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Ranking of Fuzzy Numbers on the Basis of New Fuzzy Distance

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Abstract Real-world problems often deal with uncertainties related to imprecision and vagueness, for which fuzzy modeling has provided a successful approach. Ranking fuzzy numbers is a necessary and complex task in multiple processes, such as decision making. To facilitate the task of ranking fuzzy numbers, it has been introduced an approach to construct fuzzy distances from classical interval distances because their α-cuts are continuous intervals. Usually, extant interval distances use interval endpoints or midpoints, leading to results that might not reflect the correct distance because of information loss. This paper introduces a new fuzzy distance based on a novel interval distance that considers all points within the intervals by using the concept of integration to calculate the average distance between all points belonging to two intervals, respectively. Subsequently, a series of distances between fuzzy numbers based on the proposed interval distances are defined and proved. A new method for ranking fuzzy numbers using the new fuzzy distances is then presented. Finally, the validity and effectiveness of the proposed distances will be demonstrated by a comparative analysis of numerical examples.

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1 Introduction

Since Zadeh introduced fuzzy sets [25] as an extension of crisp sets, fuzzy numbers [16] have been widely used as a special case of fuzzy sets, usually expressed as trapezoidal [25] or "quasi-trapezoidal" [8, 16] fuzzy numbers, which are approximate evaluations given by experts when more correct values are not possible or not needed. Due to the fact that ranking fuzzy numbers is a required and complex task among several steps in the decision-making process, different methods have been investigated to rank fuzzy numbers [19]:

(1) Methods based on defuzzification;
(2) Methods based on the distance between fuzzy numbers;
(3) Pairwise comparison methods.

Here, we focus on the widely used ranking method based on the distances between fuzzy numbers, in which such distances are classified into two main categories: (i) precise numerical distance [12, 20] and (ii) fuzzy distance [18, 23]. In the former, as the α-cuts of two comparative fuzzy numbers are intervals [4, 10, 11, 17], the classical interval distances are usually applied to the α-cuts of two fuzzy numbers to construct fuzzy distances [14]. Meanwhile, the latter uses the idea from Voxman [18] that fuzzy numerical values naturally lead to a family of pseudo-metrics of fuzzy numbers and introduces a fuzzy distance for fuzzy numbers in [18].

In this paper, we focus on defining numerical distances between fuzzy numbers by defining the novel interval distance between their corresponding α-cuts [4], and then ranking these fuzzy numbers by computing the distance between each fuzzy number and the selected ideal fuzzy number. Accordingly, the critical point is how to define the interval distance. In general, the classical distance between intervals is usually computed from their midpoints, lengths, or endpoints [5, 9, 10, 17], and might not correctly reflect the distance, as it should be determined by all points of two intervals. Therefore, considering that a continuous interval is a countably dense set of points, the distance between intervals should be calculated by taking the average of the distances between any two points belonging to the two corresponding intervals. To do so, we will use the concept of integral to obtain the integral expression of the distance between their corresponding α-cuts. Inspired by the outstanding work in [5, 10], a novel distance between fuzzy numbers is proposed and used for the ranking of fuzzy numbers. Therefore, the main contributions of this paper are summarized as follows:

(1) A novel distance between intervals is defined by using the concept of integral, which considers all points in the two intervals, respectively. This means that the more information used to reflect the distance, the more correct and reliable the results will be.
(2) Based on the previous interval distance and a reducing function with α as a variable, a series of new distances between fuzzy numbers is proposed, and their properties are proved. Then, a new method for ranking fuzzy numbers based on the novel fuzzy distance of each fuzzy number from the ideal fuzzy number is naturally introduced.
(3) Several numerical analyses and comparisons are provided to demonstrate the advantages of the newly proposed interval distance, fuzzy distance, and ranking methods.

The rest of this work is organized as follows. Section 2 reviews the concepts related to distance and fuzzy numbers. Section 3 first introduces the distance between intervals by using the definition of integral, then proposes a series of distances between fuzzy numbers on the basis of the new proposed interval distance, and the general properties are demonstrated. Some numerical analysis will be shown in Sect. 4. Finally, Sect. 5 points out some conclusions.

2 Preliminaries

This section reviews some basic concepts, such as distance, *L-R* shaped fuzzy numbers, and α-cuts of a fuzzy number, which will help construct the concept of distances between fuzzy numbers.

Definition 1 [14] Let $\mathbf{X}$ be a non-empty set and $\mathbf{R}^+$ be the set of all non-negative real numbers. The function $d : \mathbf{X} \times \mathbf{X} \to \mathbf{R}^+$ is a distance if and only if d satisfies the following properties:

(P1) Non-negativity: $\forall\, x, y \in \mathbf{X},\ d(x, y) \geq 0$ and $d(x, x) = 0$;
(P2) Symmetry: $\forall x, y \in \mathbf{X},\ d(x, y) = d(y, x)$;
(P3) Triangle inequality: $\forall x, y, z \in \mathbf{X},\ d(x, z) \leq d(x, y) + d(y, z)$.

There are many definitions of fuzzy numbers, and a common method is used to restrict the shape of the *L-R* membership function [16], i.e., quasi-trapezoidal fuzzy numbers [8]. Thence, a *L-R*-shaped fuzzy number, $\tilde{F} \equiv \tilde{F}(a, b, c, d)$, has the membership given as a mapping $\begin{array}{l}\mu_{\tilde{F}} : \mathbf{R} \to [0,1] \\ \quad x \mapsto \mu_{\tilde{F}}(x)\end{array}$ as follows:

$$\mu_{\tilde{F}}(x) = \begin{cases} L\left(\dfrac{b-x}{b-a}\right) & \textit{if } a \leq x \leq b; \\ 1 & \textit{if } b < x < c; \\ R\left(\dfrac{x-c}{d-c}\right) & \textit{if } c \leq x \leq d; \\ 0 & \textit{otherwise}. \end{cases} \tag{1}$$

Being $a, b, c, d \in \mathbf{R}$ real numbers such that $a \leq b \leq c \leq d$. $L(\cdot), R(\cdot) : [0, 1] \to [0, 1]$ are two non-increasing shape functions such that $R(0) = L(0) = 1$ and $R(1) = L(1) = 0$.

Accordingly, the set of all *L-R*-shaped fuzzy numbers defined on the interval $[0, 1]$ will be referred to as follows:

$$\Omega_{[0,1]} = \left\{\tilde{F}(a, b, c, d) \middle|\ \mu_{\tilde{F}} : \mathbf{R} \to [0,1] \right. \\ \left. [\text{ is a } L\text{-}R \text{ shaped fuzzy numbers}\right\}. \tag{2}$$

Here, if $L(\cdot)$ and $R(\cdot)$ are invertible functions, then its α-cut is given as follows:

$$\tilde{F^{\alpha}} = \left[b - (b-a)L^{-1}(\alpha), c + (d-c)R^{-1}(\alpha)\right],\ \alpha \in [0, 1]. \tag{3}$$

In general, let $L(x) = R(x) = 1 - x$, the *L-R* shaped fuzzy number is the trapezoidal fuzzy number (TrFN), which should be easily defined as follows:

Definition 2 [25] For real numbers $a \leq b \leq c \leq d$, the membership function of TrFN, $\tilde{T} \equiv \tilde{T}(a, b, c, d)$, is given by

$$\mu_{\tilde{T}}(x) = \begin{cases} \dfrac{x-a}{b-a} & \textit{if } a \leq x \leq b; \\ 1 & \textit{if } b < x < c; \\ \dfrac{d-x}{d-c} & \textit{if } c \leq x \leq d; \\ 0 & \textit{otherwise}. \end{cases} \tag{4}$$

And its α-cuts are computed as follows:

$$\tilde{T}^{\alpha} = [a + \alpha \cdot (b - a), d - \alpha \cdot (d - c)], \ \forall \alpha \in [0, 1], \quad (5)$$

Furthermore, if $b = c$, then $\tilde{T}$ is a triangular fuzzy number (TFN).

3 Comparison of Fuzzy Numbers Based on a Novel Fuzzy Distance

In order to rank fuzzy numbers, the distance between fuzzy numbers has been widely used [12, 20] and is usually constructed based on the α-cut of fuzzy numbers being a continuous interval. Therefore, the main aim of this section is to propose a novel interval distance to construct the distance between fuzzy numbers, and then, based on it, a new ranking of fuzzy numbers will be proposed (see Fig. 1).

3.1 A New Distance Between Intervals and Its Properties

Here, a novel distance between two intervals, which processes a finite approximation according to the fundamental properties of the integral, is proposed. First, it is necessary to state the distance measurement axioms for the intervals to complete the distance definition and then perform the interval distance calculation.

Definition 3 Let $\mathcal{I}_{\mathbf{R}}$ be the set of all possible sub-intervals of $\mathbf{R}$. A mapping $d_I : \mathcal{I}_{\mathbf{R}} \times \mathcal{I}_{\mathbf{R}} \to \mathbf{R}$ is called a distance between intervals iff it satisfies the following properties:

(1) Non-negativity: $\forall [a_l, a_r], [b_l, b_r] \in \mathcal{I}_{\mathbf{R}},\ d_I([a_l, a_r], [b_l, b_r]) \geq 0$ and $d_I([a_l, a_r], [a_l, a_r]) = 0$;

(2) Symmetry: $\forall [a_l, a_r], [b_l, b_r] \in \mathcal{I}_{\mathbf{R}},\ d_I([a_l, a_r], [b_l, b_r]) = d_I([b_l, b_r], [a_l, a_r])$;

(3) Triangle inequality: $d_I([a_l, a_r], [b_l, b_r]) \leq d_I([a_l, a_r], [c_l, c_r]) + d_I([c_l, c_r], [b_l, b_r])$ for all $[a_l, a_r], [b_l, b_r], [c_l, c_r] \in \mathcal{I}_{\mathbf{R}}$.

