

# On the Global Stability of the Wave-map Equation in Kerr Spaces with Small Angular Momentum

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**Abstract** This paper is motivated by the problem of the nonlinear stability of the Kerr solution for axially symmetric perturbations. We consider a model problem concerning the axially symmetric perturbations of a wave map  $\Phi$  defined from a fixed Kerr solution  $\mathcal{K}(M, a)$ ,  $0 \leq a \leq M$ , with values in the two dimensional hyperbolic space  $\mathbb{H}^2$ . A particular such wave map is given by the complex Ernst potential associated to the axial Killing vector-field  $\mathbf{Z}$  of  $\mathcal{K}(M, a)$ . We conjecture that this stationary solution is stable, under small axially symmetric perturbations, in the domain of outer communication (DOC) of  $\mathcal{K}(M, a)$ , for all  $0 \leq a < M$  and we provide preliminary support for its validity, by deriving convincing stability estimates for the linearized system.

**Keywords** Kerr · Stability · Nonlinear · Axially symmetric · Morawetz · Trapping · Red shift

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## 1 Introduction

According to the general expectations the Kerr family  $\mathcal{K}(a, M)$ , in the sub-extremal regime  $|a| < M$ , is stable under general perturbations. More precisely, it is expected that:

**Kerr Stability Conjecture** *An initial data set  $(\Sigma_0, g_0, k_0)$ , sufficiently close to the initial data set of a fixed sub-extremal Kerr spacetime  $\mathcal{K}(M_i, a_i)$ , admits a maximal, vacuum, future, Cauchy development  $(\mathbf{M}, \mathbf{g})$ , with a complete future null infinity  $\mathcal{I}^+$  and whose causal past  $J^-(\mathcal{I}^+)$  is bounded in the future by a smooth, complete, event horizon  $\mathcal{H}^+$ . Moreover  $(\mathbf{M}, \mathbf{g})$  remains close to  $\mathcal{K}(M_i, a_i)$  and approaches asymptotically another sub-extremal Kerr spacetime  $\mathcal{K}(M_f, a_f)$ .*

Despite its extraordinary importance, in both mathematical and astrophysical<sup>1</sup> terms, and despite half a century of sustained efforts to settle it, the conjecture remains

<sup>1</sup> If the Kerr family would turn out to be unstable under perturbations, black holes would be nothing more than mathematical artifacts. See [7] for a comprehensive account of efforts made by physicists to establish the linear stability of the Kerr family.

wide open. The main known mathematical arguments in favor of the conjecture are in fact few and, so far, not at all decisive.

- (1) We know that the Minkowski space, corresponding to  $a = 0, M = 0$  is stable, see [8].
- (2) We know that, perturbatively, the Kerr family exhausts all stationary, smooth, solutions of the Einstein vacuum equations, see [15] and [1]. In other words, any stationary solution sufficiently close to a sub-extremal Kerr must belong to the Kerr family. A full review of rigidity results in the smooth setting is discussed in [16].
- (3) We possess a significantly large class of examples of dynamical black holes, settling down to a sub-extremal Kerr, constructed from infinity, see [13].
- (4) Most importantly, we have now a satisfactory understanding of the so called *poor man linearization*. More precisely, we have a general method for establishing boundedness and quantitative decay of solutions to the scalar wave equation  $\square_{\mathbf{g}_{M,a}} \phi = 0$ , for all sub-extremal Kerr metrics  $\mathbf{g}_{M,a}$ . Such results were first established in Schwarzschild, see [3–6, 10, 19] and later extended for  $|a| \ll M$  in [2, 12, 21]. The full sub-extremal regime was recently settled in [14].
- (5) We have results establishing the non-existence of exponentially growing modes for the more realistic linearized Teukolsky equations, see [20, 23]<sup>2</sup>.

The goal of this paper is to provide additional evidence for the conjecture in the special case of axi-symmetric perturbations.

### 1.1 A Non-linear Model Problem

As well known (see [22]) the Ernst potential  $\Phi = (\Phi^1, \Phi^2)$  of a Killing vector-field  $\mathbf{Z}$  on a 3 + 1 dimensional Einstein-vacuum manifold  $(\mathbf{M}, \mathbf{g})$  can be interpreted as a wave map  $\Phi : \mathbf{M} \rightarrow \mathbb{H}^2$  where  $\mathbb{H}^2$  denotes the upper-half Poincare space with constant negative curvature  $K = -1$ . More precisely,

$$\square_{\mathbf{g}} \Phi^a + \mathbf{g}^{\mu\nu} \Gamma_{bc}^a(\Phi) \partial_\mu \Phi^b \partial_\nu \Phi^c = 0, \tag{1.1}$$

where  $\Gamma$  denotes the Christoffel symbols of the metric  $h$  of  $\mathbb{H}$ . The full, axially symmetric, space-time metric  $\mathbf{g}$  decomposes into its dynamic component  $\Phi$  and a *reduced* 1 + 2 metric  $\hat{\mathbf{g}}$  defined on the orbit space  $\hat{\mathbf{M}} = \mathbf{M}/\mathbf{Z}$  verifying,

$$\text{Ric}(\hat{\mathbf{g}})_{\alpha\beta} = \langle \partial_\alpha \Phi, \partial_\beta \Phi \rangle_h. \tag{1.2}$$

Thus, in axial symmetry, the Einstein vacuum equations are equivalent<sup>3</sup> to the coupled system (1.1)–(1.2), on the reduced space-time  $\hat{\mathbf{M}}$ . A particular, stationary, solution of the system is provided by the pair  $(\hat{\mathbf{g}}_{M,a}, \Phi_{M,a})$  denoting the decomposition of the

<sup>2</sup> Results on boundedness and decay for these equations near Schwarzschild were recently announced by Dafermos, Holzegel and Rodnianski, see [9].

<sup>3</sup> See [22] for a very clear exposition of the reduction. Note that (1.1) can also be interpreted as a wave map from  $\hat{\mathbf{M}}$  to  $\mathbb{H}$ .

Kerr metric  $\mathbf{g}_{M,a}$  of a fixed Kerr spacetime  $\mathbf{M} = \mathcal{K}(M, a)$ . The full problem of the stability of the Kerr solution, for axially symmetric perturbations, can be reformulated as a problem of stability of this special solution for the system (1.1)–(1.2). As this is still an extremely difficult problem we make one further important simplification by partially linearizing the system, that is we fix the reduced metric  $\widehat{g} = \widehat{\mathbf{g}}_{M,a}$  but allow fully nonlinear perturbations of  $\Phi_{M,a}$ . It is easy to see that this amounts to the problem of stability of axially symmetric perturbations of the stationary solution  $\Phi_{M,a}$  of the wave map system (1.1), where  $\mathbf{g}$  is fixed to be the Kerr metric  $\mathbf{g}_{M,a}$ .

**Partial Stability Conjecture** *The stationary solution  $\Phi_{M,a} : \mathcal{K}(M, a) \rightarrow \mathbb{H}$  of the wave map system (1.1) with  $\mathbf{g} = \mathbf{g}_{M,a}$  the metric of  $\mathcal{K}(M, a)$ ,  $|a| < M$ , is future asymptotically stable in the domain of outer communication of  $\mathcal{K}(M, a)$ , for all smooth, axially symmetric, admissible, perturbations<sup>4</sup>.*

*Remark 1.1* We note that the conjecture is consistent with the full nonlinear stability conjecture, for axially symmetric perturbations. More precisely the validity of the Kerr stability conjecture, for axially symmetric perturbations, implies (in principle) the validity of our partial stability conjecture, at least for initial data in the orthogonal complement of a finite dimensional space (corresponding to possible modulation). In this paper we produce convincing evidence that the conjecture is in fact true for all initial data.

We take the first step in proving the conjecture by deriving stability estimates for the linearized system. More precisely we introduce the linearized variables

$$\Phi = \Phi_{M,a} + A\Psi, \quad \Psi = (\phi, \psi).$$

and show that the linearized equations in  $\Psi$  possess a a coercive, conserved, energy quantity (for all  $|a| \leq M$ ) and verify, at least for  $a/M$  small, a Morawetz type estimate comparable to those derived in recent years, see [3–6, 10, 19], for the scalar wave equation  $\square\phi = 0$ .

*Remark 1.2* In the simplest case  $a = 0$  the system for  $\Psi = (\phi, \psi)$  is the decoupled system

$$\square\phi = 0, \quad \square\psi - \left( \frac{4}{r^2(\sin\theta)^2} - \frac{8M}{r^3} \right)\psi = 0. \tag{1.3}$$

Note the non-trivial nature of the potential for the  $\psi$  equation, singular on the axis. The precise form of the potential is important in order to derive the needed stability estimates.

### 1.2 Kerr Metric

The domain of outer communications of the Kerr spacetime  $\mathcal{K}(M, a)$ , in standard Boyer–Lindquist coordinates, is given by

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<sup>4</sup> That is: for all axially symmetric initial data, defined on a spacelike hypersurface  $\Sigma_0$ , which are sufficiently close to the corresponding data of  $\Phi_{M,a}$  and vanishing in a suitable way on the axis of symmetry.

$$\mathbf{g}_{a,M} = -\frac{q^2 \Delta}{\Sigma^2} (dt)^2 + \frac{\Sigma^2 (\sin \theta)^2}{q^2} \left( d\phi - \frac{2aMr}{\Sigma^2} dt \right)^2 + \frac{q^2}{\Delta} (dr)^2 + q^2 (d\theta)^2, \tag{1.4}$$

where

$$\begin{cases} \Delta = r^2 + a^2 - 2Mr; \\ q^2 = r^2 + a^2 (\cos \theta)^2; \\ \Sigma^2 = (r^2 + a^2)q^2 + 2Mra^2 (\sin \theta)^2 = (r^2 + a^2)^2 - a^2 (\sin \theta)^2 \Delta. \end{cases} \tag{1.5}$$

Observe that

$$(2mr - q^2)\Sigma^2 = -q^4 \Delta + 4a^2 m^2 r^2 (\sin \theta)^2. \tag{1.6}$$

Note also the useful identities,

$$\frac{\Sigma^2}{q^2} = q^2 + (p + 1)a^2 (\sin \theta)^2, \quad \Delta = q^2(1 - p) + a^2 (\sin \theta)^2, \quad p := \frac{2Mr}{q^2}. \tag{1.7}$$

Thus the metric can also be written in the form,

$$\begin{aligned} \mathbf{g}_{a,M} = & -\frac{(\Delta - a^2 \sin^2 \theta)}{q^2} dt^2 - \frac{4aMr}{q^2} \sin^2 \theta dt d\phi \\ & + \frac{q^2}{\Delta} dr^2 + q^2 d\theta^2 + \frac{\Sigma^2}{q^2} \sin^2 \theta d\phi^2 \end{aligned} \tag{1.8}$$

and,

$$\mathbf{g}_{tt} \mathbf{g}_{\phi\phi} - \mathbf{g}_{t\phi}^2 = -\Delta \sin^2 \theta.$$

The volume element  $d\mu$  of  $\mathbf{g}$  is given by

$$d\mu = q^2 |\sin \theta| dt dr d\theta d\phi$$

We also note that  $\mathbf{T} = \partial_t, \mathbf{Z} = \partial_\phi$  are both Killing and  $\mathbf{T}$  is only time-like in the complement of the ergoregion, i.e.  $q^2 > 2Mr$ .

The domain of outer communication of  $\mathcal{K}(M, a)$  is given by,

$$\mathcal{R} = \{(\theta, r, t, \varphi) \in (-\pi, \pi) \times (r_{\mathcal{H}}, \infty) \times \mathbb{R} \times \mathbb{S}^1\},$$

where  $r_{\mathcal{H}} := M + \sqrt{M^2 - a^2}$ , the larger root of  $\Delta$ , corresponds to the event horizon. The metric possesses the Killing v-fields  $\mathbf{T} = \partial_t$  and  $\mathbf{Z} = \partial_\phi$ .

The Ernst potential  $\mathring{\Phi} = (A, B)$  associated to the Killing vector-field  $\mathbf{Z} = \partial_\varphi$ , is given explicitly by the formula,

$$A + iB := \frac{\Sigma^2(\sin \theta)^2}{q^2} - i \left[ 2aM(3 \cos \theta - (\cos \theta)^3) + \frac{2a^3M(\sin \theta)^4 \cos \theta}{q^2} \right],$$

$$A = \mathbf{g}(\mathbf{Z}, \mathbf{Z}). \tag{1.9}$$

One can easily check<sup>5</sup> that  $(A, B)$  verify the system,

$$A \square A = \mathbf{D}^\mu A \mathbf{D}_\mu A - \mathbf{D}^\mu B \mathbf{D}_\mu B,$$

$$A \square B = 2 \mathbf{D}^\mu A \mathbf{D}_\mu B. \tag{1.10}$$

where  $\square = \square_{g_{M,a}}$  denotes the usual wave operator with respect to the metric. We can interpret  $\mathring{\Phi} := (A, B)$  as a stationary, axisymmetric, wave map from  $\mathcal{K}(M, a)$  to the hyperbolic space  $\mathbb{H}^2 = (\mathbb{R}_+^2, h)$  with the metric  $h$  given by,

$$ds^2 = \frac{1}{A^2} (dA^2 + dB^2)$$

### 1.3 Reinterpreting the Conjecture

As mentioned above the goal of this paper is to investigate the future global asymptotic stability, in the exterior region of  $\mathcal{K}(M, a)$ , of the special stationary map  $\mathring{\Phi} = (A, B)$ , under general axially symmetric perturbations. In other words we consider solutions  $\Phi = (X, Y)$  of the wave map system,

$$X \square X = \mathbf{D}^\mu X \mathbf{D}_\mu X - \mathbf{D}^\mu Y \mathbf{D}_\mu Y,$$

$$X \square Y = 2 \mathbf{D}^\mu X \mathbf{D}_\mu Y. \tag{1.11}$$

which are  $\mathbf{Z}$ -invariant, i.e.  $\mathbf{Z}(\Phi^1) = \mathbf{Z}(\Phi^2) = 0$ , and whose initial conditions on a given space-like hypersurface in  $\mathcal{R}$  are a small perturbation of the initial data of  $\mathring{\Phi}$ . We have to be careful however that the perturbed map  $\Phi = (X, Y)$  has the same axis of rotation as  $\mathring{\Phi} = (A, B)$ , i.e.  $\Phi = \mathring{\Phi}$  on the axis of symmetry of  $\mathcal{K}(M, a)$ , i.e.  $\sin^2 \theta = 0$ . To make sure that this latter condition is satisfied we search for solutions  $\Phi = (X, Y)$  of the form,

$$\Phi = \mathring{\Phi} + A\Psi, \quad \Psi = (\phi, \psi). \tag{1.12}$$

with  $\psi$  vanishing on the axis of symmetry  $\mathcal{A}$ . With these notation we can interpret the system (1.11) as a nonlinear system of equations for  $\Psi$ , depending also on the fixed  $\mathring{\Phi}$ , of the form,

$$\mathcal{F}(\mathring{\Phi}; \Psi) = 0. \tag{1.13}$$

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<sup>5</sup> Or derive from first principles, see [22].

Our Conjecture can thus be interpreted as a statement on the stability of the trivial solution  $\Psi \equiv 0$  for the nonlinear system (1.13).

**Conjecture** *The trivial solution  $\Psi = 0$  of the nonlinear system (1.13) is future asymptotically stable in the exterior region  $r \geq r_{\mathcal{H}}$  for arbitrary, smooth, axially symmetric, admissible (i.e. such that  $\psi = 0$  on the axis  $\mathcal{A}$ ) initial conditions on a  $\mathcal{Z}$ -invariant spacelike hypersurface.*

### 1.4 Main Difficulties

A simple comparison with the far simpler case of nonlinear systems of wave equations in Minkowski space shows that we cannot expect the conjecture to be valid without addressing the following obstacles.

- (1) *Strong linear stability.* To start with, one needs to show that the solutions to the wave map system cannot grow out of control. It does not suffice to show that the solutions to the linearized equations are simply bounded; one needs to prove quantitative decay estimates comparable to the known decay estimates for the standard wave equation in the Minkowski space  $\mathbb{R}^{1+3}$ . Moreover these estimates have to be robust, i.e. the methods used in their derivation can be extended, in principle, to the nonlinear equations.
- (2) *Nonlinear stability.* Though strong linear stability is an essential ingredient in the proof of nonlinear stability, it is by no means enough. The nonlinear terms of the equation also have to satisfy special structural conditions, such as the null condition.
- (3) *Degeneracy on the axis.* An additional difficulty is the degeneracy of our system on the axis of symmetry, i.e where  $A$  vanishes, see (1.3). Our functional analysis framework, see Definition 1.6, is adapted to handle such a situation.

The first difficulty is the most serious one. The case when the linearized equation is simply  $\square_g \Psi = 0$  has now been well understood in full generality, for all  $|a| < M$  and under no symmetry assumptions, see [14] and the references therein. Our linearized equations differ significantly, however, from this case. Indeed taking the Fréchet derivative of  $\mathcal{F}$  with respect to  $\Psi$  we obtain a linear operator with coefficients which depend on  $\mathring{\Phi} = (A, B)$  in a non-trivial fashion. The linearized equations are in fact of the form:

$$\begin{aligned}
 0 &= \square \phi + 2 \frac{\mathbf{D}^\mu B}{A} \mathbf{D}_\mu \psi - 2 \frac{\mathbf{D}^\mu B \mathbf{D}_\mu B}{A^2} \phi + 2 \frac{\mathbf{D}^\mu B \mathbf{D}_\mu A}{A^2} \psi \\
 0 &= \square \psi - 2 \frac{\mathbf{D}^\mu B}{A} \mathbf{D}_\mu \phi - \frac{\mathbf{D}^\mu A \mathbf{D}_\mu A + \mathbf{D}^\mu B \mathbf{D}_\mu B}{A^2} \psi.
 \end{aligned}
 \tag{1.14}$$

and cannot be decoupled. It is not apriori clear that such an equation possesses a well defined notion of energy, i.e. a conserved and coercive integral quantity similar to the standard energy quantity for  $\square \Psi = 0$ . Though the existence of such a quantity is by no means enough to prove strong linear stability it is an absolutely necessary first step. Our first result is the following:

**Theorem 1.3** *The linearized equations (1.14) (for axi-symmetric solutions  $\Psi$ ) admit an energy-momentum tensor type quantity  $\mathcal{Q}_{\mu\nu} = \mathcal{Q}[\Psi]_{\mu\nu}$  and a source  $\mathcal{J}_\nu$ , both quadratic in  $(\Psi, \partial\Psi)$ , depending also on  $(\dot{\Phi}, \partial\dot{\Phi})$ , verifying the following:*

- (a)  $\mathcal{Q}(X, Y) > 0$ , for any future-oriented, timelike, vector-fields  $X, Y$ ;
- (b)  $\mathbf{D}^\nu \mathcal{Q}_{\mu\nu} = \mathcal{J}_\nu$ .
- (c)  $\mathbf{g}(\mathbf{T}, \mathcal{J}) = 0$ .
- (d)  $\mathcal{Q}(Z, X) = 0$ , for any vector-field  $X$  orthogonal to  $\mathbf{Z}$ .

The underlying reason for the existence of a quantity verifying (b) and (c) is a somewhat less familiar manifestation of Noether’s principle, which we discuss below. The positivity (a), on the other hand, is a consequence of the negative curvature properties of  $\mathbb{H}$ . The property (d) can be easily derived from the form of  $\mathcal{Q}$ , displayed below, and the  $\mathbf{Z}$ -invariance of  $\Psi$ .

As a consequence of the Theorem we deduce that the current  $\mathbf{P}_\mu := \mathbf{Q}_{\mu\nu} \mathbf{T}^\nu$  is conserved, i.e.

$$\mathbf{D}^\mu \mathbf{P}_\mu = 0.$$

which leads, by integration on causal domains, to conserved energy type quantities and fluxes. In view of (d) the energy is a coercive quantity in  $\mathcal{R}$  with a mild degeneracy on the horizon  $r = r_{\mathcal{H}}$ . Theorem 1.3 is thus a strong first indication of the validity of our conjecture for all values of the Kerr parameters,  $|a| < M$ . Yet, as alluded above, the bounds provided by the energy are not by themselves enough to even prove the boundedness of solutions to the system (1.14), subject to nice initial conditions.

To actually go beyond the bounds provided by the energy and prove strong linear stability we encounter the same difficulties as for the simpler case of axially symmetric<sup>6</sup> solutions of the standard wave equation  $\square\phi = 0$  in the DOC of  $\mathcal{K}(a, m)$ , i.e. degeneracy of the energy at the horizon, presence of trapped null geodesics and slow decay at null infinity. As it is now well understood, the major ingredient for proving strong linear stability for linear systems on black holes is the derivation of an integrated decay estimate of Morawetz type. Such estimates, which degenerate in the trapping region, i.e. region of  $\mathcal{K}(M, a)$  which contain trapped null geodesics, are quite subtle, and difficult to derive.

Fortunately, in the case of axial symmetry, all trapped null geodesics are restricted to the hypersurface at  $r = r_*$ , the largest root of the polynomial equation in  $r$ ,  $r^3 - 3r^2M + a^2(r + M) = 0$ . This allows one, in principle, to use a vector-field method approach similar to that used in the derivation of the Morawetz type integrated decay estimate for solutions of the scalar wave equation in Schwarzschild. The main new difficulties are the presence of the source term  $\mathcal{J}$  in the divergence equation  $\mathbf{Div}\mathcal{Q} = \mathcal{J}$ , and the degeneracy on the axis. We overcome these difficulties in this paper, for small values of  $a/M$ . Inspired by the  $r$ -weighted estimates of Dafermos–Rodnianski<sup>7</sup>, see

<sup>6</sup> In the case of general solutions there is another major obstacle, namely the lack of coerciveness of the energy in the ergoregion. The strong linear stability of  $\square\phi = 0$  in Kerr has recently been fully resolved for all values  $|a| < m$  in [14].

<sup>7</sup> Their estimates provide similar decay information for the *outgoing energy* associated to null hypersurfaces.



[11], we also prove a stronger version of the Morawetz estimate which provides decay information for an appropriate notion of *outgoing energy* associated to space-like hypersurfaces.

A precise version of our second theorem requires a space-like  $\mathbf{Z}$ -invariant foliation  $\Sigma_t$  of the entire domain of outer communication, transversal to the horizon and whose leaves are transported by  $\mathbf{T}$ . In what follows we give a first, informal, version of the theorem, for the linearized equations (1.14) in which we do not specify the foliation. A more precise version will be given later in this section.

To state the theorem we choose a smooth, increasing function  $\chi_{\geq 4M}$  supported for  $r \geq 4M$ , equal to 1 for  $r \geq 6M$ , and define the *outgoing energy density* ( $e(\phi)$ ,  $e(\psi)$ ),

$$\begin{aligned}
 e(\phi)^2 &:= \frac{(\partial_1 \phi)^2}{r^2} + (L\phi)^2 + \frac{M^2[(\partial_2 \phi)^2 + (\partial_3 \phi)^2]}{r^2} + \frac{\phi^2}{r^2}, \\
 e(\psi)^2 &:= \frac{(\partial_1 \psi)^2 + \psi^2(\sin \theta)^{-2}}{r^2} + (L\psi)^2 + \frac{M^2[(\partial_2 \psi)^2 + (\partial_3 \psi)^2]}{r^2} + \frac{\psi^2}{r^2}.
 \end{aligned}$$

where  $L$  is the *future outgoing* vector-field,

$$L := \chi_{\geq 4M}(r) \left( \partial_r + \frac{r}{r - 2M} \partial_t \right).$$

The vector-fields  $\partial_1, \partial_2, \partial_3$  are defined precisely in (1.22), as coordinate derivatives in a new set of variables. They agree with the coordinate derivatives  $\partial_\theta, \partial_r, \partial_t$  outside a small neighborhood of the event horizon.

**Theorem 1.4** *Assume that  $(\phi, \psi)$  is an admissible  $\mathbf{Z}$ -invariant solution of the linear system (1.14). Then, for any  $\alpha \in (0, 2)$  and any  $t_1 \leq t_2$ ,*

$$\mathcal{B}_\alpha(t_1, t_2) + \int_{\Sigma_{t_2}} \frac{r^\alpha}{M^\alpha} [e(\phi)^2 + e(\psi)^2] d\mu_t \leq C_\alpha \int_{\Sigma_{t_1}} \frac{r^\alpha}{M^\alpha} [e(\phi)^2 + e(\psi)^2] d\mu_t$$

with  $d\mu_t$  the induced measure on  $\Sigma_t$  and  $\mathcal{B}_\alpha$  the bulk integral,

$$\begin{aligned}
 \mathcal{B}_\alpha(t_1, t_2) &:= \int_{\mathcal{D}_{[t_1, t_2]}} \frac{r^\alpha}{M^\alpha} \left\{ \frac{(r - r^*)^2}{r^3} \frac{|\partial_\theta \phi|^2 + |\partial_\theta \psi|^2 + \psi^2(\sin \theta)^{-2}}{r^2} \right. \\
 &\quad + \frac{1}{r} [(L\phi)^2 + (L\psi)^2] + \frac{1}{r^3} (\phi^2 + \psi^2) + \frac{M^2}{r^3} [(\partial_r \phi)^2 + (\partial_r \psi)^2] \\
 &\quad \left. + \frac{M^2(r - r^*)^2}{r^5} [(\partial_t \phi)^2 + (\partial_t \psi)^2] \right\} d\mu.
 \end{aligned}$$

Note that, as expected the integrand of the bulk integral  $\mathcal{B}_\alpha$  degenerates at  $r = r_*$ . Though the presence of the  $r^\alpha$ -weights in our Morawetz type estimate appear to be new even in the particular case of the standard scalar wave equation, they were clearly inspired by the work of Dafermos-Rodnianski [11]. The main new idea in [11] was to observe that one can replace the  $(t, r)$  weights of the classical conformal

multiplier method, along outgoing null hypersurfaces, by weights which depend only on  $r$ , provided that one has already derived a local decay estimate. The new twist in our work is to show that similar estimates can be derived on spacelike hypersurfaces. Unlike in the case of [11], where the proof of  $r$ -weighted estimates are can be neatly separated from the main local decay estimate, we are obliged in our work to prove them simultaneously. Proving a simultaneous estimate, on both the space-time integral, requires much more careful choices of the multipliers at infinity.

### 1.5 Proof of Theorem 1.3

In this section we give a first, informal, derivation of Theorem 1.3, based on first principles, which can be easily generalized to other situations. In the next section we shall re-derive the result by a straightforward verification.

Observe first that the linear system (1.14) is derivable from a Lagrangian<sup>8</sup>  $\mathcal{L}[\mathring{\Phi}, \Psi]$ ,  $\mathring{\Phi} = (\mathring{\Phi}^1, \mathring{\Phi}^2) = (A, B)$ ,  $\Psi = (\Psi^1, \Psi^2) = (\phi, \psi)$ , defined as follows:

$$\mathcal{L}[\mathring{\Phi}, \Psi] = \mathbf{g}^{\mu\nu} [\mathring{\mathbf{D}}_\mu \phi \mathring{\mathbf{D}}_\nu \phi + \mathring{\mathbf{D}}_\mu \psi \mathring{\mathbf{D}}_\nu \psi + A^{-2}(\phi \partial_\mu B - \psi \partial_\mu A)(\phi \partial_\nu B - \psi \partial_\nu A)] \tag{1.15}$$

with,

$$\mathring{\mathbf{D}}_\mu \phi = \partial_\mu \phi + A^{-1} \partial_\mu B \psi \quad \mathring{\mathbf{D}}_\mu \psi = \partial_\mu \psi - A^{-1} \partial_\mu B \phi.$$

We then define, as usual, the energy momentum tensor of the linearized field equation to be the quantity,

$$\mathcal{Q}[\mathring{\Phi}, \Psi]_{\mu\nu} := \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \frac{1}{2} \mathbf{g}_{\mu\nu} \mathcal{L}. \tag{1.16}$$

We also define the source:

$$\mathcal{J}[\mathring{\Phi}, \Psi]_\mu := 2 \frac{\partial \mathcal{L}[\psi]}{\partial \mathring{\Phi}^c} \partial_\mu \mathring{\Phi}^c, \quad c = 1, 2. \tag{1.17}$$

Note that, in view of the stationarity of  $\mathring{\Phi}$ ,

$$\mathbf{T}^\mu \mathcal{J}_\mu = 2 \frac{\partial \mathcal{L}[\psi]}{\partial \mathring{\Phi}^c} \mathbf{T}^\mu \partial_\mu \mathring{\Phi}^c = 0.$$

**Lemma 1.5** *We have the local conservation law:*

$$\mathbf{D}^\nu \mathcal{Q}_{\mu\nu} = \mathcal{J}_\mu.$$

---

<sup>8</sup> One can identify  $\mathcal{L}$  as the quadratic form in  $\Psi$  generated by the Taylor expansion at  $\mathring{\Phi}$  of the Lagrangian of the original, nonlinear, system (1.11).

*Proof* Let  $\chi_s$  be the one-parameter group of local diffeomorphisms generated by a given vector-field  $X$ . We shall use the flow  $\chi$  to vary the fields  $\Psi$  according to

$$\mathbf{g}_s = (\chi_s)_*\mathbf{g}, \quad \psi_s = (\chi_s)_*\Psi, \quad \dot{\Phi}_s = (\chi_s)_*\dot{\Phi}.$$

From the invariance of the action integral under diffeomorphisms,  $S(s) = S[\Psi_s, \mathbf{g}_s, \dot{\Phi}_s] = S[\Psi, \mathbf{g}; \dot{\Phi}]$ . Therefore,

$$\begin{aligned} 0 &= \frac{d}{ds}S(s) \Big|_{s=0} = \int \frac{\partial \mathcal{L}}{\partial \Psi^a} X(\Psi^a) dv_{\mathbf{g}} \\ &\quad + \int \left( \frac{\partial \mathcal{L}}{\mathbf{g}^{\mu\nu}} - \frac{1}{2} \mathbf{g}^{\mu\nu} (\text{lin})\mathcal{L} \right) \dot{\mathbf{g}}_{\mu\nu} dv_{\mathbf{g}} + \int \frac{\partial \mathcal{L}}{\partial \dot{\Phi}^a} \dot{\Phi}^a dv_{\mathbf{g}} \\ &= \int \mathcal{Q}^{\mu\nu} (\mathbf{D}_\mu X_\nu + \mathbf{D}_\nu X_\mu) dv_{\mathbf{g}} + 2 \int \mathcal{J}^\mu X_\mu dv_{\mathbf{g}} \\ &= -2 \int \mathbf{D}_\nu \mathcal{Q}^{\mu\nu} X_\mu dv_{\mathbf{g}} + 2 \int \mathcal{J}^\mu X_\mu dv_{\mathbf{g}} \end{aligned}$$

Since the vector-field  $X^\mu$  is arbitrary we deduce,

$$-\mathbf{D}_\nu \mathcal{Q}^{\mu\nu} + \mathcal{J}^\mu = 0$$

as desired. □

In view of the definitions of  $\mathcal{Q}$  and  $\mathcal{L}$  we can write

$$\mathcal{Q}_{\mu\nu} = \mathcal{T}_{\mu\nu} - \frac{1}{2} \mathbf{g}_{\mu\nu} (\text{tr}_{\mathbf{g}} \mathcal{T}), \quad \mathcal{T}_{\mu\nu} := E_\mu E_\nu + F_\mu F_\nu + M_\mu M_\nu \tag{1.18}$$

where,

$$\begin{aligned} E_\mu &:= \mathring{\mathbf{D}}_\mu \phi = \partial_\mu \phi + \psi A^{-1} \partial_\mu B, \quad F_\mu := \mathring{\mathbf{D}}_\mu \psi = \partial_\mu \psi - \phi A^{-1} \partial_\mu B, \\ M_\mu &:= A^{-1} (\phi \partial_\mu B - \psi \partial_\mu A). \end{aligned}$$

The positivity property (a) is now an immediate consequence of the structure (1.18) of the energy momentum tensor. Indeed, it is well known that at every point of the ergoregion where  $r > r_H$ , there exists a linear combination of  $\mathbf{T}$  and  $\mathbf{Z}$ ,  $\mathbf{T} + c\mathbf{Z}$ , which is timelike. Therefore, since  $\mathbf{T} \cdot E = \mathbf{T} \cdot F = \mathbf{T} \cdot M = 0$ ,

$$0 < \mathcal{T}(\mathbf{T} + c\mathbf{Z}, X) = \mathcal{T}(\mathbf{T}, X).$$

On the other hand, since  $X$  is orthogonal to  $\mathbf{Z}$ ,

$$\mathbf{g}(\mathbf{T} + c\mathbf{Z}, X) = \mathbf{g}(\mathbf{T}, X).$$

Hence,

$$0 < \mathcal{Q}(\mathbf{T} + c\mathbf{Z}, X) = \mathcal{Q}(\mathbf{T}, X),$$

as desired.

### 1.6 New Coordinates

As well known the Boyer-Lindquist coordinates, are singular near the horizon and as such are not appropriate for our task. To avoid this difficulty it has become standard to define a new set of variables which are well behaved across the horizon and coincide with the Boyer-Lindquist coordinates away from it.

