.7) that

$$\left\langle \nu; E(\widetilde{y}) - E(Y) - \Lambda \cdot (\widetilde{y} - Y) \right\rangle = 0$$

which yields the desired conclusion **(i)**.

(ii) Suppose that $Y = \langle \nu; \widetilde{y} \rangle \in \mathrm{Dom}[E] \cap \partial \mathrm{Dom}[E]$ or $E(Y) = 0$. In accordance with the hypothesis (3.6), we have in both cases

$$E(Y) = 0.$$

Consequently, we get from (3.7),

$$\langle \nu; E(\widetilde{y}) \rangle = 0$$

which implies that ν is supported by zero points of E as $E \geq 0$ which is the alternative **(ii)**. □

In accordance with (3.5),

$$\langle \mathcal{V}; E(\tilde{\varrho}, \tilde{\mathbf{m}}, \widetilde{S})\rangle(t,x) = E(\varrho, \mathbf{m}, S)(t,x) \text{ for a.a. } (t,x).$$

Clearly, E satisfies the hypotheses of Lemma 3.1; whence either $\mathcal{V}_{t,x}$ is a Dirac mass, specifically,

$$\mathcal{V}_{t,x} = \delta_{\varrho(t,x),\mathbf{m}(t,x),S(t,x)}, \tag{3.8}$$

or

$$\text{supp}[\mathcal{V}_{t,x}] \subset \left\{ \tilde{\varrho} = 0,\ \tilde{\mathbf{m}} = 0,\ \widetilde{S} \leq 0 \right\},$$

which, combined with (3.4), yields again (3.8). Indeed (3.4) means that the barycenter of $\mathcal{V}_{t,x}$ is located above the line $\widetilde{S} = \tilde{\varrho}\underline{s}$. As the Young measure is a Dirac mass, we conclude the sequence $\{\varrho_n, \mathbf{m}_n, S_n\}_{n=1}^{\infty}$ converges in measure; whence a suitable subsequence converges a.a. We have proved Theorem 2.6.

4 Convergence of consistent approximations to the isentropic Euler system

Our goal is to show Theorem 2.7. It turns out the proof is more complicated than that of Theorem 2.6 as the weak solution satisfies merely the field equations (2.13), (2.14).

4.1 Turbulent defect measures

In the following, we pass several times to suitable subsequences in the vanishing viscosity sequence without explicit relabeling. However, it is easy to see that it is enough to show the conclusion of Theorem 2.7 for a subsequence once the limit $[\varrho, \mathbf{m}]$ has been fixed.

It follows from the bounds imposed by the energy inequality (2.20) that we may suppose

$$\begin{aligned} (\varrho_n - \varrho_\infty) &\to (\varrho - \varrho_\infty) \text{ weakly-(*) in } L^\infty(0,T; (L^\gamma + L^2)(R^d)), \\ (\mathbf{m}_n - \mathbf{m}_\infty) &\to (\mathbf{m} - \mathbf{m}_\infty) \text{ weakly-(*) in } L^\infty(0,T; (L^{\frac{2\gamma}{\gamma+1}} + L^2)(R^d; R^d)). \end{aligned} \tag{4.1}$$

In particular, we get (2.24). Indeed, as the total energy $E(\varrho, \mathbf{m})$ is a strictly convex function of $(\varrho, \mathbf{m})$, it is easy to check that

$$\begin{aligned} E(\varrho, \mathbf{m}|\varrho_\infty, \mathbf{m}_\infty) &\gtrsim (\varrho - \varrho_\infty)^2 + (\mathbf{m} - \mathbf{m}_\infty)^2 \ \text{ for } \frac{1}{2}\varrho_\infty \leq \varrho \leq 2\varrho_\infty, \\ \frac{1}{2}|\mathbf{m}_\infty| \leq |\mathbf{m}| \leq 2|\mathbf{m}_\infty|, &\gtrsim 1 + \varrho^\gamma + \frac{|\mathbf{m}|^2}{\varrho} \text{ otherwise;} \end{aligned} \tag{4.2}$$

whence the desired bounds follow from the energy inequality (2.24).