3.1.1 A New Interval Distance in the Form of Integral

To propose new distances between intervals, we use the idea from Yager's paper [21] to consider a continuous interval as dense sets of points, and the new distances are expressed by an integral form. As a result, we propose a novel distance between intervals, defined as follows:

Definition 4 Let $[a_l, a_r]$ and $[b_l, b_r]$ be two numerical continuous intervals verifying $a_l \leq a_r$ and $b_l \leq b_r$. The new distance between these two intervals is defined as follows:

$$d_I([a_l,a_r],[b_l,b_r]) = \begin{cases} 0 \ \textit{if}\ [a_l,a_r]=[b_l,b_r]; \\ \int_0^1 \int_0^1 |a_l - b_l + (a_r - a_l)x - (b_r - b_l)y| dxdy \ \textit{otherwise}. \end{cases} \quad (6)$$

Let us illustrate it more clearly in the case of $[a_l, a_r] \neq [b_l, b_r]$. Let $[a_l, a_r]$ and $[b_l, b_r]$ be two numerical continuous intervals, the distance between intervals $[a_l, a_r]$ and $[b_l, b_r]$ is computed by the following steps:

Step 1: Setting $\delta_{[a_l,a_r]} = \frac{a_r - a_l}{n}$ and $\delta_{[b_l,b_r]} = \frac{b_r - b_l}{m}$ for $n, m \gg 1$, use $a_i = a_l + i \cdot \delta_{[a_l,a_r]} = a_l + i \cdot \frac{1}{n} \cdot (a_r - a_l)$ and $b_j = b_l + j \cdot \delta_{[b_l,b_r]} = b_l + j \cdot \frac{1}{m} \cdot (b_r - b_l)$ to represent the ordered discrete points, satisfying $a_0 \leq a_1 \leq \cdots \leq a_m$ and

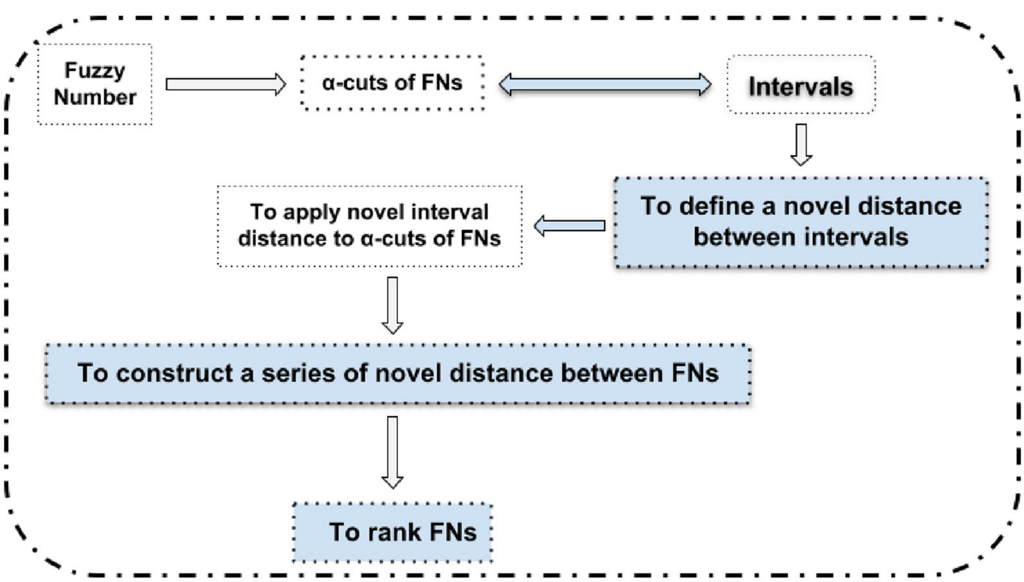

Fig. 1 Main steps of ranking fuzzy numbers

$b_0 \le b_1 \le \cdots \le b_n$, of the intervals $[a_l, a_r]$ and $[b_l, b_r]$, respectively. When $i = j = 0$, we obtain $a_0 = a_l$ and $b_0 = b_l$, respectively; and when $i = n$ and $j = m$, we obtain $a_n = a_r$ and $b_m = b_r$, respectively.

Step 2: Taking into account that the distance between two points a_i and b_j is computed directly as $d_{ij} = |a_i - b_j|$ for $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, m$, the sum of all distances d_{ij} is then computed as $\sum_{i=0}^{n}\sum_{j=0}^{m} d_{ij} = \sum_{i=0}^{n}\sum_{j=0}^{m} |a_i - b_j|$.

Step 3: Considering the simplest case in which each distance d_{ij} contributes identically to the final distance between the intervals $[a_l, a_r]$ and $[b_l, b_r]$, then such final interval distance can be approximated as the average of the sum of all distances d_{ij}, i.e., $\left(\sum_{i=0}^{n}\sum_{j=0}^{m}|a_i - b_j|\right)\frac{1}{(n+1)\cdot(m+1)}$.

Step 4: To be in accordance with the definition of the integral, then the distance can be changed slightly to be approximated as $\left(\sum_{i=1}^{n}\sum_{j=1}^{m}|a_i - b_j|\right)\frac{1}{n\cdot m}$. Thus, we can achieve that

$$d_I([a_l,a_r],[b_l,b_r]) \approx \left(\sum_{i=1}^{n}\sum_{j=1}^{m}|a_i - b_j|\right)\frac{1}{n\cdot m} = \left(\sum_{i=1}^{n}\sum_{j=1}^{m}\left|a_l + i\cdot\frac{1}{n}\cdot(a_r - a_l) - b_l - j\cdot\frac{1}{m}\cdot(b_r - b_l)\right|\right)\frac{1}{n\cdot m} \tag{7}$$

Step 5: Subsequently, letting $\Delta x = \frac{1}{n}$ and $\Delta y = \frac{1}{m}$, we obtain

$$d_I([a_l,a_r],[b_l,b_r]) \approx \left(\sum_{i=1}^{n}\sum_{j=1}^{m}|a_l + i\cdot\Delta x\cdot(a_r - a_l) - b_l - j\cdot\Delta y\cdot(b_r - b_l)|\right)\Delta x\Delta y \tag{8}$$

Step 6: Finally, letting $n \to +\infty$ and $m \to +\infty$, thus, $\Delta x \to +0$ and $\Delta y \to +0$, denoting $x = i\cdot\Delta x$ and $y = j\cdot\Delta y$, and considering that the value of i is from 1 to n and the value of j is from 1 to m, then we can obtain $x, y \in (0, 1]$ and

$$d_I([a_l,a_r],[b_l,b_r]) = \int_0^1\int_0^1 |a_l + (a_r - a_l)x - b_l - (b_r - b_l)y|\,\mathrm{d}x\mathrm{d}y. \tag{9}$$

However, it should be pointed out that if $[a_l, a_r] = [b_l, b_r]$ with $a^r \ne a^l$, using the above formula shall obtain $d_I([a_l, a_r], [a_l, a_r]) = \frac{a^r - a^l}{3} \ne 0$, which is contrary to the axiom of distance. Thus, another constraint should be added to the final formula, i.e., set $d_I([a_l, a_r], [a_l, a_r]) = 0$ when $[a_l, a_r] = [b_l, b_r]$ with $a^r \ne a^l$. Therefore, the final distance will be obtained as in Eq. (6).

Note 1 In fact, $d_I([a_l, a_r], [b_l, b_r])$ might be similarly rewritten as follows:

$$\begin{aligned} d_I([a_l,a_r],[b_l,b_r]) &= \int_0^1\int_0^1 |a_r - (a_r - a_l)x - b_l - (b_r - b_l)y|\,\mathrm{d}x\mathrm{d}y;\\ &= \int_0^1\int_0^1 |a_l + (a_r - a_l)x - b_r + (b_r - b_l)y|\,\mathrm{d}x\mathrm{d}y;\\ &= \int_0^1\int_0^1 |a_r - (a_r - a_l)x - b_r + (b_r - b_l)y|\,\mathrm{d}x\mathrm{d}y;\\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{a_l + a_r}{2} - \frac{b_r + b_l}{2} + (a_r - a_l)x - (b_r - b_l)y\right|\mathrm{d}x\mathrm{d}y. \end{aligned}$$

For sake of simplicity, in this contribution, we only use

$$d_I([a_l, a_r], [b_l, b_r]) = \int_0^1\int_0^1 |a_l + x\cdot(a_r - a_l) - b_l - (b_r - b_l)y|\,\mathrm{d}x\,\mathrm{d}y$$

to compute the distance between two unequal numerical continuous intervals.

In addtion, if set $d_I^2([a_l, a_r], [b_l, b_r]) = \int_0^1\int_0^1 |a_l + x\cdot(a_r - a_l) - b_l - (b_r - b_l)y|^2\,\mathrm{d}x\,\mathrm{d}y$, then it is the interval distance introduced in [17]:

$$\begin{aligned} d_I([a_l,a_r],[b_l,b_r]) &= \sqrt{\int_0^1\int_0^1 |a_l + x\cdot(a_r - a_l) - b_l - (b_r - b_l)y|^2\mathrm{d}x\mathrm{d}y}\\ &= \sqrt{\left(\frac{a_l + a_r}{2} - \frac{b_r + b_l}{2}\right)^2 + \frac{1}{12}\left((a_r - a_l)^2 + (b_r - b_l)^2\right)}. \end{aligned} \tag{10}$$

Note 2 There are several special cases that should be illustrated under the situation $[a_l, a_r] \ne [b_l, b_r]$.

(1) Both $[a_l, a_r]$ and $[b_l, b_r]$ are constants, i.e., $a_r = a_l$ and $b_r = b_l$, then Eq. (6) shall be rewritten as follows:

$$d_I([a_l, a_r], [b_l, b_r]) = |a_l - b_l| = |a_r - b_r|.$$

In other words, if both $[a_l, a_r]$ and $[b_l, b_r]$ are constants, then Eq. (6) is the general distance between two real numbers.

(2) Just one of $[a_l, a_r]$ and $[b_l, b_r]$ is constant.

(a) If $[a_l, a_r]$ is a constant and $[b_l, b_r]$ is an interval, i.e., $a_r = a_l$ and $b_r \neq b_l$, then Eq. (6) shall be rewritten as follows:

$$d_I([a_l,a_r],[b_l,b_r]) = \begin{cases} \frac{b_r-b_l}{2} - \frac{(b_r-a_l)(a_l-b_l)}{b_r-b_l} & \text{if } a_l \in [b_l,b_r]; \\ \left|a_l - \frac{b_r+b_l}{2}\right| & \text{otherwise}. \end{cases}$$

(b) If $[b_l, b_r]$ is a constant and $[a_l, a_r]$ is an interval, i.e., $a_r \neq a_l$ and $b_r = b_l$, then Eq. (6) shall be rewritten as follows:

$$d_I([a_l,a_r],[b_l,b_r]) = \begin{cases} \frac{a_r-a_l}{2} - \frac{(a_r-b_l)(b_l-a_l)}{a_r-a_l} & \text{if } b_l \in [a_l,a_r]; \\ \left|b_l - \frac{a_r+a_l}{2}\right| & \text{otherwise}. \end{cases}$$

(3) Both $[a_l, a_r]$ and $[b_l, b_r]$ are continuous intervals, i.e., $(a_r - a_l)(b_r - b_l) \neq 0$, let $u = a_l + (a_r - a_l)x$ and $v = b_l + (b_r - b_l)y$, then

$$\begin{aligned} \mathrm{d}u &= (a_r - a_l)\mathrm{d}x \\ \mathrm{d}v &= (b_r - b_l)\mathrm{d}y \end{aligned}.$$

Thence, $\forall x, y \in [0, 1]$, we shall obtain $\begin{cases} \mathrm{d}x = \frac{1}{(a_r - a_l)}\mathrm{d}u & \forall u \in [a_l, a_r]; \\ \mathrm{d}y = \frac{1}{(b_r - b_l)}\mathrm{d}v & \forall v \in [b_l, b_r], \end{cases}$ such that

$$d_I([a_l,a_r],[b_l,b_r]) = \frac{1}{(a_r-a_l)(b_r-b_l)} \int_{a_l}^{a_r} \int_{b_l}^{b_r} |u-v| \mathrm{d}u\mathrm{d}v. \tag{11}$$

Obviously, this formula has limitations in computing a distance between intervals compare to Eq. (6) because of its potential constraint $(a_r - a_l)(b_r - b_l) \neq 0$.

Furthermore, if $a_r \leq b_l$ or $b_r \leq a_l$, Eq. (9) can be simplified as follows:

$$d_I([a_l, a_r], [b_l, b_r]) = \left|\frac{a_l + a_r}{2} - \frac{b_l + b_r}{2}\right| = \begin{cases} \frac{a_l + a_r}{2} - \frac{b_l + b_r}{2} & \text{if} \quad b_r \leq a_l; \\ \frac{b_l + b_r}{2} - \frac{a_l + a_r}{2} & \text{if} \quad a_r \leq b_l. \end{cases} \tag{12}$$

Example 1 For sake of simplicity, let $[a_l, a_r]$ and $[b_l, b_r]$ be two randomly generated sub-intervals of $[0, 1]$, applying Eq. (6), some results of $d_I([a_l, a_r], [b_l, b_r])$ shown in Table 1.