We fix first a smooth function  $\chi : \mathbb{R} \rightarrow [0, 1]$  supported in the interval  $(-\infty, 5M/2]$  and equal to 1 in the interval  $(-\infty, 9M/4]$ , and define  $g_1, g_2 : (r_{\mathcal{H}}, \infty) \rightarrow \mathbb{R}$  such that

$$g'_1(r) = \chi(r) \frac{2Mr}{\Delta}, \quad g'_2(r) = \chi(r) \frac{a}{\Delta}. \tag{1.19}$$

We define the functions

$$t_+ := t + g_1(r), \quad \phi_+ := \phi + g_2(r). \tag{1.20}$$

Therefore

$$dt_+ = dt + \chi(r) \frac{2Mr}{\Delta} dr, \quad d\phi_+ = d\phi + \chi(r) \frac{a}{\Delta} dr.$$

In  $(\theta, r, t_+, \phi_+)$  coordinates, the metric  $\mathbf{g}$  becomes<sup>9</sup>

$$\begin{aligned} \mathbf{g} = & q^2(d\theta)^2 + \left[ \frac{q^2}{\Delta}(1 - \chi^2(r)) + \frac{2Mr + q^2}{q^2} \chi^2(r) \right] (dr)^2 \\ & + 2\chi(r) \frac{2Mr}{q^2} dr dt_+ - 2\chi(r) \frac{a(\sin \theta)^2(q^2 + 2Mr)}{q^2} dr d\phi_+ \\ & + \frac{2Mr - q^2}{q^2} (dt_+)^2 - \frac{4aMr(\sin \theta)^2}{q^2} dt_+ d\phi_+ + \frac{\Sigma^2(\sin \theta)^2}{q^2} (d\phi_+)^2. \end{aligned} \tag{1.21}$$

Let

$$\partial_1 = \partial_\theta = \frac{d}{d\theta}, \quad \partial_2 = \partial_r = \frac{d}{dr}, \quad \partial_3 = \partial_{t_+} = \frac{d}{dt_+} = \mathbf{T}, \quad \partial_4 = \partial_{\phi_+} = \frac{d}{d\phi_+} = \mathbf{Z}. \tag{1.22}$$

The nontrivial components of the metric  $\mathbf{g}$  are

$$\begin{aligned} \mathbf{g}_{11} = & q^2, \quad \mathbf{g}_{33} = \frac{2Mr - q^2}{q^2}, \quad \mathbf{g}_{34} = -\frac{2aMr(\sin \theta)^2}{q^2}, \quad \mathbf{g}_{44} = \frac{\Sigma^2(\sin \theta)^2}{q^2}, \\ \mathbf{g}_{22} = & \frac{q^2}{\Delta}(1 - \chi^2(r)) + \frac{2Mr + q^2}{q^2} \chi^2(r), \\ \mathbf{g}_{23} = & \chi(r) \frac{2Mr}{q^2}, \quad \mathbf{g}_{24} = -\chi(r) \frac{a(\sin \theta)^2(q^2 + 2Mr)}{q^2}. \end{aligned} \tag{1.23}$$

<sup>9</sup> See the appendix for more calculations in these coordinates.

The metric  $g$  extends smoothly to the larger open set

$$\tilde{\mathcal{R}} = \{(\theta, r, t_+, \phi_+) \in (-\pi, \pi) \times (0, \infty) \times \mathbb{R} \times \mathbb{S}^1\}.$$

For  $t \in \mathbb{R}$  and  $c \in (0, \infty)$  let

$$\Sigma_t^c := \{(\theta, r, t_+, \phi_+) \in \tilde{\mathcal{R}} : t_+ = t \text{ and } r > c\}. \tag{1.24}$$

The surfaces  $\Sigma_t^{r_{\mathcal{H}}}$ ,  $t \in \mathbb{R}$ , form a  $\mathbf{Z}$ -invariant foliation of spacelike surfaces of the domain of outer communications of the Kerr spacetime  $\mathcal{K}(M, a)$ . Moreover, the foliation is compatible with the Killing vector-field  $\mathbf{T}$ , i. e.  $\Phi_{t_1}(\Sigma_{t_2}^c) = \Sigma_{t_1+t_2}^c$  for any  $t_1, t_2 \in \mathbb{R}$ , where  $\Phi_t$  denotes the flow associated to  $\mathbf{T}$ .

As mentioned earlier we are interested in solutions of the form (1.12), i.e.,  $\Phi = (A', B') = (A, B) + \varepsilon(A\phi, A\psi)$  of the wave-map equation (1.10), in causal domains of the form

$$\mathcal{D}_I^c := \cup_{t \in I} \Sigma_t^c = \{(\theta, r, t_+, \phi_+) \in \tilde{\mathcal{R}} : t_+ \in I \text{ and } r > c\}, \tag{1.25}$$

where  $I \subseteq \mathbb{R}$  is an interval and  $c < r_{\mathcal{H}}$ . Notice that if  $c < r_{\mathcal{H}}$  then  $\mathcal{D}_{\mathbb{R}}^c$  contains a small neighborhood of the future event horizon  $\mathcal{H}^+$  as well as the entire domain of outer communication. For any  $c \in (0, \infty)$  and any interval  $I \subseteq \mathbb{R}$  let

$$\mathcal{N}_I^c := \{(\theta, r, t_+, \phi_+) \in \tilde{\mathcal{R}} : t_+ \in I \text{ and } r = c\}. \tag{1.26}$$

Notice that the hypersurfaces  $\mathcal{N}_I^c$  are spacelike if  $c < r_{\mathcal{H}}$ , null (and contained in the future event horizon  $\mathcal{H}^+$ ) if  $c = r_{\mathcal{H}}$ , and timelike if  $c > r_{\mathcal{H}}$ .

### 1.7 Precise Version of Our Second Theorem

We define now our main function spaces:

**Definition 1.6** For any  $m \in \mathbb{Z}^+$ ,  $c \in (0, \infty)$ , and  $t \in \mathbb{R}$  let  $H^m(\Sigma_t^c)$  denote the usual  $L^2$ -based Sobolev space of functions on the hypersurface  $\Sigma_t^c$ , with respect to the induced Kerr metric (see (1.23)). Let

$$\begin{aligned} \tilde{H}^m(\Sigma_t^c) := & \left\{ f : \Sigma_t^c \rightarrow \mathbb{R} : \|f\|_{\tilde{H}^m(\Sigma_t^c)} := \|f\|_{H^m(\Sigma_t^c)} \right. \\ & \left. + \sum_{m'+m''=1}^m \|(\tilde{\partial}_1/r)^{m'} \tilde{\partial}_2^{m''} f\|_{L^2(\Sigma_t^c)} < \infty \right\}, \end{aligned} \tag{1.27}$$

where, by definition,

$$\tilde{\partial}_1 g := \left( \partial_1 - \frac{2 \cos \theta}{\sin \theta} \right) g, \quad \tilde{\partial}_2 g := \partial_2 g. \tag{1.28}$$

For any  $g \in C^1(\Sigma_t^c)$  satisfying  $\mathbf{Z}(g) = 0$  let

$$\nabla g := (\partial_1 g/r, \partial_2 g), \quad \tilde{\nabla} g := (\tilde{\partial}_1 g/r, \tilde{\partial}_2 g). \tag{1.29}$$

Finally, let

$$\begin{aligned} \mathbf{H}^m(\Sigma_t^c) &:= \{(\phi, \psi) : \Sigma_t^c \rightarrow \mathbb{R} \times \mathbb{R} : \|(\phi, \psi)\|_{\mathbf{H}^m(\Sigma_t^c)} \\ &:= \|\phi\|_{H^m(\Sigma_t^c)} + \|\psi\|_{\tilde{H}^m(\Sigma_t^c)} < \infty\}. \end{aligned} \tag{1.30}$$

For any  $R \geq 33M/16$  let  $\chi_{\geq R} : [0, \infty) \rightarrow [0, 1]$  denote a smooth increasing function supported in  $[R, \infty)$ , equal to 1 in  $[2R - 2M, \infty)$ , and satisfying the natural differential inequalities. Let

$$L := \chi_{\geq 4M}(r) \left( \partial_2 + \frac{r}{r - 2M} \partial_3 \right), \tag{1.31}$$

For any  $t \in \mathbb{R}$  and  $(\phi, \psi) \in \mathbf{H}^1(\Sigma_t^c)$  we define the *outgoing energy density*  $(e(\phi), e(\psi))$ ,

$$\begin{aligned} e(\phi)^2 &:= \frac{(\partial_1 \phi)^2}{r^2} + (L\phi)^2 + \frac{M^2[(\partial_2 \phi)^2 + (\partial_3 \phi)^2]}{r^2} + \frac{\phi^2}{r^2}, \\ e(\psi)^2 &:= \frac{(\partial_1 \psi)^2 + \psi^2(\sin \theta)^{-2}}{r^2} + (L\psi)^2 + \frac{M^2[(\partial_2 \psi)^2 + (\partial_3 \psi)^2]}{r^2} + \frac{\psi^2}{r^2}. \end{aligned} \tag{1.32}$$

We work in the axially symmetric case, therefore the relevant trapped null geodesics are still confined to a codimension 1 set. Assuming that  $a \ll M$ , it is easy to see that the equation

$$r^3 - 3Mr^2 + a^2r + Ma^2 = 0$$

has a unique solution  $r^* \in (M, \infty)$ . Moreover,  $r^* \in [3M - a^2/M, 3M]$ .

**Theorem 1.7** *Assume that  $M \in (0, \infty)$ ,  $N_0 := 4$ ,  $a \in [0, \bar{\varepsilon}M]$  and  $c_0 \in [r_{\mathcal{H}} - \bar{\varepsilon}M, r_{\mathcal{H}}]$ , where  $\bar{\varepsilon} \in (0, 1]$  is a sufficiently small constant. Assume that  $T \geq 0$ , and  $(\phi, \psi) \in C^k([0, T] : \mathbf{H}^{N_0-k}(\Sigma_t^{c_0}))$ ,  $k \in [0, N_0]$ , is a solution of the system*

$$\begin{aligned} \square \phi + 2 \frac{\mathbf{D}^\mu B}{A} \mathbf{D}_\mu \psi - 2 \frac{\mathbf{D}^\mu B \mathbf{D}_\mu B}{A^2} \phi + 2 \frac{\mathbf{D}^\mu B \mathbf{D}_\mu A}{A^2} \psi &= \mathcal{N}_\phi, \\ \square \psi - 2 \frac{\mathbf{D}^\mu B}{A} \mathbf{D}_\mu \phi - \frac{\mathbf{D}^\mu A \mathbf{D}_\mu A + \mathbf{D}^\mu B \mathbf{D}_\mu B}{A^2} \psi &= \mathcal{N}_\psi, \end{aligned} \tag{1.33}$$

satisfying

$$\mathbf{Z}(\phi, \psi) = 0 \quad \text{in } \mathcal{D}_{[0, T]}^{c_0}. \tag{1.34}$$

Then, for any  $\alpha \in (0, 2)$  and any  $t_1 \leq t_2 \in [0, T]$ ,

$$\begin{aligned} & \mathcal{B}_\alpha^{c_0}(t_1, t_2) + \int_{\Sigma_{t_2}^{c_0}} \frac{r^\alpha}{M^\alpha} [e(\phi)^2 + e(\psi)^2] d\mu_t \\ & \leq \bar{C}_\alpha \int_{\Sigma_{t_1}^{c_0}} \frac{r^\alpha}{M^\alpha} [e(\phi)^2 + e(\psi)^2] d\mu_t \\ & \quad + \bar{C}_\alpha \int_{\mathcal{D}_{[t_1, t_2]}^{c_0}} \frac{r^\alpha}{M^\alpha} [e(\phi, \mathcal{N}_\phi) + e(\psi, \mathcal{N}_\psi)] d\mu, \end{aligned} \tag{1.35}$$

where  $\bar{C}_\alpha$  is a large constant that may depend on  $\alpha$ ,

$$\begin{aligned} \mathcal{B}_\alpha^{c_0}(t_1, t_2) := & \int_{\mathcal{D}_{[t_1, t_2]}^{c_0}} \frac{r^\alpha}{M^\alpha} \left\{ \frac{(r - r^*)^2}{r^3} \frac{|\partial_1 \phi|^2 + |\partial_1 \psi|^2 + \psi^2 (\sin \theta)^{-2}}{r^2} \right. \\ & + \frac{1}{r} [(L\phi)^2 + (L\psi)^2] + \frac{1}{r^3} (\phi^2 + \psi^2) + \frac{M^2}{r^3} [(\partial_2 \phi)^2 + (\partial_2 \psi)^2] \\ & \left. + \frac{M^2 (r - r^*)^2}{r^5} [(\partial_3 \phi)^2 + (\partial_3 \psi)^2] \right\} d\mu, \end{aligned} \tag{1.36}$$

and, for  $f \in \{\phi, \psi\}$ ,

$$e(f, \mathcal{N}_f) := |\mathcal{N}_f| \left[ (Lf)^2 + \frac{M^2 [(\partial_2 f)^2 + (\partial_3 f)^2] + f^2}{r^2} \right]^{1/2}. \tag{1.37}$$

The point of proving an energy estimate such as (1.35) involving outgoing energies is that it leads directly to decay estimates. For example, we have the following corollary:

**Corollary 1.8** *Assume that  $N_1 = 8$  and  $(\phi, \psi) \in C^k([0, T] : \mathbf{H}^{N_1-k}(\Sigma_t^{c_0}))$ ,  $k \in [0, N_1]$ , is a solution of the system (1.33) with  $\mathcal{N}_\phi = \mathcal{N}_\psi = 0$ . Then, for any  $t \in [0, T]$  and  $\beta < 2$ ,*

$$\begin{aligned} & \int_{\Sigma_t^{c_0}} [e(\phi)^2 + e(\psi)^2] d\mu_t \\ & \lesssim_\beta (1 + t/M)^{-\beta} \sum_{k=0}^4 M^{2k} \int_{\Sigma_0^{c_0}} \frac{r^2}{M^2} [e(\mathbf{T}^k \phi)^2 + e(\mathbf{T}^k \psi)^2] d\mu_t. \end{aligned} \tag{1.38}$$

The point of the corollary is the almost  $(1 + t/M)^{-2}$  decay of the outgoing energy on the hypersurface  $\Sigma_t^{c_0}$ , in terms of initial data; such a decay is not possible, of course, for the standard energy. One can further commute the equation with the vector-field  $\partial_r$  and use elliptic estimates to prove control decay of higher order outgoing energies as well. Such estimates, with improved decay, can then be combined, in principle, with a bootstrap argument to analyze globally the full semilinear system and prove the Partial Stability Conjecture in the case  $a \ll M$ . Note that the precise form of the

system is given in Proposition 2.1; the nonlinearities  $\mathcal{N}_\phi^\varepsilon$  and  $\mathcal{N}_\psi^\varepsilon$  are quadratic and appear to satisfy suitable null conditions which are needed to prove global existence.

The explicit loss of derivatives of the estimate (1.38) can be improved; however some loss is unavoidable due to the degeneracy of the bulk integral at  $r = r^*$  in (1.35). We note that the analogous decay estimate for the standard wave equation in Minkowski space follows, with  $\beta = 2$  and without the loss of derivatives, from the conservation of the conformal energy (see, for example, section 3 in [17]).

### 1.8 Conclusions

The estimates presented in this paper offer convincing evidence for the validity of our conjecture. Further work is needed to remove the smallness condition for  $a/M$ , provide sufficiently strong pointwise decay estimate in the wave zone region and implement the standard approach for proving global existence results for nonlinear wave equations which satisfy the null condition<sup>10</sup>.

### 1.9 Organization

The rest of the paper is organized as follows. In section 2 we derive the main identities in the paper, including the precise form of the system and the divergence identities; this provides an alternative explicit proof of Theorem 1.3. In section 3 we give an outline of the proof of the main theorem in the simplified case (1.3). In sections 4 and 5 we give a complete proof of the main Theorem 1.7, first in the case of the pure wave equation on the Schwarzschild space, and then for the full system on the Kerr spaces. In section 6 we provide a proof of Corollary 1.8, using Theorem 1.7 and an elliptic estimate. Finally, the appendix contains several explicit calculations in Kerr spaces, some Hardy inequalities, and some properties of the modified Sobolev spaces  $\tilde{H}^m$ .

## 2 Derivation of the Main Algebraic Identities. Theorem 1.3 Revisited

Assume that  $(A', B') = (A, B) + (\varepsilon A\phi, \varepsilon A\psi)$  is a solution of the wave-map equation (1.10) on some interval  $I$ , where  $(\phi, \psi) \in C^k(I : \mathbf{H}^{N_1-k}(\Sigma_t^{c_0}))$ ,  $k = 0, \dots, N_1$ . The functions  $(\phi, \psi)$  satisfy the system

$$A^2\Box\phi + 2A\mathbf{D}^\mu B\mathbf{D}_\mu\psi - 2\mathbf{D}^\mu B\mathbf{D}_\mu B\phi + 2\mathbf{D}^\mu B\mathbf{D}_\mu A\psi + \varepsilon[A\phi\Box(A\phi) - \mathbf{D}^\mu(A\phi)\mathbf{D}_\mu(A\phi) + \mathbf{D}^\mu(A\psi)\mathbf{D}_\mu(A\psi)] = 0,$$

<sup>10</sup> Such a program was carried out by J. Luk (in the simpler case of the nonlinear stability of the trivial solution), for semi-linear wave equations verifying the null condition, see [18].



and

$$A^2\Box\psi - 2AD^\mu BD_\mu\phi - (D^\mu AD_\mu A + D^\mu BD_\mu B)\psi + \varepsilon[A\phi\Box(A\psi) - 2D^\mu(A\phi)D_\mu(A\psi)] = 0.$$

Using the formulas (1.10) these equations become

$$A^2(1 + \varepsilon\phi)\Box\phi + 2AD^\mu BD_\mu\psi - 2D^\mu BD_\mu B\phi + 2D^\mu BD_\mu A\psi + \varepsilon[A^2D^\mu\psi D_\mu\psi + 2A\psi D^\mu AD_\mu\psi + D^\mu AD_\mu A\psi^2 - A^2D^\mu\phi D_\mu\phi - D^\mu BD_\mu B\phi^2] = 0,$$

and

$$A^2(1 + \varepsilon\phi)\Box\psi - 2AD^\mu BD_\mu\phi - (D^\mu AD_\mu A + D^\mu BD_\mu B)\psi + \varepsilon[-2A^2D^\mu\phi D_\mu\psi - D^\mu AD_\mu A\phi\psi - D^\mu BD_\mu B\phi\psi - 2A\psi D^\mu AD_\mu\phi] = 0.$$

We divide the equations by  $A^2(1 + \varepsilon\phi)$  to conclude that

$$\begin{aligned} \Box\phi + 2\frac{D^\mu B}{A}D_\mu\psi - 2\frac{D^\mu BD_\mu B}{A^2}\phi + 2\frac{D^\mu BD_\mu A}{A^2}\psi &= \varepsilon\mathcal{N}_\phi^\varepsilon, \\ \Box\psi - 2\frac{D^\mu B}{A}D_\mu\phi - \frac{D^\mu AD_\mu A + D^\mu BD_\mu B}{A^2}\psi &= \varepsilon\mathcal{N}_\psi^\varepsilon, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \mathcal{N}_\phi^\varepsilon &= \frac{A^2D^\mu\phi D_\mu\phi - A^2D^\mu\psi D_\mu\psi - 2A\psi D^\mu AD_\mu\psi + D^\mu BD_\mu B\phi^2 - D^\mu AD_\mu A\psi^2}{A^2(1 + \varepsilon\phi)} \\ &+ \frac{\phi}{A^2(1 + \varepsilon\phi)}[2AD^\mu BD_\mu\psi - 2D^\mu BD_\mu B\phi + 2D^\mu BD_\mu A\psi], \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}_\psi^\varepsilon &= \frac{2A^2D^\mu\phi D_\mu\psi + (D^\mu AD_\mu A + D^\mu BD_\mu B)\phi\psi + 2A\psi D^\mu AD_\mu\phi}{A^2(1 + \varepsilon\phi)} \\ &- \frac{\phi}{A^2(1 + \varepsilon\phi)}[2AD^\mu BD_\mu\phi + (D^\mu AD_\mu A + D^\mu BD_\mu B)\psi]. \end{aligned}$$

The formulas for the nonlinear terms  $\mathcal{N}_\phi^\varepsilon$  and  $\mathcal{N}_\psi^\varepsilon$  can be simplified, and the calculations can be reversed. To summarize, we have proved the following:

**Proposition 2.1** *Assume  $I \subseteq \mathbb{R}$  is an interval,  $\varepsilon > 0$ , and  $(\phi, \psi) \in C^k(I : \mathbf{H}^{N_1-k}(\Sigma_t^{C_0}))$ ,  $k = 0, \dots, N_1$ . Then  $(A', B') = (A, B) + (\varepsilon A\phi, \varepsilon A\psi)$  is a solution of the wave-map equation (1.10) on the interval  $I$  if and only if  $(\phi, \psi)$  satisfy the*

nonlinear system

$$\begin{aligned} \square\phi + 2\frac{\mathbf{D}^\mu B}{A}\mathbf{D}_\mu\psi - 2\frac{\mathbf{D}^\mu B\mathbf{D}_\mu B}{A^2}\phi + 2\frac{\mathbf{D}^\mu B\mathbf{D}_\mu A}{A^2}\psi &= \varepsilon\mathcal{N}_\phi^\varepsilon, \\ \square\psi - 2\frac{\mathbf{D}^\mu B}{A}\mathbf{D}_\mu\phi - \frac{\mathbf{D}^\mu A\mathbf{D}_\mu A + \mathbf{D}^\mu B\mathbf{D}_\mu B}{A^2}\psi &= \varepsilon\mathcal{N}_\psi^\varepsilon, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \mathcal{N}_\phi^\varepsilon &= \frac{A^2(\mathbf{D}^\mu\phi\mathbf{D}_\mu\phi - \mathbf{D}^\mu\psi\mathbf{D}_\mu\psi) + (\phi\mathbf{D}^\mu B - \psi\mathbf{D}^\mu A)(2A\mathbf{D}_\mu\psi - \phi\mathbf{D}_\mu B + \psi\mathbf{D}_\mu A)}{A^2(1 + \varepsilon\phi)}, \\ \mathcal{N}_\psi^\varepsilon &= \frac{2A^2\mathbf{D}^\mu\phi\mathbf{D}_\mu\psi + 2A(\psi\mathbf{D}^\mu A - \phi\mathbf{D}^\mu B)\mathbf{D}_\mu\phi}{A^2(1 + \varepsilon\phi)}. \end{aligned} \tag{2.3}$$

### 2.1 The Energy–Momentum Tensor

We study now solutions of the system

$$\begin{aligned} \square\phi + 2\frac{\mathbf{D}^\mu B}{A}\mathbf{D}_\mu\psi - 2\frac{\mathbf{D}^\mu B\mathbf{D}_\mu B}{A^2}\phi + 2\frac{\mathbf{D}^\mu B\mathbf{D}_\mu A}{A^2}\psi &= \mathcal{N}_\phi, \\ \square\psi - 2\frac{\mathbf{D}^\mu B}{A}\mathbf{D}_\mu\phi - \frac{\mathbf{D}^\mu A\mathbf{D}_\mu A + \mathbf{D}^\mu B\mathbf{D}_\mu B}{A^2}\psi &= \mathcal{N}_\psi. \end{aligned} \tag{2.4}$$

Our main goal is to construct a suitable energy-momentum tensor that verifies a good divergence equation. More precisely, let

$$E_\mu := \mathbf{D}_\mu\phi + \psi A^{-1}\mathbf{D}_\mu B, \quad F_\mu := \mathbf{D}_\mu\psi - \phi A^{-1}\mathbf{D}_\mu B, \quad M_\mu := \frac{\phi\mathbf{D}_\mu B - \psi\mathbf{D}_\mu A}{A}. \tag{2.5}$$

Using the formulas

$$A\mathbf{D}_\mu\phi = AE_\mu - \psi\mathbf{D}_\mu B, \quad A\mathbf{D}_\mu\psi = AF_\mu + \phi\mathbf{D}_\mu B, \tag{2.6}$$

the identities (2.4) and (1.10) show that

$$\begin{aligned} \mathbf{D}^\mu E_\mu + \frac{\mathbf{D}^\mu B F_\mu}{A} - \frac{\mathbf{D}^\mu B M_\mu}{A} &= \mathcal{N}_\phi, \\ \mathbf{D}^\mu F_\mu - \frac{\mathbf{D}^\mu B E_\mu}{A} + \frac{\mathbf{D}^\mu A M_\mu}{A} &= \mathcal{N}_\psi, \\ \mathbf{D}^\mu M_\mu - \frac{\mathbf{D}^\mu B E_\mu}{A} + \frac{\mathbf{D}^\mu A F_\mu}{A} &= 0. \end{aligned} \tag{2.7}$$

We also calculate

$$\begin{aligned}
 \mathbf{D}_\mu E_\nu - \mathbf{D}_\nu E_\mu &= \frac{F_\mu \mathbf{D}_\nu B - F_\nu \mathbf{D}_\mu B}{A} + \frac{M_\mu \mathbf{D}_\nu B - M_\nu \mathbf{D}_\mu B}{A}, \\
 \mathbf{D}_\mu F_\nu - \mathbf{D}_\nu F_\mu &= -\frac{E_\mu \mathbf{D}_\nu B - E_\nu \mathbf{D}_\mu B}{A} - \frac{M_\mu \mathbf{D}_\nu A - M_\nu \mathbf{D}_\mu A}{A}, \\
 \mathbf{D}_\mu M_\nu - \mathbf{D}_\nu M_\mu &= \frac{E_\mu \mathbf{D}_\nu B - E_\nu \mathbf{D}_\mu B}{A} - \frac{F_\mu \mathbf{D}_\nu A - F_\nu \mathbf{D}_\mu A}{A}.
 \end{aligned} \tag{2.8}$$

Let

$$\begin{aligned}
 T_{\mu\nu} &:= E_\mu E_\nu + F_\mu F_\nu + M_\mu M_\nu, \\
 Q_{\mu\nu} &:= T_{\mu\nu} + \mathbf{g}_{\mu\nu} \mathcal{L}, \\
 \mathcal{L} &:= -(1/2) \mathbf{g}^{\alpha\beta} T_{\alpha\beta} = -(1/2)(E_\alpha E^\alpha + F_\alpha F^\alpha + M_\alpha M^\alpha).
 \end{aligned} \tag{2.9}$$

We calculate the divergence

$$\begin{aligned}
 \mathbf{D}^\mu Q_{\mu\nu} &= E_\nu \mathbf{D}^\mu E_\mu + E^\mu (\mathbf{D}_\mu E_\nu - \mathbf{D}_\nu E_\mu) \\
 &\quad + F_\nu \mathbf{D}^\mu F_\mu + F^\mu (\mathbf{D}_\mu F_\nu - \mathbf{D}_\nu F_\mu) \\
 &\quad + M_\nu \mathbf{D}^\mu M_\mu + M^\mu (\mathbf{D}_\mu M_\nu - \mathbf{D}_\nu M_\mu),
 \end{aligned}$$

Using (2.7) and (2.8) we calculate

$$\begin{aligned}
 &E_\nu \mathbf{D}^\mu E_\mu + E^\mu (\mathbf{D}_\mu E_\nu - \mathbf{D}_\nu E_\mu) \\
 &= \frac{E_\nu (\mathbf{D}^\mu B M_\mu - \mathbf{D}^\mu B F_\mu) - F_\nu E^\mu \mathbf{D}_\mu B - M_\nu E^\mu \mathbf{D}_\mu B}{A} \\
 &\quad + \frac{\mathbf{D}_\nu B (E^\mu F_\mu + E^\mu M_\mu)}{A} + \mathcal{N}_\phi E_\nu, \\
 &F_\nu \mathbf{D}^\mu F_\mu + F^\mu (\mathbf{D}_\mu F_\nu - \mathbf{D}_\nu F_\mu) \\
 &= \frac{E_\nu F^\mu \mathbf{D}_\mu B + F_\nu (\mathbf{D}^\mu B E_\mu - \mathbf{D}^\mu A M_\mu) + M_\nu F^\mu \mathbf{D}_\mu A}{A} \\
 &\quad + \frac{-\mathbf{D}_\nu B E^\mu F_\mu - \mathbf{D}_\nu A F^\mu M_\mu}{A} + \mathcal{N}_\psi F_\nu,
 \end{aligned}$$

and

$$\begin{aligned}
 &M_\nu \mathbf{D}^\mu M_\mu + M^\mu (\mathbf{D}_\mu M_\nu - \mathbf{D}_\nu M_\mu) \\
 &= \frac{-E_\nu M^\mu \mathbf{D}_\mu B + F_\nu M^\mu \mathbf{D}_\mu A + M_\nu (\mathbf{D}^\mu B E_\mu - \mathbf{D}^\mu A F_\mu)}{A} \\
 &\quad + \frac{\mathbf{D}_\nu B M^\mu E_\mu - \mathbf{D}_\nu A M^\mu F_\mu}{A}.
 \end{aligned}$$

Therefore

$$\mathbf{D}^\mu Q_{\mu\nu} = \frac{2\mathbf{D}_\nu B M^\mu E_\mu - 2\mathbf{D}_\nu A M^\mu F_\mu}{A} + \mathcal{N}_\phi E_\nu + \mathcal{N}_\psi F_\nu. \tag{2.10}$$

### 2.2 Divergence Identities

Given a vector-field  $X$ , a function  $w$ , and 1-forms  $m, m'$  we define the form

$$\begin{aligned}
 P_\mu &= P_\mu[X, w, m, m'] := Q_{\mu\nu}X^\nu + \frac{1}{2}w(\phi E_\mu + \psi F_\mu) \\
 &\quad - \frac{1}{4}\mathbf{D}_\mu w(\phi^2 + \psi^2) + \frac{1}{4}(m_\mu\phi^2 + m'_\mu\psi^2). \tag{2.11}
 \end{aligned}$$

Then, using (2.5)–(2.7) we calculate the divergence

$$\begin{aligned}
 \mathbf{D}^\mu P_\mu &= X^\nu \mathcal{J}_\nu + \frac{1}{2}Q_{\mu\nu}^{(X)}\pi^{\mu\nu} + \frac{1}{2}\mathbf{D}^\mu w(\phi E_\mu + \psi F_\mu) + \frac{1}{2}w(\mathbf{D}^\mu\phi E_\mu + \mathbf{D}^\mu\psi F_\mu) \\
 &\quad + \frac{1}{2}w(\phi\mathbf{D}^\mu E_\mu + \psi\mathbf{D}^\mu F_\mu) - \frac{1}{4}\square w(\phi^2 + \psi^2) - \frac{1}{2}\mathbf{D}^\mu w(\phi\mathbf{D}_\mu\phi + \psi\mathbf{D}_\mu\psi) \\
 &\quad + \frac{1}{4}(\phi^2\mathbf{D}^\mu m_\mu + \psi^2\mathbf{D}^\mu m'_\mu) + \frac{1}{2}(\phi m^\mu\mathbf{D}_\mu\phi + \psi m'^\mu\mathbf{D}_\mu\psi) \\
 &= X^\nu \mathcal{J}_\nu + \frac{1}{2}Q_{\mu\nu}^{(X)}\pi^{\mu\nu} - \frac{1}{4}\square w(\phi^2 + \psi^2) \\
 &\quad + \frac{1}{4}(\phi^2\mathbf{D}^\mu m_\mu + \psi^2\mathbf{D}^\mu m'_\mu) + \frac{1}{2}(\phi m^\mu\mathbf{D}_\mu\phi + \psi m'^\mu\mathbf{D}_\mu\psi) + E',
 \end{aligned}$$

where

$$\begin{aligned}
 E' &= \frac{1}{2}\mathbf{D}^\mu w(\phi E_\mu + \psi F_\mu - \phi\mathbf{D}_\mu\phi - \psi\mathbf{D}_\mu\psi) \\
 &\quad + \frac{1}{2}w(\mathbf{D}^\mu\phi E_\mu + \mathbf{D}^\mu\psi F_\mu + \phi\mathbf{D}^\mu E_\mu + \psi\mathbf{D}^\mu F_\mu) \\
 &= 0 + \frac{1}{2}w(E^\mu E_\mu + F^\mu F_\mu + M^\mu M_\mu + \phi\mathcal{N}_\phi + \psi\mathcal{N}_\psi).
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \mathbf{D}^\mu P_\mu &= X^\nu \mathcal{J}_\nu + \frac{1}{2}Q_{\mu\nu}^{(X)}\pi^{\mu\nu} - \frac{1}{4}\square w(\phi^2 + \psi^2) - w\mathcal{L} \\
 &\quad + \frac{1}{4}(\phi^2\mathbf{D}^\mu m_\mu + \psi^2\mathbf{D}^\mu m'_\mu) + \frac{1}{2}(\phi m^\mu\mathbf{D}_\mu\phi + \psi m'^\mu\mathbf{D}_\mu\psi) \\
 &\quad + \frac{1}{2}w(\phi\mathcal{N}_\phi + \psi\mathcal{N}_\psi).
 \end{aligned}$$

### 2.3 Summary

We summarize the results of the section in the following:

**Proposition 2.2** (i) Assume that  $(\phi, \psi) \in C^k(I : \mathbf{H}^{N_0-k}(\Sigma_t^{C_0}))$ ,  $k = 0, \dots, N_0$  satisfy the system (2.4). Let

$$\begin{aligned}
 E_\mu &:= \mathbf{D}_\mu \phi + \psi A^{-1} \mathbf{D}_\mu B, & F_\mu &:= \mathbf{D}_\mu \psi - \phi A^{-1} \mathbf{D}_\mu B, & M_\mu &:= \frac{\phi \mathbf{D}_\mu B - \psi \mathbf{D}_\mu A}{A}, \\
 Q_{\mu\nu} &:= E_\mu E_\nu + F_\mu F_\nu + M_\mu M_\nu + \mathbf{g}_{\mu\nu} \mathcal{L}, \\
 \mathcal{L} &:= -\frac{1}{2}(E_\alpha E^\alpha + F_\alpha F^\alpha + M_\alpha M^\alpha).
 \end{aligned}
 \tag{2.12}$$

Then

$$\mathbf{D}^\mu Q_{\mu\nu} =: J_\nu = \frac{2\mathbf{D}_\nu B M^\mu E_\mu - 2\mathbf{D}_\nu A M^\mu F_\mu}{A} + \mathcal{N}_\phi E_\nu + \mathcal{N}_\psi F_\nu.
 \tag{2.13}$$

(ii) Let

$$\begin{aligned}
 P_\mu &= P_\mu[X, w, m, m'] := Q_{\mu\nu} X^\nu + \frac{1}{2}w(\phi E_\mu + \psi F_\mu) \\
 &\quad - \frac{1}{4}\mathbf{D}_\mu w(\phi^2 + \psi^2) + \frac{1}{4}(m_\mu \phi^2 + m'_\mu \psi^2),
 \end{aligned}
 \tag{2.14}$$

where  $X$  is a smooth vector-field,  $w$  is a smooth function, and  $m, m'$  are smooth 1-forms. Then

$$\begin{aligned}
 2\mathbf{D}^\mu P_\mu &= 2X^\nu J_\nu + Q_{\mu\nu}{}^{(X)}\pi^{\mu\nu} - 2w\mathcal{L} + (\phi m^\mu \mathbf{D}_\mu \phi + \psi m'^\mu \mathbf{D}_\mu \psi) \\
 &\quad + \frac{1}{2}\phi^2(\mathbf{D}^\mu m_\mu - \square w) + \frac{1}{2}\psi^2(\mathbf{D}^\mu m'_\mu - \square w) + w(\phi \mathcal{N}_\phi + \psi \mathcal{N}_\psi).
 \end{aligned}
 \tag{2.15}$$

Note that theorem 1.3 is an immediate consequence of the first part of the proposition. Indeed, assuming that  $(\mathcal{N}_\phi, \mathcal{N}_\psi) = 0$  it is immediate that  $J$  is orthogonal to  $\mathbf{T}$ . The positivity of the energy momentum tensor  $\mathcal{Q}$  is an immediate consequence of its form (2.12).