4.1.1 Internal energy and pressure defect

Next, recall that the sequence

$$0 \leq P(\varrho_n) - P'(\varrho_\infty)(\varrho_n - \varrho_\infty) - P(\varrho_\infty),\ n = 1, 2, \dots,$$

is bounded in $L^\infty(0, T; L^1(R^d))$ uniformly in n by (2.24). It holds

$$L^\infty(0, T; L^1(R^d)) \subset L^\infty_{w^*}(0, T, \mathcal{M}(R^d)),$$

where the symbol $\mathcal{M}(R^d)$ denotes the set of finite Borel measures on R^d and $L^\infty_{w^*}(0, T; \mathcal{M}(R^d))$ stands for the space of weak-(*)-measurable mappings $\nu : [0, T] \to \mathcal{M}(R^d)$ such that

$$\operatorname*{ess\ sup}_{\tau \in [0,T]} \|\nu(\tau)\|_{\mathcal{M}(R^d)} < \infty.$$

In addition, $L^\infty_{w^*}(0, T, \mathcal{M}(R^d))$ is the dual of $L^1(0, T, C_0(R^d))$ hence passing to a suitable subsequence as the case may be, there is $\mathcal{P} \in L^\infty_{w^*}(0, T; \mathcal{M}(R^d))$ such that

$$P(\varrho_n) - P'(\varrho_\infty)(\varrho_n - \varrho_\infty) - P(\varrho_\infty) \to \mathcal{P} \text{ weakly-(*) in } L^\infty_{w^*}(0, T; \mathcal{M}(R^d)).$$

As the function P is convex and the approximate internal energies are non–negative, we deduce by weak lower semicontinuity that

$$\mathfrak{R}_e \equiv \mathcal{P} - \left[P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty) \right] \in L^\infty_{w^*}(0, T; \mathcal{M}^+(R^d)),$$

where $\mathcal{M}^+(R^d)$ denotes the set of non–negative finite Borel measures on R^d. This defines the internal energy defect measure $\mathfrak{R}_e$. It is important to note that

$$\int_0^T \int_{R^d} \psi(t)\varphi(x)\, \mathrm{d}\mathfrak{R}_e(t)\, \mathrm{d}t = \lim_{n\to\infty} \int_0^T \int_\Omega \psi(t)\varphi(x) \left(P(\varrho_n) - P(\varrho) \right)\ \mathrm{d}x\, \mathrm{d}t \quad \text{for any } \psi \in L^1(0, T),\ \varphi \in C_c(R^d), \tag{4.3}$$

which will be used later.

4.1.2 Viscosity defect

Writing

$$\mathbb{C}_n \equiv \mathbf{1}_{\varrho_n > 0} \left[\frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} - \mathbf{u}_\infty \otimes \mathbf{m}_n - \mathbf{m}_n \otimes \mathbf{u}_\infty + \varrho_n \mathbf{u}_\infty \otimes \mathbf{u}_\infty \right]$$

we obtain the existence of $\mathbb{C} \in L^\infty_{w^*}(0,T;\mathcal{M}^+(R^d;R^{d\times d}_{\rm sym}))$, where $\mathcal{M}^+(R^d;R^{d\times d}_{\rm sym})$ is the set of finite symmetric positive semidefinite matrix–valued (signed) Borel measures, such that

$$\mathbb{C}_n \to \mathbb{C} \text{ weakly-(*) in } L^\infty_{w^*}(0,T;\mathcal{M}^+(R^d;R^{d\times d}_{\rm sym})).$$

More specifically, each component $C_{i,j}$ is a finite signed measure on R^d, $C_{i,j} = C_{j,i}$, and

$$\mathbb{C}(t) : (\xi \otimes \xi) \in \mathcal{M}^+(R^d) \text{ for any } \xi \in R^d \text{ and a.a. } t \in (0,T). \tag{4.4}$$

The viscosity defect measure is then defined by

$$\mathfrak{R}_v \equiv \mathbb{C} - \mathbf{1}_{\varrho>0}\left[\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} - \mathbf{u}_\infty \otimes \mathbf{m} - \mathbf{m} \otimes \mathbf{u}_\infty + \varrho \mathbf{u}_\infty \otimes \mathbf{u}_\infty\right] \in L^\infty_{w^*}(0,T;\mathcal{M}(R^d;R^{d\times d}_{\rm sym})).$$