Regarding the same distance values in the above table, the following interesting conclusions might be drawn:

(1) For two compared intervals $[a_{1,l}, a_{1,r}], [b_l, b_r]$ and $[a_{2,l}, a_{2,r}], [b_l, b_r]$, if one of these three conditions has been satisfied, that is, (i) $a_{1,r} - a_{1,l} = a_{2,r} - a_{2,l}$, $b_r = a_{1,r}$, $a_{2,l} = b_l$, and $b_l - a_{1,l} = a_{2,r} - b_r$; (ii) $a_{1,r} - a_{1,l} = a_{2,r} - a_{2,l}$, $a_{1,l} = b_l$, $a_{2,r} = b_r$, and $b_r - a_{1,r} = a_{2,l} - b_l$); (iii) $a_{1,r} - a_{1,l} = a_{2,r} - a_{2,l}$, $b_l - a_{1,r} = a_{2,r} = b_r$, and $a_{2,l} - b_l = b_r - a_{1,r}$, then $d_I([a_{1,l}, a_{1,r}], [b_l, b_r]) = d_I([a_{2,l}, a_{2,r}], [b_l, b_r])$.

(2) For two compared intervals $[a_{1,l}, a_{1,r}]$, $[b_{1,l}, b_{1,r}]$ and $[a_{2,l}, a_{2,r}]$, $[b_{2,l}, b_{2,r}]$, which satisfy $a_{1,l} - b_{1,r} = a_{2,l} - b_{2,r}$, $a_{1,r} - a_{1,l} = a_{2,r} - a_{2,l}$ and $b_{1,r} - b_{1,l} = b_{2,r} - b_{2,l}$ (or $a_{1,r} - a_{1,l} = b_{2,r} - b_{2,l}$ and $a_{2,r} - a_{2,l} = b_{1,r} - b_{1,l}$), then $d_I([a_{1,l}, a_{1,r}], [b_{1,l}, b_{1,r}]) = d_I([a_{2,l}, a_{2,r}], [b_{2,l}, b_{2,r}])$.

3.1.2 General Properties of the New Interval Distance

In this subsection, we briefly study the properties of the proposed interval distance.

Theorem 1 *The function given by Eq. (6) is a distance between intervals.*

Proof

(1) Necessity. This is obvious, we omit the proof here.

(2) Sufficiency. Obviously, this definition satisfies the first two properties: *Non-negativity* and *Symmetry*. For the *Triangle inequality* property, let $[c_l, c_r]$ satisfying $c_l \leq c_r$, then

$$d_I([a_l,a_r],[b_l,b_r]) = \int_0^1 \int_0^1 |a_l - b_l + (a_r - a_l)x - (b_r - b_l)y| \mathrm{d}x\mathrm{d}y;$$

$$d_I([a_l,a_r],[c_l,c_r]) = \int_0^1 \int_0^1 |a_l - c_l + (a_r - a_l)x - (c_r - c_l)z| \mathrm{d}x\mathrm{d}z;$$

$$d_I([c_l,c_r],[b_l,b_r]) = \int_0^1 \int_0^1 |c_l - b_l + (c_r - c_l)z - (b_r - b_l)y| \mathrm{d}z\mathrm{d}y.$$

Thence, by using the classical inequality $|x + y| \leq |x| + |y|$, $\forall x, y \in \mathbf{R}$

Table 1 Results of the proposed interval distance

$[a_l,a_r]$ $[b_l,b_r]$	$[0.2,0.7]$ $[0.5,0.9]$	$[0.2,0.4]$ $[0.5,0.7]$	$[0.2,0.4]$ $[0.2,0.7]$	$[0.5,0.7]$ $[0.2,0.7]$	$[0.4,0.7]$ $[0.6,0.9]$	$[0.5,0.8]$ $[0.7,1.0]$
$d_I([a_l,a_r],[b_l,b_r])$	0.2633	0.3	0.1767	0.1767	0.2037	0.2037

$$\begin{aligned}&|a_l-b_l+(a_r-a_l)x-(b_r-b_l)y|\\&\quad=|(a_l-c_l+(a_r-a_l)x-(c_r-c_l)z)\\&\quad+(c_l-b_l+(c_r-c_l)z-(b_r-b_l)y)|\\&\quad\leq|a_l-c_l+(a_r-a_l)x-(c_r-c_l)z|\\&\quad+|c_l-b_l+(c_r-c_l)z-(b_r-b_l)y|,\end{aligned}$$

and

$$\begin{aligned}&d_I([a_l,a_r],[b_l,b_r])\\&=\int_0^1\int_0^1|a_l-b_l+(a_r-a_l)x-(b_r-b_l)y|\mathrm{d}x\mathrm{d}y\\&\leq\int_0^1\int_0^1|a_l-c_l+(a_r-a_l)x-(c_r-c_l)z|\mathrm{d}x\mathrm{d}z\\&+\int_0^1\int_0^1|c_l-b_l+(c_r-c_l)z-(b_r-b_l)y|\mathrm{d}z\mathrm{d}y.\end{aligned}$$

Thus,

$$\begin{aligned}&d_I([a_l,a_r],[b_l,b_r])\leq d_I([a_l,a_r],[c_l,c_r])\\&\quad+d_I([c_l,c_r],[b_l,b_r]).\end{aligned}$$

Therefore, this interval distance is a distance measure. □

3.2 A Ranking for Fuzzy Numbers Based on the Novel Distance Between Fuzzy Numbers

The proposed novel interval distances could be applied to the corresponding α-cuts of the two fuzzy numbers being compared, which is due to the fact that the α-cuts of the fuzzy numbers are continuous intervals. Therefore, the distances between fuzzy numbers can be well defined by using the reducing functions and applying the new interval distance to their corresponding α-cuts. Thus, a new ranking metric for fuzzy numbers is proposed.

To define distances between fuzzy numbers, it is necessary to introduce the following distance axioms for fuzzy numbers first. Note that the axioms (1)–(3) from Definition 1 in terms of fuzzy numbers in $\Omega_{[0,1]}$ and $d_f:\Omega_{[0,1]}\times\Omega_{[0,1]}\rightarrow\mathbf{R}$ are as follows:

(1) Non-negativity: $\forall\tilde{F}_1,\tilde{F}_2\in\Omega_{[0,1]},\ d_f(\tilde{F}_1,\tilde{F}_2)\geq0$ and $d_f(\tilde{F}_1,\tilde{F}_1)=0$;

(2) Symmetry: $\forall\tilde{F}_1,\tilde{F}_2\in\Omega_{[0,1]},\ d_f(\tilde{F}_1,\tilde{F}_2)=d_f(\tilde{F}_2,\tilde{F}_1)$;

(3) Triangle inequality: $d_f(\tilde{F}_1,\tilde{F}_2)\leq d_f(\tilde{F}_1,\tilde{F}_3)+d_f(\tilde{F}_3,\tilde{F}_2)$ for all $\tilde{F}_1,\tilde{F}_2,\tilde{F}_3\in\Omega_{[0,1]}$.

3.2.1 Novel Distance Between Fuzzy Numbers

Before defining a distance between fuzzy numbers, we should review the concept of a reducing function with α as a variable, which will help define a series of new distances between fuzzy numbers.

Definition 5 [18] A function

$s:[0,1]\longrightarrow[0,1]$

is a reducing function if it is increasing and satisfies $s(0)=0$ and $s(1)=1$. If another constraint $\int_0^1 s(\alpha)\mathrm{d}\,\alpha=\frac{1}{2}$ is satisfied, then it is called a regular reducing function.

For example, $s(\alpha)=\sin\left(\frac{\pi}{2}\alpha\right)$ and $s(\alpha)=\alpha^\kappa,\ \kappa>0$ are reducing functions; $s(\alpha)=\frac{\pi}{4}sin\left(\frac{\pi}{2}\alpha\right)$ and $s(\alpha)=\alpha$ are regular reducing functions.

Definition 6 Let $\tilde{F}_1$ and $\tilde{F}_2$ be two arbitrary fuzzy numbers in $\Omega_{[0,1]}$, let d_I be a distance between intervals, then the distance between two fuzzy numbers with respect to the reducing function $s(\cdot)$ is defined as follows:

$$d_{f,s(\cdot)}(\tilde{F}_1,\tilde{F}_2)=\frac{\int_0^1 s(\alpha)\cdot\left[d_I\left(\tilde{F}_1^\alpha,\tilde{F}_2^\alpha\right)\right]\mathrm{d}\,\alpha}{\int_0^1 s(\alpha)\mathrm{d}\,\alpha}\tag{13}$$

where $\tilde{F}_1^\alpha,\tilde{F}_2^\alpha$ are α-cuts of fuzzy numbers $\tilde{F}_1,\tilde{F}_2$ for all $\alpha\in[0,1]$, respectively.

Note 3 Let $s(\alpha)=\alpha^\kappa$ be a reducing function with parameter $\kappa>0$, then Eq. (13) should be rewritten as follows:

$$d_{f,\alpha^\kappa}(\tilde{F}_1,\tilde{F}_2)=(\kappa+1)\int_0^1\alpha^\kappa\cdot\left[d_I\left(\tilde{F}_1^\alpha,\tilde{F}_2^\alpha\right)\right]\mathrm{d}\,\alpha.\tag{14}$$

Furthermore, if $s(\cdot)$ is a regular reducing function $s(\alpha)=\alpha$, i.e., $\kappa=1$, then Eq. (13) should be simplified as follows:

$$d_{f,\alpha^1}(\tilde{F}_1,\tilde{F}_2)=2\int_0^1\alpha\cdot\left[d_I\left(\tilde{F}_1^\alpha,\tilde{F}_2^\alpha\right)\right]\mathrm{d}\,\alpha.\tag{15}$$

Without loss of generality, in this proposal, we will use Eq. (14) to calculate the distance between fuzzy numbers.

Theorem 2 *The distance $d_{f,s(\cdot)}$ satisfies the distance axioms for fuzzy numbers.*

Table 2 The distances between TrFNs/TFNs

	$s(\alpha)=\alpha^{\kappa}$	$\tilde{T}_1\ \tilde{T}_2$	$\tilde{T}_1\ \tilde{T}_3$	$\tilde{T}_1\ \tilde{T}_4$	$\tilde{T}_2\ \tilde{T}_3$	$\tilde{T}_2\ \tilde{T}_4$	$\tilde{T}_3\ \tilde{T}_4$
$d_{f,\alpha^{\kappa}}$	$\kappa=0.1$	0.2505	0.1154	0.1170	0.1676	0.3081	0.1529
	$\kappa=0.5$	0.2503	0.1110	0.1129	0.1673	0.3148	0.1557
	$\boldsymbol{\kappa=1}$	**0.2502**	**0.1067**	**0.1093**	**0.1672**	**0.3214**	**0.1592**
	$\kappa=1.5$	0.2501	0.1036	0.1069	0.1671	0.3264	0.1623
	$\kappa=2$	0.2500	0.1013	0.1054	0.1670	0.3301	0.1649

Proof Properties (I) and (II) are obvious. We only need to prove the property (III).