### 3 Main Ideas in the Proof of Theorem 1.7

In this section we provide main ideas and motivation for the various choices we need to make in the proof of theorem 1.7. Our proof follows the well established pattern of proving integrated local energy decay estimates on black holes, such as Schwarzschild, for which the ergoregion is trivial and the trapped region is contained to a level surface  $r = r^* > r_{\mathcal{H}}$ . It is quite fortunate that our axially symmetric linearized system can be treated in the same manner. Though our treatment follows the clear and efficient approach of [19], we should point out that many of the ideas go back to other authors such as [5, 6, 10]. An essential ingredient in the proof is to take into account the red shift effects of the horizon, idea which goes back to [10].

In our problem we need to make two important modifications. Most importantly, to get any estimate at all, we need to account for the source term  $\mathcal{J}$ . This requires, in particular, a serious modification of the current  $P_\mu$  in (2.15), modification which adds considerably to the complexity of the proof.

The second important modification has to do with the presence of weights in our main estimate. Typically, integrated decay estimates are designed to deal with the region close to the black hole, most importantly the trapping region. They are then complemented by weighted estimates in the asymptotic region. Thus, for example, J. Luk (see [18]), relies on an integrated local decay estimate (proved earlier by Dafermos-Rodnianski (see [12]) for small  $a/M$ ), which he combines with weighted estimates in the asymptotic region based on a straightforward adaptation of the classical conformal method. The use of conformal method, however, is quite awkward in the black hole region, because the weights involved in the conformal method lead to errors which grow linearly in  $t$ . This problem was later fixed by a different method of Dafermos-Rodnianski in [11], who replace the conformal method by  $r$ -weighted estimates. The new method allows one to prove decay estimate for the energy associated to hypersurfaces which are spacelike near the black hole region but become null in the asymptotic region. This depends, however, on having first derived an integrated local decay estimate<sup>11</sup>. In our work here we refine the analysis significantly by deriving  $r$ -weighted estimates for the *outgoing energy* across spacelike hypersurfaces, *simultaneously* with the integrated local decay estimates.

### 3.1 Outline of the Proof

We discuss now the main ideas in the proof. For simplicity, we consider only the equation for  $\psi$  in the Schwarzschild case  $a = 0$ , which carries most of the conceptual difficulties of the problem. In this case  $B = 0$ ,  $A = r^2(\sin \theta)^2$ , and the equation is

$$\square\psi - \frac{4 - 8(M/r)(\sin \theta)^2}{r^2(\sin \theta)^2}\psi = 0. \tag{3.1}$$

As in (2.2) we define

$$F_\mu := \mathbf{D}_\mu \psi, \quad M_\mu := \frac{-\psi \mathbf{D}_\mu A}{A}, \quad Q_{\mu\nu} := F_\mu F_\nu + M_\mu M_\nu - \frac{1}{2} \mathbf{g}_{\mu\nu} (F_\alpha F^\alpha + M_\alpha M^\alpha). \tag{3.2}$$

For suitable triplets  $(X, w, m')$  we define

$$\tilde{P}_\mu = \tilde{P}_\mu[X, w, m'] := Q_{\mu\nu} X^\nu + \frac{w}{2} \psi F_\mu - \frac{\psi^2}{4} \mathbf{D}_\mu w + \frac{\psi^2}{4} m'_\mu - \frac{X^\nu \mathbf{D}_\nu A}{A} \frac{\mathbf{D}_\mu A}{A} \psi^2. \tag{3.3}$$

Notice the correction  $-\frac{X^\nu \mathbf{D}_\nu A}{A} \frac{\mathbf{D}_\mu A}{A} \psi^2$ , compared to the definition of  $P$  in (2.14), which is needed to partially compensate for the source term  $J$ . Then we have the divergence identity

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<sup>11</sup> The  $r$ -weighted estimates produce boundary terms which are estimated with the help of the integrated decay estimate. Because of the degenerate nature of this latter, the method leads to an overall a loss of derivatives.

$$2\mathbf{D}^\mu \tilde{P}_\mu = \sum_{j=1}^5 L^j, \tag{3.4}$$

where

$$\begin{aligned} L^1 &= L^1[X, w, m'] := Q_{\mu\nu}^{(X)} \pi^{\mu\nu} + w(F_\alpha F^\alpha + M_\alpha M^\alpha), \\ L^2 &= L^2[X, w, m'] := \psi m'^\mu \mathbf{D}_\mu \psi, \\ L^3 &= L^3[X, w, m'] := \frac{1}{2} \psi^2 (\mathbf{D}^\mu m'_\mu - \square w), \\ L^4 &= L^4[X, w, m'] := -2\mathbf{D}^\mu \left[ \frac{X^\nu \mathbf{D}_\nu A}{A} \frac{\mathbf{D}_\mu A}{A} \right] \psi^2. \end{aligned} \tag{3.5}$$

The divergence identity gives

$$\int_{\Sigma_{t_1}^c} \tilde{P}_\mu n_0^\mu d\mu_{t_1} = \int_{\Sigma_{t_2}^c} \tilde{P}_\mu n_0^\mu d\mu_{t_2} + \int_{\mathcal{N}_{[t_1, t_2]}^c} \tilde{P}_\mu k_0^\mu d\mu_c + \int_{\mathcal{D}_{[t_1, t_2]}^c} \mathbf{D}^\mu \tilde{P}_\mu d\mu, \tag{3.6}$$

where  $t_1, t_2 \in [0, T]$ ,  $c \in (c_0, 2M]$ ,  $n_0 := n/|\mathbf{g}^{33}|^{1/2}$ ,  $k_0 := k/|\mathbf{g}^{22}|^{1/2}$ , and the integration is with respect to the natural measures induced by the metric  $\mathbf{g}$ . To prove the main theorem we need to choose a suitable multiplier triplet  $(X, w, m')$  in a such a way that all the terms in the identity above are nonnegative. This is the *method of simultaneous inequalities* of Marzuola–Metcalf–Tataru–Tohaneanu [19].

To accomplish our task we need to superimpose four different choices of multiplier triplets  $(X, w, m')$ , denoted  $(X_{(k)}, w_{(k)}, m'_{(k)})$ ,  $k \in \{1, 2, 3, 4\}$ . The first multiplier ( $k = 1$ ) is important in a neighborhood of the trapped set  $\{r = 3M\}$ ; the second multiplier ( $k = 2$ ) is important in a neighborhood of the horizon  $\{r = r_{\mathcal{H}}\}$ ; the third multiplier ( $k = 3$ ) is important at infinity, in the construction of outgoing energies at infinity; the fourth multiplier is important to control the term  $L^4$ , which is connected to the presence of the nontrivial potential in (3.1).

### 3.1.1 The Multipliers $(X_{(1)}, w_{(1)}, m'_{(1)})$ and $(X_{(2)}, w_{(2)}, m'_{(2)})$ .

The first two multipliers are similar to the multipliers used in [19] in the case of the homogeneous wave equation. Set

$$\begin{aligned} X_{(1)} &:= f_1(r)\partial_2 + g_1(r)\partial_3, & f_1(r) &:= \frac{a_1(r)\Delta}{r^2}, & g_1(r) &:= \frac{a_1(r)\chi(r)2M}{r} + 1, \\ w_{(1)}(r, \theta) &:= f'_1(r) + f_1(r)\partial_r \log(r^4/\Delta) - \epsilon_1 \tilde{w}(r), \\ \tilde{w}(r) &:= M^2(r - 33M/16)^3 (r - r^*)^2 r^{-8} \mathbf{1}_{[33M/16, \infty)}(r), \\ m'_{(1)} &:= 0, \end{aligned}$$

where  $r^* = 3M$ ,  $\epsilon_1 \in (0, 1]$  is a small constant, and  $a_1 : (0, \infty) \rightarrow \mathbb{R}$  is a smooth function. The important function  $a_1$ , which vanishes on the trapped region  $\{r = r^*\}$ , is defined by

$$R(r) := (r - r^*)(r + 2M) + 6M^2 \log \left( \frac{r - 2M}{r^* - 2M} \right),$$

$$a_1(r) := r^{-2} \delta^{-1} \kappa(\delta R(r)) + \left[ \frac{r^* - 2M}{r} - \frac{6M^2}{r^2} \log \left( \frac{r - r_{\mathcal{H}}}{r^* - r_{\mathcal{H}}} \right) \right] \chi_{\geq DM}(r),$$

where  $D$  is a sufficiently large constant,  $\delta = \epsilon_2^2 M^{-2}$  for a small positive constant  $\epsilon_2$ , and  $\kappa : \mathbb{R} \rightarrow \mathbb{R}$  is an increasing smooth function satisfying  $\kappa(y) = y$  on  $[-1, \infty)$  and  $\kappa(y) = -2$  on  $(-\infty, -3]$ . This is essentially the choice of [19], except for the correction at infinity, containing the cutoff function  $\chi_{\geq DM}$ ; this correction is needed in order to match properly with the third multiplier at infinity to produce outgoing energies.

In a small neighborhood of the horizon we need to use the redshift effect. We define the second multiplier

$$X_{(2)} := f_2(r) \partial_2 + g_2(r) \partial_3, \quad f_2(r) := -\epsilon_2 a_2(r), \quad g_2(r) := \epsilon_2 a_2(r) (1 - \epsilon_2),$$

$$w_{(2)}(r) := -2\epsilon_2 a_2(r) / r, \quad m'_{(2)2} = m'_{(2)3} := \epsilon_2 M^{-2} \gamma(r), \quad m'_{(2)1} = m'_{(2)4} := 0,$$

where

$$a_2(r) := \begin{cases} M^{-3} (9M/4 - r)^3 & \text{if } r \leq 9M/4, \\ 0 & \text{if } r \geq 9M/4, \end{cases}$$

and  $\gamma : [c_0, \infty) \rightarrow [0, 1]$  is a function supported in  $[c_0, 17M/8]$ , satisfying  $\gamma(2M) = 1/2$  and the more technical property (4.38). As in [19], the multipliers  $(X_{(1)}, w_{(1)}, m'_{(1)})$  and  $(X_{(2)}, w_{(2)}, m'_{(2)})$  cooperate well to generate mostly positive bulk contributions. More precisely, the constants  $\epsilon_1, \epsilon_2$  can be chosen such that, for some absolute constant  $\epsilon_3 > 0$ ,

$$\sum_{j=1}^4 (L^j_{(1)} + L^j_{(2)}) \geq \epsilon_3 \sum_{Y \in \{F, M\}} \left[ \frac{(r - r^*)^2}{r^3} (Y_1/r)^2 + \frac{M^2}{r^3} (Y_2)^2 + \frac{M^2 (r - r^*)^2}{r^5} (Y_3)^2 \right]$$

$$+ \epsilon_3 \frac{M}{r^4} \psi^2 - \epsilon_3^{-1} \frac{M}{r^4} \mathbf{1}_{[DM, \infty)}(r) \psi^2 + \tilde{L}, \tag{3.7}$$

where

$$\tilde{L} := \frac{8\Delta(r^2 - 4Mr)}{r^7} a_1(r) \psi^2 + (1 - 2C_1 \epsilon_1) \mathbf{1}_{[r^*, \infty)}(r) \left\{ \frac{M}{r^4} \left( 7 - \frac{44M}{r} + \frac{72M^2}{r^2} \right) \psi^2 \right.$$

$$\left. + \frac{8a_1(r)(r - r^*)}{r^4} \frac{(\cos \theta)^2}{(\sin \theta)^2} \psi^2 + \frac{2a_1(r)(r - r^*)}{r^4} (F_1)^2 + 2a'_1(r) \frac{\Delta^2}{r^4} (F_2)^2 \right\}. \tag{3.8}$$



Moreover, letting  $\tilde{P}_{(j)} := \tilde{P}_\mu[X_{(j)}, w_{(j)}, m'_{(j)}]$ ,  $j = 1, 2$ , we have

$$2(\tilde{P}_{(1)\mu} + \tilde{P}_{(2)\mu})k^\mu \geq \epsilon_3 \sum_{Y \in \{F, M\}} [(Y_1/r)^2 + (Y_2)^2(2 - c/M)] + \epsilon_3 M^{-2} \psi^2 - \epsilon_3^{-1} (F_3)^2, \tag{3.9}$$

along  $\mathcal{N}_{[t_1, t_2]}^c$ . Also, with  $p = 2M/r$ ,

$$2(\tilde{P}_{(1)\mu} + \tilde{P}_{(2)\mu})n^\mu \geq -\epsilon_3^{-1} \{ \tilde{e}_0 + \mathbf{1}_{[8M, 2DM]}(r)(F_3)^2 \} - \frac{\chi_{\geq 8M}(r)(1-p)}{r^2} \partial_2(r\psi^2) + \epsilon_3 (F_2)^2 \mathbf{1}_{(c_0, 17M/8]}(r), \tag{3.10}$$

and

$$2(\tilde{P}_{(1)\mu} + \tilde{P}_{(2)\mu})n^\mu \leq \epsilon_3^{-1} \{ \tilde{e}_0 + \mathbf{1}_{[8M, 2DM]}(r)(F_3)^2 \} - \frac{\chi_{\geq 8M}(r)(1-p)}{r^2} \partial_2(r\psi^2) + \epsilon_3^{-1} (F_2)^2 \mathbf{1}_{(c_0, 17M/8]}(r), \tag{3.11}$$

where

$$\tilde{e}_0 = \frac{(F_1)^2 + (M_1)^2}{r^2} + (L\psi)^2 + \frac{M^2|r - 2M|}{r^3} (F_2)^2 + \frac{M^2}{r^2} (F_3)^2 + \frac{1}{r^2} \psi^2.$$

Notice that the bulk terms in (3.7) are mostly positive, with the exception of the term  $\tilde{L}$ . The terms along  $\mathcal{N}_{[t_1, t_2]}^c$  are also mostly positive. On the other hand, the bounds (3.10) and (3.11) we have so far on the integrals along the hypersurfaces  $\Sigma_t^c$  are very weak; these bounds will be improved by choosing a suitable multiplier  $(X_{(3)}, w_{(3)}, m'_{(3)})$  at infinity.

### 3.1.2 The Multiplier $(X_{(4)}, w_{(4)}, m'_{(4)})$

Our next goal is to control the term  $\tilde{L}$  in (3.8). This is a new term, when compared to solutions of the homogeneous wave equation, connected to the nontrivial potential in (3.1) and the bulk term  $L^4$  in (3.5). Since  $a'_1(r) \geq 0$  and  $a_1(r)(r - r^*) \geq 0$ , this term can only be problematic in the region  $\{r \in [r^*, 4M]\}$ . We define

$$\begin{aligned} X_{(4)} &:= 0, & w_{(4)} &:= 0, \\ \tilde{m}'_{(4)1}(r, \theta) &:= -(1 - 2C_1\epsilon_1) \frac{8(r - r^*)a_1(r)\chi_{\leq 6R}(r) \cos \theta}{r^2} \frac{1}{\sin \theta} \mathbf{1}_{[r^*, \infty)}(r), \\ \tilde{m}'_{(4)2}(r) &:= (1 - 2C_1\epsilon_1) \frac{2b(r)}{\Delta}, & \tilde{m}'_{(4)3} &:= 0, & \tilde{m}'_{(4)4} &:= 0, \end{aligned}$$

for a suitable function  $b$  supported in  $[r^*, 4M]$ . Careful estimates, as in Lemma 5.3, and completion of squares show that one can choose the function  $b$  in such a way that

$$L^1_{(4)} = L^4_{(4)} = 0, \quad \tilde{L} + L^2_{(4)} + L^3_{(4)} \geq -C_2|2M - c_0|r^{-4}\psi^2 \quad (3.12)$$

for some constant  $C_2$  sufficiently large, and

$$|2\tilde{P}_{(4)\mu}n^\mu| \lesssim \epsilon_3^{-1}\psi^2/r^2 \quad \text{and} \quad 2\tilde{P}_{(4)\mu}k^\mu = 0 \quad \text{along} \quad \mathcal{N}^c_{[t_1, t_2]}. \quad (3.13)$$

These two bounds can be combined with (3.7)–(3.11) to effectively remove the contribution of the term  $\tilde{L}$ .

### 3.1.3 The Multiplier $(X_{(3)}, w_{(3)}, m'_{(3)})$

Finally, we are ready to define the multiplier at infinity and close the estimate. First of all, to obtain any simultaneous estimate at all, we need to make sure that the contributions of the integrals of  $2\tilde{P}_\mu n^\mu$  on the hypersurfaces  $\Sigma_t^c$  are positive. So far, these integrals are far from positive, in view of the estimates (3.10), (3.11), and (3.13).

The formula (A.16) shows that

$$\begin{aligned} 2n^\mu \tilde{P}_\mu [K \partial_3, 0, 0] &= 2n^\mu Q_{\mu\nu} (K \partial_3)^\nu \\ &= K \sum_{Y \in \{F, M\}} [\mathbf{g}^{11}(Y_1)^2 + \mathbf{g}^{22}(Y_2)^2 + (-\mathbf{g}^{33})(Y_3)^2]. \end{aligned}$$

Therefore, one could make the integrals of  $2\tilde{P}_\mu n^\mu$  along the hypersurfaces  $\Sigma_t^c$  positive by adding a multiplier of the form  $(K \partial_3, 0, 0)$ , for some positive constant  $K$  sufficiently large, and using a Hardy estimate to control the integral of the 0's order term in terms of the first order terms. Notice that such a multiplier does not affect the bulk integrals. This is precisely the argument used in [19] to close the simultaneous estimate for the standard energy for the wave equation.

In our case, however, we are looking to prove stronger estimates involving outgoing energies. A multiplier of the form  $(K \partial_3, 0, 0)$  is not allowed, since this would create contributions at infinity of the form  $(F_2)^2 + (F_3)^2$ , which are unacceptable in view of the definition (1.32). Instead, we choose the last multiplier of the form

$$\begin{aligned} X_{(3)} &:= f_3 \partial_2 + \left( \frac{f_3}{1-p} + g_3 \right) \partial_3, & w_{(3)} &:= \frac{2f_3}{r}, \\ m'_{(3)1} &:= m'_{(3)4} := 0, & m'_{(3)2} &:= \frac{2h_3}{r(1-p)}, & m'_{(3)3} &:= -\frac{2h_3}{r}, \end{aligned} \quad (3.14)$$

for some suitable functions  $f_3, g_3, h_3$ . The function  $f_3$  should behave like  $(r/M)^\alpha$  for large  $r$ , in order to produce the desired power in the outgoing energy. To make sure that it does not interfere with the crucial trapping region we have to choose it to vanish for  $r \leq 8M$ . The role of the function  $g_3$  is to match, to some extent, the role played by the multiplier  $K\mathbf{T}$  in the boundary estimate discussed earlier. Thus we choose  $g_3$  to

be a very large constant when  $r \leq C_4M$ , for some large constant  $C_4$ , but we choose it to decay as  $r \rightarrow \infty$ , at the rate  $r^{\alpha-2}$ , such that it does not interfere with the outgoing energy. Precise choices are provided in (5.59)–(5.61),

$$f_3(r) := \epsilon_4 \chi_{\geq 8M}(r) e^{\beta(r)}, \quad g_3(r) := \int_r^\infty \left[ \rho(s) + \frac{\epsilon_4 M^2}{s^3} f_3(s) \right] ds,$$

where

$$\beta(8M) := 0, \quad \beta'(r) := \left( \frac{4M}{r^2} + \frac{1}{r} \right) (1 - \chi_{\geq C_4 M}(r)) + \frac{\alpha}{r} \chi_{\geq C_4 M}(r),$$

and

$$\rho(r) := \delta M^{-1} \left[ \chi_{\geq C_4 M}(r) + \chi_{\geq 4C_4 M}(r) \left( C_4^7 e^{\beta(r)} \frac{M^3}{r^3} - 1 \right) \right].$$

The constants  $\epsilon_4, C_4$  satisfy  $\epsilon_4 = \epsilon_3^2$  and  $C_4 \geq \epsilon_4^{-4} \alpha^{-1} (2 - \alpha)^{-1}$ , while  $\delta \in [10^{-4} C_4^{-3}, 10^4 C_4^{-3}]$  is such that  $\int_{C_4 M}^\infty \rho(s) ds = C_4$ .

The function  $h_3$  can be chosen explicitly in terms of  $f_3$  and  $g_3$ , in such a way to complete squares and create positive 0's order contributions. The positivity of the bulk terms in (3.7) and (3.12), together with the choice  $\epsilon_4 \ll \epsilon_3$ , is used to show positivity of the total bulk contribution in the transition region. Overall, we derive the desired lower bound on the bulk term,

$$\begin{aligned} \sum_{j=1}^4 (L_{(1)}^j + L_{(2)}^j + L_{(4)}^j + L_{(3)}^j) &\gtrsim_\alpha e^\beta \left\{ \frac{(r - r^*)^2}{r^2} \frac{(\partial_1 \psi)^2 + (\psi / \sin \theta)^2}{r^3} \right. \\ &\left. + \frac{M^2}{r^3} (\partial_2 \psi)^2 + \frac{M^2 (r - r^*)^2}{r^5} (\partial_3 \phi)^2 + \frac{\psi^2}{r^3} + \frac{(L\psi)^2}{r} \right\}. \end{aligned} \tag{3.15}$$

At the same time one can estimate precisely the size of the term  $2\tilde{P}_{(3)\mu} n^\mu$  at infinity, and use positivity of the function  $g_3$  in the transition region to absorb the contributions of the other terms  $2\tilde{P}_{(j)\mu} n^\mu, j \in \{1, 2, 4\}$ . Overall, we find that

$$\int_{\Sigma_t^c} 2[\tilde{P}_{(1)\mu} + \tilde{P}_{(2)\mu} + \tilde{P}_{(3)\mu} + \tilde{P}_{(4)\mu}] n_0^\mu d\mu_t \approx_\alpha \int_{\Sigma_t^c} e^\beta [e(\phi)^2 + e(\psi)^2] d\mu_t. \tag{3.16}$$

Finally we find that

$$2[\tilde{P}_{(1)\mu} + \tilde{P}_{(2)\mu} + \tilde{P}_{(3)\mu} + \tilde{P}_{(4)\mu}] k^\mu \geq 0 \quad \text{along } \mathcal{N}_{[0, T]}^c. \tag{3.17}$$

The theorem follows from (3.15)–(3.17), and the divergence identity (3.6).

### 4 The Wave Equation in the Schwarzschild Spacetime

We show first how to prove Theorem 1.7 in the simplest case:  $a = 0$  (the Schwarzschild spacetime) and  $\psi = 0$ . In this case  $B = 0$  and we are simply considering  $\mathbf{Z}$ -invariant solutions of the wave equation

$$\square\phi = 0.$$

In the rest of this section we use the coordinates  $(\theta, r, u_+, \phi_+)$  and the induced vector-fields  $\partial_1 = \partial_\theta, \partial_2 = \partial_r, \partial_3 = \partial_t, \partial_4 = \partial_\phi$ , see (1.21)–(1.22). For simplicity of notation, we identify functions that depend on  $r$  (or on some of the other variables) with the corresponding functions defined on the spacetime.

Notice that

$$q^2 = r^2, \quad p = \frac{2M}{r}, \tag{4.1}$$

with  $p$  introduced in (1.7). The nontrivial components of the metric are

$$\begin{aligned} \mathbf{g}^{11} &= r^{-2}, \\ \mathbf{g}^{22} &= 1 - p, \\ \mathbf{g}^{23} &= p\chi, \\ \mathbf{g}^{33} &= \frac{-1 + p^2\chi^2}{1 - p}, \\ \mathbf{g}^{44} &= \frac{1}{r^2(\sin\theta)^2}. \end{aligned} \tag{4.2}$$

Given a function  $H$  that depends only on  $r$ , the formula (A.9) shows that

$$\square H = \mathbf{g}^{22}\partial_2^2 H + D^2\partial_2 H = \frac{r - 2M}{r}\partial_2^2 H + \frac{2r - 2M}{r^2}\partial_2 H. \tag{4.3}$$

Similarly, if  $m$  is a 1-form with  $m_4 = 0, \mathcal{L}_T m = 0, \mathcal{L}_Z m = 0$ , then

$$\begin{aligned} \mathbf{D}^\mu m_\mu &= \mathbf{g}^{\alpha\beta}\partial_\alpha m_\beta + [\partial_\mu \mathbf{g}^{\mu\nu} + (1/2)\mathbf{g}^{\mu\nu}\partial_\mu \log |r^4(\sin\theta)^2|]m_\nu \\ &= \frac{1}{r^2}\left[\partial_1 m_1 + \frac{\cos\theta}{\sin\theta}m_1\right] + \left[(1 - p)\partial_2 m_2 + \frac{2r - 2M}{r^2}m_2\right] \\ &\quad + p\left[\chi\partial_2 m_3 + (\chi' + \chi/r)m_3\right]. \end{aligned} \tag{4.4}$$

Therefore, given a vector-field

$$X = f(r)\partial_2 + g(r)\partial_3, \tag{4.5}$$

as in (A.12), and a 1-form  $Y$  with  $Y_4 = 0$ , and letting

$${}^{(Y)}Q_{\mu\nu} = Y_\mu Y_\nu - (1/2)\mathbf{g}_{\mu\nu}(Y_\rho Y^\rho),$$

we have, see (A.15)–(A.17),

$$\begin{aligned}
 {}^{(Y)}Q_{\mu\nu}{}^{(X)}\pi^{\mu\nu} &= (Y_1)^2 \frac{-f'(r)}{r^2} + (Y_2)^2 \frac{-f(r)(2r - 2M) + f'(r)(r^2 - 2Mr)}{r^2} \\
 &\quad + (Y_3)^2 \left[ -f(r)\partial_2 \mathbf{g}^{33} + 2g'(r)\mathbf{g}^{23} - f'(r)\mathbf{g}^{33} - \frac{2rf(r)\mathbf{g}^{33}}{r^2} \right] \\
 &\quad + 2Y_2Y_3 \frac{-2Mrf(r)\chi'(r) - 2Mf(r)\chi(r) + g'(r)(r^2 - 2Mr)}{r^2},
 \end{aligned} \tag{4.6}$$

$$\begin{aligned}
 2{}^{(Y)}Q(n, X) &= (Y_1)^2 \frac{g(r)}{r^2} + (Y_2)^2 [g(r)(1 - p) - 2f(r)\mathbf{g}^{23}] \\
 &\quad + (Y_3)^2 [-g(r)\mathbf{g}^{33}] + 2Y_2Y_3 [-f(r)\mathbf{g}^{33}],
 \end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
 2{}^{(Y)}Q(k, X) &= (Y_1)^2 \frac{-f(r)}{r^2} + (Y_2)^2 [f(r)(1 - p)] \\
 &\quad + (Y_3)^2 [-f(r)\mathbf{g}^{33} + 2g(r)\mathbf{g}^{23}] + 2Y_2Y_3 [g(r)(1 - p)].
 \end{aligned} \tag{4.8}$$

Here  $f'$  and  $g'$  denote the derivatives with respect to  $r$  of the functions  $f$  and  $g$ , and

$$n = -\mathbf{g}^{3\mu}\partial_\mu = -\mathbf{g}^{23}\partial_2 - \mathbf{g}^{33}\partial_3, \quad k = \mathbf{g}^{2\mu}\partial_\mu = (1 - p)\partial_2 + \mathbf{g}^{23}\partial_3. \tag{4.9}$$

In this section we prove the following:

**Theorem 4.1** *Assume that  $M \in (0, \infty)$ ,  $N_0 = 4$ ,  $a = 0$  and  $c_0 := 2M - \bar{\varepsilon}M$ , where  $\bar{\varepsilon} \in [0, 1)$  is a sufficiently small constant. Assume that  $T \geq 0$ , and  $\phi \in C^k([0, T] : H^{N_0-k}(\Sigma_t^{c_0}))$ ,  $k \in [0, N_0]$ , is a  $\mathbf{Z}$ -invariant real-valued solution of the wave equation*

$$\square\phi = 0. \tag{4.10}$$

Then, for any  $\alpha \in (0, 2)$  and any  $t_1 \leq t_2 \in [0, T]$ ,

$$\mathcal{E}_\alpha^{c_0}(t_2) + \mathcal{B}_\alpha^{c_0}(t_1, t_2) \leq \bar{C}_\alpha \mathcal{E}_\alpha^{c_0}(t_1), \tag{4.11}$$

where  $\bar{C}_\alpha$  is a large constant that may depend on  $\alpha$ ,

$$E_\mu := \mathbf{D}_\mu\phi, \quad L\phi := \chi_{\geq 4M}(r) \left( \partial_2 + \frac{1}{1-p}\partial_3 \right) \phi = \chi_{\geq 4M}(r) \left( E_2 + \frac{1}{1-p}E_3 \right), \tag{4.12}$$

$$\mathcal{E}_\alpha^{c_0}(t) := \int_{\Sigma_t^{c_0}} \frac{r^\alpha}{M^\alpha} \left[ (E_1/r)^2 + (L\phi)^2 + M^2 r^{-2} [(E_2)^2 + (E_3^2)] + r^{-2}\phi^2 \right] d\mu_t, \tag{4.13}$$

$$\begin{aligned} \mathcal{B}_\alpha^{c_0}(t_1, t_2) := & \int_{\mathcal{D}_{[t_1, t_2]}^{c_0}} \frac{r^\alpha}{M^\alpha} \left\{ \frac{(r - 3M)^2 (E_1)^2}{r^3} + \frac{1}{r} (L\phi)^2 \right. \\ & \left. + \frac{1}{r^3} \phi^2 + \frac{M^2}{r^3} \left[ (E_2)^2 + \frac{(r - 3M)^2}{r^2} (E_3)^2 \right] \right\} d\mu. \end{aligned} \tag{4.14}$$

The rest of the section is concerned with the proof of Theorem 4.1. Let

$$Q_{\mu\nu} := E_\mu E_\nu - (1/2)\mathbf{g}_{\mu\nu}(E_\rho E^\rho), \quad J_\nu := \mathbf{D}^\mu Q_{\mu\nu} = \mathcal{N} E_\nu. \tag{4.15}$$

For any vector-field  $X$ , real scalar function  $w$ , and 1-form  $m$  we define

$$P_\mu = P_\mu[X, w, m] := Q_{\mu\nu} X^\nu + \frac{1}{2} w \phi E_\mu - \frac{1}{4} \phi^2 \mathbf{D}_\mu w + \frac{1}{4} m_\mu \phi^2. \tag{4.16}$$

The formula (2.15) becomes

$$\begin{aligned} 2\mathbf{D}^\mu P_\mu = \mathcal{T}[X, w, m] := & \overset{(X)}{\pi}{}^{\mu\nu} Q_{\mu\nu} + w E^\mu E_\mu + \phi m^\mu E_\mu \\ & + \frac{1}{2} \phi^2 (\mathbf{D}^\mu m_\mu - \square w). \end{aligned} \tag{4.17}$$

We use the divergence identity

$$\int_{\Sigma_{t_1}^c} P_\mu n_0^\mu d\mu_{t_1} = \int_{\Sigma_{t_2}^c} P_\mu n_0^\mu d\mu_{t_2} + \int_{\mathcal{N}_{[t_1, t_2]}^c} P_\mu k_0^\mu d\mu_c + \int_{\mathcal{D}_{[t_1, t_2]}^c} \mathbf{D}^\mu P_\mu d\mu, \tag{4.18}$$

where  $t_1, t_2 \in [0, T]$ ,  $c \in (c_0, 2M]$ ,  $n_0 := n/|\mathbf{g}^{33}|^{1/2}$ ,  $k_0 := k/|\mathbf{g}^{22}|^{1/2}$ , and the integration is with respect to the natural measures induced by the metric  $\mathbf{g}$ . We would like to find multipliers  $(X, w, m)$  in such a way that the contributions of the integrals in (4.18) are all nonnegative.

### 4.1 The Multipliers $(X_{(k)}, w_{(k)}, m_{(k)})$ , $k \in \{1, 2\}$

In this subsection we define three multipliers  $(X_{(k)}, w_{(k)}, m_{(k)})$ ,  $k \in \{1, 2, 3\}$ , which are used to generate positive terms in the divergence identity (4.18). The first multiplier  $(X_{(1)}, w_{(1)}, m_{(1)})$  is relevant in a neighborhood of the trapped set  $\{r = 3M\}$  and the second multiplier  $(X_{(2)}, w_{(2)}, m_{(2)})$  is relevant in a neighborhood of the horizon  $\{r = 2M\}$ . The third multiplier  $(X_{(3)}, w_{(3)}, m_{(3)})$  generates outgoing energies at infinity; at the same time it contains a large multiple of the vector-field  $\partial_3$  which helps with the positivity of the boundary integrals  $P_\mu n^\mu$ .