Now, a simple but crucial observation is that the $\mathfrak{R}_v$ is positive semidefinite. To see this, we compute

$$\begin{aligned}\mathfrak{R}_v : (\xi\otimes\xi) &= \lim_{n\to\infty} \mathbf{1}_{\varrho_n>0}\frac{\mathbf{m}_n\otimes\mathbf{m}_n}{\varrho_n} : (\xi\otimes\xi) - \mathbf{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} : (\xi\otimes\xi)\\ &= \lim_{n\to\infty} \frac{|\mathbf{m}_n\cdot\xi|^2}{\varrho_n} - \frac{|\mathbf{m}\cdot\xi|^2}{\varrho} \text{ in } \mathcal{D}'((0,T)\times B)\end{aligned}$$

for any bounded ball $B \subset R^d$; whence the desired conclusion follows from the weak lower semicontinuity of the convex function $[\varrho,\mathbf{m}] \mapsto \frac{|\mathbf{m}\cdot\xi|^2}{\varrho}$, $\xi \in R^d$. We conclude that

$$\mathfrak{R}_v \in L^\infty_{w^*}(0,T;\mathcal{M}^+(R^d;R^{d\times d}_{\rm sym})).$$

Finally, similarly to (4.3), we note that

$$\begin{aligned}&\int_0^T\int_{R^d}\psi(t)\boldsymbol{\varphi}(x) : {\rm d}\mathfrak{R}_v(t)\,{\rm d}t\\ &= \lim_{n\to\infty}\int_0^T\int_\Omega \psi(t)\boldsymbol{\varphi}(x) : \left(\mathbf{1}_{\varrho_n>0}\frac{\mathbf{m}_n\otimes\mathbf{m}_n}{\varrho_n} - \mathbf{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right)\,{\rm d}x\,{\rm d}t\\ &\text{for any } \psi \in L^1(0,T),\ \boldsymbol{\varphi} \in C_c(R^d;R^{d\times d})).\end{aligned} \tag{4.5}$$

4.1.3 Total defect

We introduce the *total defect measure*

$$\mathbb{D} \equiv \mathfrak{R}_v + (\gamma-1)\mathfrak{R}_e\mathbb{I} \in L^\infty_{w^*}(0,T;\mathcal{M}^+(R^d;R^{d\times d}_{\rm sym})), \tag{4.6}$$

which describes the defect in the momentum equation. Moreover, we get for the total energy

$$E(\varrho_n, \mathbf{m}_n | \varrho_\infty, \mathbf{m}_\infty) \to E(\varrho, \mathbf{m} | \varrho_\infty, \mathbf{m}_\infty) + \frac{1}{2}\text{trace}[\mathfrak{R}_v] + \mathfrak{R}_e \tag{4.7}$$

weakly-(*) in $L^\infty_{w^*}(0, T; \mathcal{M}^+(R^d; R^{d\times d}_{\text{sym}}))$. In other words, we have a precise relation of the defect in the momentum equation and the defect of the energy. Finally, we get from (4.7) that

$$\int_0^T \int_\Omega \psi(t)\varphi(x) \left(\frac{1}{2}\frac{|\mathbf{m}_n|^2}{\varrho_n} - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho_n) - P(\varrho) \right) \, \mathrm{d}x \, \mathrm{d}t$$
$$\to \int_0^T \psi(t)\varphi(x) \, \mathrm{d} \left(\frac{1}{2}\text{trace}[\mathfrak{R}_v(t)] + \mathfrak{R}_e(t) \right) \, \mathrm{d}t$$

for any $\psi \in L^1(0, T)$ and any $\varphi \in C_c(R^d)$.

4.1.4 Bounded domain

The above construction of the turbulent defect measure $\mathbb{D}$ as well as the proof of its properties can be carried out the same way on a bounded domain $\Omega \subset R^d$, while using the dualities

$$L^1(0, T; C(\overline{\Omega}))^* \cong L^\infty_{w^*}(0, T; \mathcal{M}(\overline{\Omega})) \text{ and } L^1(0, T; C_0(\overline{\Omega}; R^{d\times d}))^* \cong L^\infty_{w^*}(0, T; \mathcal{M}(\overline{\Omega}; R^{d\times d})),$$

respectively, where $\mathcal{M}(\overline{\Omega})$ is the set of bounded Borel measures on $\overline{\Omega}$ (and similarly for the matrix–valued case).