$\forall \tilde{F}_1,\tilde{F}_2,\tilde{F}_3\in\Omega_{[0,1]}$, applying Eq. (13), for a given reducing function $s(\alpha)$, $\int_0^1 s(\alpha)d\alpha$ is a constant, then the following results shall be obtained:

$$d_{f,s(\cdot)}(\tilde{F}_1,\tilde{F}_2)=\frac{1}{\int_0^1 s(\alpha)\mathrm{d}\alpha}\int_0^1 s(\alpha)\cdot\left[d_I(\tilde{F}_1^{\alpha},\tilde{F}_2^{\alpha})\right]\mathrm{d}\alpha;$$

$$d_{f,s(\cdot)}(\tilde{F}_1,\tilde{F}_3)=\frac{1}{\int_0^1 s(\alpha)\mathrm{d}\alpha}\int_0^1 s(\alpha)\cdot\left[d_I(\tilde{F}_1^{\alpha},\tilde{F}_3^{\alpha})\right]\mathrm{d}\alpha;$$

$$d_{f,s(\cdot)}(\tilde{F}_3,\tilde{F}_2)=\frac{1}{\int_0^1 s(\alpha)\mathrm{d}\alpha}\int_0^1 s(\alpha)\cdot\left[d_I(\tilde{F}_3^{\alpha},\tilde{F}_2^{\alpha})\right]\mathrm{d}\alpha.$$

Considering Theorem 1, i.e., for $\alpha\in[0,1]$, $d_I(\tilde{F}_1^{\alpha},\tilde{F}_2^{\alpha})\le d_I(\tilde{F}_1^{\alpha},\tilde{F}_3^{\alpha})+d_I(\tilde{F}_3^{\alpha},\tilde{F}_2^{\alpha})$, then

$$\begin{aligned}&\int_0^1 s(\alpha)\cdot\left[d_I(\tilde{F}_1^{\alpha},\tilde{F}_2^{\alpha})\right]\mathrm{d}\alpha\\ &\le\int_0^1 s(\alpha)\cdot\left[d_I(\tilde{F}_1^{\alpha},\tilde{F}_3^{\alpha})+d_I(\tilde{F}_3^{\alpha},\tilde{F}_2^{\alpha})\right]\mathrm{d}\alpha\\ &=\int_0^1 s(\alpha)\cdot\left[d_I(\tilde{F}_1^{\alpha},\tilde{F}_3^{\alpha})\right]d\alpha+\int_0^1 s(\alpha)\cdot\left[d_I(\tilde{F}_3^{\alpha},\tilde{F}_2^{\alpha})\right]\mathrm{d}\alpha.\end{aligned}$$

Finally, we shall obtain

$$d_{f,s(\cdot)}(\tilde{F}_1,\tilde{F}_2)\le d_{f,s(\cdot)}(\tilde{F}_1,\tilde{F}_3)+d_{f,s(\cdot)}(\tilde{F}_3,\tilde{F}_2).$$

□

To simplify the computation process, the following example is introduced to show the performance of the distance based on α-cuts of TrFNs.

Example 2 Let $\tilde{T}_1=\tilde{T}_1(0.5,0.625,0.875,1)$ and $\tilde{T}_2=\tilde{T}_2(0.375,0.5,0.5,0.625)$ be two TrFNs, and let $\tilde{T}_3=\tilde{T}_3(0.5,0.667,0.834)$ and $\tilde{T}_4=\tilde{T}_4(0.5,0.852,1)$ be two TFNs, then applying Eq. (14), the obtained distances with different $\kappa>0$ are shown in Table 2.

(1) As we can see, the results are reasonable. For example, as κ increases, $d_{f,\alpha^{\kappa}}(\tilde{T}_2,\tilde{T}_4)$ and $d_{f,\alpha^{\kappa}}(\tilde{T}_3,\tilde{T}_4)$ increases, and the rest decreases with it. The underlying reason should be due to the fact that $f(\alpha,\kappa)=\frac{\alpha^{\kappa}}{\int_0^1\alpha^{\kappa}\mathrm{d}\alpha}$ plays the role of a weight, because the interval distance between their α-cuts of a pair of TrFNs is the same for different κ.

(2) Based on Fig. 2, the impacts of parameters α,κ on $f(\alpha,\kappa)=\frac{\alpha^{\kappa}}{\int_0^1\alpha^{\kappa}\mathrm{d}\alpha}$ will be further investigated in-depth. In the Fig. 2, the x-axis represents $\alpha\in[0,1]$ and the y-axis shows the value of $f(\alpha,\kappa)$ for a given $\kappa\in\{0.1,0.2,\ldots,1,\ldots,1.9,2\}$. The solid curves represent the given parameter $\kappa\in\{0.1,0.2,\ldots,1\}$; and the dotted curves represent the parameter $\kappa\in\{1.1,1.2,\ldots,1.9,2\}$. Therefore, on the basis of Fig. 2, we can conclude the following:

 (a) For a given κ, as α increase, the value of $f(\alpha,\kappa)$ is increasing.
 (b) At $\alpha=1$, the maximum value of $f(\alpha,\kappa)$, i.e., $\kappa+1$, increases as κ increase.
 (c) When $\kappa=0.1$, the asymptote of $f(\alpha,0.1)$ is $y=1.1$, i.e., Eq. (14) almost perform as an average aggregation operator.
 (d) For the rest $\kappa>0.1$, the interval distance between the α-cuts will become increasingly significant at higher α value for a pair of TrFNs.

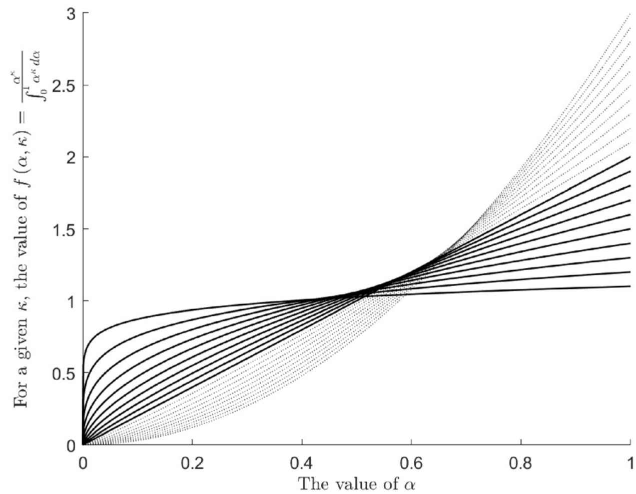

Fig. 2 The image of function $f(\alpha,\kappa)=\frac{\alpha^{\kappa}}{\int_0^1\alpha^{\kappa}\mathrm{d}\alpha}$ for each given $\kappa\in\{0.1,0.2,\ldots,1,\ldots,1.9,2\}$

In summary, based on these conclusions, the distance between α-cuts increases for the cases $\tilde{T}_2, \tilde{T}_4$ and $\tilde{T}_3, \tilde{T}_4$, but decreases for the remaining cases.

3.2.2 New Ranking of Fuzzy Numbers Based on the Proposed Distance

Once the distance between fuzzy numbers has been proposed, the ranking index of fuzzy numbers will appear naturally. Let $\{\tilde{F}_1, \tilde{F}_2, \ldots, \tilde{F}_n\}(n \geq 2)$ be a set of n arbitrary fuzzy numbers that need to be ordered, and $\forall \alpha \in [0,1]$, let $\tilde{F}_i = \tilde{F}(a_i, b_i, c_i, d_i)$, and $\tilde{F}_i^{\alpha} = \left[b_i - (b_i - a_i)L_i^{-1}(\alpha), c_i + (d_i - c_i)R_i^{-1}(\alpha)\right]$ be the α-cut of $\tilde{F}_i$ and $b_i - (b_i - a_i)L_i^{-1}(\alpha), c_i + (d_i - c_i)R_i^{-1}(\alpha)$ be two endpoints of $\tilde{F}_i^{\alpha}$ for all $i = 1, 2, \cdots, n$, then these two ideal fuzzy numbers are given by

$$\begin{cases} \tilde{F}^{*} = \bigcup_{\alpha \in [0,1]} \left[\sup_i \left\{b_i - (b_i - a_i)L_i^{-1}(\alpha)\right\}, \sup_i \left\{c_i + (d_i - c_i)R_i^{-1}(\alpha)\right\}\right] \\ \tilde{F}_{*} = \bigcup_{\alpha \in [0,1]} \left[\inf_i \left\{b_i - (b_i - a_i)L_i^{-1}(\alpha)\right\}, \inf_i \left\{c_i + (d_i - c_i)R_i^{-1}(\alpha)\right\}\right] \end{cases} \tag{16}$$

Therefore, $\tilde{F}_i$ has at least three basic ranking indices, defined as follows.

(1) Compute the distance to $\tilde{F}_{*}$, given as

$$RI_1(\tilde{F}_i) = d_{f,s(\cdot)}(\tilde{F}_i, \tilde{F}_{*}), \tag{17}$$

(2) Compute the negative of the distance to $\tilde{F}^{*}$, given as

$$RI_2(\tilde{F}_i) = -d_{f,s(\cdot)}(\tilde{F}_i, \tilde{F}^{*}), \tag{18}$$

(3) Inspired by the TOPSIS [13, 15, 26] method, and considering both the two distances of $\tilde{F}_i$ from $\tilde{F}^{*}$ and $\tilde{F}_i$ from $\tilde{F}_{*}$, the ranking value is calculated as follows:

$$RI_3(\tilde{F}_i) = \frac{d_{f,s(\cdot)}(\tilde{F}_i, \tilde{F}_{*})}{d_{f,s(\cdot)}(\tilde{F}_i, \tilde{F}^{*}) + d_{f,s(\cdot)}(\tilde{F}_i, \tilde{F}_{*})}. \tag{19}$$

From the ranking values of $\tilde{F}_i$, it suggests that the larger the value, the higher the ranking position of all fuzzy numbers.

Example 3 Continuing with Example 2, applying Eqs. (17), (18), and (19), it is easy to compute the three ranking index for these TrFNs, respectively, as shown in Figs. 3, 4, and 5. In these figures, the y-axis represents the ranking index value; the x-axis indicates the value of $\kappa \in [0.1, 2]$. It is an obvious conclusion from the figures that the order of these TrFNs is $\tilde{T}_4 \succ \tilde{T}_1 \succ \tilde{T}_3 \succ \tilde{T}_2$.