4.1.1 Analysis Around the Trapped Set  $r = 3M$

This is similar to the construction in [19]. We define the first multiplier  $(X_{(1)}, w_{(1)}, m_{(1)})$  by the formulas

$$\begin{aligned} X_{(1)} &:= f_1(r)\partial_2 + g_1(r)\partial_3, & f_1(r) &:= \frac{a_1(r)\Delta}{r^2}, & g_1(r) &:= \frac{a_1(r)\chi(r)2M}{r} + 1, \\ w_{(1)}(r) &:= f'_1(r) + f_1(r)\partial_r \log(r^4/\Delta) - \epsilon_1 \tilde{w}(r), & m_{(1)} &\equiv 0, \\ \tilde{w}(r) &:= M^2(r - 33M/16)^3(r - 3M)^2 r^{-8} \mathbf{1}_{[33M/16, \infty)}(r), \end{aligned} \tag{4.19}$$

where  $a_1 : (0, \infty) \rightarrow \mathbb{R}$  is a smooth increasing function to be fixed,  $\lim_{r \rightarrow \infty} a_1(r) = 1$ , and  $\epsilon_1 \in (0, 1]$  is a small constant. Using (4.6),

$$Q_{\mu\nu}^{(X_{(1)})} \pi^{\mu\nu} + w_{(1)} E_\mu E^\mu = [K_{(1)}^{11} (E_1)^2 + K_{(1)}^{22} (E_2)^2 + K_{(1)}^{33} (E_3)^2 + 2K_{(1)}^{23} E_2 E_3],$$

where

$$\begin{aligned} K_{(1)}^{11} &= \frac{-f'_1(r)}{r^2} + w_{(1)}(r)\mathbf{g}^{11} = \frac{2(r - 3M)}{r^4} a_1 - \epsilon_1 \tilde{w} \mathbf{g}^{11}, \\ K_{(1)}^{22} &= \frac{-f_1(r)(2r - 2M) + f'_1(r)\Delta}{r^2} + w_{(1)}(r)\mathbf{g}^{22} = \frac{2\Delta^2}{r^4} a'_1 - \epsilon_1 \tilde{w} \mathbf{g}^{22}, \\ K_{(1)}^{33} &= -f_1(r)\partial_2 \mathbf{g}^{33} + 2g'_1(r)\mathbf{g}^{23} - f'_1(r)\mathbf{g}^{33} - \frac{2rf_1(r)\mathbf{g}^{33}}{r^2} + w_{(1)}(r)\mathbf{g}^{33} \\ &= \frac{8M^2\chi^2}{r^2} a'_1 - \epsilon_1 \tilde{w} \mathbf{g}^{33}, \\ K_{(1)}^{23} &= \frac{-2Mrf_1(r)\chi'(r) - 2Mf_1(r)\chi(r) + g'_1(r)\Delta}{r^2} + w_{(1)}(r)\mathbf{g}^{23} \\ &= \frac{4M\Delta\chi}{r^3} a'_1 - \epsilon_1 \tilde{w} \mathbf{g}^{23}. \end{aligned}$$

where  $a'_1$  denotes the  $r$  derivative of the function  $a_1$ . Therefore

$$\begin{aligned} Q_{\mu\nu}^{(X_{(1)})} \pi^{\mu\nu} + w_{(1)} E_\mu E^\mu &= \frac{2(r - 3M)a_1 - \epsilon_1 \tilde{w} r^2}{r^4} (E_1)^2 \\ &\quad + \left( 2a'_1 - \frac{\epsilon_1 \tilde{w}}{1-p} \right) \left( \frac{\Delta}{r^2} E_2 + \frac{2M\chi}{r} E_3 \right)^2 + \frac{\epsilon_1 \tilde{w}}{1-p} (E_3)^2. \end{aligned} \tag{4.20}$$

Moreover

$$\phi m_{(1)}^\mu E_\mu + \frac{1}{2} \phi^2 (\mathbf{D}^\mu m_{(1)\mu} - \square w_{(1)}) = -\frac{1}{2} \square w_{(1)} \phi^2. \tag{4.21}$$

We define now the important function  $a_1(r)$ . Assume  $\kappa : \mathbb{R} \rightarrow \mathbb{R}$  is an increasing smooth function satisfying  $\kappa(y) = y$  on  $[-1, \infty)$  and  $\kappa(y) = -2$  on  $(-\infty, -3]$ . We set

$$\begin{aligned}
 R(r) &:= (r - 3M)(r + 2M) + 6M^2 \log \left[ \frac{r - 2M}{M} \right], \\
 a_1(r) &:= r^{-2} \delta^{-1} \kappa(\delta R(r)) + \left[ \frac{M}{r} - \frac{6M^2}{r^2} \log \left( \frac{r - 2M}{M} \right) \right] \chi_{\geq DM}(r),
 \end{aligned}
 \tag{4.22}$$

where  $\delta := \epsilon_2^2 M^{-2}$  is a small constant and  $D \gg 1$  is a large constant. The function  $a_1$  is well defined, using the formula above, for  $r > 2M$ . Clearly  $a_1(r) = -2r^{-2} \delta^{-1}$  for  $r$  sufficiently close to  $2M$ . Therefore  $a_1$  can be extended smoothly by this formula to the full interval  $r \in (c_0, \infty)$ .

Clearly

$$R'(r) = 2r - M + \frac{6M^2}{r - 2M}.
 \tag{4.23}$$

The function  $R$  is increasing on  $(2M, \infty)$ . Let  $r_\delta$  denote the unique number in  $(2M, \infty)$  with the property that  $R(r_\delta) = -1/\delta$ , and notice that

$$\frac{r_\delta - 2M}{M} \approx e^{-(6\delta M^2)^{-1}}.$$

Clearly  $a_1(3M) = 0$ ,

$$a'_1(r) = r^{-2} \left[ R'(r) \kappa'(\delta R(r)) - \frac{2\kappa(\delta R(r))}{\delta r} \right]
 \tag{4.24}$$

if  $r \leq DM$ , and

$$\begin{aligned}
 a'_1(r) &= \frac{12M^2}{r^3} + \left[ \frac{M}{r} - \frac{6M^2}{r^2} \log \left( \frac{r - 2M}{M} \right) \right] \chi'_{\geq DM}(r) \\
 &\quad + \left[ \frac{M}{r^2} - \frac{12M^2}{r^3} \log \left( \frac{r - 2M}{M} \right) + \frac{6M^2}{r^2(r - 2M)} \right] (1 - \chi_{\geq DM}(r))
 \end{aligned}
 \tag{4.25}$$

if  $r \geq r_\delta$ . In view of (4.24), if  $r \in (c_0, r_\delta]$  then  $a'_1(r) \geq 2\delta^{-1} r^{-3}$ . On the other hand, if  $r \in [r_\delta, \infty)$  then  $a'_1(r) \geq 12M^2 r^{-3}$ . Therefore

$$a_1(3M) = 0 \quad \text{and} \quad a'_1(r) \geq 12M^2 r^{-3} \quad \text{for } r \in (c_0, \infty),
 \tag{4.26}$$

provided that  $\delta \leq (10M)^{-2}$ .

Let

$$h_1(r) := f'_1(r) + f_1(r) \partial_r \log(r^4/\Delta) = \frac{r - 2M}{r^3} \partial_r (r^2 a_1(r)).
 \tag{4.27}$$



We calculate, as before,

$$h_1(r) = \frac{r - 2M}{r^3} R'(r) \kappa'(\delta R(r)) \tag{4.28}$$

if  $r \leq DM$ , and

$$h_1(r) = \frac{r - 2M}{r^3} \left\{ 2r - \left[ M - \frac{6M^2}{r - 2M} \right] (1 - \chi_{\geq DM}(r)) + \left[ Mr - 6M^2 \log \left( \frac{r - 2M}{M} \right) \right] \chi'_{\geq DM}(r) \right\} \tag{4.29}$$

if  $r \geq r_\delta$ . Letting

$$\tilde{R}(r) := \frac{r - 2M}{r^3} R'(r) = \frac{2}{r} - \frac{5M}{r^2} + \frac{8M^2}{r^3},$$

we have

$$\begin{aligned} (\square h_1)(r) &= \frac{\partial_2(\Delta \cdot \partial_2 h_1)}{r^2} \\ &= r^{-2} \left\{ \kappa'(\delta R(r)) \partial_r [\Delta \tilde{R}'(r)] + \delta^2 \kappa'''(\delta R(r)) r^7 \tilde{R}(r)^3 (r - 2M)^{-1} + \delta \kappa''(\delta R(r)) [3r^4 \tilde{R}(r) \tilde{R}'(r) + 4r^3 \tilde{R}(r)^2] \right\} \end{aligned}$$

if  $r \leq DM$ , and

$$\begin{aligned} (\square h_1)(r) &= \frac{\partial_2(\Delta \cdot \partial_2 h_1)}{r^2} \\ &= r^{-2} \partial_r [\Delta \tilde{R}'(r)] + O(Mr^{-4}) \mathbf{1}_{[DM, \infty)}(r) \\ &= -\frac{2M}{r^4} \left( 7 - \frac{44M}{r} + \frac{72M^2}{r^2} \right) + O(Mr^{-4}) \mathbf{1}_{[DM, \infty)}(r). \end{aligned}$$

if  $r \geq r_\delta$ . Therefore, the last two identities show that

$$\begin{aligned} (\square h_1)(r) &= -\frac{2M}{r^4} \left( 7 - \frac{44M}{r} + \frac{72M^2}{r^2} \right) + O(Mr^{-4}) \mathbf{1}_{[DM, \infty)}(r) \\ &\quad + M^{-3} O(1) \mathbf{1}_{(c_0, r_\delta]}(r) + O\left(\frac{\delta^2 M^2}{r - 2M}\right) \mathbf{1}_{[r'_\delta, r_\delta]}(r), \end{aligned} \tag{4.30}$$

where  $r'_\delta$  denotes the unique number in  $(2M, \infty)$  with the property that  $R(r'_\delta) = -2/\delta$ . Notice that

$$7 - \frac{44M}{r} + \frac{72M^2}{r^2} \geq 1/10 \quad \text{for any } r \geq M. \tag{4.31}$$

Therefore, since  $w_{(1)} = h_1 - \epsilon_1 \tilde{w}$ , it follows that

$$-\frac{1}{2}(\square w_{(1)})(r) \geq \frac{M}{10r^4} - \frac{C_1 M}{r^4} \mathbf{1}_{[DM, \infty)}(r) - \frac{C_1}{M^3} \mathbf{1}_{(c_0, r_\delta]}(r) - \frac{C_1 \delta^2 M^2}{r - 2M} \mathbf{1}_{[r'_\delta, r_\delta]}(r), \tag{4.32}$$

for a sufficiently large constant  $C_1$ , provided that the constant  $\epsilon_1$  is sufficiently small. Using also (4.20)–(4.21) and (4.26),

$$\begin{aligned} \mathcal{T}[X_{(1)}, w_{(1)}, m_{(1)}] &\geq \frac{(2 - C_1 \epsilon_1)(r - 3M)a_1(r)}{r^4} (E_1)^2 \\ &+ (2 - C_1 \epsilon_1)a'_1(r)((1 - p)E_2 + p\chi(r)E_3)^2 + \epsilon_1 \tilde{w}(r)(E_3)^2 + \frac{M}{10r^4} \phi^2 \\ &- \frac{C_1 M}{r^4} \mathbf{1}_{[DM, \infty)}(r) \phi^2 - \frac{C_1}{M^3} \mathbf{1}_{(c_0, r_\delta]}(r) \phi^2 - \frac{C_1 \delta^2 M^2}{r - 2M} \mathbf{1}_{[r'_\delta, r_\delta]}(r) \phi^2, \end{aligned} \tag{4.33}$$

for a sufficiently large constant  $C_1$ , provided that the constant  $\epsilon_1$  is sufficiently small.

The remaining contributions  $2P_\mu n_0^\mu$  and  $2P_\mu k_0^\mu$  in the divergence identity (4.18) cannot be estimated effectively at this time. We will prove partial estimates for these terms in Lemma 4.2 below, after we construct the second multiplier  $(X_{(2)}, w_{(2)}, m_{(2)})$  and show how to fix some of the parameters.

#### 4.1.2 Analysis in a Neighborhood of the Horizon

In a small neighborhood of the horizon we need to use the redshift effect. For this we define the second multiplier  $(X_{(2)}, w_{(2)}, m_{(2)})$  by the formulas

$$\begin{aligned} X_{(2)} &:= f_2(r)\partial_2 + g_2(r)\partial_3, & f_2(r) &:= -\epsilon_2 a_2(r), & g_2(r) &:= \epsilon_2 a_2(r)(1 - \epsilon_2), \\ w_{(2)}(r) &:= -2\epsilon_2 a_2(r)/r, & m_{(2)2} &:= \epsilon_2 M^{-2} \gamma(r), & m_{(2)3} &:= \epsilon_2 M^{-2} \gamma(r) \end{aligned} \tag{4.34}$$

where  $\epsilon_2$  is a small positive constant (recall that  $\delta = \epsilon_2^2 M^{-2}$ ),

$$a_2(r) := \begin{cases} M^{-3}(9M/4 - r)^3 & \text{if } r \leq 9M/4, \\ 0 & \text{if } r \geq 9M/4, \end{cases} \tag{4.35}$$

and  $\gamma : [c_0, \infty) \rightarrow [0, 1]$  is a suitable function (to be fixed later) supported in  $[c_0, 17M/8]$  and satisfying  $\gamma(2M) = 1/2$ .

Notice that  $\chi = 1$  in the support of the functions  $a_2$  and  $\gamma$ . As before, we calculate

$$\mathcal{Q}_{\mu\nu}^{(X_{(2)})} \pi^{\mu\nu} + w_{(2)} E_\mu E^\mu = [K_{(2)}^{11}(E_1)^2 + K_{(2)}^{22}(E_2)^2 + K_{(2)}^{33}(E_3)^2 + 2K_{(2)}^{23} E_2 E_3],$$

where

$$\begin{aligned}
 K_{(2)}^{11} &= \frac{-f_2'(r)}{r^2} + w_{(2)}(r)\mathbf{g}^{11} = \epsilon_2 \frac{ra_2' - 2a_2}{r^3}, \\
 K_{(2)}^{22} &= \frac{-f_2(r)(2r - 2M) + f_2'(r)\Delta}{r^2} + w_2(r)\mathbf{g}^{22} = \epsilon_2 \left[ -\frac{r - 2M}{r}a_2' + \frac{2M}{r^2}a_2 \right], \\
 K_{(2)}^{33} &= -f_2(r)\partial_2\mathbf{g}^{33} + 2g_2'(r)\mathbf{g}^{23} - f_2'(r)\mathbf{g}^{33} - \frac{2rf_2(r)\mathbf{g}^{33}}{r^2} + w_2(r)\mathbf{g}^{33} \\
 &= \epsilon_2 \left[ -\frac{r - 2M + 4\epsilon_2 M}{r}a_2' + \frac{2M}{r^2}a_2 \right], \\
 K_{(2)}^{23} &= \frac{-2Mrf_2(r)\chi'(r) - 2Mf_2(r)\chi(r) + g_2'(r)\Delta}{r^2} + w_2(r)\mathbf{g}^{23} \\
 &= -\epsilon_2 \left[ -\frac{(1 - \epsilon_2)(r - 2M)}{r}a_2'(r) + \frac{2M}{r^2}a_2(r) \right].
 \end{aligned}$$

Using the explicit formula (4.35), it is easy to see that

$$\begin{aligned}
 Q_{\mu\nu}^{(X_{(2)})} \pi^{\mu\nu} + w_{(2)}E_\mu E^\mu &\geq \mathbf{1}_{(c_0, 9M/4)}(r)(9M/4 - r)^2 M^{-3} \\
 &\times \left[ C_2^{-1} \epsilon_2 (E_2 - E_3)^2 + C_2^{-1} \epsilon_2^2 (E_3)^2 - C_2 \epsilon_2 (E_1)^2 / r^2 \right], \tag{4.36}
 \end{aligned}$$

for a sufficiently large constant  $C_2$ , provided that  $\epsilon_2$  is sufficiently small and  $c_0$  is sufficiently close to  $2M$ . Moreover, using the definitions and (4.3)–(4.4),

$$\begin{aligned}
 \phi m_{(2)}^\mu E_\mu + \frac{1}{2} \phi^2 (\mathbf{D}^\mu m_{(2)\mu} - \square w_{(2)}) \\
 = \frac{\epsilon_2 \gamma}{M^2} \phi (E_2 - E_3) + \frac{\epsilon_2}{2} \phi^2 \left( \frac{1}{M^2} \gamma' + \frac{2}{rM^2} \gamma + 2\square(a_2/r) \right).
 \end{aligned}$$

Therefore, recalling also that  $\gamma \in [0, 1]$  and completing the square,

$$\begin{aligned}
 T[X_{(2)}, w_{(2)}, m_{(2)}] &\geq \frac{\epsilon_2}{2M^2} \phi^2 \gamma' + M^{-1} \epsilon_2^4 \mathbf{1}_{(c_0, 17M/8)}(r) [(E_2)^2 + (E_3)^2] \\
 &\quad - C_2 \epsilon_2 \mathbf{1}_{(c_0, 9M/4)}(r) [M^{-1} (E_1)^2 / r^2 + M^{-3} \phi^2], \tag{4.37}
 \end{aligned}$$

provided that  $\epsilon_2$  is sufficiently small and  $c_0$  is sufficiently close to  $2M$ .

We examine now (4.33) and (4.37) and fix the constant  $\epsilon_1, \epsilon_2$  and the function  $\gamma$  such that the sum  $T[X_{(1)}, w_{(1)}, m_{(1)}] + T[X_{(2)}, w_{(2)}, m_{(2)}]$  is nonnegative when  $r \in (c_0, DM]$ . For the positivity of the zero order term we need that

$$\frac{M}{20r^4} + \frac{\epsilon_2 \gamma'(r)}{2M^2} \geq \frac{C_1}{M^3} \mathbf{1}_{(c_0, r_\delta)}(r) + \frac{C_1 \delta^2 M^2}{r - 2M} \mathbf{1}_{[r'_\delta, r_\delta]}(r) + \frac{C_2 \epsilon_2}{M^3} \mathbf{1}_{(c_0, 9M/4)}(r). \tag{4.38}$$

Recall that  $\delta = \epsilon_2^2 M^{-2}$ . The point is that

$$\int_{c_0}^{\infty} \frac{C_1}{M^3} \mathbf{1}_{(c_0, r_\delta]}(r) + \frac{C_1 \delta^2 M^2}{r - 2M} \mathbf{1}_{[r'_\delta, r_\delta]}(r) dr \leq \frac{C_1^2 \epsilon_2^2}{M^2},$$

provided that  $2M - c_0 \leq \epsilon_2^2$ . This is easy to see if one recalls the definitions of  $r_\delta$  and  $r'_\delta$ . Therefore, assuming that  $\epsilon_2$  is sufficiently small, one can fix the function  $\gamma$  to achieve the inequality (4.38), while still preserving the other properties of  $\gamma$ , namely

$$\gamma : [c_0, \infty) \rightarrow [0, 1] \text{ is supported in } [c_0, 17M/8] \text{ and satisfies } \gamma(2M) = 1/2. \tag{4.39}$$

Indeed, the function  $\gamma$  can be chosen to increase on the interval  $(c_0, r_\delta]$  and then decrease for  $r \geq 2r_\delta - 2M$  in a way to satisfy both (4.38) and (4.39).

Notice that the sum of the first order terms in  $\mathcal{T}[X_{(1)}, w_{(1)}, m_{(1)}] + \mathcal{T}[X_{(2)}, w_{(2)}, m_{(2)}]$  is nonnegative and nondegenerate if we simply have  $\epsilon_1, \epsilon_2 > 0$  sufficiently small. Therefore, one can fix the parameters  $\epsilon_1, \epsilon_2$  and the function  $\gamma$  in such a way that

$$\begin{aligned} & \mathcal{T}[X_{(1)}, w_{(1)}, m_{(1)}] + \mathcal{T}[X_{(2)}, w_{(2)}, m_{(2)}] \\ & \geq \epsilon_3 \left[ \frac{(r - 3M)^2}{r^3} (E_1/r)^2 + \frac{M^2}{r^3} (E_2)^2 + \frac{M^2 (r - 3M)^2}{r^5} (E_3)^2 + \frac{M}{r^4} \phi^2 \right] \\ & \quad - \epsilon_3^{-1} \frac{M}{r^4} \mathbf{1}_{[DM, \infty)}(r) \phi^2, \end{aligned} \tag{4.40}$$

for a constant  $\epsilon_3 > 0$  sufficiently small (relative to  $\epsilon_1$  and  $\epsilon_2$ ). The parameter  $D$  will be fixed later, sufficiently large depending on  $\epsilon_3$ .

We can prove now some partial bounds on the remaining terms

$$2(P_{(1)\mu} + P_{(2)\mu})n_0^\mu, \quad 2(P_{(1)\mu} + P_{(2)\mu})k_0^\mu,$$

in the divergence identity (4.18), where  $P_{(k)} := P[X_{(k)}, w_{(k)}, m_{(k)}], k \in \{1, 2\}$ .

**Lemma 4.2** *There is a sufficiently small absolute constant  $\epsilon_3$  such that*

$$2(P_{(1)\mu} + P_{(2)\mu})k^\mu \geq \epsilon_3 [(E_1/r)^2 + (E_2)^2 (2 - c/M) + M^{-2} \phi^2] - \epsilon_3^{-1} (E_3)^2. \tag{4.41}$$

along  $\mathcal{N}_{[t_1, t_2]}^c$ . Also

$$\begin{aligned} 2(P_{(1)\mu} + P_{(2)\mu})n^\mu & \geq -\epsilon_3^{-1} [\tilde{F}_0 + \mathbf{1}_{[8M, 2DM]}(r) (E_3)^2] \\ & \quad - \frac{\chi_{\geq 8M}(r)(1-p)}{r^2} \partial_2(r\phi^2) + \epsilon_3 (E_2)^2 \mathbf{1}_{(c_0, 17M/8]}(r), \end{aligned} \tag{4.42}$$

and

$$2(P_{(1)\mu} + P_{(2)\mu})n^\mu \leq \epsilon_3^{-1} [\tilde{F}_0 + \mathbf{1}_{[8M, 2DM]}(r)(E_3)^2 + (E_2)^2 \mathbf{1}_{(c_0, 17M/8]}(r)] - \frac{\chi_{\geq 8M}(r)(1-p)}{r^2} \partial_2(r\phi^2), \tag{4.43}$$

where

$$\tilde{F}_0 = (E_1/r)^2 + (L\phi)^2 + M^2 r^{-2} [(E_2)^2 |1-p| + (E_3^2)] + r^{-2} \phi^2. \tag{4.44}$$

*Proof* We start with the term  $2(P_{(1)\mu} + P_{(2)\mu})k^\mu$ ,

$$2(P_{(1)\mu} + P_{(2)\mu})k^\mu = 2k^\mu Q_{\mu\nu} (X_{(1)}^\nu + X_{(2)}^\nu) + (w_{(1)} + w_{(2)})\phi E_\mu k^\mu - \frac{1}{2} \phi^2 k^\mu (\mathbf{D}_\mu w_{(1)} + \mathbf{D}_\mu w_{(2)}) + \frac{1}{2} k^\mu m_{(2)\mu} \phi^2.$$

When  $r = c \in (c_0, 2M]$  and assuming that  $2M - c$  is sufficiently small, we use the definitions, the identity  $m_{(2)3}(2M) \geq \epsilon_2/(2M^2)$ , and the identities (4.8). We have

$$2k^\mu Q_{\mu\nu} (X_{(1)}^\nu + X_{(2)}^\nu) \geq \epsilon_3 [(E_1/r)^2 + (E_2)^2 (2 - c/M)] - \epsilon_3^{-1} (E_3)^2 - \epsilon_3^{-1} |E_2 E_3| (2 - c/M),$$

$$|(w_{(1)} + w_{(2)})\phi E_\mu k^\mu| \leq \epsilon_3^{-1} M^{-1} |\phi| [(2 - c/M)|E_2| + |E_3|],$$

and

$$-\frac{1}{2} \phi^2 k^\mu (\mathbf{D}_\mu w_{(1)} + \mathbf{D}_\mu w_{(2)}) + \frac{1}{2} k^\mu m_{(2)\mu} \phi^2 \geq \epsilon_3 M^{-2} \phi^2,$$

provided that  $\epsilon_3$  is sufficiently small. The bound (4.41) follows by further reducing  $\epsilon_3$  and assuming that  $2M - c$  is sufficiently small.

We consider now the term  $2(P_{(1)\mu} + P_{(2)\mu})n^\mu$ ,

$$2(P_{(1)\mu} + P_{(2)\mu})n^\mu = 2n^\mu Q_{\mu\nu} (X_{(1)}^\nu + X_{(2)}^\nu) + (w_{(1)} + w_{(2)})\phi E_\mu n^\mu - \frac{1}{2} \phi^2 n^\mu (\mathbf{D}_\mu w_{(1)} + \mathbf{D}_\mu w_{(2)}) + \frac{1}{2} n^\mu m_{(2)\mu} \phi^2.$$

Using the definitions and the identities (4.7) we estimate

$$|\phi^2 n^\mu (\mathbf{D}_\mu w_{(1)} + \mathbf{D}_\mu w_{(2)})| + |n^\mu m_{(2)\mu} \phi^2| \leq \epsilon_3^{-1} M^2 r^{-4} \phi^2. \tag{4.45}$$

Moreover, with  $\tilde{F}_0$  as in (4.44), we write

$$\begin{aligned} & 2n^\mu Q_{\mu\nu} X_{(1)}^\nu + w_{(1)}\phi E_\mu n^\mu \\ &= (E_1)^2 \frac{g_1(r)}{r^2} + (E_2)^2 [g_1(r)(1-p) - 2f_1(r)\mathbf{g}^{23}] \\ &\quad + (E_3)^2 [-g_1(r)\mathbf{g}^{33}] + 2E_2 E_3 [-f_1(r)\mathbf{g}^{33}] + w_1\phi (-\mathbf{g}^{33} E_3 - \mathbf{g}^{23} E_2) \\ &\geq -(10\epsilon_3)^{-1} \tilde{F}_0 + [(E_2)^2(1-p) + (E_3)^2(1-p)^{-1} \\ &\quad + 2E_2 E_3 a_1 + \phi E_3 w_1(1-p)^{-1}] \chi_{\geq 8M}(r). \end{aligned}$$

Using the definitions and the formula (4.29),

$$\begin{aligned} |a_1 - 1| \chi_{\geq 8M}(r) &\lesssim M r^{-1} \mathbf{1}_{[8M, 2DM]}(r) + M^2 r^{-2} \mathbf{1}_{[8M, \infty)}(r), \\ |w_1 - 2(1-p)/r| \chi_{\geq 8M}(r) &\lesssim M r^{-2} \mathbf{1}_{[8M, 2DM]}(r) + M^2 r^{-3} \mathbf{1}_{[8M, \infty)}(r). \end{aligned}$$

Therefore

$$\begin{aligned} & 2n^\mu Q_{\mu\nu} X_{(1)}^\nu + w_{(1)}\phi E_\mu n^\mu \\ &\geq -(9\epsilon_3)^{-1} [\tilde{F}_0 + \mathbf{1}_{[8M, 2DM]}(r)(E_3)^2] - \left[ \frac{\phi^2}{r^2} + \frac{2\phi}{r} E_2 \right] \chi_{\geq 8M}(r)(1-p) \\ &\geq -(8\epsilon_3)^{-1} [\tilde{F}_0 + \mathbf{1}_{[8M, 2DM]}(r)(E_3)^2] - \frac{\chi_{\geq 8M}(r)(1-p)}{r^2} \partial_2(r\phi^2). \end{aligned}$$

Similarly, using also the observation that  $-f_2(2M) \gtrsim 1$ ,

$$2n^\mu Q_{\mu\nu} X_{(2)}^\nu + w_{(2)}\phi E_\mu n^\mu \geq -(8\epsilon_3)^{-1} \tilde{F}_0 + \epsilon_3 (E_2)^2 \mathbf{1}_{(c_0, 17M/8]}(r).$$

The bound (4.42) follows using the last two inequalities and (4.45).

The proof of the upper bound (4.43) follows in a similar way. □

*Remark 4.3* At this point one can recover the energy estimate of Marzuola–Metcalf–Tataru–Tohaneanu [19, Theorem 1.2],

$$\mathcal{E}^{c_0}(t_2) + \mathcal{B}^{c_0}(t_1, t_2) \leq \bar{C} \mathcal{E}^{c_0}(t_1),$$

where

$$\begin{aligned} \mathcal{E}^{c_0}(t) &:= \int_{\Sigma_t^{c_0}} \left[ (E_1/r)^2 + (E_2)^2 + (E_3^2) \right] d\mu_t, \\ \mathcal{B}^{c_0}(t_1, t_2) &:= \int_{\mathcal{D}_{[t_1, t_2]}^{c_0}} \left[ \frac{(r-3M)^2}{r^3} (E_1/r)^2 + \frac{M^2}{r^3} (E_2)^2 \right. \\ &\quad \left. + \frac{M^2(r-3M)^2}{r^5} (E_3)^2 + \frac{M}{r^4} \phi^2 \right] d\mu. \end{aligned}$$

To see this, we simply set  $D := \infty$  and add in a very large multiple of the Killing vector-field  $\partial_3$ . The spacetime integral  $\mathcal{B}^{c_0}(t_1, t_2)$  is generated by the right-hand side of (4.40) (some of the powers of  $r$  in the spacetime integral could in fact be improved by reexamining the proof). The formulas for the nondegenerate energies  $\mathcal{E}^{c_0}(t)$  follow from the bounds (4.42) and (4.43), the identity (4.7), and the Hardy inequality in Lemma A.1 (i). The contribution of  $P_\mu k^\mu$  along  $\mathcal{N}_{[t_1, t_2]}^c$  becomes nonnegative, in view of (4.8), and can be neglected.

### 4.2 Outgoing Energies

To prove the stronger estimates in Theorem 4.1 we consider now a multiplier  $(X_{(3)}, w_{(3)}, m_{(3)})$  of the form

$$\begin{aligned} X_{(3)} &= f_3 \partial_2 + \left( \frac{f_3}{1-p} + g_3 \right) \partial_3, & w_{(3)} &= \frac{2f_3}{r}, \\ m_{(3)1} = m_{(3)4} &= 0, & m_{(3)2} &= \frac{2h_3}{r(1-p)}, & m_{(3)3} &= -\frac{2h_3}{r}, \end{aligned} \tag{4.46}$$

where  $f_3, g'_3, h_3$  are smooth functions supported in  $\{r \geq 8M\}$ , which depend only on  $r$ . The function  $g_3$  is not supported in  $\{r \geq 8M\}$ , it is in fact a very large constant in the region  $r \in [c, 10M]$ .