4.2 Asymptotic limit

Using (4.3), (4.5) we may perform the asymptotic limit in the momentum equation (2.19) obtaining

$$\int_0^T \int_\Omega \left[\partial_t \psi \mathbf{m} \cdot \boldsymbol{\varphi} + \psi \mathbf{1}_{\varrho > 0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + \psi p(\varrho) \text{div}_x \boldsymbol{\varphi} \right] \, \mathrm{d}x \, \mathrm{d}t$$
$$= - \int_0^T \psi \left[\nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}_v(t) + (\gamma - 1) \text{div}_x \boldsymbol{\varphi} \, \mathrm{d}\mathfrak{R}_e(t) \right] \mathrm{d}t \tag{4.8}$$
$$\text{for any } \psi \in C^1_c(0, T), \ \boldsymbol{\varphi} \in C^1_c(R^d; R^d).$$

Thus, if the limit is a weak solution of the Euler system, then the left hand side of (4.8) vanishes. Hence, in view of the definition of the total defect measure (4.6), we obtain

$$\int_{R^d} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathbb{D}(t) = 0 \text{ for any } \varphi \in C^1_c(R^d; R^d) \text{ for a.a. } t \in (0, T)$$

which is nothing else than (1.2).

4.2.1 Equation $\mathrm{div}_x \mathbb{D} = 0$ in R^d

The following result, which can be regarded as a version of Liouville's theorem, is crucial in the proof of Theorem 2.7.

Proposition 4.1 *Let* $\mathbb{D} \in \mathcal{M}^+(R^d; R^{d\times d}_{\mathrm{sym}})$ *satisfy*

$$\int_{R^d} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} = 0 \textit{ for any } \varphi \in C^1_c(R^d; R^d). \tag{4.9}$$

Then $\mathbb{D} \equiv 0$.

Remark 4.2 The assumption that the matrix $\mathbb{D}$ is positive semidefinite (or alternatively negative semidefinite, as a matter of fact), is absolutely essential. Indeed, De Lellis and Székelyhidi in their proof of the so-called oscillatory lemma in [16] showed the existence of infinitely many smooth fields $\mathbb{D} \in C^\infty_c(R^d; R^{d\times d}_{\mathrm{sym}})$ satisfying $\mathrm{div}_x \mathbb{D} = 0$.

Proof of Proposition 4.1 The proof relies on the extension of (4.9) to all functions $\boldsymbol{\varphi} \in C^1(R^d; R^d)$ with $\nabla_x \boldsymbol{\varphi} \in L^\infty(R^d; R^{d\times d})$, which is possible since $\mathbb{D}$ is a finite measure. This then permits to test (4.9) by linear functions $\boldsymbol{\varphi}$ and the conclusion follows from the positive semidefinitness of $\mathbb{D}$.

To this end, let us consider a sequence of cut–off functions

$$\psi_n \in C^\infty_c(R^d),\ 0 \le \psi \le 1,\ \psi_n(x) = 1 \text{ for } |x| \le n,\ \psi_n(x) = 0 \text{ for } |x| \ge 2n,\ |\nabla_x \psi| \stackrel{<}{\sim} \frac{1}{n}$$

uniformly for $n \to \infty$.

For $\boldsymbol{\varphi} \in C^1(R^d; R^d)$, with $\nabla_x \boldsymbol{\varphi} \in L^\infty(R^d; R^{d\times d})$, we have

$$|\boldsymbol{\varphi}(x)| \stackrel{<}{\sim} (1+n) \text{ for all } x \in \mathrm{supp}\, \psi_n;$$

whence

$$\begin{aligned} 0 &= \int_{R^d} \nabla_x(\psi_n \boldsymbol{\varphi}) : \mathrm{d}\mathbb{D} = \int_{R^d} \psi_n \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} + \int_{R^d} (\nabla_x \psi_n) \otimes \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} \\ &= \int_{|x|\le n} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} + \int_{n<|x|<2n} \psi_n \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} + \int_{n<|x|<2n} (\nabla_x \psi_n) \otimes \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} \end{aligned}$$

Seeing that

$$|\psi_n \nabla_x \boldsymbol{\varphi}(x)| + |(\nabla_x \psi_n) \otimes \boldsymbol{\varphi}| \stackrel{<}{\sim} 1 \text{ whenever } n \le |x| \le 2n$$

we may use the fact that $\mathbb{D}$ is a finite (signed) measure together with Lebesgue's dominated convergence theorem to let $n \to \infty$ and conclude that

$$\int_{R^d} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} = 0 \text{ for any } \varphi \in C^1(R^d; R^d),\ \nabla_x \boldsymbol{\varphi} \in L^\infty(R^d; R^{d\times d}). \tag{4.10}$$