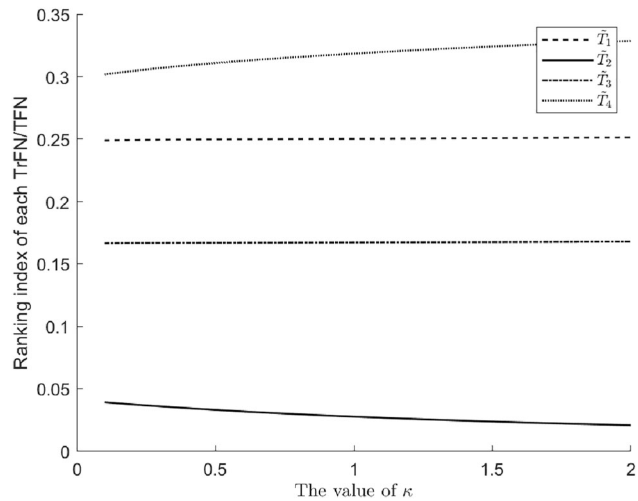

Fig. 3 Ranking index using Eq. (17)

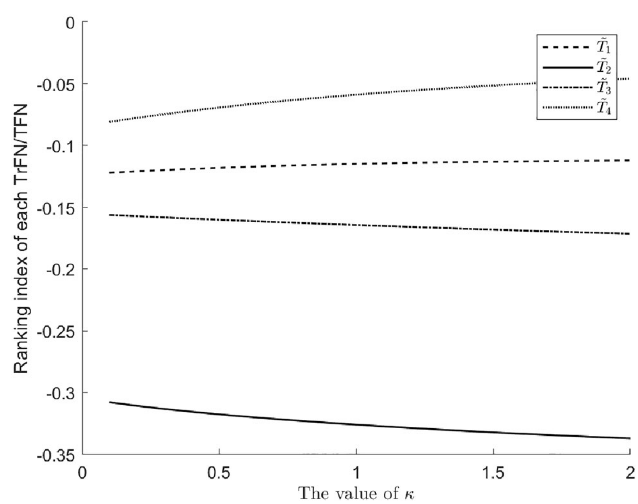

Fig. 4 Ranking index using Eq. (18)

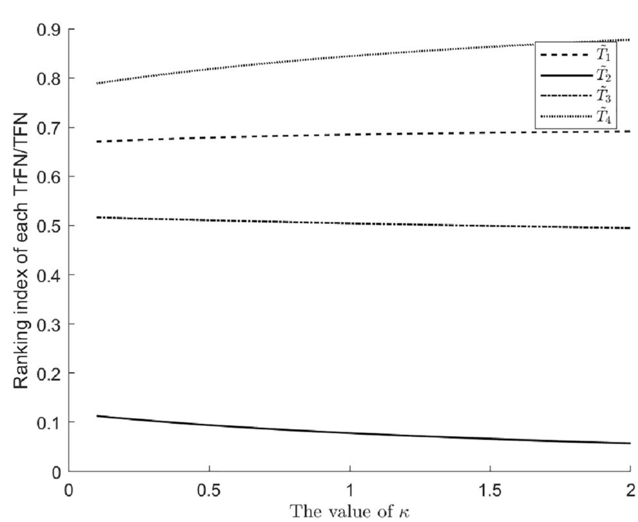

Fig. 5 Ranking index using Eq. (19)

3.2.3 Time Complexity Analyses

The proposed ranking method is performing by two key steps:

Step 1: applying the proposed interval distance to the α-cuts of the compared fuzzy numbers;
Step 2: using one of ranking index to compute the ranking index;

Based on the concept of defining an interval distance, it takes $O(n^2)$ time to calculate the distance because of the performing of integral. Once the interval distance between α-cuts is computed, it should take $O(n)$ time to compute the fuzzy distance on the basis of processing of α-cuts. The ranking index then takes $O(1)$ time to finish the final computation. It can be seen that the ranking method total takes $O(n^3)$, i.e., $O(n^3) = O(n^2) \times O(n) \times O(1)$. Therefore, the time complexity of ranking fuzzy numbers is shown as Theorem 3.

Theorem 3 *The time complexity of ranking fuzzy numbers is $O(n^3)$.*

Note 4 In spite of the time complexity, the results will be more reliable in many practical cases than those using only endpoints or midpoints of their corresponding α-cuts.

4 Numerical Comparisons and Analysis

This section focuses on demonstrating the effectiveness and generality of the proposed interval distance because the key to the distance between fuzzy numbers is actually the distance between intervals. First, a comparison and analysis of the interval distance proposed in this paper with several other classical methods are introduced. Then, the new ranking index of fuzzy numbers will be compared and analyzed with some other fuzzy number ranking methods.

4.1 Comparative Analysis Between the Proposed Interval Distance and the Classical Ones

In fact, the fundamental difference for the distance between fuzzy numbers presented in this paper and other distances lies in the different interval distances between their corresponding α-cuts. Thence, the comparative analysis of the distance between intervals is actually a comparative analysis of the distance between fuzzy numbers.

Let $[a_l, a_r]$ and $[b_l, b_r]$ be two numerical continuous intervals satisfying $a_l \le a_r$ and $b_l \le b_r$, there are three classical distances between intervals given as follows:

(1) Hamming distance [24]:

$$d_{Hm}([a_l, a_r], [b_l, b_r]) = \frac{|a_l - b_l| + |a_r - b_r|}{2} \tag{20}$$

(2) Euclidean distance [6]:

$$d_E([a_l, a_r], [b_l, b_r]) = \sqrt{|a_l - b_l|^2 + |a_r - b_r|^2} \tag{21}$$

(3) L_1 Hausdorff metric distance [3, 10]:

$$d_{\mathbf{H}}^{L_1}([a_l, a_r], [b_l, b_r]) = \max\{|a_l - b_l|, |a_r - b_r|\} \tag{22}$$

Note 5 It is worth pointing out that the proposed distance d_I uses all points in the interval to calculate the distance, while the classical distances are calculated specifically from the endpoints of the interval. Thus, for two intervals $[a_l, a_r]$ and $[b_l, b_r]$, several special interesting conclusions can be derived:

(1) If $a_l = b_l$ and $a_r \neq b_r$, the classical distances are only determined by the right endpoints of intervals as: $d_E = d_H^{L_1} = 2d_{Hm} = |a_r - b_r|$.
(2) If $a_l \neq b_l$ and $a_r = b_r$, these classical distances are determined by the left endpoints of intervals, which leads to the results: $d_E = d_H^{L_1} = 2d_{Hm} = |a_l - b_l|$.
(3) If $\frac{a_l + a_r}{2} = \frac{b_l + b_r}{2}$, then $|a_l - b_l| = |b_r - a_r|$ and these classical distances are related as follows: $d_{Hm} = d_H^{L_1} = \frac{\sqrt{2}}{2} d_E = |a_l - b_l| = |a_r - b_r|$.

Example 4 Four special cases are selected, where the two comparison intervals are both subsets of $[0, 1]$. The results by using different distances are shown in Table 3.

The bold is the result of the proposed distance d_I, which uses all points in the intervals to compute the distance between intervals, whereas the remaining ones are computed on the endpoints of intervals.

It is known that the expected value and the standard variance value are two important statistical concepts, so for the four special cases in Table 3, the following Figs. 6, 7, 8, and 9 will be used to further illustrate that the proposed method has certain advantages. These figures consider the following four cases:

(i) If the compared intervals are randomly generated with the same left endpoint $a_l = b_l = 0.6787$, then we obtain Fig. 6.
(ii) If the compared intervals are randomly generated with the same right endpoint $a_r = b_r = 0.6787$, then the expected and standard variance values are obtained Fig. 7.
(iii) If the compared intervals are randomly generated with the same midpoint $\frac{a_l + a_r}{2} = \frac{b_l + b_r}{2} = 0.5060$, then it is obtained Fig. 8.

Table 3 The comparative results of the distances between intervals

$[a_l, a_r]$	$[0.6787, 0.7577]$	$[0.1712, 0.6555]$	$[0.3171, 0.6948]$	$[0.0462, 0.2769]$
$[b_l, b_r]$	$[0.6787, 0.7431]$	$[0.0318, 0.6555]$	$[0.0617, 0.9502]$	$[0.0971, 0.8235]$
$\mathbf{d_I}$	**0.0248**	**0.1951**	**0.2355**	**0.3103**
d_{Hm}	0.0073	0.0697	0.2554	0.2987
d_E	0.0146	0.1394	0.3612	0.5490
$d_H^{L_1}$	0.0146	0.1394	0.2554	0.5465

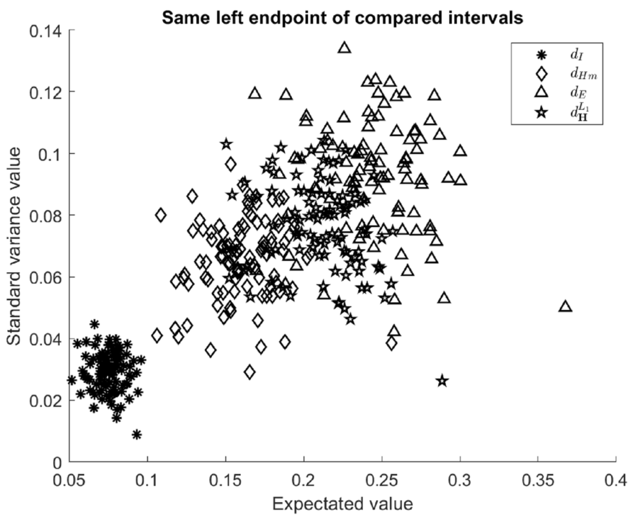

Fig. 6 The expected and standard variance values of different distances for 100 randomly pairs of intervals on $[0, 1]$

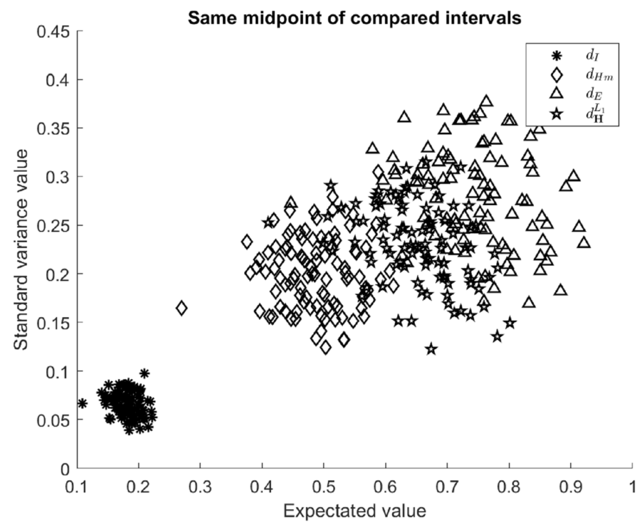

Fig. 8 The expected and standard variance values of different distances for 100 randomly pairs of intervals on $[0, 1]$

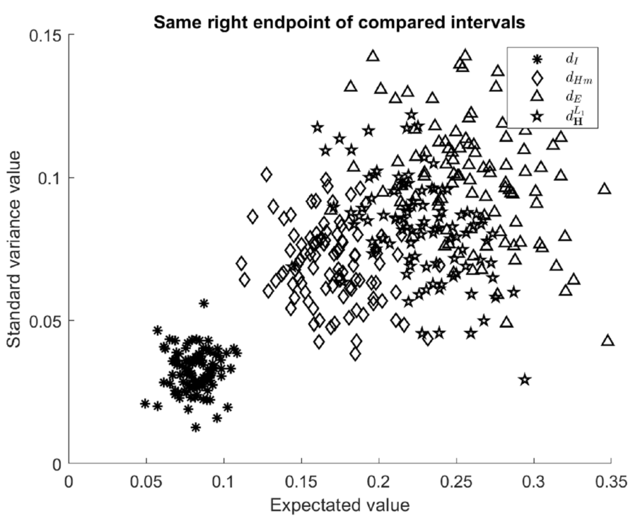

Fig. 7 The expected and standard variance values of different distances for 100 randomly pairs of intervals on $[0, 1]$

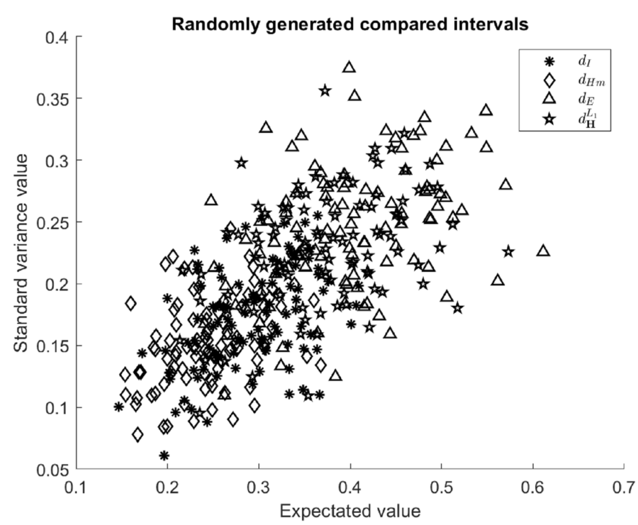

Fig. 9 The expected and standard variance values of different distances for 100 randomly pairs of intervals on $[0, 1]$

(iv) If the compared intervals are randomly generated, then we obtain Fig. 9.