As before, using (4.6), we calculate

$$Q_{\mu\nu}^{(X_{(3)})} \pi^{\mu\nu} + w_{(3)} E_\mu E^\mu = [K_{(3)}^{11}(E_1)^2 + K_{(3)}^{22}(E_2)^2 + K_{(3)}^{33}(E_3)^2 + 2K_{(3)}^{23}E_2E_3],$$

where

$$\begin{aligned} K_{(3)}^{11} &= \frac{-f'_3(r)}{r^2} + w_{(3)}(r)\mathbf{g}^{11} = \frac{2f_3 - rf'_3}{r^3}, \\ K_{(3)}^{22} &= \frac{-f_3(r)(2r - 2M) + f'_3(r)\Delta}{r^2} + w_{(3)}(r)\mathbf{g}^{22} = (1-p)f'_3 - \frac{2Mf_3}{r^2}, \\ K_{(3)}^{33} &= -f_3(r)\partial_2\mathbf{g}^{33} - f'_3(r)\mathbf{g}^{33} - \frac{2f_3(r)\mathbf{g}^{33}}{r} + w_{(3)}(r)\mathbf{g}^{33} = \frac{f'_3}{1-p} - \frac{2Mf_3}{r^2(1-p)^2}, \\ K_{(3)}^{23} &= (1-p)\left(\frac{f_3}{1-p} + g_3\right)' = f'_3 - \frac{2Mf_3}{r^2(1-p)} + (1-p)g'_3. \end{aligned}$$

Moreover

$$\begin{aligned} &\phi m_{(3)}^\mu E_\mu + \frac{1}{2}\phi^2(\mathbf{D}^\mu m_{(3)\mu} - \square w_{(3)}) \\ &= 2h_3 \frac{\phi}{r} \left( E_2 + \frac{E_3}{1-p} \right) + \phi^2 \left[ \frac{h'_3}{r} + \frac{h_3}{r^2} - \frac{(1-p)f'_3}{r} - \frac{2Mf'_3}{r^3} + \frac{2Mf_3}{r^4} \right]. \end{aligned}$$

Set

$$\begin{aligned}
 H_3 &:= (1 - p)f'_3 - \frac{2Mf_3}{r^2} + (1 - p)^2g'_3, \\
 h_3 &:= H_3 \cdot (1 - \tilde{\alpha}),
 \end{aligned}
 \tag{4.47}$$

where  $\tilde{\alpha} = (2 - \alpha)/10 > 0$ . The identities above show that

$$\begin{aligned}
 T[X_{(3)}, w_{(3)}, m_{(3)}] &= \frac{(E_1)^2}{r^2} \frac{2f_3 - rf'_3}{r} + H_3 \left( E_2 + \frac{E_3}{1 - p} \right)^2 \\
 &\quad - (1 - p)^2g'_3 \left[ (E_2)^2 + \frac{(E_3)^2}{(1 - p)^2} \right] \\
 &\quad + 2h_3 \frac{\phi}{r} \left( E_2 + \frac{E_3}{1 - p} \right) \\
 &\quad + \phi^2 \left[ \frac{h_3}{r^2} + \frac{h'_3}{r} - \frac{(1 - p)f''_3}{r} - \frac{2Mf'_3}{r^3} + \frac{2Mf_3}{r^4} \right].
 \end{aligned}$$

After completing the square this becomes

$$\begin{aligned}
 T[X_{(3)}, w_{(3)}, m_{(3)}] &= \frac{(E_1)^2}{r^2} \frac{2f_3 - rf'_3}{r} + H_3 \left( L\phi + \frac{(1 - \tilde{\alpha})\phi}{r} \right)^2 \\
 &\quad - (1 - p)^2g'_3 \left[ (E_2)^2 + \frac{(E_3)^2}{(1 - p)^2} \right] \\
 &\quad + \phi^2 \left[ \frac{(\tilde{\alpha} - \tilde{\alpha}^2)H_3 - \tilde{\alpha}rH'_3}{r^2} + \frac{6Mf_3}{r^4} \right. \\
 &\quad \left. - \frac{2Mf'_3}{r^3} + \frac{(1 - p)^2g''_3}{r} + \frac{4M(1 - p)g'_3}{r^3} \right].
 \end{aligned}
 \tag{4.48}$$

Using (4.7) we calculate

$$\begin{aligned}
 2P_{(3)\mu}n^\mu &= 2Q_{\mu\nu}X_{(3)}^\nu n^\mu + w_{(3)}\phi E_\mu n^\mu - \frac{1}{2}\phi^2 n^\mu \mathbf{D}_\mu w_{(3)} + \frac{1}{2}n^\mu m_{(3)\mu}\phi^2 \\
 &= \frac{(E_1)^2}{r^2} \left[ \frac{f_3}{1 - p} + g_3 \right] + (E_2)^2 [f_3 + g_3(1 - p)] \\
 &\quad + (E_3)^2 \left[ \frac{f_3}{(1 - p)^2} + \frac{g_3(1 - p^2\chi^2)}{1 - p} \right] \\
 &\quad + 2E_2E_3 \frac{f_3}{1 - p} + \frac{2f_3}{r(1 - p)}\phi E_3 + \frac{m_{(3)3}}{2(1 - p)}\phi^2 \\
 &= \frac{(E_1)^2}{r^2} \left[ \frac{f_3}{1 - p} + g_3 \right] + f_3 \left[ E_2 + \frac{E_3}{1 - p} + \frac{\phi}{r} \right]^2 - f_3 \frac{\phi^2}{r^2} \\
 &\quad - 2f_3E_2 \frac{\phi}{r} + g_3(1 - p) \left[ (E_2)^2 + \frac{(E_3)^2(1 - p^2\chi^2)}{(1 - p)^2} \right] - \frac{h_3}{r(1 - p)}\phi^2.
 \end{aligned}$$



Therefore

$$\begin{aligned}
 2P_{(3)\mu}n^\mu &= \frac{(E_1)^2}{r^2} \left[ \frac{f_3}{1-p} + g_3 \right] + f_3 \left[ L\phi + \frac{\phi}{r} \right]^2 \\
 &\quad + g_3(1-p) \left[ (E_2)^2 + \frac{(E_3)^2(1-p^2\chi^2)}{(1-p)^2} \right] \\
 &\quad - \frac{1}{r^2} \partial_2 [f_3 r \phi^2] + \phi^2 \left[ \frac{\tilde{\alpha} H_3}{r(1-p)} + \frac{2Mf_3}{r^3(1-p)} - \frac{(1-p)g'_3}{r} \right]. \tag{4.49}
 \end{aligned}$$

### 4.3 Proof of the Theorem 4.1

We compare now the expressions (4.48) and (4.49) with the lower bounds in (4.40) and (4.42). We would like to fix the functions  $f_3$  and  $g_3$  and the constant  $D$  in such a way that the sum of the corresponding expressions is bounded from below. More precisely, the sum of the spacetime integrals is pointwise bounded from below, while the sum of the integrals on the surfaces  $\Sigma_r^c$  is bounded from below after integration by parts and the use of a simple Hardy-type inequality.

One should think of the functions  $f_3$  and  $g_3$  in the following way: the function  $f_3$  vanishes when  $r \leq 8M$  and behaves like  $r^\alpha$  as  $r \rightarrow \infty$ . On the other hand the function  $g_3$  is an extremely large constant when  $r \leq C_4M$ , for some large constant  $C_4$  but vanishes as  $r \rightarrow \infty$  at a rate of  $r^{\alpha-2}$ . More precisely, we are looking for functions  $f_3, g_3$  of the form

$$f_3(r) = \epsilon_4 \chi_{\geq 8M}(r) e^{\beta(r)}, \quad g_3(r) = \int_r^\infty \rho(s) ds, \tag{4.50}$$

where  $\epsilon_4 = \epsilon_3^2$  is a small constant,  $C_4 = C_4(\alpha) \geq \epsilon_4^{-4} \alpha^{-1} (2-\alpha)^{-1}$  is a large constant (to be fixed), and  $\beta, \rho : (c, \infty) \rightarrow [0, \infty)$  are smooth functions satisfying

$$\begin{aligned}
 &\beta(r) \in [-10, 0] \text{ and } M\beta'(r) \in [1/10, 10] && \text{if } r \in (c, 8M], \\
 \max \left( \frac{\alpha}{100r}, \frac{4M}{r^2} + \frac{1}{r} \mathbf{1}_{[8M, C_4M]}(r) \right) \leq \beta'(r) \leq \frac{2}{r} && \text{if } r \in [8M, \infty), \\
 &\rho(r) = 0 \text{ and } g_3(r) \in [C_4/2, 2C_4] && \text{if } r \leq C_4M, \\
 \rho(r) \leq \frac{\epsilon_4}{100} \beta'(r) e^{\beta(r)} \text{ and } \rho'(r) \leq \frac{\epsilon_4 M}{100r^3} e^{\beta(r)} && \text{if } r \geq C_4M, \\
 &\frac{e^\beta M^2}{r^2} \leq g_3(r) \leq \frac{C_4^{10} e^\beta M^2}{r^2} && \text{if } r \geq C_4M, \\
 &(1 - 2\tilde{\alpha})H_3(r) - rH'_3(r) \geq 0 && \text{if } r \in [16M, \infty).
 \end{aligned} \tag{4.51}$$

A specific choice satisfying these conditions is given in (4.58)–(4.59). As a result of these conditions, we clearly have

$$\begin{aligned}
 g'_3 &= -\rho, \\
 H_3 &\geq \frac{\epsilon_4}{100} e^\beta \beta' \chi_{\geq 8M}, \\
 e^{\beta(r)} &\in [r/(100M), r^2/M^2] \quad \text{for } r \in (c, C_4M].
 \end{aligned}
 \tag{4.52}$$

Let

$$\begin{aligned}
 (X, w, m) &:= (X_{(1)}, w_{(1)}, m_{(1)}) + (X_{(2)}, w_{(2)}, m_{(2)}) + (X_{(3)}, w_{(3)}, m_{(3)}), \\
 \mathcal{T}[X, w, m] &:= \mathcal{T}[X_{(1)}, w_{(1)}, m_{(1)}] + \mathcal{T}[X_{(2)}, w_{(2)}, m_{(2)}] + \mathcal{T}[X_{(3)}, w_{(3)}, m_{(3)}], \\
 P_\mu &:= P_{(1)\mu} + P_{(2)\mu} + P_{(3)\mu}.
 \end{aligned}$$

Our next lemma contains the main bounds on the terms in the divergence identity (4.18).

**Lemma 4.4** *Assume that the conditions (4.51) hold and that  $C_4$  sufficiently large (depending on  $\epsilon_4$ ). Then there is an absolute constant  $\epsilon_5 = \epsilon_5(\alpha) > 0$  sufficiently small such that*

$$\begin{aligned}
 \mathcal{T}[X, w, m] &\geq \epsilon_5 \left[ \left( \frac{2}{r} e^\beta - \beta' e^\beta + \frac{100}{r} \right) \frac{(r - 3M)^2}{r^2} \frac{(E_1)^2}{r^2} + e^\beta \beta' (L\phi)^2 \right. \\
 &\quad \left. + \left( \rho + \frac{M^2}{r^3} \right) (E_2)^2 + \left( \rho + \frac{M^2(r - 3M)^2}{r^5} \right) (E_3)^2 + \frac{e^\beta \beta'}{r^2} \phi^2 \right].
 \end{aligned}
 \tag{4.53}$$

Moreover, for any  $t \in [0, T]$ ,

$$\int_{\Sigma_t^c} 2P_\mu n_0^\mu \, d\mu_t \geq \epsilon_5 \int_{\Sigma_t^c} e^\beta \frac{(E_1)^2}{r^2} + e^\beta (L\phi)^2 + g_3 [(E_2)^2 + (E_3)^2] + \frac{e^\beta \beta'}{r} \phi^2 \, d\mu_t
 \tag{4.54}$$

and

$$\int_{\Sigma_t^c} 2P_\mu n_0^\mu \, d\mu_t \leq \epsilon_5^{-1} \int_{\Sigma_t^c} e^\beta \frac{(E_1)^2}{r^2} + e^\beta (L\phi)^2 + g_3 [(E_2)^2 + (E_3)^2] + \frac{e^\beta \beta'}{r} \phi^2 \, d\mu_t.
 \tag{4.55}$$

Finally,

$$2P_\mu k^\mu \geq \epsilon_5 \left[ \frac{(E_1)^2}{M^2} + (E_2)^2 \frac{2M - c}{M} + (E_3)^2 + \frac{\phi^2}{M^2} \right] \quad \text{along } \mathcal{N}_{[t_1, t_2]}^c,
 \tag{4.56}$$

*Proof* We start with the proof of (4.53). Using the definitions we have

$$\begin{aligned} \frac{2f_3 - rf'_3}{r} &= \epsilon_4 e^\beta [(2/r - \beta')\chi_{\geq 8M} - \chi'_{\geq 8M}], \\ \frac{6Mf_3}{r^4} - \frac{2Mf'_3}{r^3} + \frac{(1-p)^2 g''_3}{r} + \frac{4M(1-p)g'_3}{r^3} &\geq \frac{\epsilon_4 M}{100r^4} e^\beta \chi_{\geq 8M} - \frac{2\epsilon_4 M}{r^3} e^\beta \chi'_{\geq 8M}. \end{aligned} \tag{4.57}$$

We combine the formulas (4.40) and (4.48) to estimate

$$T[X, w, m] \geq I_1 + I_2 + I'_2 + I_3,$$

where

$$\begin{aligned} I_1 &:= \frac{(E_1)^2}{r^2} \frac{2f_3 - rf'_3}{r} + \epsilon_3 \frac{(E_1)^2}{r^2} \frac{(r - 3M)^2}{r^3}, \\ I_2 &:= H_3 \left( L\phi + \frac{(1 - \tilde{\alpha})\phi}{r} \right)^2, \\ I'_2 &:= -(1 - p)^2 g'_3 \left[ (E_2)^2 + \frac{(E_3)^2}{(1 - p)^2} \right] + \epsilon_3 \left[ \frac{M^2}{r^3} (E_2)^2 + \frac{M^2(r - 3M)^2}{r^5} (E_3)^2 \right], \\ I_3 &:= \phi^2 \left[ \frac{(\tilde{\alpha} - \tilde{\alpha}^2)H_3 - \tilde{\alpha}rH'_3}{r^2} + \frac{\epsilon_4 M}{100r^4} e^\beta \chi_{\geq 8M} \right. \\ &\quad \left. - \frac{2\epsilon_4 M}{r^3} e^\beta \chi'_{\geq 8M} + \frac{\epsilon_3 M}{r^4} - \epsilon_3^{-1} \frac{M}{r^4} \mathbf{1}_{[DM, \infty)}(r) \right]. \end{aligned}$$

Using (4.51), (4.52), and (4.57) it is easy to see that, for some sufficiently small constant  $\epsilon_5$  (which may depend on  $\alpha$ ),

$$\begin{aligned} I_1 &\geq \epsilon_5 \left[ e^\beta \left( \frac{2}{r} - \beta' \right) \chi_{\geq 8M} + \frac{(r - 3M)^2}{r^3} \right] \frac{(E_1)^2}{r^2}, \\ I_2 + I'_2 &\geq \epsilon_5 e^\beta \beta' \chi_{\geq 8M} \left( L\phi + \frac{(1 - \tilde{\alpha})\phi}{r} \right)^2 + \epsilon_5 \left( \rho + \frac{M^2}{r^3} \right) (E_2)^2 \\ &\quad + \epsilon_5 \left( \rho + \frac{M^2(r - 3M)^2}{r^5} \right) (E_3)^2, \\ I_3 &\geq \epsilon_5 \left( \frac{M}{r^4} e^\beta \chi_{\geq 8M} + \frac{e^\beta \beta'}{r^2} \right) \phi^2, \end{aligned}$$

provided that  $\epsilon_4$  is fixed (sufficiently small relative to  $\epsilon_3$ ), and  $D$  is sufficiently large depending on  $\epsilon_4$  such that  $e^{\beta(DM)} \geq \epsilon_4^{-4}$ . The bound (4.53) follows.

To prove (4.54) we combine now the formulas (4.49), (4.42), and (4.44) to estimate

$$2P_\mu n^\mu \geq I_4 + I_5 + I_6 - \frac{1}{r^2} \partial_2 [f_3 r \phi^2] - \frac{\chi_{\geq 8M}(r)(1 - p)}{r^2} \partial_2 (r \phi^2),$$

where

$$\begin{aligned}
 I_4 &:= \frac{(E_1)^2}{r^2} \left[ \frac{f_3}{1-p} + g_3 \right] - \epsilon_3^{-1} \frac{(E_1)^2}{r^2}, \\
 I_5 &:= f_3 \left( L\phi + \frac{\phi}{r} \right)^2 + g_3(1-p) \left[ (E_2)^2 + \frac{(E_3)^2(1-p^2\chi^2)}{(1-p)^2} \right] \\
 &\quad + \epsilon_3(E_2)^2 \mathbf{1}_{(c_0, 17M/8]}(r) - \epsilon_3^{-1} \left[ (L\phi)^2 + \frac{M^2}{r^2} \left[ (E_2)^2|1-p| + (E_3)^2 \right] \right], \\
 I_6 &:= \phi^2 \left[ \frac{\tilde{\alpha}F_3}{r(1-p)} + \frac{2Mf_3}{r^3(1-p)} - \frac{(1-p)g'_3}{r} \right] - \epsilon_3^{-1} \frac{1}{r^2} \phi^2.
 \end{aligned}$$

Using (4.51), (4.52), and (4.57) it follows that

$$\begin{aligned}
 I_4 &\geq \epsilon_5 e^\beta \frac{(E_1)^2}{r^2}, \\
 I_5 + I_6 &\geq \epsilon_5 e^\beta \left( L\phi + \frac{\phi}{r} \right)^2 + [g_3(1-p) + \epsilon_3 \mathbf{1}_{(c_0, 17M/8]}(r)] \frac{(E_2)^2}{2} + \epsilon_5 g_3 (E_3)^2 \\
 &\quad + \epsilon_4 \frac{\tilde{\alpha}e^\beta \beta'}{1000r} \phi^2 - 10\epsilon_3^{-1} \frac{1}{r^2} \phi^2,
 \end{aligned}$$

provided that  $C_4$  is sufficiently large (relative to  $\epsilon_4$ ) and  $|c_0 - 2M| \leq C_4^{-10}$  is sufficiently small. Using the Hardy inequalities in Lemma A.1 (i) and (ii) it is easy to see that the integral on the negative term  $-10\epsilon_3^{-1}r^{-2}\phi^2$  in  $I_6$  along  $\Sigma_t^c$  can be absorbed by the integrals of the positive terms  $\epsilon_4 \frac{\tilde{\alpha}e^\beta \beta'}{1000r} \phi^2$  and  $g_3(1-p) \frac{(E_2)^2}{2}$ , provided that the constant  $C_4$  is sufficiently large.

Moreover, notice that for any  $t \in [0, T]$

$$\int_{\Sigma_t^c} 2P_\mu n_0^\mu d\mu_t = C \int_{\mathbb{S}^2} \int_c^\infty 2P_\mu n^\mu r^2(\sin \theta) dr d\theta.$$

After integration by parts in  $r$  it follows that

$$\begin{aligned}
 &\left| \int_{\mathbb{S}^2} \int_c^\infty \frac{1}{r^2} \partial_2 [f_3 r \phi^2] r^2(\sin \theta) dr d\theta \right| \\
 &\quad + \left| \int_{\mathbb{S}^2} \int_c^\infty \frac{\chi_{\geq 8M}(r)(1-p)}{r^2} \partial_2 (r\phi^2) r^2(\sin \theta) dr d\theta \right| \leq \epsilon_4^{-1} \int_{\Sigma_t^c} \frac{1}{r^2} \phi^2 d\mu_t,
 \end{aligned}$$

so these terms can also be absorbed. The desired bound (4.54) follows.

The proof of (4.55) is similar, starting from the inequality (4.43) and the identity (4.49). To prove (4.56) we start from the bound (4.41),

$$2(P_{(1)\mu} + P_{(2)\mu})k^\mu \geq \epsilon_3 \left[ (E_1/r)^2 + (E_2)^2(2 - c/M) + M^{-2}\phi^2 \right] - \epsilon_3^{-1} (E_3)^2.$$

The identity (4.8) shows that

$$2P_{(3)\mu}k^\mu = 2k^\mu Q_{\mu\nu}X_{(3)}^\nu = 2g_3(c)p(E_3)^2 + 2g_3(c)(1 - p)E_2E_3.$$

The lower bound (4.56) follows since  $g_3(c) \in [C_4/2, 2C_4]$ , provided that  $C_4$  is sufficiently large and  $|c - 2M|/M$  is sufficiently small.  $\square$

*Proof of Theorem 4.1* We can now complete the proof of Theorem 4.1, using Lemma 4.4 and the divergence identity. We have to fix functions  $\beta$  and  $\rho$  satisfying (4.51). With  $\alpha$  as in the statement of the theorem, we define first the smooth function  $\beta$  by setting  $\beta(8M) = 0$  and

$$\beta'(r) = \left(\frac{4M}{r^2} + \frac{1}{r}\right)(1 - \chi_{\geq C_4^4 M}(r)) + \frac{\alpha}{r}\chi_{\geq C_4^4 M}(r). \tag{4.58}$$

This choice clearly satisfies the first two conditions in (4.51). Then we define

$$\rho(r) = \delta M^{-1} \left[ \chi_{\geq C_4 M}(r) + \chi_{\geq 4C_4^4 M}(r) \left( C_4^7 e^{\beta(r)} \frac{M^3}{r^3} - 1 \right) \right], \tag{4.59}$$

where  $\delta \in [10^{-4}C_4^{-3}, 10^4C_4^{-3}]$  is such that  $\int_{C_4 M}^\infty \rho(s) ds = C_4$ .

Notice that

$$e^{\beta(r)} \approx \frac{r}{M} \text{ if } r \leq 10C_4^4 M \quad \text{and} \quad e^{\beta(r)} \approx C_4^4 \left( \frac{r}{C_4^4 M} \right)^\alpha \text{ if } r \geq (1/10)C_4^4 M. \tag{4.60}$$

The other bounds in (4.51) follow easily. Moreover, the definitions show that

$$\begin{aligned} e^{\beta(r)} &\approx_{C_5} \frac{r^\alpha}{M^\alpha}, & \beta'(r) &\approx_{C_5} \frac{1}{r}, & \left(\frac{2}{r} - \beta'(r)\right) &\approx_{C_5} \frac{1}{r}, \\ \rho(r) &\approx_{C_5} \chi_{\geq C_4 M}(r) \frac{M^{2-\alpha}}{r^{3-\alpha}}, & g_3(r) &\approx_{C_5} \frac{r^{\alpha-2}}{M^{\alpha-2}}. \end{aligned}$$

for some large constant  $C_5$ , where  $A \approx_{C_5} B$  means  $A \in [C_5^{-1}B, C_5B]$ . The desired conclusion of the theorem follows from Lemma 4.4 and the divergence identity.  $\square$

### 5 Proof of Theorem 1.7

In this section we prove Theorem 1.7. We still use some of the ideas from the previous section. We use the more complicated divergence identities (2.14) and (2.15),

$$\begin{aligned} 2\mathbf{D}^\mu P_\mu &= 2X^\nu J_\nu + Q_{\mu\nu}{}^{(X)}\pi^{\mu\nu} + w(E_\alpha E^\alpha + F_\alpha F^\alpha + M_\alpha M^\alpha) \\ &\quad + (\phi m^\mu \mathbf{D}_\mu \phi + \psi m'^\mu \mathbf{D}_\mu \psi) + \frac{1}{2}\phi^2(\mathbf{D}^\mu m_\mu - \square w) \\ &\quad + \frac{1}{2}\psi^2(\mathbf{D}^\mu m'_\mu - \square w) + w(\phi \mathcal{N}_\phi + \psi \mathcal{N}_\psi), \end{aligned} \tag{5.1}$$

where

$$E_\mu = \mathbf{D}_\mu \phi + \psi A^{-1} \mathbf{D}_\mu B, \quad F_\mu = \mathbf{D}_\mu \psi - \phi A^{-1} \mathbf{D}_\mu B, \quad M_\mu = \frac{\phi \mathbf{D}_\mu B - \psi \mathbf{D}_\mu A}{A}, \tag{5.2}$$

$$Q_{\mu\nu} := E_\mu E_\nu + F_\mu F_\nu + M_\mu M_\nu - (1/2) \mathbf{g}_{\mu\nu} (E_\alpha E^\alpha + F_\alpha F^\alpha + M_\alpha M^\alpha), \tag{5.3}$$

$$P_\mu = P_\mu[X, w, m, m'] = Q_{\mu\nu} X^\nu + \frac{1}{2} w (\phi E_\mu + \psi F_\mu) - \frac{1}{4} \mathbf{D}_\mu w (\phi^2 + \psi^2) + \frac{1}{4} (m_\mu \phi^2 + m'_\mu \psi^2), \tag{5.4}$$

and

$$J_\nu = \frac{2\mathbf{D}_\nu B M^\mu E_\mu - 2\mathbf{D}_\nu A M^\mu F_\mu}{A} + \mathcal{N}_\phi E_\nu + \mathcal{N}_\psi F_\nu. \tag{5.5}$$

Recall (see (1.9)) that

$$A = \frac{\Sigma^2 (\sin \theta)^2}{q^2}, \quad B = - \left[ 2aM(3 \cos \theta - (\cos \theta)^3) + \frac{2a^3 M (\sin \theta)^4 \cos \theta}{q^2} \right]. \tag{5.6}$$

These formulas show that

$$\begin{aligned} A^{-1} \mathbf{D}_1 B &= \frac{6aMq^2 \sin \theta}{\Sigma^2} - \frac{2a^3 M [4 \sin \theta q^2 - 5(\sin \theta)^3 q^2 + 2a^2 (\sin \theta)^3 (\cos \theta)^2]}{\Sigma^2 q^2}, \\ A^{-1} \mathbf{D}_2 B &= \frac{4ra^3 M (\sin \theta)^2 \cos \theta}{q^2 \Sigma^2}, \\ A^{-1} \mathbf{D}_1 A &= \frac{2 \cos \theta}{\sin \theta} - \frac{2a^2 \Delta \sin \theta \cos \theta}{\Sigma^2} - \frac{2a^2 \sin \theta \cos \theta}{q^2}, \\ A^{-1} \mathbf{D}_2 A &= \frac{4r(r^2 + a^2) - a^2 (\sin \theta)^2 (2r - 2M)}{\Sigma^2} - \frac{2r}{q^2}. \end{aligned} \tag{5.7}$$

Notice that

$$r^{-1} \left| \frac{\mathbf{D}_1 B}{A} \right| + \left| \frac{\mathbf{D}_2 B}{A} \right| + r^{-1} \left| \frac{\mathbf{D}_1 A}{A} - \frac{2 \cos \theta}{\sin \theta} \right| + \left| \frac{\mathbf{D}_2 A}{A} - \frac{2}{r} \right| \lesssim aMr^{-3}. \tag{5.8}$$

and

$$\begin{aligned} \frac{|E_1 - \mathbf{D}_1 \phi|}{r} + |E_2 - \mathbf{D}_2 \phi| + |E_3 - \mathbf{D}_3 \phi| &\lesssim aMr^{-3} (|\phi| + |\psi|), \\ \frac{|F_1 - \mathbf{D}_1 \psi|}{r} + |F_2 - \mathbf{D}_2 \psi| + |F_3 - \mathbf{D}_3 \psi| &\lesssim aMr^{-3} (|\phi| + |\psi|), \\ \left| \frac{M_1}{r} + \frac{2 \cos \theta}{r \sin \theta} \psi \right| + \left| M_2 + \frac{2\psi}{r} \right| + |M_3| &\lesssim aMr^{-3} (|\phi| + |\psi|). \end{aligned} \tag{5.9}$$

Letting  ${}^0\mathbf{g}^{\alpha\beta}$  denote the Schwarzschild components of the metric, see (4.2), and  $\mathbf{g}^{\alpha\beta}$  the Kerr components, we notice that

$$\begin{aligned} \mathbf{g}^{11} &= {}^0\mathbf{g}^{11} + O(a^2r^{-4}), & \mathbf{g}^{22} &= {}^0\mathbf{g}^{22} + O(a^2r^{-2}), \\ \mathbf{g}^{23} &= {}^0\mathbf{g}^{23} + O(a^2M^2r^{-4}), & \mathbf{g}^{33} &= {}^0\mathbf{g}^{33} + O(a^2r^{-2}). \end{aligned} \tag{5.10}$$

We notice that the term  $J_1$  in (5.5) is singular when  $\theta = 0$ , due to the fraction  $\mathbf{D}_1A/A$ . To eliminate this singularity we work with a modification of the 1-form  $P$ , namely

$$\tilde{P}_\mu = \tilde{P}_\mu[X, w, m, m'] := P_\mu - \frac{X^\nu \mathbf{D}_\nu A}{A} \frac{\mathbf{D}_\mu A}{A} \psi^2. \tag{5.11}$$

Then

$$2\mathbf{D}^\mu \tilde{P}_\mu = 2\mathbf{D}^\mu P_\mu - 4 \frac{X^\nu \mathbf{D}_\nu A}{A} \frac{\mathbf{D}_\mu A}{A} \psi \mathbf{D}^\mu \psi - 2\mathbf{D}^\mu \left[ \frac{X^\nu \mathbf{D}_\nu A}{A} \frac{\mathbf{D}_\mu A}{A} \right] \psi^2 = \sum_{j=1}^5 L^j, \tag{5.12}$$

where

$$\begin{aligned} L^1 &= L^1[X, w, m, m'] := Q_{\mu\nu}^{(X)} \pi^{\mu\nu} + w(E_\alpha E^\alpha + F_\alpha F^\alpha + M_\alpha M^\alpha), \\ L^2 &= L^2[X, w, m, m'] := \phi m^\mu \mathbf{D}_\mu \phi + \psi m'^\mu \mathbf{D}_\mu \psi, \\ L^3 &= L^3[X, w, m, m'] := \frac{1}{2} \phi^2 (\mathbf{D}^\mu m_\mu - \square w) + \frac{1}{2} \psi^2 (\mathbf{D}^\mu m'_\mu - \square w), \\ L^4 &= L^4[X, w, m, m'] := -2\mathbf{D}^\mu \left[ \frac{X^\nu \mathbf{D}_\nu A}{A} \frac{\mathbf{D}_\mu A}{A} \right] \psi^2, \\ L^5 &= L^5[X, w, m, m'] := 2X^\nu J_\nu - 4 \frac{X^\nu \mathbf{D}_\nu A}{A} \frac{\mathbf{D}_\mu A}{A} \psi \mathbf{D}^\mu \psi + w(\phi \mathcal{N}_\phi + \psi \mathcal{N}_\psi). \end{aligned} \tag{5.13}$$

The terms  $L^1, L^2, L^3$  are similar to the corresponding terms we estimated in the proof of Theorem 4.1. The main new terms are  $L^4$  and the quadratic part of  $L^5$ . We describe these terms below.

**Lemma 5.1** *Assuming that  $X = f\partial_2 + g\partial_3$ , where  $f$  may depend only on  $r$ , we have*

$$L^4 = -8 \frac{\mathbf{g}^{22}}{r} \partial_2 [r^{-1} f] \psi^2 + O(a^2r^{-5}) [ |f| + r|f'| ] \psi^2 \tag{5.14}$$

and

$$\begin{aligned} |L^5| &\lesssim \frac{aM}{r^4} |f| (|\phi| + |\psi|) \left\{ \sum_{Y \in \{E, F\}} \left( \frac{|Y_1|}{r} + \frac{M}{r} |Y_2| + \frac{M}{r} |Y_3| \right) + \frac{1}{r} (|\phi| + |\psi|) \right\} \\ &\quad + |\mathcal{N}_\phi| |2fE_2 + 2gE_3 + w\phi| + |\mathcal{N}_\psi| |2fF_2 + 2gF_3 + w\psi|. \end{aligned} \tag{5.15}$$

*Proof* We rewrite

$$L^4 = -2\mathbf{D}^\mu \left[ \frac{X^\nu \mathbf{D}_\nu A}{A} \frac{\mathbf{D}_\mu A}{A} \right] \psi^2 = -2\mathbf{D}^\mu \left[ \frac{f \mathbf{D}_2 A}{A} \frac{\mathbf{D}_\mu A}{A} \right] \psi^2.$$

In view of (1.10) and (5.7),

$$\left| \frac{f \mathbf{D}_2 A}{A} \mathbf{D}^\mu \left[ \frac{\mathbf{D}_\mu A}{A} \right] \right| = \left| \frac{f \mathbf{D}_2 A}{A} \frac{\mathbf{D}_\mu B \mathbf{D}^\mu B}{A^2} \right| \lesssim \frac{a^2 M^2}{r^7} |f|.$$

Also

$$\left| \mathbf{g}^{11} \partial_1 \left[ \frac{f \partial_2 A}{A} \right] \frac{\partial_1 A}{A} \right| \lesssim \frac{a^2}{r^5} |f|.$$

and

$$\left| \mathbf{g}^{22} \partial_2 \left[ \frac{f \partial_2 A}{A} \right] \frac{\partial_2 A}{A} - \mathbf{g}^{22} \partial_2 \left[ \frac{2f}{r} \right] \frac{2}{r} \right| \lesssim \frac{a^2}{r^5} |f| + \frac{a^2}{r^4} |f'|.$$

The desired formula (5.14) follows.

We estimate now the term  $L^5$ . We start by rewriting

$$L^5 = 2X^\nu \left[ \frac{2\mathbf{D}_\nu B M^\mu E_\mu - 2\mathbf{D}_\nu A M^\mu F_\mu}{A} \right] - 4 \frac{X^\nu \mathbf{D}_\nu A}{A} \frac{\mathbf{D}_\mu A}{A} \psi \mathbf{D}^\mu \psi + \mathcal{N}_\phi (2X^\nu E_\nu + w\phi) + \mathcal{N}_\psi (2X^\nu F_\nu + w\psi).$$

Using (5.7) and (5.2), we estimate

$$\left| 2X^\nu \frac{2\mathbf{D}_\nu B M^\mu E_\mu}{A} \right| \lesssim \frac{a^2 M}{r^5} |f| (|\phi| + |\psi|) [|E_1/r| + Mr^{-1}|E_2| + Mr^{-1}|E_3|],$$

and

$$\begin{aligned} & \left| 2X^\nu \left[ \frac{-2\mathbf{D}_\nu A M^\mu F_\mu}{A} \right] - 4 \frac{X^\nu \mathbf{D}_\nu A}{A} \frac{\mathbf{D}_\mu A}{A} \psi \mathbf{D}^\mu \psi \right| \\ & \lesssim \frac{aM}{r^4} |f| |\phi| \left[ |F_1/r| + \frac{M}{r} |F_2| + \frac{M}{r} |F_3| + \frac{1}{r} |\psi| \right]. \end{aligned}$$

The desired formula (5.15) follows. □

As in the proof of Theorem 4.1, our goal is to choose suitable multipliers  $(X, w, m, m')$  in a such a way that the quadratic terms in the divergence formula

$$\int_{\Sigma_{t_1}^c} \tilde{P}_\mu n_0^\mu d\mu_{t_1} = \int_{\Sigma_{t_2}^c} \tilde{P}_\mu n_0^\mu d\mu_{t_2} + \int_{\mathcal{N}_{[t_1, t_2]}^c} \tilde{P}_\mu k_0^\mu d\mu_c + \int_{\mathcal{D}_{[t_1, t_2]}^c} \mathbf{D}^\mu \tilde{P}_\mu d\mu \tag{5.16}$$



are nonnegative, where  $t_1, t_2 \in [0, T], c \in (c_0, r_{\mathcal{H}}], n_0 := n/|\mathbf{g}^{33}|^{1/2}, k_0 := k/|\mathbf{g}^{22}|^{1/2}$ , and the integration is with respect to the natural measures induced by the metric  $\mathbf{g}$ .

### 5.1 The Multipliers $(X_{(k)}, w_{(k)}, m_{(k)}, m'_{(k)}), k \in \{1, 2, 3, 4\}$

In this subsection we introduce the main multipliers. The multipliers  $(X_{(k)}, w_{(k)}, m_{(k)}, m'_{(k)}), k \in \{1, 2, 3\}$  are analogous to the multipliers  $(X_{(k)}, w_{(k)}, m_{(k)}), k \in \{1, 2, 3\}$ , used in the analysis of the wave equation in Schwarzschild spacetime in the previous section. On the other hand, the multiplier  $(X_{(4)}, w_{(4)}, m_{(4)}, m'_{(4)})$ , which is supported in a small region close to the trapped set, is new and is used mostly to control the contribution of the new term  $L^4$  in (5.13).