Finally, given a vector $\xi \in R^d$, we may use

$$\boldsymbol{\varphi}(x) = \xi(\xi \cdot x)$$

as a test function in (4.10) to obtain

$$\int_{R^d} (\xi \otimes \xi) : \mathrm{d}\mathbb{D} = 0 \text{ for any } \xi \in R^d.$$

As $\mathbb{D}$ is positive semidefinite in the sense of (4.4), i.e. $(\xi \otimes \xi) : \mathbb{D}$ is a non–negative finite measure on R^d, this yields $(\xi \otimes \xi) : \mathbb{D} = 0$ for any $\xi \in R^d$. Thus for any $g \in C_b(R^d)$, $g \geq 0$, and the matrix $\int_{R^d} g \,\mathrm{d}\mathbb{D}$ is positive semidefinite and we may infer

$$\int_{R^d} g \,\mathrm{d}D_{i,j} = 0 \text{ for any } i, j.$$

As g was arbitrary, this yields the desired conclusion $\mathbb{D} \equiv 0$. □

4.2.2 Equation $\mathrm{div}_x \mathbb{D} = 0$ in a bounded domain

A trivial example of a constant–valued matrix shows that Proposition 4.1 does not hold if R^d is replaced by a bounded domain Ω unless some extra restrictions are imposed. In addition to the hypotheses of Proposition 4.1, we shall assume that $\mathbb{D}$ vanishes sufficiently fast near the boundary $\partial\Omega$.

Proposition 4.3 *Let $\Omega \subset R^d$ be a bounded domain. Let $\mathbb{D} \in \mathcal{M}^+(\overline{\Omega}; R^{d\times d}_{\mathrm{sym}})$ satisfying*

$$\int_{R^d} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} = 0 \textit{ for any } \varphi \in C^1_c(\Omega; R^d), \tag{4.11}$$

and

$$\frac{1}{\delta} \int_{\{x\in\Omega;\, \mathrm{dist}[x,\partial\Omega]\leq\delta\}} \mathrm{d(trace)}[\mathbb{D}] \to 0 \textit{ as } \delta \to 0. \tag{4.12}$$

Then $\mathbb{D} \equiv 0$.

Proof Similarly to the proof of Proposition 4.1, it is enough to show that (4.11) can be extended to a suitable function $\varphi \in C^1(\overline{\Omega}; R^d)$, whose gradient is constant.

It is a routine matter, cf. e.g. Galdi [27], to construct a sequence of cut–off functions ψ_n enjoying the following properties:

$$\psi_n \in C^1_c(\Omega),\ 0 \leq \psi_n \leq 1,\ \psi_n(x) = 1 \text{ whenever } \mathrm{dist}[x, \partial\Omega] > \frac{1}{n},\ |\nabla_x \psi_n| \stackrel{<}{\sim} n.$$

Thus, plugging $\psi_n \boldsymbol{\varphi}$, $\boldsymbol{\varphi} \in C^1(\overline{\Omega}; R^d)$ in (4.11) we get

$$\begin{aligned}
0 &= \int_\Omega \nabla_x (\psi_n \boldsymbol{\varphi}) : \mathrm{d}\mathbb{D} = \int_\Omega \psi_n \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} + \int_\Omega (\nabla_x \psi_n) \otimes \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} \\
&= \int_{\mathrm{dist}[x,\partial\Omega] > \frac{1}{n}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} + \int_{\mathrm{dist}[x,\partial\Omega] \leq \frac{1}{n}} \psi_n \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} + \int_{\mathrm{dist}[x,\partial\Omega] \leq \frac{1}{n}} (\nabla_x \psi_n) \otimes \boldsymbol{\varphi} : \mathrm{d}\mathbb{D}
\end{aligned}$$

Now, we observe that

$$|\psi_n \nabla_x \boldsymbol{\varphi}(x)| + |(\nabla_x \psi_n) \otimes \boldsymbol{\varphi}(x)| \stackrel{<}{\sim} n \text{ whenever } \mathrm{dist}[x, \partial\Omega] \leq \frac{1}{n},$$

which due to (4.12) allows to pass to the limit as $n \to \infty$ in the second and the third term on the right hand side. The convergence of the first term follows from the fact that by (4.12) the defect vanishes on the boundary, i.e.

$$\int_{\partial\Omega} \mathrm{d}|\mathbb{D}| = 0,$$

and in the interior of Ω we have pointwise convergence of the corresponding integrand.