As seen in Figs. 6, 7, and 8, if hundreds of compared intervals have the same endpoint or midpoint, then the proposed interval distances will have smaller expected and standard variance values, indicating that the rest of the classical distances are highly susceptible to the endpoint or midpoint values, which is extremely detrimental.

Table 4 Time complexity comparison between proposed and classical interval distance

	d_I	d_{Hm} [24]	d_E [6]	$d_{\mathbf{H}}^{L_1}$ [3, 10]
Time complexity	$O(n^2)$	$O(4)$	$O(4^2)$	$O(4)$

Furthermore, in Fig. 9, even if these distances use randomly generated intervals, we can see that the points of d_I are randomly scattered in the rest of the point range. This further suggests that the novel distances are valid.

Our model is more time consuming, but much more reliable because it takes all the information of the interval into account (see Table 4).

The time complex order of computing fuzzy distance is then added as follows:

$$O(4n) < O(16n) < O\left(n^3\right). \tag{23}$$

Additionally, the new interval distance can be directly applied to the real example in [12], which uses the hesitant interval-valued fuzzy sets (HIVFSs) [7] to express expert opinions, and will, therefore, be further investigated. In this example, an investment company wants to invest a sum of money in one of the five alternative companies with the lowest risk, the best return on investment, and market prospects. The five possible alternative companies are $\mathcal{A}_1$, an automobile company; $\mathcal{A}_2$, a food company; $\mathcal{A}_3$, a computer company; $\mathcal{A}_4$, a weapons company; and $\mathcal{A}_5$, a television company. Three experts are invited to evaluate these companies in terms of risk, return on investment, and market prospects, with a value of 1 indicating the greatest return on investment and market prospects and the lowest risk, and 0 indicating the opposite.

Example 5 We take the numerical example introduced in [12], in which the distance is given as follows:

$$D^*\left(\tilde{A_i}, \tilde{A_j}\right) = \frac{1}{m}\sum_{k=1}^{m}\left|\frac{1}{\#\tilde{H}_{\tilde{\mathcal{A}}_j(e_k)} \times \#\tilde{H}_{\tilde{\mathcal{A}}_i(e_k)}} \times \left[\sum_{t_j=1}^{\#\tilde{H}_{\tilde{\mathcal{A}}_j(e_k)}} \sum_{t_i=1}^{\#\tilde{H}_{\tilde{\mathcal{A}}_i(e_k)}} \delta^*\left(\tilde{H}_{\tilde{\mathcal{A}}_i(e_k,t_i)}, \tilde{H}_{\tilde{\mathcal{A}}_j\left(e_k,t_j\right)}\right)\right]\right|,$$

where $\#\tilde{H}_{\tilde{\mathcal{A}}_i(e_k)}$ and $\#\tilde{H}_{\tilde{\mathcal{A}}_j(e_k)}$ are the cardinality of the sets $\tilde{H}_{\tilde{\mathcal{A}}_i(e_k)}$ and $\tilde{H}_{\tilde{\mathcal{A}}_j(e_k)}$, respectively. And $\tilde{H}_{\tilde{\mathcal{A}}_i(e_k,t_i)} \in \tilde{H}_{\tilde{\mathcal{A}}_i(e_k)}$ and $\tilde{H}_{\tilde{\mathcal{A}}_j\left(e_k,t_j\right)} \in \tilde{H}_{\tilde{\mathcal{A}}_j(e_k)}$ represent two sub-intervals of $[0,1]$, respectively. $\delta^*(\cdot)$ computes the deviation between two intervals.

In fact, the key part of $D^*(\cdot)$ is defined by the deviation $\delta^*(\cdot)$ between intervals $\tilde{a}$ and $\tilde{b}$ with an additional potential constraint $(a_r - a_l)(b_r - b_l) \neq 0$, the computation formula is defined as follows:

$$\delta^*\left(\tilde{a}, \tilde{b}\right) = \frac{\int_{a_l}^{a_r}\int_{b_l}^{b_r}(x-y)\,\mathrm{d}x\mathrm{d}y}{(a_r - a_l)(b_r - b_l)} = \begin{cases} \dfrac{(a_l + a_r)}{2} - \dfrac{(b_l + b_r)}{2} & \textit{if } (a_r - a_l)(b_r - b_l) \neq 0; \\ \textit{invalid} & \textit{otherwise}. \end{cases} \tag{24}$$

Actually, only if $(a_r - a_l)(b_r - b_l) \neq 0$, the deviation δ^* makes sense, and it is defined by the deviation of the midpoints of the corresponding intervals. And if at least one of $\tilde{a}$ and $\tilde{b}$ is a constant, i.e., $(a_r - a_l)(b_r - b_l) = 0$, the formula is invalid. In addition, if $\frac{(a_l+a_r)}{2} = \frac{(b_l+b_r)}{2}$, i.e., $\tilde{a}$ and $\tilde{b}$ are symmetric but satisfy $\tilde{a}$ is not equal to $\tilde{b}$, i.e., $\tilde{a} \neq \tilde{b}$, the deviation δ^* does not identify their difference, that is, there exists a distance between them. Thus, the application of the deviation δ^* is limited. It is worth pointing out that this formulation is quite similar to Eq. (11), which is a special case of our proposed distance.

Therefore, to apply Eq. (6), then for $j \in \{1,2,3,4,5\}$, the new distance between HIVFSs $\tilde{A^*}$ and $\tilde{A_j}$ is given

$$D\left(\tilde{A^*}, \tilde{A_j}\right) = \frac{1}{m}\sum_{k=1}^{m}\left\{\frac{1}{\#\tilde{H}_{\tilde{\mathcal{A}}_j(e_k)} \times \#\tilde{H}_{\tilde{\mathcal{A}}^*(e_k)}} \times \left[\sum_{t_j=1}^{\#\tilde{H}_{\tilde{\mathcal{A}}_j(e_k)}} \sum_{t^*=1}^{\#\tilde{H}_{\tilde{\mathcal{A}}^*(e_k)}} d_I\left(\tilde{H}_{\tilde{\mathcal{A}}^*(e_k,t^*)}, \tilde{H}_{\tilde{\mathcal{A}}_j\left(e_k,t_j\right)}\right)\right]\right\}. \tag{25}$$

Finally, we will obtain the distance with $m = 3$ and ranking results are shown in Table 5.

From the table, we can see that both have the same top alternative $\mathcal{A}_4$, the medium one $\mathcal{A}_5$, and the bottom one $\mathcal{A}_3$, and the ranking is totally different between $\mathcal{A}_1$ and $\mathcal{A}_2$, which indicates that our proposed method is effective. It should be noted that the proposed approach in [12] uses the difference between the midpoints of the interval to identify the deviation of the intervals; especially when multiple intervals are involved in the computation process, there will be positive and negative offsets or even zero, which results in different results. Whereas our proposed distance between HIVFSs, $D\left(\tilde{A^*}, \tilde{A_j}\right)$, uses all points in the intervals to compute the distance between the corresponding intervals, which uses more information and thus the results will be more reliable than the results obtained with the distance introduced in [12].

Table 5 Distance and ranking results

	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	*Rankings*
Distance method in [12]	0.3528	0.3306	0.2528	0.4361	0.3528	$\mathcal{A}_4 \succ \mathcal{A}_1 \sim \mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_3$
Proposed $\mathbf{D}(\mathbf{A}^*, \mathbf{A_j})$	**0.3097**	**0.3867**	**0.2583**	**0.4336**	**0.3505**	$\mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_5 \succ \mathcal{A}_1 \succ \mathcal{A}_3$

4.2 Comparison and Analysis of New Ranking Index for Fuzzy Numbers

This section focuses on the comparison and analysis of our proposed new ranking index of fuzzy numbers.

Example 6 Considering the examples shown in [22] with the following sets of TrFN/TFN:

Set 1: $\tilde{T_1} = \tilde{T_1}(0.4, 0.5, 1)$, $\tilde{T_2} = \tilde{T_2}(0.4, 0.7, 1)$, $\tilde{T_3} = \tilde{T_3}(0.4, 0.9, 1)$

Set 2: $\tilde{T_1} = \tilde{T_1}(0.3, 0.4, 0.7, 0.9)$, $\tilde{T_2} = \tilde{T_2}(0.3, 0.7, 0.9)$, $\tilde{T_3} = \tilde{T_3}(0.5, 0.7, 0.9)$

Set 3: $\tilde{T_1} = \tilde{T_1}(0.3, 0.5, 0.7)$, $\tilde{T_2} = \tilde{T_2}(0.3, 0.5, 0.8, 0.9)$, $\tilde{T_3} = \tilde{T_3}(0.3, 0.5, 0.9)$

Set 4: $\tilde{T_1} = \tilde{T_1}(0, 0.4, 0.7, 0.8)$, $\tilde{T_2} = \tilde{T_2}(0.2, 0.5, 0.9)$, $\tilde{T_3} = \tilde{T_3}(0.1, 0.6, 0.8)$

Set 5: $\tilde{T_1} = \tilde{T_1}(0.2, 0.4, 0.6, 0.8)$, $\tilde{T_2} = \tilde{T_2}(0.3, 0.5, 0.7)$, $\tilde{T_3} = \tilde{T_3}(0.35, 0.5, 0.65)$

Set 6: $\tilde{T_1} = \tilde{T_1}(13, 14, 17)$, $\tilde{T_2} = \tilde{T_2}(12, 14, 17)$, $\tilde{T_3} = \tilde{T_3}(10, 15, 16.2)$

In Figs. 10, 11, 12, 13, 14, and 15, $\tilde{T_1}$, $\tilde{T_2}$ and $\tilde{T_3}$ in each group are displayed in the same way. We shall obtain the new bold results shown in Table 6 by applying Eqs. (17), (18) and (19) with $s(\alpha) = \alpha^1$.

For the sets 1–4 and set 6, the new proposal achieves efficient results ranking these TrFNs, which is almost the same as the other ranking methods. In addition, it should be noted that Eqs. (18) and (19) rank the set 6 in the same way as the other methods, while Eq. (17) ranks them slightly different depending on the choice of κ (see Figs. 16, 17 and 18). In these figures, the y-axis represents the ranking index value; the x-axis indicates the value of $\kappa \in [0.1, 2]$. And the vertical line is $\kappa = 1$, and its crossing points with the curves are the ranking of TrFNs. In addition, the different choices to κ affect the ranking result, as shown in Figs. 16, 17 and 18.