#### 5.1.1 Analysis Around the Trapped Set

As in the previous section, we start by constructing the multiplier  $(X_{(1)}, w_{(1)}, m_{(1)}, m'_{(1)})$ , which is relevant in a neighborhood of the trapped set. For now our main concern is the positivity of the spacetime integral  $\mathbf{D}^\mu \tilde{P}_\mu$ ; as in the proof of Theorem 4.1, the positivity of the surfaces integrals along  $\Sigma_i^c$  and  $\mathcal{N}_{[t_1, t_2]}^c$  can only be addressed after the other multipliers are introduced.

It is important to recall that we are in the axially symmetric case. Therefore the relevant trapped null geodesics are still confined to a codimension 1 set. More precisely, recalling that  $a \ll M$ , it is easy to see that the equation  $r^3 - 3Mr^2 + a^2r + Ma^2 = 0$  has a unique solution  $r^* \in (c_0, \infty)$ . Moreover,  $r^* \in [3M - a^2/M, 3M]$  and

$$|r^3 - 3Mr^2 + a^2r + Ma^2 - (r - r^*)r^2| \lesssim (a^2/M)r|r - r^*| \quad \text{if } r \in (c_0, \infty). \tag{5.17}$$

We start by setting, as before,

$$\begin{aligned} X_{(1)} &:= f_1(r)\partial_2 + g_1(r)\partial_3, & f_1(r) &:= \frac{a_1(r)\Delta}{r^2}, & g_1(r) &:= \frac{a_1(r)\chi(r)2M}{r} + 1, \\ w_{(1)}(r, \theta) &:= f'_1(r) + f_1(r)\partial_r \log(\Sigma^2/\Delta) - \epsilon_1 \tilde{w}(r), & & & & (5.18) \\ \tilde{w}(r) &:= M^2(r - 33M/16)^3(r - r^*)^2 r^{-8} \mathbf{1}_{[33M/16, \infty)}(r), \\ m_{(1)} = m'_{(1)} &:= 0, \end{aligned}$$

where  $a_1 : (0, \infty) \rightarrow \mathbb{R}$  is a smooth function to be fixed,  $\lim_{r \rightarrow \infty} a_1(r) = 1, \epsilon_1 \in (0, 1]$  is a small constant and  $\Sigma^2 = (r^2 + a^2)^2 - a^2(\sin \theta)^2 \Delta$  is as in (1.5).

Let

$$L^j_{(1)} := L^j[X_{(1)}, w_{(1)}, m_{(1)}, m'_{(1)}],$$

for  $j \in \{1, 2, 3, 4, 5\}$ , see (5.13). Notice that

$$L^2_{(1)} = 0, \quad L^3_{(1)} = -\frac{1}{2} \square w_{(1)} (\phi^2 + \psi^2). \tag{5.19}$$

Using (A.15),

$$L^1_{(1)} = \sum_{Y \in \{E, F, M\}} [K^{11}_{(1)}(Y_1)^2 + K^{22}_{(1)}(Y_2)^2 + K^{33}_{(1)}(Y_3)^2 + 2K^{23}_{(1)}Y_2Y_3],$$

where

$$\begin{aligned} K^{11}_{(1)} &= \frac{-f'_1(r)}{q^2} + w_{(1)}(r, \theta) \mathbf{g}^{11}, \\ K^{22}_{(1)} &= \frac{-f_1(r)(2r - 2M) + f'_1(r)\Delta}{q^2} + w_{(1)}(r, \theta) \mathbf{g}^{22}, \\ K^{33}_{(1)} &= -f_1(r)\partial_2 \mathbf{g}^{33} + 2g'_1(r)\mathbf{g}^{23} - f'_1(r)\mathbf{g}^{33} - \frac{2rf_1(r)\mathbf{g}^{33}}{q^2} + w_{(1)}(r, \theta) \mathbf{g}^{33}, \\ K^{23}_{(1)} &= \frac{-2Mrf_1(r)\chi'(r) - 2Mf_1(r)\chi(r) + g'_1(r)\Delta}{q^2} + w_{(1)}(r, \theta) \mathbf{g}^{23}. \end{aligned}$$

Simple calculations, using also (A.6), show that

$$\begin{aligned} \partial_r \log(\Sigma^2/\Delta) &= \frac{\Delta \partial_r \Sigma^2 - \Sigma^2 \partial_r \Delta}{\Delta \Sigma^2} = \frac{2(r^2 + a^2)(r^3 - 3Mr^2 + a^2r + Ma^2)}{\Delta \Sigma^2}, \\ \mathbf{g}^{33} &= -\frac{\Sigma^2}{q^2 \Delta} + \frac{4M^2 r^2}{q^2 \Delta} \chi(r)^2. \end{aligned} \tag{5.20}$$

Using also the formulas (4.19) and (A.6) we calculate

$$\begin{aligned} K^{11}_{(1)} &= a_1(r) \frac{2(r^2 + a^2)(r^3 - 3Mr^2 + a^2r + Ma^2)}{r^2 q^2 \Sigma^2} - \epsilon_1 \tilde{w}(r) \mathbf{g}^{11}, \\ K^{22}_{(1)} &= \frac{2\Delta^2}{q^2 r^2} \left[ a'_1(r) + a_1(r) \frac{-2a^2(r^2 + a^2) + a^2(\sin \theta)^2(r^2 - 3Mr + 2a^2)}{\Sigma^2 r} \right] \\ &\quad - \epsilon_1 \tilde{w}(r) \mathbf{g}^{22}, \\ K^{33}_{(1)} &= \frac{8M^2 \chi(r)^2}{q^2} \left[ a'_1(r) + a_1(r) \frac{-2a^2(r^2 + a^2) + a^2(\sin \theta)^2(r^2 - 3Mr + 2a^2)}{\Sigma^2 r} \right] \\ &\quad - \epsilon_1 \tilde{w}(r) \mathbf{g}^{33}, \\ K^{23}_{(1)} &= \frac{4M \Delta \chi(r)}{q^2 r} \left[ a'_1(r) + a_1(r) \frac{-2a^2(r^2 + a^2) + a^2(\sin \theta)^2(r^2 - 3Mr + 2a^2)}{\Sigma^2 r} \right] \\ &\quad - \epsilon_1 \tilde{w}(r) \mathbf{g}^{23}. \end{aligned}$$

Therefore

$$\begin{aligned}
 L_{(1)}^1 \geq & \sum_{Y \in \{E, F, M\}} \left\{ \frac{(2 - a/M)a_1(r)(r - r^*) - \epsilon_1 r^4 \tilde{w}(r) q^{-2}}{r^4} (Y_1)^2 \right. \\
 & + \left[ (2 - a/M)a_1'(r) - \epsilon_1 \tilde{w}(r) \frac{r^4}{q^2 \Delta} \right] \left( \frac{\Delta}{r^2} Y_2 + \frac{2M\chi(r)}{r} Y_3 \right)^2 \\
 & \left. + \epsilon_1 \tilde{w}(r) \frac{\Sigma^2}{q^2 \Delta} (Y_3)^2 \right\}, \tag{5.21}
 \end{aligned}$$

provided that  $a$  is sufficiently small and

$$a_1(r^*) = 0 \quad \text{and} \quad a_1'(r) \geq a^{1/2} M^{3/2} r^{-3} |a_1(r)| \text{ for } r \in (c_0, \infty). \tag{5.22}$$

This condition is clearly satisfied by the function  $a_1$  defined below.

The important function  $a_1$  is defined as in the proof of Theorem 4.1, see (4.22),

$$\begin{aligned}
 R(r) & := (r - r^*)(r + 2M) + 6M^2 \log \left( \frac{r - r_{\mathcal{H}}}{r^* - r_{\mathcal{H}}} \right), \\
 a_1(r) & := r^{-2} \delta^{-1} \kappa(\delta R(r)) + \left[ \frac{r^* - 2M}{r} - \frac{6M^2}{r^2} \log \left( \frac{r - r_{\mathcal{H}}}{r^* - r_{\mathcal{H}}} \right) \right] \chi_{\geq DM}(r), \tag{5.23}
 \end{aligned}$$

where  $\delta := \epsilon_2^2 M^{-2}$  is a small constant and  $D \gg 1$  is a large constant. This function can be analyzed as in section 4, see (4.23)–(4.32), once we observe that

$$r_{\mathcal{H}} = 2M + O(a^2/M), \quad r^* = 3M + O(a^2/M), \quad \Sigma^2 = r^4 + O(a^2 r^2).$$

Recalling also the identities (5.10) and defining

$$h_1(r, \theta) := f_1'(r) + f_1(r) \partial_r \log(\Sigma^2/\Delta) = \frac{\Delta}{\Sigma^2} \partial_r [a_1(r) \Sigma^2 r^{-2}], \tag{5.24}$$

we estimate, as in (4.30),

$$\begin{aligned}
 (\square h_1)(r, \theta) & = -\frac{2M}{r^4} \left( 7 - \frac{44M}{r} + \frac{72M^2}{r^2} \right) + O(ar^{-4}) + O(Mr^{-4}) \mathbf{1}_{[DM, \infty)}(r) \\
 & \quad + M^{-3} O(1) \mathbf{1}_{(c_0, r_\delta]}(r) + O\left( \frac{\delta^2 M^2}{r - r_{\mathcal{H}}} \right) \mathbf{1}_{[r'_\delta, r_\delta]}(r), \tag{5.25}
 \end{aligned}$$

where  $r_\delta$  and  $r'_\delta$  denote the unique numbers in  $(r_{\mathcal{H}}, \infty)$  with the property that  $R(r_\delta) = -1/\delta$  and  $R(r'_\delta) = -2/\delta$ . We also have, compare with (4.26),

$$a_1(r^*) = 0 \quad \text{and} \quad a_1'(r) \geq 10M^2 r^{-3} \quad \text{for } r \in (c_0, \infty), \tag{5.26}$$

if  $\delta$  is sufficiently small. In particular, this implies (5.22) if  $a$  is sufficiently small relative to  $\epsilon_2$ .

The bound (5.21) shows that

$$L^1_{(1)} \geq \sum_{Y \in \{E, F, M\}} \left\{ \frac{(2 - C_1 \epsilon_1) a_1(r)(r - r^*)}{r^4} (Y_1)^2 + \epsilon_1 \tilde{w}(r) (Y_3)^2 + (2 - C_1 \epsilon_1) a'_1(r) \left( \frac{\Delta}{r^2} Y_2 + \frac{2M\chi(r)}{r} Y_3 \right)^2 \right\}, \tag{5.27}$$

for a sufficiently large constant  $C_1$ , provided that the constant  $\epsilon_1$  is sufficiently small and  $a/M \leq \epsilon_1$ . Moreover, the identities (5.19) and (5.25) show that

$$L^3_{(1)} \geq \frac{M(1 - C_1 \epsilon_1)}{r^4} \left( 7 - \frac{44M}{r} + \frac{72M^2}{r^2} \right) (\phi^2 + \psi^2) - \frac{C_1 M}{r^4} \mathbf{1}_{[DM, \infty)}(r) (\phi^2 + \psi^2) - \frac{C_1}{M^3} \mathbf{1}_{(c_0, r_\delta]}(r) (\phi^2 + \psi^2) - \frac{C_1 \delta^2 M^2}{r - r_{\mathcal{H}}} \mathbf{1}_{[r'_\delta, r_\delta]}(r) (\phi^2 + \psi^2). \tag{5.28}$$

The bounds (5.14) and (5.15) and the definitions show that

$$L^2_{(1)} = 0, \tag{5.29}$$

$$L^4_{(1)} = -8 \frac{g^{22}}{r} \partial_2 [r^{-1} f_1] \psi^2 + O(a^2 r^{-5}) [ |f_1| + r |f'_1| ] \psi^2 = \left[ -\frac{8\Delta^2}{q^2 r^4} a'_1(r) + \frac{8\Delta(r^2 - 4Mr)}{q^2 r^5} a_1(r) \right] \psi^2 + O(a^2 r^{-5}) [ |a_1| + |r - r_{\mathcal{H}}| |a'_1| ] \psi^2. \tag{5.30}$$

and

$$|L^5_{(1)}| \lesssim \frac{aM}{r^4} |f| (|\phi| + |\psi|) \left\{ \sum_{Y \in \{E, F\}} \left( \frac{|Y_1|}{r} + \frac{M}{r} |Y_2| + \frac{M}{r} |Y_3| \right) + \frac{1}{r} (|\phi| + |\psi|) \right\} + |\mathcal{N}_\phi| |2f_1 E_2 + 2g_1 E_3 + w_1 \phi| + |\mathcal{N}_\psi| |2f_1 F_2 + 2g_1 F_3 + w_1 \psi|. \tag{5.31}$$

Using (5.25) and (5.30), together with the inequalities in the last line of (5.9), after possibly increasing the constant  $C_1$  we have

$$L^1_{(1)} + L^4_{(1)} \geq \sum_{Y \in \{E, F\}} \left\{ \frac{(2 - C_1 \epsilon_1) a_1(r)(r - r^*)}{r^4} (Y_1)^2 + \epsilon_1 \tilde{w}(r) (Y_3)^2 + (2 - C_1 \epsilon_1) a'_1(r) \left( \frac{\Delta}{r^2} Y_2 + \frac{2M\chi(r)}{r} Y_3 \right)^2 \right\} + \frac{8\Delta(r^2 - 4Mr)}{r^7} a_1(r) \psi^2 + \frac{(2 - C_1 \epsilon_1) a_1(r)(r - r^*)}{r^4} \frac{4(\cos \theta)^2 \psi^2}{(\sin \theta)^2} - C_1 \frac{a^2 |a_1(r)| + \epsilon_1 r^2 |r - r_{\mathcal{H}}| |a'_1(r)|}{r^5} (\phi^2 + \psi^2). \tag{5.32}$$

### 5.1.2 Analysis in a Neighborhood of the Horizon

In a small neighborhood of the horizon we need to use the redshift effect. As in subsection 4.1, we define

$$\begin{aligned}
 X_{(2)} &:= f_2(r)\partial_2 + g_2(r)\partial_3, & f_2(r) &:= -\epsilon_2 a_2(r), & g_2(r) &:= \epsilon_2 a_2(r)(1 - \epsilon_2), \\
 w_{(2)}(r) &:= -2\epsilon_2 a_2(r)/r, & m_{(2)2} &= m_{(2)3} = m'_{(2)2} = m'_{(2)3} &:= \epsilon_2 M^{-2} \gamma(r), \\
 m_{(2)1} &= m_{(2)4} = m'_{(2)1} = m'_{(2)4} &:= 0,
 \end{aligned}
 \tag{5.33}$$

where  $\epsilon_2$  is a small positive constant (recall that  $\delta = \epsilon_2^2 M^{-2}$ ),

$$a_2(r) := \begin{cases} M^{-3}(9M/4 - r)^3 & \text{if } r \leq 9M/4, \\ 0 & \text{if } r \geq 9M/4, \end{cases}
 \tag{5.34}$$

and  $\gamma : [c_0, \infty) \rightarrow [0, 1]$  is a function supported in  $[c_0, 17M/8]$ , and satisfying  $\gamma(r_{\mathcal{H}}) = 1/2$  and a property similar to (4.38).

Let  $L^j_{(2)} := L^j[X_{(2)}, w_{(2)}, m_{(2)}, m'_{(2)}]$ ,  $j \in \{1, 2, 3, 4, 5\}$ . As in the proof of Theorem 4.1, see Lemma 4.2 and (4.40), the multipliers  $(X_{(1)}, w_{(1)}, m_{(1)}, m'_{(1)})$  and  $(X_{(2)}, w_{(2)}, m_{(2)}, m'_{(2)})$  can be combined to prove the following:

**Lemma 5.2** *The constants  $\epsilon_1, \epsilon_2$  can be fixed small enough such that there is a sufficiently small absolute constant  $\epsilon_3 > 0$  with the property that*

$$\begin{aligned}
 &\sum_{j=1}^4 (L^j_{(1)} + L^j_{(2)}) \\
 &\geq \epsilon_3 \sum_{Y \in \{E, F, M\}} \left[ \frac{(r - r^*)^2}{r^3} (Y_1/r)^2 + \frac{M^2}{r^3} (Y_2)^2 + \frac{M^2(r - r^*)^2}{r^5} (Y_3)^2 \right] \\
 &\quad + \epsilon_3 \frac{M}{r^4} (\phi^2 + \psi^2) - \epsilon_3^{-1} \frac{M}{r^4} \mathbf{1}_{[DM, \infty)}(r) (\phi^2 + \psi^2) + \tilde{L},
 \end{aligned}
 \tag{5.35}$$

where

$$\begin{aligned}
 \tilde{L} &:= \frac{8\Delta(r^2 - 4Mr)}{r^7} a_1(r) \psi^2 + (1 - 2C_1 \epsilon_1) \mathbf{1}_{[r^*, \infty)}(r) \left\{ \frac{M}{r^4} \left( 7 - \frac{44M}{r} + \frac{72M^2}{r^2} \right) \psi^2 \right. \\
 &\quad \left. + \frac{8a_1(r)(r - r^*)}{r^4} \frac{(\cos \theta)^2}{(\sin \theta)^2} \psi^2 + \frac{2a_1(r)(r - r^*)}{r^4} (F_1)^2 + 2a'_1(r) \frac{\Delta^2}{r^4} (F_2)^2 \right\},
 \end{aligned}
 \tag{5.36}$$

provided that  $a/M$  and  $(r_{\mathcal{H}} - c_0)/M$  are very small relative to  $\epsilon_3$ . Moreover

$$2(\tilde{P}_{(1)\mu} + \tilde{P}_{(2)\mu})k^\mu \geq \epsilon_3 \sum_{Y \in \{E, F, M\}} [(Y_1/r)^2 + (Y_2)^2(r_{\mathcal{H}} - c)/M] + \epsilon_3 M^{-2}(\phi^2 + \psi^2) - \epsilon_3^{-1}[(E_3)^2 + (F_3)^2], \tag{5.37}$$

along  $\mathcal{N}_{[t_1, t_2]}^c$ . Also

$$2(\tilde{P}_{(1)\mu} + \tilde{P}_{(2)\mu})n^\mu \geq -\epsilon_3^{-1}\{\tilde{e}_0 + \mathbf{1}_{[8M, 2DM]}(r)[(E_3)^2 + (F_3)^2]\} - \frac{\chi_{\geq 8M}(r)(1-p)}{r^2} \partial_2(r\phi^2 + r\psi^2) + \epsilon_3[(E_2)^2 + (F_2)^2]\mathbf{1}_{(c_0, 17M/8]}(r), \tag{5.38}$$

and

$$2(\tilde{P}_{(1)\mu} + \tilde{P}_{(2)\mu})n^\mu \leq \epsilon_3^{-1}\{\tilde{e}_0 + \mathbf{1}_{[8M, 2DM]}(r)[(E_3)^2 + (F_3)^2]\} - \frac{\chi_{\geq 8M}(r)(1-p)}{r^2} \partial_2(r\phi^2 + r\psi^2) + \epsilon_3^{-1}[(E_2)^2 + (F_2)^2]\mathbf{1}_{(c_0, 17M/8]}(r), \tag{5.39}$$

where

$$\tilde{e}_0 = \frac{(E_1)^2 + (F_1)^2 + (M_1)^2}{r^2} + (L\phi)^2 + (L\psi)^2 + \frac{M^2|r - r_{\mathcal{H}}|}{r^3} [(E_2)^2 + (F_2)^2] + \frac{M^2}{r^2} [(E_3)^2 + (F_3)^2] + \frac{1}{r^2}(\phi^2 + \psi^2). \tag{5.40}$$

Finally,

$$|L_{(1)}^5| + |L_{(2)}^5| \leq \frac{\epsilon_3^{-1}aM|r - r^*|}{r^5} (|\phi| + |\psi|) \times \left\{ \sum_{Y \in \{E, F\}} \left( \frac{|Y_1|}{r} + \frac{M(|Y_2| + |Y_3|)}{r} \right) + \frac{1}{r} (|\phi| + |\psi|) \right\} + \epsilon_3^{-1}[e(\phi, \mathcal{N}_\phi) + e(\psi, \mathcal{N}_\psi)]. \tag{5.41}$$

*Proof* The order of the constants to keep in mind is

$$\max(a/M, (r_{\mathcal{H}} - c_0)/M) \ll \epsilon_3 \ll \min(\epsilon_1, \epsilon_2) \leq \max(\epsilon_1, \epsilon_2) \ll C_1^{-1} \ll 1. \tag{5.42}$$

Most of the proof follows in the same way as in Lemma 4.2, using the identities/inequalities (A.16), (A.17), (5.25), (5.31), and (5.32)

The term  $\tilde{L}$  is new, when compared to the corresponding inequality (4.40) in the case of the pure wave equation. It is necessary to have this term because of the term  $L^4_{(1)}$  in (5.30), which leads to the term

$$\frac{8\Delta(r^2 - 4Mr)}{r^7} a_1(r)\psi^2$$

in (5.32). This term is clearly nonnegative if  $r \leq r^*$  or  $r \geq 4M$ ; however, for  $r \in [r^*, 4M]$  we need an additional multiplier to control this term. The other terms in (5.36) are coming from corresponding terms in (5.32) and (5.25), and their role is to help  $\tilde{L}$  become positive. We show how to control this term below.  $\square$

### 5.1.3 The New Multiplier $(X_{(4)}, w_{(4)}, m_{(4)}, m'_{(4)})$

We define, with  $a_1$  as in (5.23),

$$\begin{aligned} X_{(4)} &:= 0, & w_{(4)} &= 0, & m_{(4)} &= 0, \\ \tilde{m}'_{(4)1}(r, \theta) &:= -(1 - 2C_1\epsilon_1) \frac{8(r - r^*)a_1(r)\chi_{\leq 6R}(r) \cos \theta}{r^2 \sin \theta} \mathbf{1}_{[r^*, \infty)}(r), & (5.43) \\ \tilde{m}'_{(4)2}(r) &:= (1 - 2C_1\epsilon_1) \frac{2b(r)}{\Delta}, & \tilde{m}'_{(4)3} &:= 0, & \tilde{m}'_{(4)4} &:= 0, \end{aligned}$$

for some function  $b$  supported in  $[r^*, 4M]$  to be fixed. We prove the following:

**Lemma 5.3** *Letting  $L^j_{(4)} := L^j[X_{(4)}, w_{(4)}, m_{(4)}, m'_{(4)}]$ ,  $j \in \{1, 2, 3, 4, 5\}$ , we have*

$$L^1_{(4)} = L^4_{(4)} = L^5_{(4)} = 0 \tag{5.44}$$

and, for some constant  $C_2$  sufficiently large,

$$\tilde{L} + L^2_{(4)} + L^3_{(4)} \geq -C_2(a + |r_{\mathcal{H}} - c_0|)r^{-4}(\phi^2 + \psi^2). \tag{5.45}$$

Moreover,

$$|2\tilde{P}_{(4)\mu}n^\mu| \lesssim \epsilon_3^{-1}\psi^2/r^2 \quad \text{and} \quad 2\tilde{P}_{(4)\mu}k^\mu = 0 \quad \text{along } \mathcal{N}^c_{[t_1, t_2]}. \tag{5.46}$$

*Proof* The identities in (5.44) are clear. The inequality in (5.45) is also clear in the regions  $\{r \leq r^*\}$  and  $\{r \geq 12M\}$ .

Using the formula (A.10) we calculate, in the region  $\{r \in [r^*, 12M]\}$ ,

$$\begin{aligned} \frac{1}{2}\mathbf{D}^\mu \tilde{m}'_{(4)\mu} &= (1 - 2C_1\epsilon_1) \left[ \frac{4(r - r^*)a_1(r)\chi_{\leq 6R}(r)}{q^2r^2} + \frac{b'(r)}{q^2} \right], \\ \psi \tilde{m}'_{(4)\mu} \mathbf{D}_\mu \psi &= (1 - 2C_1\epsilon_1) \left[ - \frac{8(r - r^*)a_1(r)\chi_{\leq 6R}(r) \cos \theta}{q^2r^2 \sin \theta} \psi \mathbf{D}_1 \psi + \frac{2b(r)}{q^2} \psi \mathbf{D}_2 \psi \right]. \end{aligned}$$

Therefore, in the region  $\{r \in [r^*, 12M]\}$ ,

$$\begin{aligned} &L_{(4)}^2 + L_{(4)}^3 + \tilde{L} \\ &= \frac{8\Delta(r^2 - 4Mr)}{r^7} a_1(r)\psi^2 + (1 - 2C_1\epsilon_1) \left\{ \frac{M}{r^4} \left( 7 - \frac{44M}{r} + \frac{72M^2}{r^2} \right) \psi^2 \right. \\ &\quad \left. + \frac{8a_1(r)(r - r^*)}{r^4} \frac{(\cos \theta)^2}{(\sin \theta)^2} \psi^2 + \frac{2a_1(r)(r - r^*)}{r^4} (F_1)^2 + 2a'_1(r) \frac{\Delta^2}{r^4} F_2^2 \right\} \\ &\quad + (1 - 2C_1\epsilon_1) \left[ \frac{4(r - r^*)a_1(r)\chi_{\leq 6R}(r)}{q^2 r^2} + \frac{b'(r)}{q^2} \right] \psi^2 \\ &\quad + (1 - 2C_1\epsilon_1) \left[ -\frac{8(r - r^*)a_1(r)\chi_{\leq 6R}(r)}{q^2 r^2} \frac{\cos \theta}{\sin \theta} \psi \mathbf{D}_1 \psi + \frac{2b(r)}{q^2} \psi \mathbf{D}_2 \psi \right]. \end{aligned}$$

Recalling (5.9), we may replace  $\mathbf{D}_1 \psi$  and  $\mathbf{D}_2 \psi$  with  $F_1$  and  $F_2$ , up to acceptable errors. Then we divide by  $(1 - 2C_1\epsilon_1)$  and complete squares. For (5.45) it suffices to prove that

$$\begin{aligned} -C_2 a &\leq \frac{8\Delta(r^2 - 4Mr)a_1(r)}{r^7(1 - 2C_1\epsilon_1)} + \frac{M}{r^4} \left( 7 - \frac{44M}{r} + \frac{72M^2}{r^2} \right) \\ &\quad + \left[ \frac{4(r - r^*)a_1(r)\chi_{\leq 6R}(r)}{r^4} + \frac{b'(r)}{r^2} \right] - \frac{b(r)^2}{2\Delta^2 a'_1(r)}, \end{aligned}$$

for any  $r \in [r^*, 12M]$ , for some function  $b$  supported in  $[r^*, 4M]$  to be fixed. After algebraic simplifications, it suffices to prove that, for any  $r \in [r^*, 4M]$ ,

$$\begin{aligned} 0 &\leq \frac{M}{r^4} \left( 7 - \frac{44M}{r} + \frac{72M^2}{r^2} \right) + \frac{8\Delta(r - 4M)a_1(r)}{r^6(1 - 2C_1\epsilon_1)} \\ &\quad + \left[ \frac{4(r - r^*)a_1(r)}{r^4} + \frac{b'(r)}{r^2} \right] - \frac{b(r)^2}{2a'_1(r)\Delta^2}. \end{aligned} \tag{5.47}$$

We multiply both sides of (5.47) by  $r^6/M^3$ . It suffices to find a function  $b$  supported in  $[r^*, 4M]$  such that, for  $r \in [r^*, 4M]$ ,

$$\begin{aligned} 1 &\lesssim \frac{r^4 b'(r)}{M^3} + \left( \frac{7r^2}{M^2} - \frac{44r}{M} + 72 \right) - \frac{r^4 b(r)^2}{2M^3 a'_1(r)(r - 2M)^2} \\ &\quad + 4a_1(r) \left( \frac{3r^3}{M^3} - \frac{15r^2}{M^2} + \frac{16r}{M} \right). \end{aligned} \tag{5.48}$$

Let

$$r = (3 + s)M, \quad \tilde{b}(s) := b((3 + s)M).$$

Notice also that, for  $s \in [0, 1]$ ,

$$|a_1((3 + s)M) - \tilde{a}_1(s)| + |Ma'_1((3 + s)M) - \tilde{a}'_1(s)| \lesssim a,$$



where

$$\tilde{a}_1(s) := \frac{5s + s^2 + 6 \log(1 + s)}{(3 + s)^2}, \quad \tilde{a}'_1(s) := \frac{33 + s - 12 \log(1 + s) - 12 \frac{s}{s+1}}{(3 + s)^3}. \tag{5.49}$$

For (5.48) it suffices to prove that, for  $s \in [0, 1]$ ,

$$1 \lesssim \tilde{b}'(s) - \frac{\tilde{b}(s)^2}{2\tilde{a}'_1(s)(1 + s)^2} + \frac{7s^2 - 2s + 3}{(3 + s)^4} + 4\tilde{a}_1(s) \frac{3s^2 + 3s - 2}{(3 + s)^3}. \tag{5.50}$$

Notice that  $\tilde{a}'_1(s)(1 + s)^2 \geq 1$  for any  $s \in [0, 1]$ . Indeed, using (5.49),

$$\begin{aligned} (3 + s)^3 [\tilde{a}'_1(s)(1 + s)^2 - 1] &= (1 + s)[33 + 10s - 11s^2 + 12(1 + s)(s - \log(1 + s))] \\ &\quad - (3 + s)^3 \\ &= 12(1 + s)^2(s - \log(1 + s)) + 6 + 16s - 10s^2 - 12s^3 \\ &\geq 0. \end{aligned}$$

Therefore, for (5.50) it suffices to prove that, for  $s \in [0, 1]$ ,

$$1 \lesssim \tilde{b}'(s) - \frac{\tilde{b}(s)^2}{2} + \frac{7s^2 - 2s + 3}{(3 + s)^4} + 4\tilde{a}_1(s) \frac{3s^2 + 3s - 2}{(3 + s)^3}. \tag{5.51}$$

Moreover, for  $s \in [0, 1]$ ,

$$\begin{aligned} \frac{7s^2 - 2s + 3}{(3 + s)^4} + 4\tilde{a}_1(s) \frac{3s^2 + 3s - 2}{(3 + s)^3} &= \frac{9 - 91s + 167s^2 + 115s^3 - 24s^4}{(3 + s)^5} \\ &\quad + \frac{24(-2 + 3s + 3s^2)[\log(1 + s) - s + s^2/2]}{(3 + s)^5} \\ &\geq \frac{9 - 91s + 167s^2 + 91s^3}{(3 + s)^5} \\ &\geq \frac{9(1 - 10s + 18s^2)}{(3 + s)^5} \mathbf{1}_{[1/10, 1]}(s) + \frac{4s^2}{(3 + s)^5} + 10^{-10}. \end{aligned}$$

Therefore, to prove (5.51) it suffices to find a function  $\tilde{b}$  supported in  $[1/10, 1]$  such that

$$\tilde{b}'(s) + \frac{9(1 - 10s + 18s^2)}{(3 + s)^5} \geq 0 \quad \text{and} \quad |\tilde{b}(s)| \leq \frac{\sqrt{2}s}{16} \tag{5.52}$$

for any  $s \in [1/10, 1]$ .

Notice that  $1 - 10s + 18s^2 = 18(s - s_1)(s - s_2)$  where  $s_1 = (5 - \sqrt{7})/18$ ,  $s_2 = (5 + \sqrt{7})/18$ . We define  $\tilde{b}(s) = 0$  for  $s \leq s_1$  and

$$\tilde{b}(s) := \int_{s_1}^s \frac{9(10\rho - 1 - 18\rho^2)}{3^5} d\rho$$

for  $s \in [s_1, s_2]$ . The desired inequalities (5.52) are easy to verify for  $s \in [1/10, s_2]$ , and, moreover,  $\tilde{b}(s_2) = 7^{3/2}9^{-4} \leq 3 \cdot 10^{-3}$ .

On the other hand, for  $s \geq s_2$ , we would like to define the function  $\tilde{b}$  decreasing, still satisfying (5.52), and vanishing for  $s \geq 1$ . The only condition for this to be possible is the inequality

$$\int_{s_2}^1 \frac{9(1 - 10\rho + 18\rho^2)}{4^5} d\rho \geq \tilde{b}(s_2),$$

which is easy to verify. This completes the proof of the main inequality (5.45).

The identity and the inequality in (5.46) follow from definitions. □

As a consequence of Lemma 5.2 and Lemma 5.3 we have:

**Corollary 5.4** *There is a sufficiently small absolute constant  $\epsilon_3 > 0$  with the property that*

$$\begin{aligned} & \sum_{j=1}^4 (L^j_{(1)} + L^j_{(2)} + L^j_{(4)}) \\ & \geq \epsilon_3 \sum_{Y \in \{E, F, M\}} \left[ \frac{(r - r^*)^2}{r^3} (Y_1/r)^2 + \frac{M^2}{r^3} (Y_2)^2 + \frac{M^2(r - r^*)^2}{r^5} (Y_3)^2 \right] \\ & \quad + \epsilon_3 \frac{M}{r^4} (\phi^2 + \psi^2) - \epsilon_3^{-1} \frac{M}{r^4} \mathbf{1}_{[DM, \infty)}(r) (\phi^2 + \psi^2), \end{aligned} \tag{5.53}$$

and

$$\begin{aligned} 2(\tilde{P}_{(1)\mu} + \tilde{P}_{(2)\mu} + \tilde{P}_{(4)\mu})k^\mu & \geq \epsilon_3 \sum_{Y \in \{E, F, M\}} [(Y_1/r)^2 + (Y_2)^2(r\mathcal{H} - c)/M] \\ & \quad + \epsilon_3 M^{-2} (\phi^2 + \psi^2) - \epsilon_3^{-1} [(E_3)^2 + (F_3)^2], \end{aligned} \tag{5.54}$$

along  $\mathcal{N}^c_{[t_1, t_2]}$ . Moreover, with  $\tilde{e}_0$  as in (5.40),

$$\begin{aligned} 2(\tilde{P}_{(1)\mu} + \tilde{P}_{(2)\mu} + \tilde{P}_{(4)\mu})n^\mu & \geq -\epsilon_3^{-1} \{ \tilde{e}_0 + \mathbf{1}_{[8M, 2DM)}(r) [(E_3)^2 + (F_3)^2] \} \\ & \quad - \frac{\chi_{\geq 8M}(r)(1 - p)}{r^2} \partial_2(r\phi^2 + r\psi^2) \\ & \quad + \epsilon_3 [(E_2)^2 + (F_2)^2] \mathbf{1}_{(c_0, 17M/8)}(r), \end{aligned} \tag{5.55}$$

and

$$\begin{aligned}
 2(\tilde{P}_{(1)\mu} + \tilde{P}_{(2)\mu} + \tilde{P}_{(4)\mu})n^\mu &\leq \epsilon_3^{-1} \{ \tilde{e}_0 + \mathbf{1}_{[8M, 2DM]}(r) [(E_3)^2 + (F_3)^2] \} \\
 &\quad - \frac{\chi_{\geq 8M}(r)(1-p)}{r^2} \partial_2(r\phi^2 + r\psi^2) \\
 &\quad + \epsilon_3^{-1} [(E_2)^2 + (F_2)^2] \mathbf{1}_{(c_0, 17M/8]}(r). \tag{5.56}
 \end{aligned}$$

Finally,

$$\begin{aligned}
 |L_{(1)}^5| + |L_{(2)}^5| + |L_{(4)}^5| &\leq \frac{\epsilon_3^{-1} aM|r - r^*|}{r^5} (|\phi| + |\psi|) \\
 &\quad \times \left\{ \sum_{Y \in \{E, F\}} \left( \frac{|Y_1|}{r} + \frac{M(|Y_2| + |Y_3|)}{r} \right) + \frac{1}{r} (|\phi| + |\psi|) \right\} \\
 &\quad + \epsilon_3^{-1} [e(\phi, \mathcal{N}_\phi) + e(\psi, \mathcal{N}_\psi)]. \tag{5.57}
 \end{aligned}$$

These inequalities should be compared with the inequalities (4.40) and the corresponding inequalities in Lemma 4.2.