□

4.3 Strong convergence

Applying Proposition 4.1 in the situation of Theorem 2.7 we obtain that $\mathfrak{R}_v \equiv 0$ and $\mathfrak{R}_e \equiv 0$. In accordance with (4.7), this yields

$$E(\varrho_n, \mathbf{m}_n | \varrho_\infty, \mathbf{m}_\infty) \to E(\varrho, \mathbf{m} | \varrho_\infty, \mathbf{m}_\infty) \tag{4.13}$$

weakly-(*) in $L^\infty_{w^*}(0, T; \mathcal{M}^+(R^d))$. We show that this implies the strong convergence claimed in Theorem 2.7.

First, we recall that both kinetic and internal energy are convex functions of the density and the momentum so from (4.13) we obtain

$$\begin{aligned}
&\int_B \left[\frac{|\boldsymbol{m}_n|^2}{\varrho_n} - 2\mathbf{m}_n \cdot \mathbf{u}_\infty + \varrho_n |\mathbf{u}_\infty|^2 \right] \mathrm{d}x\, \mathrm{d}t \to \int_B \left[\frac{|\boldsymbol{m}|^2}{\varrho} - 2\mathbf{m} \cdot \mathbf{u}_\infty + \varrho |\mathbf{u}_\infty|^2 \right] \mathrm{d}x\, \mathrm{d}t, \\
&\int_B P(\varrho_n) - P'(\varrho_\infty)(\varrho_n - \varrho_\infty) - P(\varrho_\infty)\, \mathrm{d}x\, \mathrm{d}t \\
&\qquad \to \int_B P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty)\, \mathrm{d}x\, \mathrm{d}t,
\end{aligned} \tag{4.14}$$

for every compact set $B \subset [0, T] \times R^d$.

Accordingly, choosing $B = [0, T] \times K$ for a compact set $K \subset R^d$, we obtain the convergence of the norms of ϱ_n in $L^\gamma([0, T] \times K)$, hence the strong convergence

$$\varrho_n \to \varrho \ \text{ in } L^\gamma([0, T] \times K).$$

Let us now establish the strong convergence of the momenta on $[0, T] \times K$. To this end, we recall that by the energy bounds it holds (up to a subsequence)

$$\boldsymbol{h}_n \equiv \frac{\mathbf{m}_n}{\sqrt{\varrho_n}} \to \boldsymbol{h} \text{ weakly in } L^2([0, T] \times K; R^d)$$

for some $\boldsymbol{h} \in L^2([0, T] \times K; R^d)$, and by (4.1)

$$\mathbf{m}_n \to \mathbf{m} \text{ weakly in } (L^{\frac{2\gamma}{\gamma+1}})([0, T] \times K; R^d).$$

We shall show that

$$\boldsymbol{h} = 1_{\varrho>0} \frac{\mathbf{m}}{\sqrt{\varrho}} \text{ a.a. in } [0, T] \times K.$$

Combining the weak convergence of $\boldsymbol{h}_n$ with the strong convergence of ϱ_n and the weak convergence of $\mathbf{m}_n$ we obtain

$$\sqrt{\varrho_n}\boldsymbol{h}_n = \mathbf{m}_n \to \mathbf{m} = \sqrt{\varrho}\boldsymbol{h} \text{ weakly in } L^1([0, T] \times K; R^d);$$

whence it is enough to prove that $\boldsymbol{h} = 0$ whenever $\varrho = 0$. By weak lower semicontinuity of the L^2-norm together with (4.14), we obtain

$$\int_{\varrho<\delta} \mathbf{1}_K |\boldsymbol{h}|^2 \, dx \, dt \le \lim_{n\to\infty} \int_{\varrho<\delta} \mathbf{1}_K \frac{|\mathbf{m}_n|^2}{\varrho_n} \, dx \, dt = \int_{\varrho<\delta} \mathbf{1}_K \frac{|\mathbf{m}|^2}{\varrho} \, dx \, dt.$$

Now, it is enough to observe that in the limit $\delta \to 0$, the left hand side converges to

$$\int_{\varrho=0} \mathbf{1}_K |\boldsymbol{h}|^2 \, dx \, dt,$$

whereas the right hand side vanishes, since due to the integrability of the kinetic energy $\frac{|\mathbf{m}|^2}{\varrho}$ it holds that the set, where $\varrho = 0$ and $\mathbf{m} \neq 0$, is of zero Lebesgue measure. Thus $\boldsymbol{h} = 0$ whenever $\varrho = 0$.