To better illustrate the case $\kappa = 1$, i.e., $f(\alpha, \kappa) = \frac{\alpha}{\int_0^1 \alpha\, d\alpha} = 2\alpha$, we compute the values of $\frac{\alpha \times d_I(\tilde{T_1^{\alpha}}, \tilde{F_*^{\alpha}})}{\int_0^1 \alpha\, d\alpha} = 2\alpha d_I (\tilde{T_1^{\alpha}}, \tilde{F_*^{\alpha}})$ and $-\frac{\alpha \times d_I(\tilde{T_1^{\alpha}}, \tilde{F^{*,\alpha}})}{\int_0^1 \alpha\, d\alpha} = -2\alpha d_I(\tilde{T_1^{\alpha}}, \tilde{F^{*,\alpha}})$ for $i = 1, 2, 3$, which are parts of Eqs. (17) and (18), respectively; and shown in Figs. 19 and 20.

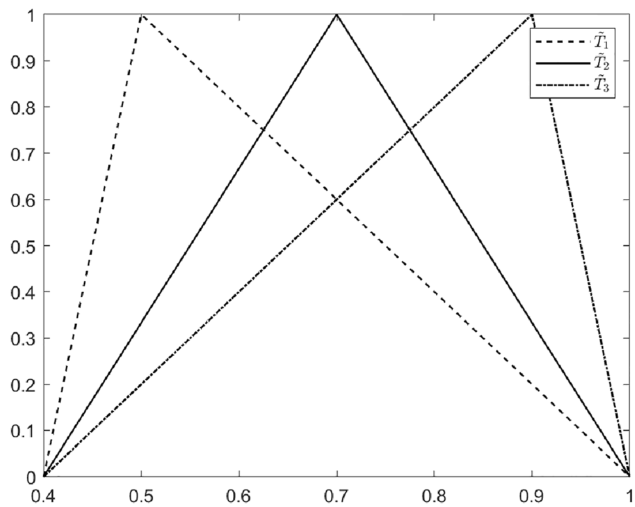

Fig. 10 TrFNs/TFNs in Set 1

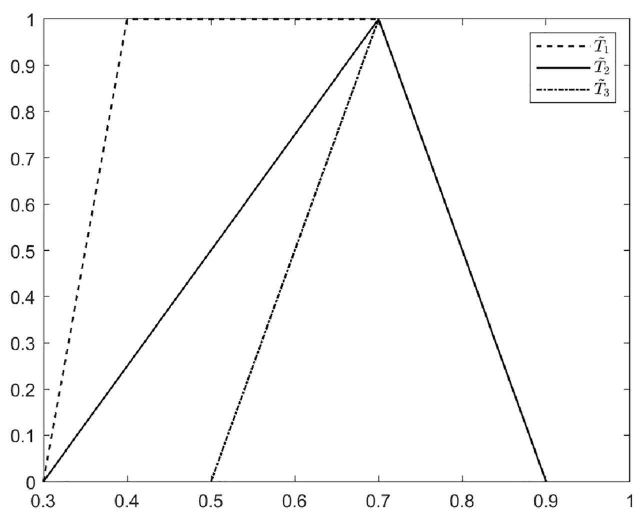

Fig. 11 TrFNs/TFNs in Set 2

As we can see, Figs. 19 and 20 indicates that parameter α has significant effects on the values of $2\alpha d_I(\tilde{T_1^{\alpha}}, \tilde{F_*^{\alpha}})$ and $-2\alpha d_I(\tilde{T_1^{\alpha}}, \tilde{F^{*,\alpha}})$ for $i = 1, 2, 3$. Furthermore, the final ranking result can be easily obtained as $\tilde{T_3} \succ \tilde{T_2} \succ \tilde{T_1}$ by using Eq. (17) and $\tilde{T_3} \succ \tilde{T_1} \succ \tilde{T_2}$ by using Eq. (18), respectively.

However, for the set 5, these fuzzy numbers are symmetric at the vertical line 0.5, no ranking is given for Eq. (19) and other methods, because they treat fuzzy numbers $\tilde{T_1}$, $\tilde{T_2}$, and $\tilde{T_3}$ as equivalent. Whereas, for

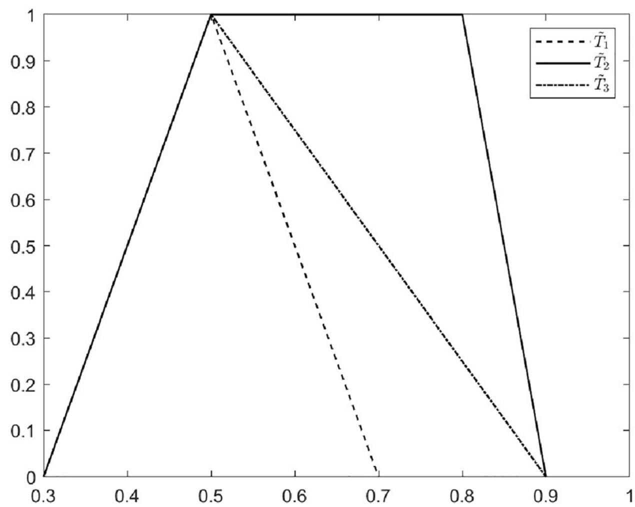

Fig. 12 TrFNs/TFNs in Set 3

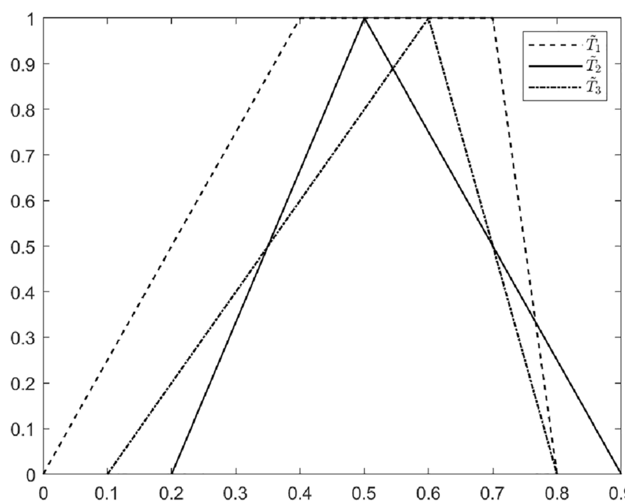

Fig. 13 TrFNs/TFNs in Set 4

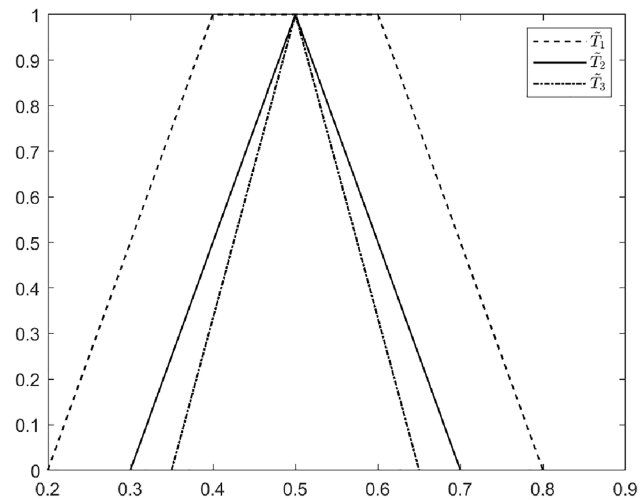

Fig. 14 TrFNs/TFNs in Set 5

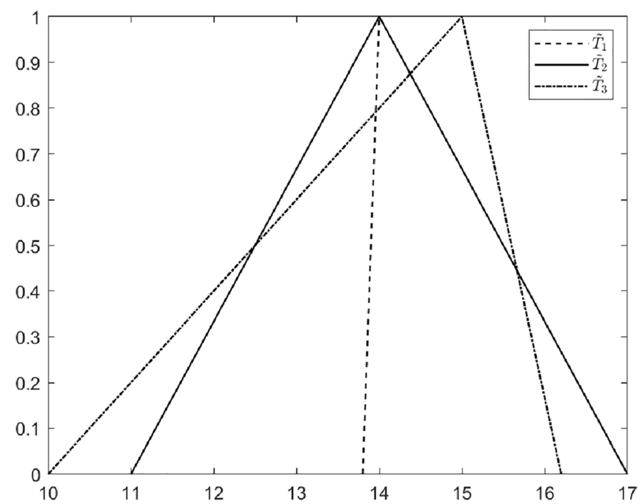

Fig. 15 TrFNs/TFNs in Set 6

Eqs. (17) and (18), the ranking are $\tilde{T}_1 \succ \tilde{T}_2 \succ \tilde{T}_3$ and $\tilde{T}_3 \succ \tilde{T}_2 \succ \tilde{T}_1$, respectively. The reason is that the new proposal makes $d_{f,\alpha^1}(\tilde{T}_1, \tilde{T}_*) = d_{f,\alpha^1}(\tilde{T}_1, \tilde{T}^*) = 0.1053$, $d_{f,\alpha^1}(\tilde{T}_2, \tilde{T}_*) = d_{f,\alpha^1}(\tilde{T}_2, \tilde{T}^*) = 0.0785$, and $d_{f,\alpha^1}(\tilde{T}_3, \tilde{T}_*) = d_{f,\alpha^1}(\tilde{T}_3, \tilde{T}^*) = 0.0752$, while the methods presented in [1, 2, 22] are mainly affected by the medium value of the α-cut of the fuzzy number.

5 Conclusions

This paper focuses on the ranking of fuzzy numbers determined by the distance between them, and the key idea is to define the distance between their α-cuts. Considering that the α-cut of a fuzzy number is a continuous interval, this paper has proposed a novel distance between continuous intervals using the concept of integral. The advantage of this approach is that it uses all points in the interval to compute the distance, and its results should be more reliable and correct than the existing classical methods, which usually use the endpoints or midpoints of the interval to compute it, which may lose some information and not reflect the distance correctly. Therefore, using the reduction function with α as the variable, a series of distances between fuzzy numbers have been proposed based on the proposed interval distances, and it is shown that these distances satisfy the distance axiom for fuzzy numbers. The validity and effectiveness of the proposed interval distances are demonstrated by comparative analysis of numerical examples. Based on these findings, we have introduced a new ranking index for fuzzy numbers and proved its validity by using the set of TrFNs as numerical examples. Furthermore, regarding the