### 5.1.4 Outgoing Energies

Finally, as in subsection 4.2, we define  $(X_{(3)}, w_{(3)}, m_{(3)}, m'_{(3)})$  by

$$\begin{aligned}
 X_{(3)} &:= f_3 \partial_2 + \left( \frac{f_3}{1 - \tilde{p}} + g_3 \right) \partial_3, & w_{(3)} &:= \frac{2f_3}{r}, & m'_{(3)} &:= m_{(3)}, \\
 m_{(3)1} &:= m_{(3)4} := 0, & m_{(3)2} &:= \frac{2h_3}{r(1 - \tilde{p})}, & m_{(3)3} &:= -\frac{2h_3}{r}, \tag{5.58}
 \end{aligned}$$

where  $\tilde{p} := 2M/r$ , and  $f_3, g_3$  are defined by

$$f_3(r) := \epsilon_4 \chi_{\geq 8M}(r) e^{\beta(r)}, \quad g_3(r) := \int_r^\infty \left[ \rho(s) + \frac{\epsilon_4 M^2}{s^3} f_3(s) \right] ds, \tag{5.59}$$

where

$$\beta(8M) := 0, \quad \beta'(r) := \left( \frac{4M}{r^2} + \frac{1}{r} \right) (1 - \chi_{\geq 4M}(r)) + \frac{\alpha}{r} \chi_{\geq 4M}(r), \tag{5.60}$$

and

$$\rho(r) := \delta M^{-1} \left[ \chi_{\geq 4M}(r) + \chi_{\geq 4C_4 M}(r) \left( C_4^7 e^{\beta(r)} \frac{M^3}{r^3} - 1 \right) \right]. \tag{5.61}$$

The constants  $\epsilon_4, C_4$  satisfy  $\epsilon_4 = \epsilon_3^2$  and  $C_4 \geq \epsilon_4^{-4} \alpha^{-1} (2 - \alpha)^{-1}$ , while  $\delta \in [10^{-4} C_4^{-3}, 10^4 C_4^{-3}]$  is such that  $\int_{C_4 M}^\infty \rho(s) ds = C_4$ . Recall (4.60),

$$e^{\beta(r)} \approx \frac{r}{M} \text{ if } r \leq 10C_4 M \quad \text{and} \quad e^{\beta(r)} \approx C_4^4 \left( \frac{r}{C_4 M} \right)^\alpha \text{ if } r \geq (1/10)C_4 M. \tag{5.62}$$

Notice the additional term  $M^2 s^{-3} f_3(s)$  in the definition of the function  $g_3$ ; this term is needed in order to be able to estimate the contributions of the new terms containing the small coefficient  $a$ , in a way that is uniform as  $\alpha \rightarrow 0$  or  $\alpha \rightarrow 2$ .

Also let

$$H_3 := (1 - \tilde{p}) f_3' - \frac{2M f_3}{r^2} - (1 - \tilde{p})^2 \rho - \frac{\epsilon_4 M^2 f_3}{r^3} (1 - \tilde{p})^2, \quad h_3 := H_3 \cdot (1 - \tilde{\alpha}), \tag{5.63}$$

where  $\tilde{\alpha} := (2 - \alpha)/10$ . Recall the bounds (4.51) and (4.52),

$$\begin{aligned} & \beta(r) \in [-10, 0] \text{ and } M\beta'(r) \in [1/10, 10] && \text{if } r \in (c, 8M], \\ & \max \left( \frac{\alpha}{100r}, \frac{4M}{r^2} + \frac{1}{r} \mathbf{1}_{[8M, C_4 M]}(r) \right) \leq \beta'(r) \leq \frac{2}{r} && \text{if } r \in [8M, \infty), \\ & \rho(r) = 0 \text{ and } g_3(r) \in [C_4/2, 2C_4] && \text{if } r \leq C_4 M, \\ & \rho(r) \leq \frac{\epsilon_4}{100} \beta'(r) e^{\beta(r)} \text{ and } \rho'(r) \leq \frac{\epsilon_4 M}{100r^3} e^{\beta(r)} && \text{if } r \geq C_4 M, \\ & \frac{e^\beta M^2}{r^2} \leq g_3(r) \leq \frac{C_4^{10} e^\beta M^2}{r^2} && \text{if } r \geq C_4 M, \\ & (1 - 2\tilde{\alpha})H_3(r) - rH_3'(r) \geq 0 && \text{if } r \in [16M, \infty), \end{aligned} \tag{5.64}$$

$$\begin{aligned} & g_3' = -\rho - \epsilon_4 M^2 r^{-3} f_3, \\ & |H_3 - (1 - \tilde{p}) f_3'| \leq \frac{(2 + \epsilon_4) M f_3}{r^2} + \rho, \\ & e^{\beta(r)} \in [r/(100M), r^2/M^2] \quad \text{for } r \in (c, C_4 M), \end{aligned} \tag{5.65}$$

and

$$\begin{aligned} & \frac{2f_3 - r f_3'}{r} = \epsilon_4 e^\beta [(2/r - \beta') \chi_{\geq 8M} - \chi'_{\geq 8M}], \\ & \frac{6M f_3}{r^4} - \frac{2M f_3'}{r^3} + \frac{(1 - \tilde{p})^2 g_3''}{r} + \frac{4M(1 - \tilde{p}) g_3'}{r^3} \geq \frac{\epsilon_4 M}{100r^4} e^\beta \chi_{\geq 8M} - \frac{2\epsilon_4 M}{r^3} e^\beta \chi'_{\geq 8M}. \end{aligned} \tag{5.66}$$

Notice that

$$g^{33} = -\frac{r^2 + a^2}{\Delta} + O(a^2 M r^{-3}) \quad \text{if } r \geq 5M/2. \tag{5.67}$$

*Proof of Theorem 1.7* Let  $L_{(3)}^j := L^j[X_{(3)}, w_{(3)}, m_{(3)}, m'_{(3)}]$ ,  $j \in \{1, 2, 3, 4, 5\}$ . As in the proof of (4.48), we have

$$L_{(3)}^1 = \sum_{Y \in \{E, F, M\}} [K_{(3)}^{11}(Y_1)^2 + K_{(3)}^{22}(Y_2)^2 + K_{(3)}^{33}(Y_3)^2 + 2K_{(3)}^{23}Y_2Y_3],$$

where, with  $O' := O[a^2r^{-2}(f_3/r + f'_3)]$ ,

$$\begin{aligned} K_{(3)}^{11} &= \frac{-f'_3(r)}{q^2} + w_{(3)}(r)\mathbf{g}^{11} = \frac{2f_3 - rf'_3}{rq^2}, \\ K_{(3)}^{22} &= \frac{-f_3(r)(2r - 2M) + f'_3(r)\Delta}{q^2} + w_{(3)}(r)\mathbf{g}^{22} = (1 - \tilde{p})f'_3 - \frac{2Mf_3}{r^2} + O', \\ K_{(3)}^{33} &= -f_3(r)\partial_2\mathbf{g}^{33} - f'_3(r)\mathbf{g}^{33} - \frac{2rf_3(r)\mathbf{g}^{33}}{q^2} + w_{(3)}(r)\mathbf{g}^{33} = \frac{f'_3}{1 - \tilde{p}} - \frac{2Mf_3}{r^2(1 - \tilde{p})^2} \\ &\quad + O', \\ K_{(3)}^{23} &= \left(\frac{f_3}{1 - \tilde{p}} + g_3\right)' \frac{\Delta}{q^2} = f'_3 - \frac{2Mf_3}{r^2(1 - \tilde{p})} - \rho(1 - \tilde{p}) - \frac{\epsilon_4 M^2 f_3}{r^3}(1 - \tilde{p}) + O'. \end{aligned}$$

Therefore

$$\begin{aligned} L_{(3)}^1 &\geq \sum_{Y \in \{E, F, M\}} \left\{ \frac{2f_3 - rf'_3}{rq^2}(Y_1)^2 + H_3 \left[ Y_2 + \frac{Y_3}{1 - \tilde{p}} \right]^2 \right. \\ &\quad \left. + \left[ \rho + \frac{\epsilon_4 M^2 f_3}{2r^3} \right] [(1 - \tilde{p})^2(Y_2)^2 + (Y_3)^2] \right\} \\ &\quad - aMr^{-3}e^{\beta(r)}\chi_{\geq 5M}(r)[(Y_2)^2 + (Y_3)^2]. \end{aligned}$$

Also, using also (A.9), (A.10) the definitions (5.13), and Lemma 5.1,

$$\begin{aligned} L_{(3)}^2 &\geq \frac{2h_3}{r}\phi \left[ \mathbf{D}_2\phi + \frac{\mathbf{D}_3\phi}{1 - \tilde{p}} \right] + \frac{2h_3}{r}\psi \left[ \mathbf{D}_2\psi + \frac{\mathbf{D}_3\psi}{1 - \tilde{p}} \right] \\ &\quad - aMr^{-4}e^{\beta(r)}\chi_{\geq 5M}(r)[|\phi|\mathbf{D}_2\phi + |\phi|\mathbf{D}_3\phi + |\psi|\mathbf{D}_2\psi + |\psi|\mathbf{D}_3\psi], \\ L_{(3)}^3 &\geq (\phi^2 + \psi^2) \left[ \frac{h_3}{r^2} + \frac{h'_3}{r} + \frac{2Mf_3}{r^4} - \frac{2Mf'_3}{r^3} - \frac{(1 - \tilde{p})f''_3}{r} \right] \\ &\quad - (\phi^2 + \psi^2)ar^{-4}e^{\beta(r)}\chi_{\geq 5M}(r), \end{aligned}$$

and

$$L_{(3)}^4 \geq \frac{8(1 - \tilde{p})}{r^3}(f_3 - rf'_3)\psi^2 - \psi^2ar^{-4}e^{\beta(r)}\chi_{\geq 5M}(r).$$

We combine now the  $M_2^2$  term in the right-hand side of  $L_{(3)}^1$  and  $L_{(3)}^4$ . Recalling also the definition and (5.9) we have  $(M_2)^2 \geq 4r^{-2}\psi^2 - (\phi^2 + \psi^2)aMr^{-4}$ . Therefore,

$$H_3(M_2)^2 + L_{(3)}^4 \geq -(\phi^2 + \psi^2)ar^{-4}e^{\beta(r)}\chi_{\geq 5M}(r),$$

using the second inequality in (5.65) and the definitions.

We add up the estimates above and complete the square to conclude that

$$\begin{aligned} &L_{(3)}^1 + L_{(3)}^2 + L_{(3)}^3 + L_{(3)}^4 \\ &\geq \sum_{Y \in \{E, F, M\}} \frac{2f_3 - rf_3'}{2r^3}(Y_1)^2 \\ &\quad + H_3\left(E_2 + \frac{E_3}{1-\tilde{p}} + \frac{(1-\tilde{\alpha})\phi}{r}\right)^2 + H_3\left(F_2 + \frac{F_3}{1-\tilde{p}} + \frac{(1-\tilde{\alpha})\psi}{r}\right)^2 \\ &\quad + \left[\rho + \frac{\epsilon_4 M^2 f_3}{2r^3}\right] [(1-\tilde{p})^2(E_2)^2 + (E_3)^2 + (1-\tilde{p})^2(F_2)^2 + (F_3)^2] \\ &\quad + (\phi^2 + \psi^2) \left[ \frac{(\tilde{\alpha} - \tilde{\alpha}^2)H_3 - \tilde{\alpha}rH_3'}{r^2} + \frac{H_3'}{r} + \frac{2Mf_3}{r^4} - \frac{2Mf_3'}{r^3} - \frac{(1-\tilde{p})f_3''}{r} \right] \\ &\quad - \epsilon_3^{-1}ar^{-4}e^{\beta(r)}\chi_{\geq 5M}(r)(\phi^2 + \psi^2). \end{aligned}$$

Combining this with (5.53) and estimating as in the proof of Lemma 4.4 we conclude that

$$\begin{aligned} &\sum_{j=1}^4 (L_{(1)}^j + L_{(2)}^j + L_{(4)}^j + L_{(3)}^j) \\ &\geq \sum_{Y \in \{E, F, M\}} \epsilon_4^2 \left( \frac{e^\beta(2-r\beta')}{r} + \frac{100}{r} \right) \frac{(r-r^*)^2}{r^2} \frac{(Y_1)^2}{r^2} \\ &\quad + \epsilon_4^2 \left( \frac{\tilde{\alpha}^2 e^{\beta\beta'}}{r^2} + \frac{Me^\beta}{r^4} \right) (\phi^2 + \psi^2) + \sum_{Y \in \{E, F\}} \epsilon_4^2 \frac{M^2 e^\beta}{100r^3} \left[ (Y_2)^2 + \frac{(r-r^*)^2}{r^2} (Y_3)^2 \right] \\ &\quad + \epsilon_4^2 e^{\beta\beta'} \left[ \left( E_2 + \frac{E_3}{1-\tilde{p}} + \frac{(1-\tilde{\alpha})\phi}{r} \right)^2 + \left( F_2 + \frac{F_3}{1-\tilde{p}} + \frac{(1-\tilde{\alpha})\psi}{r} \right)^2 \right], \tag{5.68} \end{aligned}$$

provided that  $D$  is taken large enough and  $\epsilon_4$  is sufficiently small.

Moreover, using Lemma 5.1,

$$\begin{aligned} |L_{(3)}^5| &\leq \frac{aM}{r^4} \epsilon_4 e^\beta \chi_{\geq 8M} (|\phi| + |\psi|) \left\{ \sum_{Y \in \{E, F\}} \frac{|Y_1| + M|Y_2| + M|Y_3|}{r} + \frac{1}{r} (|\phi| + |\psi|) \right\} \\ &\quad + e^\beta e(\phi, \mathcal{N}_\phi) + e^\beta e(\psi, \mathcal{N}_\psi). \end{aligned}$$

Combining this with (5.57), (5.68), and (5.9), we obtain the final lower bound on the space-time term, for some small constant  $\epsilon_5 = \epsilon_5(\alpha)$ ,

$$\begin{aligned} & \sum_{j=1}^5 (L_{(1)}^j + L_{(2)}^j + L_{(4)}^j + L_{(3)}^j) \\ & \geq \epsilon_5 e^\beta \left\{ \frac{(r - r^*)^2}{r^2} \frac{(\partial_1 \phi)^2 + (\partial_1 \psi)^2 + (\psi / \sin \theta)^2}{r^3} \right. \\ & \quad + \frac{M^2}{r^3} [(\partial_2 \phi)^2 + (\partial_2 \psi)^2] + \frac{M^2(r - r^*)^2}{r^5} [(\partial_3 \phi)^2 + (\partial_3 \psi)^2] \\ & \quad \left. + \frac{\phi^2 + \psi^2}{r^3} + \frac{(L\phi)^2 + (L\psi)^2}{r} \right\} - e^\beta [e(\phi, \mathcal{N}_\phi) + e(\psi, \mathcal{N}_\psi)]. \end{aligned} \tag{5.69}$$

We consider now the contribution of  $\tilde{P}_{(3)\mu} n^\mu$ . Using (A.16) and the definitions we write

$$\begin{aligned} 2\tilde{P}_{(3)\mu} n^\mu &= 2Q_{\mu\nu} X_{(3)}^\nu n^\mu + w_{(3)}(\phi E_\mu + \psi F_\mu) n^\mu - \frac{n^\mu \mathbf{D}_\mu w_{(3)}}{2} (\phi^2 + \psi^2) \\ & \quad + \frac{n^\mu}{2} (m_{(3)\mu} \phi^2 + m'_{(3)\mu} \psi^2) - 2 \frac{X_{(3)}^\nu \mathbf{D}_\nu A}{A} \frac{n^\mu \mathbf{D}_\mu A}{A} \psi^2 \\ &= \frac{m_{(3)3}(-\mathbf{g}^{33})}{2} (\phi^2 + \psi^2) \\ & \quad + \sum_{Y \in \{E, F, M\}} \left[ \frac{(Y_1)^2}{q^2} \left( \frac{f_3}{1 - \tilde{p}} + g_3 \right) + \frac{\Delta(Y_2)^2}{q^2} \left( \frac{f_3}{1 - \tilde{p}} + g_3 \right) \right. \\ & \quad \left. + (Y_3)^2 (-\mathbf{g}^{33}) \left( \frac{f_3}{1 - \tilde{p}} + g_3 \right) + 2Y_2 Y_3 (-\mathbf{g}^{33}) f_3 \right] \\ & \quad + \frac{2f_3}{r} (-\mathbf{g}^{33}) (\phi E_3 + \psi F_3). \end{aligned}$$

As before, the main point is that the function  $g_3$  is extremely large when  $r$  is small. We can combine this last identity with the bounds (5.55) and (5.56), as in the proof of Lemma 4.4 to conclude that, for any  $t \in [0, T]$ ,

$$\int_{\Sigma_t^c} 2[\tilde{P}_{(1)\mu} + \tilde{P}_{(2)\mu} + \tilde{P}_{(3)\mu} + \tilde{P}_{(4)\mu}] n_0^\mu d\mu_t \approx_\alpha \int_{\Sigma_t^c} e^\beta [e(\phi)^2 + e(\psi)^2] d\mu_t. \tag{5.70}$$

Finally, using (A.17), the contribution of  $\tilde{P}_{(3)\mu} k^\mu$  along  $\mathcal{N}_{[0, T]}^c$  is

$$\begin{aligned} 2\tilde{P}_{(3)\mu} k^\mu &= 2Q_{\mu\nu} X_{(3)}^\nu k^\mu + w_{(3)}(\phi E_\mu + \psi F_\mu) k^\mu - \frac{k^\mu \mathbf{D}_\mu w_{(3)}}{2} (\phi^2 + \psi^2) \\ & \quad + \frac{k^\mu}{2} (m_{(3)\mu} \phi^2 + m'_{(3)\mu} \psi^2) - 2 \frac{X_{(3)}^\nu \mathbf{D}_\nu A}{A} \frac{k^\mu \mathbf{D}_\mu A}{A} \psi^2 \end{aligned}$$

$$= \sum_{Y \in \{E, F\}} [2g_3(c)\mathbf{g}^{23}(Y_3)^2 + 2Y_2Y_3g_3(c)\mathbf{g}^{22}].$$

Combining with (5.54) we obtain

$$2[\tilde{P}_{(1)\mu} + \tilde{P}_{(2)\mu} + \tilde{P}_{(3)\mu} + \tilde{P}_{(4)\mu}]k^\mu \geq 0 \quad \text{along } \mathcal{N}_{[0, T]}^c. \tag{5.71}$$

The theorem follows from (5.69), (5.70), (5.71), and the divergence identity (5.16).  $\square$

### 6 Proof of Corollary 1.8

In this section we provide a proof of Corollary 1.8. The main issue is the degeneracy of the weights in the bulk term at  $r = r^*$ . We compensate for this by losing derivatives. More precisely:

**Lemma 6.1** *Assume that  $(\phi, \psi) \in C^k([0, T] : \mathbf{H}^{6-k}(\Sigma_t^{c_0}))$ ,  $k \in [0, 6]$ , is a solution of the system (1.33) with  $\mathcal{N}_\phi = \mathcal{N}_\psi = 0$ . Then*

$$\begin{aligned} & \mathcal{B}\mathcal{B}_\alpha^{c_0}(t_1, t_2) + \sum_{k=0}^2 \int_{\Sigma_{t_2}^{c_0}} \frac{r^\alpha}{M^\alpha} [e(\phi_k)^2 + e(\psi_k)^2] d\mu_t \\ & \lesssim_\alpha \sum_{k=0}^2 \int_{\Sigma_{t_1}^{c_0}} \frac{r^\alpha}{M^\alpha} [e(\phi_k)^2 + e(\psi_k)^2] d\mu_t, \end{aligned} \tag{6.1}$$

for any  $\alpha \in (0, 2)$  and any  $t_1 \leq t_2 \in [0, T]$ , where  $\phi_k := M^k \mathbf{T}^k \phi$ ,  $\psi_k := M^k \mathbf{T}^k \psi$ , and

$$\begin{aligned} \mathcal{B}\mathcal{B}_\alpha^{c_0}(t_1, t_2) := & \int_{\mathcal{D}_{[t_1, t_2]}^{c_0}} \frac{r^\alpha}{M^\alpha} \left\{ \frac{|\partial_1 \phi|^2 + |\partial_1 \psi|^2 + \psi^2 (\sin \theta)^{-2}}{r^3} + \frac{1}{r} [(L\phi)^2 + (L\psi)^2] \right. \\ & \left. + \frac{1}{r^3} (\phi^2 + \psi^2) + \frac{M^2}{r^3} [(\partial_2 \phi)^2 + (\partial_2 \psi)^2 + (\partial_3 \phi)^2 + (\partial_3 \psi)^2] \right\} d\mu. \end{aligned} \tag{6.2}$$

Assuming Lemma 6.1, it is not hard to complete the proof of Corollary 1.8

*Proof of Corollary 1.8* We prove the estimate in two steps. Notice first that the inequality (6.2) is equivalent to

$$\begin{aligned} & \int_{t_1}^{t_2} \left( \int_{\Sigma_s^{c_0}} \frac{r^{\alpha-1}}{M^\alpha} [e(\phi)^2 + e(\psi)^2] d\mu_s \right) ds + \sum_{k=0}^2 \int_{\Sigma_{t_2}^{c_0}} \frac{r^\alpha}{M^\alpha} [e(\phi_k)^2 + e(\psi_k)^2] d\mu_t \\ & \lesssim_\alpha \sum_{k=0}^2 \int_{\Sigma_{t_1}^{c_0}} \frac{r^\alpha}{M^\alpha} [e(\phi_k)^2 + e(\psi_k)^2] d\mu_t, \end{aligned}$$



for any  $t_1 \leq t_2 \in [0, T]$  and  $\alpha \in (0, 2)$ . Let

$$I_{\beta,l}(s) := \sum_{k=0}^l \int_{\Sigma_s^{c_0}} \frac{r^\beta}{M^\beta} [e(\phi_k)^2 + e(\psi_k)^2] d\mu_s. \tag{6.3}$$

Therefore, for any  $\alpha \in (0, 2)$ ,  $l \in \{0, 1, 2\}$ , and  $t_1 \leq t_2 \in [0, T]$ , we have

$$I_{\alpha,l+2}(t_2) + \int_{t_1}^{t_2} \frac{1}{M} I_{\alpha-1,l}(s) ds \lesssim_\alpha I_{\alpha,l+2}(t_1). \tag{6.4}$$

We apply (6.4) first with  $\alpha$  close to 2 and  $l = 2, 4$ ; the result is

$$\int_0^T \frac{1}{M} I_{\alpha-1,2}(s) ds \lesssim_\alpha I_{\alpha,4}(0) \quad \text{and} \quad I_{\alpha-1,2}(s') \lesssim_\alpha I_{\alpha-1,2}(s) \quad \text{if } s \leq s'.$$

These inequalities show easily that

$$I_{\alpha-1,2}(s) \lesssim_\alpha I_{\alpha,4}(0) \frac{M}{M+s} \quad \text{for any } s \in [0, T] \quad \text{and} \quad \alpha \in (0, 2). \tag{6.5}$$

To apply this argument again we need to improve slightly on (6.5). More precisely, we'd like to show that

$$I_{1+\epsilon,2}(s) \lesssim_\epsilon I_{2,4}(0) \frac{M^{1-2\epsilon}}{(M+s)^{1-2\epsilon}} \quad \text{for any } s \in [0, T] \quad \text{and} \quad \epsilon \in (0, 1/10]. \tag{6.6}$$

Indeed, we estimate

$$I_{1+\epsilon,2}(s) \lesssim II(s) + III(s),$$

where, using (6.5) and (6.4),

$$\begin{aligned} II(s) &:= \sum_{k=0}^l \int_{\Sigma_s^{c_0}, r \leq M+s} \frac{r^{1+\epsilon}}{M^{1+\epsilon}} [e(\phi_k)^2 + e(\psi_k)^2] d\mu_s \\ &\lesssim I_{1-\epsilon/2,2} \frac{(M+s)^{7\epsilon/4}}{M^{7\epsilon/4}} \lesssim_\epsilon I_{2,4}(0) \frac{M^{1-2\epsilon}}{(M+s)^{1-2\epsilon}} \end{aligned}$$

and

$$\begin{aligned} III(s) &:= \sum_{k=0}^l \int_{\Sigma_s^{c_0}, r \geq M+s} \frac{r^{1+\epsilon}}{M^{1+\epsilon}} [e(\phi_k)^2 + e(\psi_k)^2] d\mu_s \\ &\lesssim I_{2-\epsilon/2,2} \frac{M^{1-3\epsilon/2}}{(M+s)^{1-3\epsilon/2}} \lesssim_\epsilon I_{2,2}(0) \frac{M^{1-3\epsilon/2}}{(M+s)^{1-3\epsilon/2}}. \end{aligned}$$

The bound (6.6) follows.

We can now repeat the argument at the beginning of the proof, starting from the bounds,

$$\int_{t_1}^T \frac{1}{M} I_{\epsilon,0}(s) ds \lesssim_{\alpha} I_{1+\epsilon,2}(t_1) \quad \text{and} \quad I_{\epsilon,0}(s') \lesssim_{\alpha} I_{\epsilon,0}(s) \quad \text{if } s \leq s',$$

which follow from (6.4) and Theorem 1.7. Using now (6.6) it follows easily that

$$I_{\epsilon,0}(s) \lesssim_{\epsilon} I_{2,4}(0) \frac{M^{2-2\epsilon}}{(M+s)^{2-2\epsilon}} \quad \text{for any } s \in [0, T] \quad \text{and} \quad \epsilon \in (0, 1/10],$$

which gives the conclusion of Corollary 1.8. □

We turn now to the proof of Lemma 6.1.

*Proof of Lemma 6.1* In view of Theorem 1.7, with the notation (6.3), we know that

$$\begin{aligned} I_{\alpha,2}(t_2) + \sum_{k=0}^2 \int_{\mathcal{D}_{[t_1,t_2]}^{c_0}} \frac{r^{\alpha}}{M^{\alpha}} \left\{ \frac{(r-r^*)^2}{r^3} \frac{(\partial_1 \phi_k)^2 + (\partial_1 \psi_k)^2 + \psi_k^2 (\sin \theta)^{-2}}{r^2} \right. \\ \left. + \frac{1}{r^3} (\phi_k^2 + \psi_k^2) + \frac{M^2}{r^3} [(\partial_2 \phi_k)^2 + (\partial_2 \psi_k)^2] \right\} d\mu \lesssim_{\alpha} I_{\alpha,2}(t_1), \end{aligned} \tag{6.7}$$

for any  $t_1 \leq t_2 \in [0, T]$  and  $\alpha \in (0, 2)$ . It suffices to prove that

$$\int_{\mathcal{D}_{[t_1,T]}^{c_0}} \frac{r^{\alpha}}{M^{\alpha}} \tilde{\chi}(r) \frac{(\partial_1 \phi)^2 + (\partial_1 \psi)^2 + \psi^2 (\sin \theta)^{-2}}{r^3} d\mu \lesssim_{\alpha} I_{\alpha,2}(t_1), \tag{6.8}$$

where  $\tilde{\chi} := \chi_{\geq 9M/4} - \chi_{\geq 4M}$ . For this we use elliptic estimate and (6.7).

The equation for  $\phi$  and the formula (A.9) show that

$$\mathbf{g}^{11} \left[ \partial_1^2 \phi + \frac{\cos \theta}{\sin \theta} \partial_1 \phi \right] + \mathbf{g}^{22} \partial_2^2 \phi + \frac{2\mathbf{g}^{11} \mathbf{D}_1 B}{A} \partial_1 \psi = -F_{\phi}, \tag{6.9}$$

where

$$\begin{aligned} F_{\phi} := & \mathbf{g}^{33} \partial_3^2 \phi + 2\mathbf{g}^{23} \partial_2 \partial_3 \phi + D^2 \partial_2 \phi + D^3 \partial_3 \phi \\ & + 2 \frac{\mathbf{D}^2 \mathbf{B} \mathbf{D}_2 \psi + \mathbf{D}^3 \mathbf{B} \mathbf{D}_3 \psi}{A} - 2 \frac{\mathbf{D}^{\mu} \mathbf{B} \mathbf{D}_{\mu} B}{A^2} \phi + 2 \frac{\mathbf{D}^{\mu} \mathbf{B} \mathbf{D}_{\mu} A}{A^2} \psi. \end{aligned}$$

It follows from (6.7) that

$$\int_{\mathcal{D}_{[t_1,T]}^{c_0}} \frac{M^4}{r^3} |F_{\phi}|^2 d\mu \lesssim_{\alpha} I_{\alpha,2}(t_1). \tag{6.10}$$

Using then integration by parts and (6.9), we have

$$\begin{aligned} & \int_{\mathcal{D}_{[t_1, T]}^{c_0}} \frac{r^\alpha}{M^\alpha} \tilde{\chi}(r) \frac{(\partial_1 \phi)^2}{r^3} d\mu \\ & \lesssim \int_{[t_1, T] \times (0, \pi) \times (c_0, \infty)} \tilde{\chi}(r) \frac{(\partial_1 \phi)^2}{M^3} r^2 (\sin \theta) dr d\theta dt \\ & \lesssim \left| \int_{[t_1, T] \times (0, \pi) \times (c_0, \infty)} \tilde{\chi}(r) \phi \cdot \left[ \partial_1^2 \phi + \frac{\cos \theta}{\sin \theta} \partial_1 \phi \right] \frac{r^2}{M^3} (\sin \theta) dr d\theta dt \right| \\ & \lesssim \left| \int_{[t_1, T] \times (0, \pi) \times (c_0, \infty)} \tilde{\chi}(r) \phi \cdot \left[ \Delta \partial_2^2 \phi + \frac{2\mathbf{D}_1 B}{A} \partial_1 \psi + \frac{F_\phi}{\mathbf{g}^{11}} \right] \frac{r^2}{M^3} (\sin \theta) dr d\theta dt \right|. \end{aligned}$$

Using (6.7), (6.10), and integration by parts it follows that

$$\int_{\mathcal{D}_{[t_1, T]}^{c_0}} \frac{r^\alpha}{M^\alpha} \tilde{\chi}(r) \frac{(\partial_1 \phi)^2}{r^3} d\mu \lesssim_\alpha I_{\alpha, 2}(t_1) + [I_{\alpha, 2}(t_1)]^{1/2} \left( \int_{\mathcal{D}_{[t_1, T]}^{c_0}} \tilde{\chi}(r) \frac{(\partial_1 \psi)^2}{r^3} d\mu \right)^{1/2}. \tag{6.11}$$

Similarly, the equation for  $\psi$  and the formula (A.9) show that

$$\mathbf{g}^{11} \left[ \partial_1^2 \psi + \frac{\cos \theta}{\sin \theta} \partial_1 \psi - \frac{4(\cos \theta)^2}{(\sin \theta)^2} \psi \right] + \mathbf{g}^{22} \partial_2^2 \psi - \frac{2\mathbf{g}^{11} \mathbf{D}_1 B}{A} \partial_1 \phi = -F_\psi,$$

where  $F_\psi$  satisfies the same bound (6.10) as  $F_\phi$ , and the additional term in the left-hand side comes from the fraction  $\frac{2\cos \theta}{\sin \theta}$  in  $A^{-1} \mathbf{D}_1 A$  (see (5.7)). Integrating by parts as before we have

$$\begin{aligned} & \int_{\mathcal{D}_{[t_1, T]}^{c_0}} \frac{r^\alpha}{M^\alpha} \tilde{\chi}(r) \frac{(\partial_1 \psi)^2 + \psi^2 (\sin \theta)^{-2}}{r^3} d\mu \\ & \lesssim_\alpha I_{\alpha, 2}(t_1) + [I_{\alpha, 2}(t_1)]^{1/2} \left( \int_{\mathcal{D}_{[t_1, T]}^{c_0}} \tilde{\chi}(r) \frac{(\partial_1 \phi)^2}{r^3} d\mu \right)^{1/2}. \end{aligned}$$

The desired bound (6.8) follows using also (6.11). □

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### Appendix 1: Explicit Formulas in Kerr Spaces

Recall the Kerr spacetimes  $\mathcal{K}(m, a)$ , in standard Boyer–Lindquist coordinates,

$$\mathbf{g} = -\frac{q^2 \Delta}{\Sigma^2} (dt)^2 + \frac{\Sigma^2 (\sin \theta)^2}{q^2} \left( d\phi - \frac{2aMr}{\Sigma^2} dt \right)^2 + \frac{q^2}{\Delta} (dr)^2 + q^2 (d\theta)^2, \tag{A.1}$$

where

$$\begin{cases} \Delta = r^2 + a^2 - 2Mr; \\ q^2 = r^2 + a^2(\cos \theta)^2; \\ \Sigma^2 = (r^2 + a^2)q^2 + 2Mra^2(\sin \theta)^2 = (r^2 + a^2)^2 - a^2(\sin \theta)^2\Delta. \end{cases} \quad (\text{A.2})$$

Observe that

$$(2Mr - q^2)\Sigma^2 = -q^4\Delta + 4a^2M^2r^2(\sin \theta)^2. \quad (\text{A.3})$$

Recall the change of variables (1.19)–(1.20) and let

$$p := \frac{2Mr}{q^2}.$$

Therefore

$$\frac{\Sigma^2}{q^2} = q^2 + (p+1)a^2(\sin \theta)^2, \quad \Delta = q^2(1-p) + a^2(\sin \theta)^2.$$

Recall that

$$\partial_1 = \partial_\theta = \frac{d}{d\theta}, \quad \partial_2 = \partial_r = \frac{d}{dr}, \quad \partial_3 = \partial_t = \frac{d}{dt_+} = \mathbf{T}, \quad \partial_4 = \partial_\phi = \frac{d}{d\phi_+} = \mathbf{Z}. \quad (\text{A.4})$$

The nontrivial components of the metric  $\mathbf{g}$  become

$$\begin{aligned} \mathbf{g}_{11} &= q^2, & \mathbf{g}_{33} &= p-1, & \mathbf{g}_{34} &= -a(\sin \theta)^2 p, & \mathbf{g}_{44} &= q^2(\sin \theta)^2 + (p+1)a^2(\sin \theta)^4, \\ \mathbf{g}_{22} &= \frac{q^2}{\Delta}(1-\chi^2) + (p+1)\chi^2, & \mathbf{g}_{23} &= p\chi, & \mathbf{g}_{24} &= -a(\sin \theta)^2(p+1)\chi, \end{aligned} \quad (\text{A.5})$$

and, letting  $\text{Det} := -q^2(\sin \theta)^2$ ,

$$\begin{aligned}
 \mathbf{g}^{11} &= \frac{1}{\mathbf{g}_{11}} = \frac{1}{q^2}, \\
 \mathbf{g}^{22} &= \frac{\mathbf{g}_{33}\mathbf{g}_{44} - \mathbf{g}_{34}^2}{\text{Det}} = \frac{\Delta}{q^2}, \\
 \mathbf{g}^{23} &= \frac{\mathbf{g}_{24}\mathbf{g}_{34} - \mathbf{g}_{23}\mathbf{g}_{44}}{\text{Det}} = p\chi, \\
 \mathbf{g}^{24} &= \frac{\mathbf{g}_{23}\mathbf{g}_{34} - \mathbf{g}_{33}\mathbf{g}_{24}}{\text{Det}} = \frac{a\chi}{q^2}, \\
 \mathbf{g}^{33} &= \frac{\mathbf{g}_{22}\mathbf{g}_{44} - \mathbf{g}_{24}^2}{\text{Det}} = -(p+1)\chi^2 - \frac{q^2 + (p+1)a^2(\sin\theta)^2}{\Delta}(1-\chi^2), \\
 \mathbf{g}^{34} &= \frac{\mathbf{g}_{24}\mathbf{g}_{23} - \mathbf{g}_{22}\mathbf{g}_{34}}{\text{Det}} = \frac{-ap}{\Delta}(1-\chi^2), \\
 \mathbf{g}^{44} &= \frac{\mathbf{g}_{22}\mathbf{g}_{33} - \mathbf{g}_{23}^2}{\text{Det}} = \frac{\Delta - a^2(\sin\theta)^2(1-\chi^2)}{q^2\Delta(\sin\theta)^2}.
 \end{aligned} \tag{A.6}$$

The metric  $\mathbf{g}$  extends to the larger open set

$$\tilde{R} = \{(\theta, r, t_+, \phi_+) \in (-\pi, \pi) \times (0, \infty) \times \mathbb{R} \times \mathbb{S}^1\}.$$

Recall also the sets, see (1.24)–(1.26),

$$\begin{aligned}
 \mathcal{D}_I^c &= \{(\theta, r, t_+, \phi_+) \in \tilde{R} : t_+ \in I \text{ and } r > c\}, \\
 \Sigma_I^c &:= \{(\theta, r, t_+, \phi_+) \in \tilde{R} : t_+ = t \text{ and } r > c\}, \\
 \mathcal{N}_I^c &:= \{(\theta, r, t_+, \phi_+) \in \tilde{R} : t_+ \in I \text{ and } r = c\},
 \end{aligned} \tag{A.7}$$

defined for  $c \in (0, \infty)$ ,  $t \in \mathbb{R}$ , and intervals  $I \subseteq \mathbb{R}$ .