To summarize, we have shown that

$$\frac{\boldsymbol{m}_n}{\sqrt{\varrho_n}} \to \mathbf{1}_{\varrho>0} \frac{\boldsymbol{m}}{\sqrt{\varrho}} \text{ weakly in } L^2([0, T] \times K; R^d)$$

and hence strongly due to (4.14), which implies the strong convergence

$$\boldsymbol{m}_n = \sqrt{\varrho_n}\frac{\boldsymbol{m}_n}{\sqrt{\varrho_n}} \to \boldsymbol{m} \text{ in } L^{\frac{2\gamma}{\gamma+1}}([0,T]\times K; R^d).$$

Finally, a tightness argument as for the density above implies the desired strong convergence

$$\mathbf{m}_n \to \mathbf{m} \text{ in } L^{\frac{2\gamma}{\gamma+1}}_{\rm loc}([0,T]\times R^d; R^d).$$

The convergence of the energies in L^1 is then a consequence of the strong convergence of $\frac{|\mathbf{m}_n|}{\sqrt{\varrho_n}}$ and ϱ_n together with (3.2) and Vitali's theorem. This completes the proof of Theorem 2.7.

5 Concluding remarks

We conclude the paper by a short discussion on possible extensions of Theorem 2.7. As indicated in Proposition 4.3, the conclusion of Theorem 2.7 remains valid on bounded Lipschitz domains provided some extra assumptions about the behavior of the consistent approximation near the boundary is assumed. The relevant result can be stated as follows.

Theorem 5.1 (Asymptotic limit of consistent approximation in bounded domains) *Let $\{\varrho_n, \mathbf{m}_n\}_{n=1}^\infty$ be a consistent approximation of the isentropic Euler system in $(0,T)\times\Omega$ in the sense of Definition 2.5, where $\Omega\subset R^d$ is a bounded Lipschitz domain, and where we have set $\varrho_\infty = \mathbf{u}_\infty = 0$ in the relative energy.*

Then there exists a subsequence (not relabeled for simplicity) enjoying the following properties:

$$\varrho_n \to \varrho \textit{ weakly-(*) in } L^\infty(0,T; L^\gamma(\Omega)),$$
$$\mathbf{m}_n \to \mathbf{m} \textit{ weakly-(*) in } L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d)).$$

Suppose, in addition, that

$$\limsup_{n\to\infty} \int_{x\in\Omega;\, {\rm dist}[x,\partial\Omega]<\delta} \left(E(\varrho_n,\mathbf{m}_n) - E(\varrho,\mathbf{m})\right)(\tau,\cdot)\,{\rm d}x$$

is of order $o(\delta)$ as $\delta\to 0$. Then if $[\varrho,\mathbf{m}]$ is an admissible weak solution to the isentropic Euler system, then

$$\int_0^T \int_\Omega E\left(\varrho_n, \mathbf{m}_n \,\middle|\, \varrho, \mathbf{m}\right)\,{\rm d}x\,{\rm d}t \to 0, \tag{5.1}$$

in particular,

$$\varrho_n \to \varrho \text{ in } L^q(0, T; L^\gamma(\Omega)),$$
$$\mathbf{m}_n \to \mathbf{m} \text{ in } L^q(0, T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d)),$$

for any $1 \le q < \infty$. *Thus, for a suitable subsequence,*

$$\varrho_n \to \varrho, \ \mathbf{m}_n \to \mathbf{m} \text{ a.a. in } (0, T) \times \Omega.$$

The hypothesis (5.1) is satisfied if, for instance,

$$\lim_{n\to\infty} \|E(\varrho_n, \mathbf{m}_n) - E(\varrho, \mathbf{m})\|_{L^1((0,T)\times\mathcal{U})} = 0,$$

where $\mathcal{U}$ is an open neighborhood of $\partial\Omega$.

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