Table 6 The comparative of the ranking results

Sets	TrFNs	Eq. (17)	Eq. (18)	Eq. (19)	Abbasbandy and Hajjari [2]	Sign distance [22]	Sign distance method $p = 1$ [1]
Set 1	$\tilde{T}_1$	**0.0660**	$-$**0.2798**	**0.1908**	0.5335	0.6	1.2
	$\tilde{T}_2$	**0.1563**	$-$**0.1563**	**0.5000**	0.7	0.7	1.4
	$\tilde{T}_3$	**0.2798**	$-$**0.0660**	**0.8092**	0.8666	0.8	1.6
		$\tilde{T}_3 \succ \tilde{T}_2 \succ \tilde{T}_1$	$\tilde{T}_3 \succ \tilde{T}_2 \succ \tilde{T}_1$	$\tilde{T}_3 \succ \tilde{T}_2 \succ \tilde{T}_1$	$\tilde{T}_3 \succ \tilde{T}_2 \succ \tilde{T}_1$	$\tilde{T}_3 \succ \tilde{T}_2 \succ \tilde{T}_1$	$\tilde{T}_3 \succ \tilde{T}_2 \succ \tilde{T}_1$
Set 2	$\tilde{T}_1$	**0.1330**	$-$**0.1521**	**0.4664**	0.5584	0.575	1.15
	$\tilde{T}_2$	**0.1424**	$-$**0.0623**	**0.6956**	0.6334	0.65	1.3
	$\tilde{T}_3$	**0.1521**	$-$**0.0440**	**0.7757**	0.7	0.7	1.4
		$\tilde{T}_3 \succ \tilde{T}_2 \succ \tilde{T}_1$	$\tilde{T}_3 \succ \tilde{T}_2 \succ \tilde{T}_1$	$\tilde{T}_3 \succ \tilde{T}_2 \succ \tilde{T}_1$	$\tilde{T}_3 \succ \tilde{T}_2 \succ \tilde{T}_1$	$\tilde{T}_3 \succ \tilde{T}_2 \succ \tilde{T}_1$	$\tilde{T}_3 \succ \tilde{T}_2 \succ \tilde{T}_1$
Set 3	$\tilde{T}_1$	**0.0440**	$-$**0.1521**	**0.2243**	0.5	0.5	1
	$\tilde{T}_2$	**0.1521**	$-$**0.1330**	**0.5336**	0.6416	0.625	1.25
	$\tilde{T}_3$	**0.0623**	$-$**0.1424**	**0.3044**	0.5166	0.55	1.1
		$\tilde{T}_2 \succ \tilde{T}_3 \succ \tilde{T}_1$	$\tilde{T}_2 \succ \tilde{T}_3 \succ \tilde{T}_1$	$\tilde{T}_2 \succ \tilde{T}_3 \succ \tilde{T}_1$	$\tilde{T}_2 \succ \tilde{T}_3 \succ \tilde{T}_1$	$\tilde{T}_2 \succ \tilde{T}_3 \succ \tilde{T}_1$	$\tilde{T}_2 \succ \tilde{T}_3 \succ \tilde{T}_1$
Set 4	$\tilde{T}_1$	**0.1507**	$-$**0.1511**	**0.4994**	0.525	0.475	0.95
	$\tilde{T}_2$	**0.1176**	$-$**0.1277**	**0.4795**	0.5084	0.525	1.05
	$\tilde{T}_3$	**0.1484**	$-$**0.0974**	**0.6038**	0.575	0.525	1.05
		$\tilde{T}_3 \succ \tilde{T}_2 \succ \tilde{T}_1$	$\tilde{T}_3 \succ \tilde{T}_2 \succ \tilde{T}_1$	$\tilde{T}_3 \succ \tilde{T}_2 \succ \tilde{T}_1$	$\tilde{T}_3 \succ \tilde{T}_1 \succ \tilde{T}_2$	$\tilde{T}_3 \succ \tilde{T}_1 \sim \tilde{T}_2$	$\tilde{T}_3 \succ \tilde{T}_1 \sim \tilde{T}_2$
Set 5	$\tilde{T}_1$	**0.1053**	$-$**0.1053**	**0.5000**	0.5	0.5	1
	$\tilde{T}_2$	**0.0785**	$-$**0.0785**	**0.5000**	0.5	0.5	1
	$\tilde{T}_3$	**0.0752**	$-$**0.0752**	**0.5000**	0.5	0.5	1
		$\tilde{T}_1 \succ \tilde{T}_2 \succ \tilde{T}_3$	$\tilde{T}_3 \succ \tilde{T}_2 \succ \tilde{T}_1$	$\tilde{T}_1 \sim \tilde{T}_2 \sim \tilde{T}_3$	$\tilde{T}_1 \sim \tilde{T}_2 \sim \tilde{T}_3$	$\tilde{T}_1 \sim \tilde{T}_2 \sim \tilde{T}_3$	$\tilde{T}_1 \sim \tilde{T}_2 \sim \tilde{T}_3$
Set 6	$\tilde{T}_1$	**0.6566**	$-$**0.7041**	**0.4826**	14.1667	0.25	0.5
	$\tilde{T}_2$	**0.6711**	$-$**0.9152**	**0.4230**	14.0000	0.125	0.25
	$\tilde{T}_3$	**0.9421**	$-$**0.6744**	**0.5828**	14.6833	0.9	1.8
		$\tilde{T}_3 \succ \tilde{T}_2 \succ \tilde{T}_1$	$\tilde{T}_3 \succ \tilde{T}_1 \succ \tilde{T}_2$	$\tilde{T}_3 \succ \tilde{T}_1 \succ \tilde{T}_2$	$\tilde{T}_3 \succ \tilde{T}_1 \succ \tilde{T}_2$	$\tilde{T}_3 \succ \tilde{T}_1 \succ \tilde{T}_2$	$\tilde{T}_3 \succ \tilde{T}_1 \succ \tilde{T}_2$

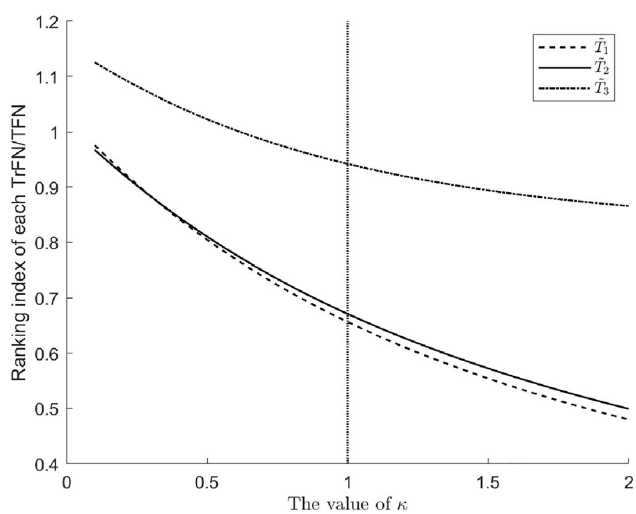

Fig. 16 Ranking index using Eq. (17)

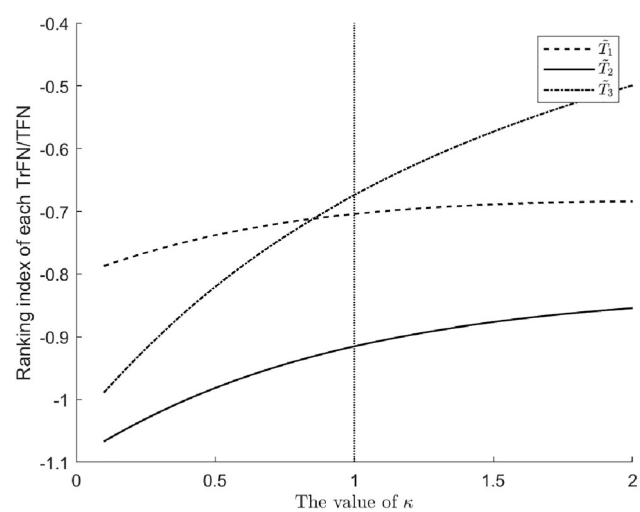

Fig. 17 Ranking index using Eq. (18)

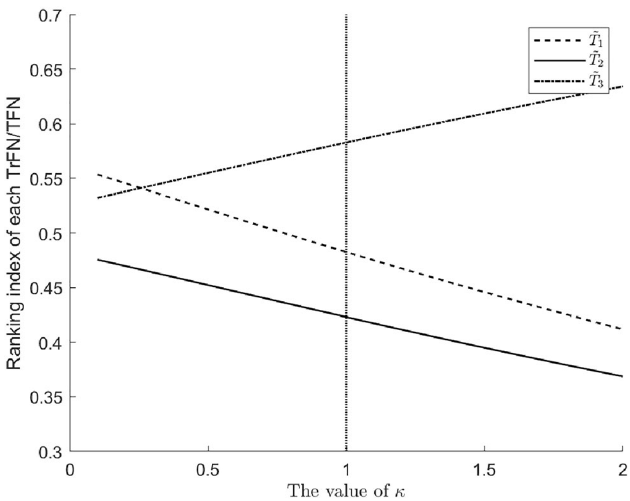

Fig. 18 Ranking index using Eq. (19)

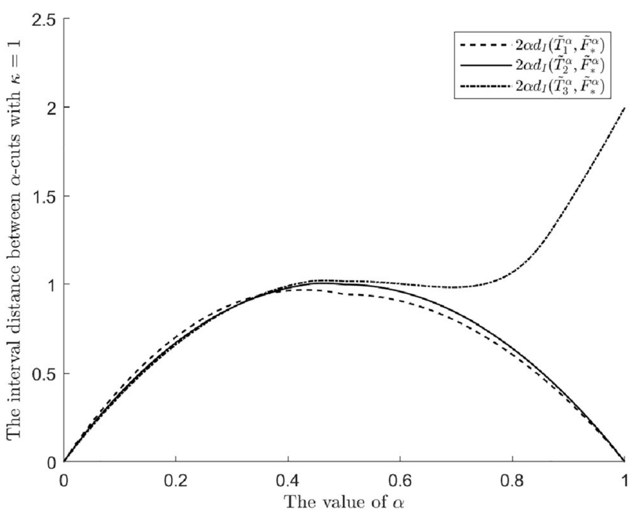

Fig. 19 The values of $2\alpha d_I(\tilde{T}_1^\alpha, \tilde{F}_*^\alpha)$ for $i = 1, 2, 3$

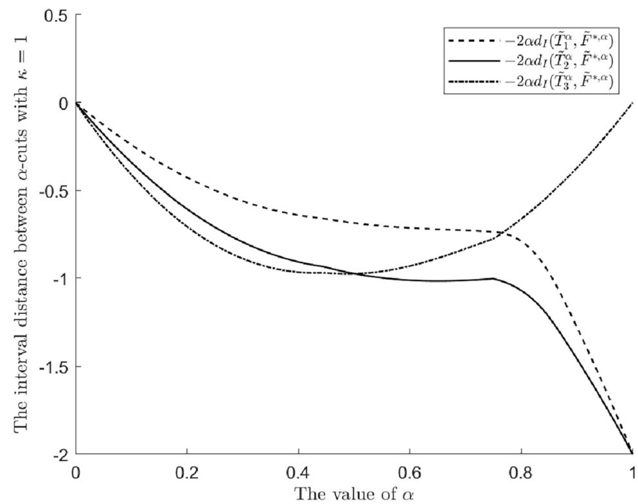

Fig. 20 The values of $-2\alpha d_I(\tilde{T}_1^\alpha, \tilde{F}^{*,\alpha})$ for $i = 1, 2, 3$

implications and insights of the research, the results of this study show that the choice of a parameter κ in the reduction function has a significant impact on the final fuzzy distance and the ranking index of the fuzzy number. This study helps us to understand the design of more appropriate fuzzy distances.

It should also be pointed out that the proposed interval distance has the limitation that when two compared intervals are the same, the integral part cannot be equal to 0, but to one third of the length of the interval. For future research, interval distances can be defined by eliminating the intersection of two comparison intervals, even extending to the p-norm form to define new distances, or using some other aggregation functions to generalize them, rather than just using the integral concept. Moreover, the proposed distances can be interestingly extended to linguistics since linguistic information can be transformed into fuzzy numbers by fuzzy linguistic methods to increase the flexibility of modeling linguistic information.

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Declarations

Conflict of interest The authors declare that there are no competing financial interests or personal relationships that might influence the work reported herein.

Ethical Approval This article does not contain any studies conducted by any of the authors on human participants or animals.

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