Notice that

$$\begin{aligned}
 \partial_1(q^2) &= -2a^2 \sin\theta \cos\theta, & \partial_2(q^2) &= 2r, \\
 \partial_1 p &= \frac{4Mr a^2 \sin\theta \cos\theta}{q^4}, & \partial_2 p &= -\frac{2M(r^2 - a^2(\cos\theta)^2)}{q^4}.
 \end{aligned} \tag{A.8}$$

Recall the general formula

$$\Gamma_{\mu\alpha\beta} = \mathbf{g}(\mathbf{D}_{\partial_\beta} \partial_\alpha, \partial_\mu) = \frac{1}{2}(\partial_\alpha \mathbf{g}_{\beta\mu} + \partial_\beta \mathbf{g}_{\alpha\mu} - \partial_\mu \mathbf{g}_{\alpha\beta}).$$

In the case of  $\mathbf{Z}$ -invariant functions  $f$ , i.e. if  $\mathbf{Z}(f) = 0$ , we have the general formula

$$\begin{aligned}
 \square f &= \mathbf{g}^{\alpha\beta} \partial_\alpha \partial_\beta f - \mathbf{g}^{\alpha\beta} \mathbf{g}^{\mu\nu} \Gamma_{\mu\alpha\beta} \partial_\nu f \\
 &= \mathbf{g}^{\alpha\beta} \partial_\alpha \partial_\beta f + [\partial_\mu \mathbf{g}^{\mu\nu} + (1/2)\mathbf{g}^{\mu\nu} \partial_\mu \log |q^4(\sin\theta)^2|] \partial_\nu f \\
 &= \mathbf{g}^{11} \partial_1^2 f + \mathbf{g}^{22} \partial_2^2 f + \mathbf{g}^{33} \partial_3^2 f + 2\mathbf{g}^{23} \partial_2 \partial_3 f \\
 &\quad + [\partial_1 \mathbf{g}^{11} + (1/2)\mathbf{g}^{11} \partial_1 \log |q^4(\sin\theta)^2|] \partial_1 f
 \end{aligned}$$

$$\begin{aligned}
 &+ [\partial_2 \mathbf{g}^{22} + (1/2)\mathbf{g}^{22} \partial_2 \log |q^4(\sin \theta)^2|] \partial_2 f \\
 &+ [\partial_2 \mathbf{g}^{23} + (1/2)\mathbf{g}^{23} \partial_2 \log |q^4(\sin \theta)^2|] \partial_3 f \\
 = &\mathbf{g}^{11} \left[ \partial_1^2 f + \frac{\cos \theta}{\sin \theta} \partial_1 f \right] + \mathbf{g}^{22} \partial_2^2 f + \mathbf{g}^{33} \partial_3^2 f \\
 &+ 2\mathbf{g}^{23} \partial_2 \partial_3 f + D^2 \partial_2 f + D^3 \partial_3 f,
 \end{aligned} \tag{A.9}$$

where

$$D^2 := \partial_2 \mathbf{g}^{22} + \mathbf{g}^{22} \frac{2r}{q^2} = \frac{2r - 2M}{q^2}, \quad D^3 := \partial_2 \mathbf{g}^{23} + \mathbf{g}^{23} \frac{2r}{q^2} = \frac{2M\chi(r) + 2Mr\chi'(r)}{q^2}.$$

Also, if  $m$  is a 1-form satisfying  $m_4 = 0$  and  $\partial_4 m_\alpha = 0, \alpha \in \{1, 2, 3, 4\}$ , then

$$\begin{aligned}
 \mathbf{D}^\alpha m_\alpha &= \mathbf{g}^{\alpha\beta} \partial_\alpha m_\beta - \mathbf{g}^{\alpha\beta} \mathbf{g}^{\mu\nu} \Gamma_{\mu\alpha\beta} m_\nu \\
 &= \mathbf{g}^{11} \left[ \partial_1 m_1 + \frac{\cos \theta}{\sin \theta} m_1 \right] + \mathbf{g}^{22} \partial_2 m_2 \\
 &\quad + \mathbf{g}^{33} \partial_3 m_3 + \mathbf{g}^{23} (\partial_2 m_3 + \partial_3 m_2) + D^2 m_2 + D^3 m_3.
 \end{aligned} \tag{A.10}$$

### Vector-fields

Letting

$$\pi_{\alpha\beta} = (\mathcal{L}_{\partial_2} \mathbf{g})_{\alpha\beta} = \Gamma_{\alpha 2\beta} + \Gamma_{\beta 2\alpha},$$

we calculate

$$\begin{aligned}
 \pi_{\alpha\beta} &= \partial_2 \mathbf{g}_{\alpha\beta}, \\
 \pi^{\alpha\beta} &= \mathbf{g}^{\alpha\mu} \mathbf{g}^{\beta\nu} \pi_{\mu\nu} = \mathbf{g}^{\alpha\mu} \mathbf{g}^{\beta\nu} \partial_2 \mathbf{g}_{\mu\nu} = -\mathbf{g}^{\beta\nu} \mathbf{g}_{\mu\nu} \partial_2 \mathbf{g}^{\alpha\mu} = -\partial_2 \mathbf{g}^{\alpha\beta}, \\
 \pi_{\alpha\beta} \mathbf{g}^{\alpha\beta} &= \partial_2 \log |q^4(\sin \theta)^2| = 4r/q^2.
 \end{aligned} \tag{A.11}$$

Therefore, for any vector field

$$X = f(r) \partial_2 + g(r) \partial_3, \tag{A.12}$$

we calculate

$$\begin{aligned}
 {}^{(X)}\pi^{\mu\nu} &:= \mathbf{D}^\mu X^\nu + \mathbf{D}^\nu X^\mu \\
 &= f \pi^{\mu\nu} + (\mathbf{D}^\mu f \delta_2^\nu + \mathbf{D}^\nu f \delta_2^\mu) + (\mathbf{D}^\mu g \delta_3^\nu + \mathbf{D}^\nu g \delta_3^\mu) \\
 &= f \pi^{\mu\nu} + f'(r) (\mathbf{g}^{\mu 2} \delta_2^\nu + \mathbf{g}^{\nu 2} \delta_2^\mu) + g'(r) (\mathbf{g}^{\mu 2} \delta_3^\nu + \mathbf{g}^{\nu 2} \delta_3^\mu).
 \end{aligned} \tag{A.13}$$

For any 1-form  $Y$  with  $Y_4 = 0$  let

$${}^{(Y)}Q_{\mu\nu} = Y_\mu Y_\nu - (1/2)\mathbf{g}_{\mu\nu} (Y_\rho Y^\rho). \tag{A.14}$$

We calculate the contraction

$$\begin{aligned}
 {}^{(Y)}Q_{\mu\nu}{}^{(X)}\pi^{\mu\nu} &= {}^{(X)}\pi^{\mu\nu}Y_\mu Y_\nu - (1/2)\mathbf{g}_{\mu\nu}(Y_\rho Y^\rho){}^{(X)}\pi^{\mu\nu} \\
 &= f(r)\pi^{\mu\nu}Y_\mu Y_\nu + 2f'(r)Y^2Y_2 + 2g'(r)Y^2Y_3 \\
 &\quad - (Y_\rho Y^\rho)[2rf(r)/q^2 + f'(r)] \\
 &= (Y_1)^2\left[f(r)\pi^{11} - \frac{2rf(r)\mathbf{g}^{11}}{q^2} - f'(r)\mathbf{g}^{11}\right] \\
 &\quad + (Y_2)^2\left[f(r)\pi^{22} + f'(r)\mathbf{g}^{22} - \frac{2rf(r)\mathbf{g}^{22}}{q^2}\right] \\
 &\quad + (Y_3)^2\left[f(r)\pi^{33} + 2g'(r)\mathbf{g}^{23} - f'(r)\mathbf{g}^{33} - \frac{2rf(r)\mathbf{g}^{33}}{q^2}\right] \\
 &\quad + 2Y_2Y_3\left[f(r)\pi^{23} + g'(r)\mathbf{g}^{22} - \frac{2rf(r)\mathbf{g}^{23}}{q^2}\right].
 \end{aligned}$$

Using also the formulas (A.6) and (A.11) this simplifies to

$$\begin{aligned}
 {}^{(Y)}Q_{\mu\nu}{}^{(X)}\pi^{\mu\nu} &= (Y_1)^2\frac{-f'(r)}{q^2} + (Y_2)^2\frac{-f(r)(2r - 2M) + f'(r)\Delta}{q^2} \\
 &\quad + (Y_3)^2\left[-f(r)\partial_2\mathbf{g}^{33} + 2g'(r)\mathbf{g}^{23} - f'(r)\mathbf{g}^{33} - \frac{2rf(r)\mathbf{g}^{33}}{q^2}\right] \\
 &\quad + 2Y_2Y_3\frac{-2Mrf(r)\chi'(r) - 2Mf(r)\chi(r) + g'(r)\Delta}{q^2}. \tag{A.15}
 \end{aligned}$$

Recall the vector-fields  $n = -\mathbf{g}^{\mu\nu}\partial_\nu u + \partial_\mu = -\mathbf{g}^{3\mu}\partial_\mu$  and  $k = \mathbf{g}^{\mu\nu}\partial_\nu r \partial_\mu = \mathbf{g}^{2\mu}\partial_\mu$ , defined in  $\tilde{R}$ , which are normal to the hypersurfaces  $\Sigma_t^c$  and  $\mathcal{N}_t^c$  respectively. We calculate

$$\begin{aligned}
 {}^{(Y)}Q(n, \partial_2) &= -\mathbf{g}^{3\mu}Y_\mu Y_2 = -\mathbf{g}^{32}(Y_2)^2 - \mathbf{g}^{33}Y_2Y_3, \\
 {}^{(Y)}Q(n, \partial_3) &= -\mathbf{g}^{3\mu}Y_\mu Y_3 + (1/2)(Y_\rho Y^\rho) = (1/2)[\mathbf{g}^{11}(Y_1)^2 + \mathbf{g}^{22}(Y_2)^2 - \mathbf{g}^{33}(Y_3)^2], \\
 {}^{(Y)}Q(k, \partial_2) &= \mathbf{g}^{2\mu}Y_\mu Y_2 - (1/2)(Y_\rho Y^\rho) = (1/2)[- \mathbf{g}^{11}(Y_1)^2 + \mathbf{g}^{22}(Y_2)^2 - \mathbf{g}^{33}(Y_3)^2],
 \end{aligned}$$

and

$${}^{(Y)}Q(k, \partial_3) = \mathbf{g}^{2\mu}Y_\mu Y_3 = \mathbf{g}^{23}(Y_3)^2 + \mathbf{g}^{22}Y_2Y_3.$$

Therefore, if  $X = f(r)\partial_2 + g(r)\partial_3$  as in (A.12) then

$$\begin{aligned}
 2{}^{(Y)}Q(n, X) &= (Y_1)^2[g(r)\mathbf{g}^{11}] + (Y_2)^2[g(r)\mathbf{g}^{22} - 2f(r)\mathbf{g}^{23}] \\
 &\quad + (Y_3)^2[-g(r)\mathbf{g}^{33}] + 2Y_2Y_3[-f(r)\mathbf{g}^{33}] \tag{A.16}
 \end{aligned}$$

and

$$2^{(Y)} Q(k, X) = (Y_1)^2[-f(r)\mathbf{g}^{11}] + (Y_2)^2[f(r)\mathbf{g}^{22}] + (Y_3)^2[-f(r)\mathbf{g}^{33} + 2g(r)\mathbf{g}^{23}] + 2Y_2Y_3[g(r)\mathbf{g}^{22}]. \tag{A.17}$$

### Hardy Inequalities

In this subsection we prove the following lemma:

**Lemma A.1** (i) *If  $c \geq c_0$  and  $f \in H_{\text{loc}}^1((c, \infty))$  satisfies  $\lim_{D \rightarrow \infty} \int_D^{2D} |f(r)|^2 dr = 0$  then*

$$\int_c^\infty |f/r|^2 \cdot r^2 dr \lesssim \int_c^\infty |f'|^2 \cdot r^2 dr. \tag{A.18}$$

(ii) *If  $g \in H_{\text{loc}}^1((0, \pi))$  and  $p \in [0, 10]$  then*

$$\int_0^\pi |g|^2(\sin \theta)^p d\theta \lesssim \int_0^\pi |g'|^2(\sin \theta)^{p+2} d\theta + \int_0^\pi |g|^2(\sin \theta)^{p+2} d\theta. \tag{A.19}$$

(iii) *If  $f \in H_{\text{loc}}^1((0, \pi))$  then*

$$\begin{aligned} & \int_0^\pi |f'|^2 \sin \theta d\theta + \int_0^\pi |f|^2(\sin \theta)^{-1} d\theta \\ & \approx \int_0^\pi \left| f' - \frac{2 \cos \theta}{\sin \theta} f \right|^2 \sin \theta d\theta + \int_0^\pi |f|^2 \sin \theta d\theta. \end{aligned} \tag{A.20}$$

(iv) *If  $g \in L_{\text{loc}}^2((0, \pi))$  then*

$$\int_0^\pi |g|^2(\sin \theta)^{-1} d\theta \lesssim \int_0^\pi \left| g' + \frac{\cos \theta}{\sin \theta} g \right|^2 \sin \theta d\theta + \int_0^\pi |g|^2 \sin \theta d\theta. \tag{A.21}$$

(v) *If  $f \in H_{\text{loc}}^1((0, \pi))$  then*

$$\begin{aligned} & \int_0^\pi |f''|^2 \sin \theta + |f'|^2(\sin \theta)^{-1} + |f|^2(\sin \theta)^{-3} d\theta \\ & \lesssim \int_0^\pi \left| f'' + \frac{\cos \theta}{\sin \theta} f' - \frac{4(\cos \theta)^2}{(\sin \theta)^2} f \right|^2 \sin \theta d\theta \\ & \quad + \int_0^\pi |f'|^2 \sin \theta + |f|^2(\sin \theta)^{-1} d\theta. \end{aligned} \tag{A.22}$$

*Proof* The inequalities in this lemma are standard Hardy-type inequalities, and we provide the proofs mostly for sake of completeness.



For (i) we may assume that  $f$  is real-valued and

$$\int_c^\infty |f'(r)|^2 r^2 dr = 1.$$

Given  $\delta > 0$  small and  $D \gg 1$  we fix a smooth function  $K = K_{\delta,D} : \mathbb{R} \rightarrow \mathbb{R}$  supported in the interval  $[c + \delta/2, 2D]$  with the properties

$$\begin{aligned} K'(r) &= 1 && \text{if } r \in [c + \delta, D], \\ |K'(r)| &\lesssim 1 && \text{if } r \in [D, 2D], \\ K' &\text{ is increasing on the interval } [c + \delta/2, c + \delta]. \end{aligned} \tag{A.23}$$

By taking  $D$  sufficiently large, we may assume that

$$\int_D^{2D} |f(r)|^2 dr \leq 1.$$

Notice that  $K(r) \lesssim rK'(r)^{1/2}$  for any  $r \in [c, D]$ , which follows easily from (A.23). Then we estimate, using integration by parts

$$\begin{aligned} \left| \int_c^D f(r)^2 K'(r) dr \right| &\lesssim \left| \int_{\mathbb{R}} f(r)^2 K'(r) dr \right| + \int_D^{2D} |f(r)|^2 dr \\ &\lesssim \int_{\mathbb{R}} |f(r)| |f'(r)| |K(r)| dr + 1 \\ &\lesssim \int_c^D |f(r)| |f'(r)| |K(r)| dr + 1 \\ &\lesssim \left| \int_c^D |f(r)|^2 K'(r) dr \right|^{1/2} \left| \int_c^D |f'(r)| r^2 dr \right|^{1/2} + 1. \end{aligned}$$

Therefore

$$\left| \int_c^D f(r)^2 K'(r) dr \right| \lesssim 1,$$

and the desired inequality follows by letting  $\delta \rightarrow 0$  and  $D \rightarrow \infty$ .

To prove (ii) we may assume that  $g$  is real-valued and

$$\int_0^\pi |g'(\theta)|^2 (\sin \theta)^{p+2} d\theta + \int_0^\pi |g(\theta)|^2 (\sin \theta)^{p+2} d\theta = 1.$$

As before, given  $\delta > 0$  small we fix a smooth function  $K = K_\delta : \mathbb{R} \rightarrow \mathbb{R}$  supported in the interval  $[\delta/2, 1/2]$  with the properties

$$\begin{aligned} K'(\theta) &= (\sin \theta)^p && \text{if } \theta \in [\delta, 1/4], \\ |K'(\theta)| &\lesssim 1 && \text{if } \theta \in [1/4, 1/2], \\ K' &\text{ is increasing on the interval } [\delta/2, \delta]. \end{aligned} \tag{A.24}$$

As before, we notice that these assumptions imply that  $K(\theta) \lesssim (\sin \theta)^{(p+2)/2} K'(\theta)^{1/2}$  for any  $\theta \in [0, 1/4]$ . Then we estimate, using integration by parts,

$$\begin{aligned} \left| \int_0^{1/4} g(\theta)^2 K'(\theta) d\theta \right| &\lesssim \left| \int_0^{1/2} g(\theta)^2 K'(\theta) d\theta \right| + 1 \\ &\lesssim \int_0^{1/2} |g(\theta)| |g'(\theta)| |K(\theta)| d\theta + 1 \\ &\lesssim \int_0^{1/4} |g(\theta)| |g'(\theta)| |K(\theta)| d\theta + 1 \\ &\lesssim \left| \int_0^{1/4} |g(\theta)|^2 K'(\theta) d\theta \right|^{1/2} \left| \int_0^{1/4} |g'(\theta)|^2 (\sin \theta)^{p+2} d\theta \right|^{1/2} + 1. \end{aligned}$$

Therefore

$$\left| \int_0^{1/4} g(\theta)^2 K'(\theta) d\theta \right| \lesssim 1.$$

Letting  $\delta \rightarrow 0$  it follows that

$$\int_0^{1/4} |g(\theta)|^2 (\sin \theta)^p d\theta \lesssim 1.$$

The change of variables  $\theta \rightarrow \pi - \theta$  now shows that

$$\int_{\pi-1/4}^{\pi} |g(\theta)|^2 (\sin \theta)^p d\theta \lesssim 1,$$

and the desired estimate follows.

To prove (iii), we notice first that the right-hand side of (A.20) is clearly dominated by the left-hand side. To prove the reverse inequality, let  $f(\theta) = (\sin \theta)^2 g(\theta)$  and notice that

$$f'(\theta) - \frac{2 \cos \theta}{\sin \theta} f(\theta) = (\sin \theta)^2 g'(\theta).$$

The desired bound follows from the inequality

$$\int_0^{\pi} |g(\theta)|^2 (\sin \theta)^3 d\theta \lesssim \int_0^{\pi} |g'(\theta)|^2 (\sin \theta)^5 d\theta + \int_0^{\pi} |g(\theta)|^2 (\sin \theta)^5 d\theta,$$

which is a consequence of (A.19).

To prove (iv), we may assume that  $g \in H^1_{\text{loc}}((0, \pi))$  is real-valued and let  $g(\theta) = h(\theta)/\sin \theta$ . Then

$$g'(\theta) + \frac{\cos \theta}{\sin \theta} g(\theta) = \frac{h'(\theta)}{\sin \theta}.$$

The inequality to prove becomes

$$\int_0^\pi \frac{h(\theta)^2}{(\sin \theta)^3} d\theta \lesssim \int_0^\pi \frac{h'(\theta)^2}{\sin \theta} d\theta + \int_0^\pi \frac{h(\theta)^2}{\sin \theta} d\theta. \tag{A.25}$$

This is nontrivial only if  $h' \in L^2((0, \pi))$ , which shows that  $h' \in L^1((0, \pi))$ . Therefore, in proving (A.25) we may assume that  $h$  extends to a continuous function on the interval  $[0, \pi]$  and  $h(0) = h(\pi) = 0$  (otherwise the right-hand side of (A.25) is equal to  $\infty$ ). In particular, for any  $\theta \in [0, \pi/2]$ ,

$$h(\theta) = \int_0^\theta h'(\mu) d\mu. \tag{A.26}$$

For  $k \leq 0$  let  $c_k := 2^{-k/2} \left[ \int_{[2^{k-1}, 2^k]} |h'(\mu)|^2 d\mu \right]^{1/2}$ . The formula (A.26) above shows that

$$|h(\theta)| \lesssim \sum_{k' \leq k} 2^{k'} c_{k'} \quad \text{if } k \leq 0 \text{ and } \theta \in [2^{k-1}, 2^k].$$

Therefore

$$\int_0^1 \frac{h(\theta)^2}{(\sin \theta)^3} d\theta \lesssim \sum_{k \leq 0} \left( \sum_{k' \leq k} 2^{k'-k} c_{k'} \right)^2 \lesssim \sum_{k \leq 0} c_k^2 \lesssim \int_0^1 \frac{h'(\theta)^2}{\sin \theta} d\theta.$$

The change of variables  $\theta \rightarrow \pi - \theta$  shows that

$$\int_{\pi-1}^\pi \frac{h(\theta)^2}{(\sin \theta)^3} d\theta \lesssim \int_{\pi-1}^\pi \frac{h'(\theta)^2}{\sin \theta} d\theta + \int_0^\pi \frac{h(\theta)^2}{\sin \theta} d\theta,$$

and the desired bound (A.25) follows.

To prove (v), we may assume that  $f \in H_{\text{loc}}^2((0, \pi))$  is real-valued and let  $f(\theta) = g(\theta)(\sin \theta)^2$ . Then

$$\begin{aligned} f''(\theta) + \frac{\cos \theta}{\sin \theta} f'(\theta) - \frac{4(\cos \theta)^2}{(\sin \theta)^2} f(\theta) \\ = (\sin \theta)^2 g''(\theta) + 3 \sin \theta \cos \theta g'(\theta) - 2(\sin \theta)^2 g(\theta). \end{aligned}$$

The inequality (A.22) becomes

$$\begin{aligned} \int_0^\pi |g''|^2 (\sin \theta)^5 + |g'|^2 (\sin \theta)^3 + |g|^2 \sin \theta d\theta \\ \lesssim \int_0^\pi |(\sin \theta)^2 g'' + 3 \sin \theta \cos \theta g'|^2 \sin \theta d\theta \\ + \int_0^\pi |g'|^2 (\sin \theta)^5 + |g|^2 (\sin \theta)^3 d\theta. \end{aligned}$$

In view of the inequality (A.19) with  $p = 1$  it suffices to prove that

$$\int_0^\pi |g'|^2 (\sin \theta)^3 d\theta \lesssim \int_0^\pi |(\sin \theta)^2 g'' + 3 \sin \theta \cos \theta g'|^2 \sin \theta d\theta + \int_0^\pi |g'|^2 (\sin \theta)^5 d\theta.$$

Letting  $h(\theta) = (\sin \theta)^3 g'(\theta)$  this is equivalent to

$$\int_0^\pi \frac{h(\theta)^2}{(\sin \theta)^3} d\theta \lesssim \int_0^\pi \frac{h'(\theta)^2}{\sin \theta} d\theta + \int_0^\pi \frac{h(\theta)^2}{\sin \theta} d\theta,$$

which was proved earlier, see (A.25). □

### The Main Function Spaces

We summarize now some of the main properties of the spaces  $H^m(\Sigma_t^c)$  and  $\tilde{H}^m(\Sigma_t^c)$ :

**Lemma A.2** *Assume  $t \in \mathbb{R}$  and  $c \geq c_0$ .*

(i) *If  $f \in H^1(\Sigma_t^c)$  satisfies  $\mathbf{Z}(f) = 0$  then*

$$\|f\|_{\tilde{H}^1(\Sigma_t^c)} \approx \|f\|_{H^1(\Sigma_t^c)} + \|(r \sin \theta)^{-1} f\|_{L^2(\Sigma_t^c)}. \tag{A.27}$$

(ii) *If  $f \in H^2(\Sigma_t^c)$ , satisfies  $\mathbf{Z}(f) = 0$  then*

$$\|f\|_{\tilde{H}^2(\Sigma_t^c)} \approx \|f\|_{H^2(\Sigma_t^c)} + \|(r \sin \theta)^{-1} f\|_{H^1(\Sigma_t^c)} + \|(r \sin \theta)^{-2} f\|_{L^2(\Sigma_t^c)}. \tag{A.28}$$

*Proof of Lemma A.2* The bound (A.27) follows easily from the definitions and (A.20). We prove now part (ii) and the bounds (A.28) for  $m = 2$ . In view of the definition, and using also (A.27),

$$\begin{aligned} \|f\|_{\tilde{H}^2(\Sigma_t^c)} &\approx \|f\|_{H^2(\Sigma_t^c)} + \|\tilde{\partial}_2 f\|_{\tilde{H}^1(\Sigma_t^c)} + \|(\tilde{\partial}_1/r)^2 f\|_{L^2(\Sigma_t^c)} + \|(\tilde{\partial}_1/r) f\|_{L^2(\Sigma_t^c)} \\ &\approx \|f\|_{H^2(\Sigma_t^c)} + \|(r \sin \theta)^{-1} \partial_2 f\|_{L^2(\Sigma_t^c)} + \|(\tilde{\partial}_1/r)^2 f\|_{L^2(\Sigma_t^c)} \\ &\quad + \|(\tilde{\partial}_1/r) f\|_{L^2(\Sigma_t^c)}. \end{aligned} \tag{A.29}$$

Using (A.18), we have

$$\|(r \sin \theta)^{-1} \partial_2 f\|_{L^2(\Sigma_t^c)} \lesssim \|\partial_2[(r \sin \theta)^{-1} f]\|_{L^2(\Sigma_t^c)} \lesssim \|(r \sin \theta)^{-1} f\|_{H^1(\Sigma_t^c)}.$$

Moreover, using the definition,

$$\begin{aligned} & \|(\tilde{\partial}_1/r)^2 f\|_{L^2(\Sigma_f^c)} + \|(\tilde{\partial}_1/r) f\|_{L^2(\Sigma_f^c)} \\ & \lesssim \|(\partial_1/r)^2 f\|_{L^2(\Sigma_f^c)} + \|[1 + (r \sin \theta)^{-1}](\partial_1/r) f\|_{L^2(\Sigma_f^c)} \\ & \quad + \|[1 + (r \sin \theta)^{-2}]f\|_{L^2(\Sigma_f^c)} \\ & \lesssim \|f\|_{H^2(\Sigma_f^c)} + \|(r \sin \theta)^{-1} f\|_{H^1(\Sigma_f^c)} + \|(r \sin \theta)^{-2} f\|_{L^2(\Sigma_f^c)}. \end{aligned}$$

Using also (A.29), it follows that

$$\|f\|_{\tilde{H}^2(\Sigma_f^c)} \lesssim \|f\|_{H^2(\Sigma_f^c)} + \|(r \sin \theta)^{-1} f\|_{H^1(\Sigma_f^c)} + \|(r \sin \theta)^{-2} f\|_{L^2(\Sigma_f^c)},$$

as desired.

For the reverse inequality, using (A.29), it remains to prove that

$$\begin{aligned} & \|(r \sin \theta)^{-2} f\|_{L^2(\Sigma_f^c)} + \|(r \sin \theta)^{-1}(\partial_1/r) f\|_{L^2(\Sigma_f^c)} \\ & \lesssim \|f\|_{H^2(\Sigma_f^c)} + \|(r \sin \theta)^{-1} \partial_2 f\|_{L^2(\Sigma_f^c)} + \|(\tilde{\partial}_1/r)^2 f\|_{L^2(\Sigma_f^c)} \\ & \quad + \|(\tilde{\partial}_1/r) f\|_{L^2(\Sigma_f^c)}. \end{aligned} \tag{A.30}$$

Using (A.20),

$$\|(r \sin \theta)^{-1}(\tilde{\partial}_1/r) f\|_{L^2(\Sigma_f^c)} \lesssim \|(\tilde{\partial}_1/r)^2 f\|_{L^2(\Sigma_f^c)} + \|(\tilde{\partial}_1/r) f\|_{L^2(\Sigma_f^c)}.$$

Also, using (A.22) and then (A.27),

$$\begin{aligned} & \|(r \sin \theta)^{-2} f\|_{L^2(\Sigma_f^c)} \\ & \lesssim \left\| r^{-2} \left[ \partial_1^2 + \frac{\cos \theta}{\sin \theta} \partial_1 - \frac{4(\cos \theta)^2}{(\sin \theta)^2} \right] f \right\|_{L^2(\Sigma_f^c)} \\ & \quad + \|(\partial_1/r) f\|_{L^2(\Sigma_f^c)} + \|(r \sin \theta)^{-1} f\|_{L^2(\Sigma_f^c)} \\ & \lesssim \|(\tilde{\partial}_1/r)^2 f\|_{L^2(\Sigma_f^c)} + \|(r \sin \theta)^{-1}(\tilde{\partial}_1/r) f\|_{L^2(\Sigma_f^c)} \\ & \quad + \|(\tilde{\partial}_1/r) f\|_{L^2(\Sigma_f^c)} + \|f\|_{H^1(\Sigma_f^c)}. \end{aligned}$$

The desired bound (A.30) follows from these two estimates. □

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